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CONSTRUCTION OF THE MULTI-SOLITONS FOR A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

PHAN VAN TIN

Abstract. We consider a derivative nonlinear Schrödinger equation with general nonlinearlity:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0,$$

In [17], the authors prove the stability of two solitary waves in energy space for $\sigma \in (1,2)$. As a consequence, there exists a solution of the above equation which is close arbitrary to sum of two solitons in energy space when $\sigma \in (1,2)$. Our goal in this paper is proving the existence of multi-solitons in energy space for $\sigma \geqslant \frac{5}{2}$. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

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1. Introduction

In this paper, we consider the following generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \tag{1.1}$$

where $\sigma \in \mathbb{R}^+$ is a given constant and $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$.

The equation (1.1) was studied in many works. In the special case $\sigma = 1$, local well-posedness, global well posedness, stability of solitary waves and stability of multi-solitons have been investigated. In [15], Ozawa gave a sufficient condition for global well posedness of (1.1) in the energy space by using a Gauge transformation to remove the derivative terms. In [2], Colin-Ohta showed that the equation has a two parameters family of solitary waves and proved the stability of these particular solutions by using variational methods. In [8], Kwon-Wu gave a result on stability of solitary waves when the parameters are at the threshold between existence and non-existence. In [11], Le Coz-Wu proved stability of multi-solitons in the energy space under some conditions on the parameters of the composing solitons.

In the general case, the local well-posedness and global well- posedness of (1.1) was studied in [6] when the initial data is in the Sobolev space $H_0^1(\Omega)$, where Ω is any unbounded interval of $\mathbb{R}$. In this work, Hayashi-Ozawa used an approximation argument. In [16], Santos proved the local well-posedness for small size initial data in weighted Sobolev spaces. The arguments used in this work follow parabolic regularization approach introduced by Kato [7].

The equation (1.1) has a two parameters family of solitons. The stability of the solitons has attracted the attention of many researchers. In [12], by using the abstract theory of Grillakis-Shatah-Strauss [3, 4], Liu-Simpson-Sulem proved that in the case $\sigma \geqslant 2$, the solitons of (1.1) are

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orbitally unstable; in the case $0 < \sigma < 1$, they are orbitally stable and in the case $\sigma \in (1,2)$ they are orbitally stable if $c < 2z_0\sqrt{\omega}$ and orbitally unstable if $c > 2z_0\sqrt{\omega}$ for some constant $z_0 \in (0,1)$. In the critical case $c = 2z_0\sqrt{\omega}$, Guo-Ning-Wu [5] proved that solitons are always orbitally unstable. In [1], Bai-Wu-Xue proved that when $\sigma \geqslant 2$, the solution is global and scattering when the initial data small in $H^s(\mathbb{R})$, $\frac{1}{2} \leqslant s \leqslant 1$. Moreover, the authors showed that when $\sigma < 2$, the scattering may not occur even under smallness conditions on the initial data. Therefore, in this model, the exponent $\sigma \geqslant 2$ is optimal for small data scattering. In [17], in the case $\sigma \in (1,2)$, Tang and Xu proved the stability of the sum of two solitary waves in the energy space using perturbation arguments, modulational analysis and an energy argument as in [13, 14]. In this chapter, we show the existence of multi-solitons in energy space in the case $\sigma \geqslant \frac{5}{2}$. Before stating the main result, we give some preliminaries on multi-solitons of (1.1).

As mentioned in [12], the equation (1.1) admits a two-parameters family of solitary waves solutions given by

$$\psi_{\omega,c}(t,x) = \varphi_{\omega,c}(x-ct)\exp\left(i\left(\omega t + \frac{c}{2}(x-ct) - \frac{1}{2\sigma+2}\int_{-\infty}^{x-ct}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta\right)\right), \tag{1.2}$$

where $\omega > \frac{c^2}{4}$ and

$$\varphi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma+1)(4\omega-c^2)}{2\sqrt{\omega}\left(\cosh(\sigma\sqrt{4\omega-c^2}y) - \frac{c}{2\sqrt{\omega}}\right)}. \tag{1.3}$$

The profile $\varphi_{\omega,c}$ is a positive solution of

$$-\partial_y^2\varphi_{\omega,c} + \left(\omega - \frac{c^2}{4}\right)\varphi_{\omega,c} + \frac{c}{2}|\varphi_{\omega,c}|^{2\sigma}\varphi_{\omega,c} - \frac{2\sigma+1}{(2\sigma+2)^2}|\varphi_{\omega,c}|^{4\sigma}\varphi_{\omega,c} = 0. \tag{1.4}$$

Define

$$\phi_{\omega,c}(y) = \varphi_{\omega,c}(y)e^{i\theta_{\omega,c}(y)}, \tag{1.5}$$

where

$$\theta_{\omega,c}(y) = \frac{c}{2}y - \frac{1}{2\sigma+2}\int_{-\infty}^{y}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta. \tag{1.6}$$

Clearly, we have

$$\psi_{\omega,c}(x,t) = e^{i\omega t}\phi_{\omega,c}(x-ct). \tag{1.7}$$

and $\phi_{\omega,c}$ solves

$$-\partial_y^2\phi_{\omega,c} + \omega\phi_{\omega,c} + ic\partial_y\phi_{\omega,c} - i|\phi_{\omega,c}|^{2\sigma}\partial_y\phi_{\omega,c} = 0, \quad y \in \mathbb{R}. \tag{1.8}$$

Let $K \in \mathbb{N}$. For each $1 \leqslant j \leqslant K$, let $(\omega_j, c_j, x_j, \theta_j) \in \mathbb{R}^4$ be parameters such that $\omega_j > \frac{c_j^2}{4}$. Define, for each $j = 1, ..., K$

$$R_j(t,x) = e^{i\theta_j}\psi_{\omega_j,c_j}(t, x-x_j)$$

and define the multi-soliton profile by

$$R = \sum_{j=1}^{K} R_j. \tag{1.9}$$

For convenience, define $h_j = \sqrt{4\omega_j - c_j^2}$, for each $j = 1, ..., K$. Our main result is the following.

Theorem 1.1. *Let $\sigma \geqslant \frac{5}{2}$, $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $(\theta_j, \omega_j, c_j, x_j)$ be a sequence of parameters such that $x_j \in \mathbb{R}$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k$, for $j \neq k$. The multi-soliton profile R is given as in (1.9). There exists a certain positive constant C_* such that if the parameters (ω_j, c_j) satisfy*

$$C_*\left((1+\|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1+\|R\|_{L^\infty H^1}^2)(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1})\right) \leqslant v_* = \inf_{j\neq k} h_j|c_j - c_k|, \tag{1.10}$$

then there exists a solution u of (1.1) such that

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}, \quad \forall t \geqslant T_0,$$

for positive constants C, T_0 depending only on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$ and $\lambda = \frac{1}{16}v_*$.

We have the following comment about the restriction $\sigma \geqslant \frac{5}{2}$.

Remark 1.2. The following inequality holds for $\sigma \geqslant 2$:

$$(a+b)^{2(\sigma-2)} - a^{2(\sigma-2)} \lesssim b^{2(\sigma-2)} + ba^{2(\sigma-2)-1}, \qquad \text{for all } a, b > 0. \tag{1.11}$$

The condition $\sigma \geqslant \frac{5}{2}$ ensures that the order of b on the right hand side of (1.11) is larger than 1. This is used in the proof of Lemma 3.3.

The condition (1.10) is an implicit condition on the parameters. Below, we show that for large, negative and enough separated velocities, the condition (1.10) holds.

Remark 1.3. We prove that there exist parameters $(\omega_j, c_j, \theta_j, x_j)$ for $1 \leqslant j \leqslant K$ such at the condition (1.10) is satisfied. Let $M > 0$, $h_j > 0$, $d_j < 0$, for each $1 \leqslant j \leqslant K$. We chose $(c_j, \omega_j) = \big(Md_j, \frac{1}{4}(h_j^2 + M^2 d_j^2)\big)$. We verify that this choice satisfies the condition (1.10) for M large enough. Indeed, we see that $c_j < 0$ and $h_j \ll |c_j|$ for M large enough. We have

$$\varphi_{\omega_j,c_j}^{2\sigma} \approx \frac{h_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)}$$

$$\partial_x \varphi_{\omega_j,c_j} \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.$$

Using $|\sinh(x)| \leqslant |\cosh(x)|$ for all $x \in \mathbb{R}$ we have

$$|\partial_x \varphi_{\omega_j,c_j}| \leqslant \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{1}{(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}})^{\frac{1}{2\sigma}}} \lesssim |\varphi_{\omega_j,c_j}|.$$

Thus,

$$\begin{aligned}
\|R_j\|_{L^\infty L^\infty} &= \|\varphi_{\omega_j,c_j}\|_{L^\infty} \lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \ll 1 \\
\|\partial_x R_j\|_{L^\infty L^\infty} &= \|\partial_x \phi_{\omega_j,c_j}\|_{L^\infty L^\infty} \\
&\lesssim \|\partial_x \varphi_{\omega_j,c_j}\|_{L^\infty} + \left\| \frac{c_j}{2} \varphi_{\omega_j,c_j} - \frac{1}{2\sigma+2} \varphi_{\omega_j,c_j}^{2\sigma+1} \right\|_{L^\infty} \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^\infty} + |c_j| \|\varphi_{\omega_j,c_j}\|_{L^\infty} \\
&\lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}.
\end{aligned}$$

Hence,

$$\|R\|_{L^\infty L^\infty} \lesssim \sum_j \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \lesssim 1$$

$$\|\partial_x R\|_{L^\infty L^\infty} \lesssim \sum_j \left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right).$$

Furthermore,

$$\begin{aligned}
\|R_j\|_{L^\infty H^1}^2 &= \|R_j\|_{L^\infty L^2}^2 + \|\partial_x R_j\|_{L^\infty L^2}^2 = \|\varphi_{\omega_j,c_j}\|_{L^2}^2 + \|\partial_x \varphi_{\omega_j,c_j}\|_{L^2}^2 \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \left\| \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \right\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \|e^{-\frac{h_j}{2}|y|}\|_{L^2}^2 \\
&\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \frac{1}{h_j} \lesssim h_j^{\frac{1}{\sigma}} h_j^{-1} = h_j^{\frac{1}{\sigma}-1},
\end{aligned}$$

where we use $h_j \leqslant 2\sqrt{\omega_j}$. Thus,

$$\|R\|_{L^\infty H^1}^2 \lesssim \sum_j h_j^{\frac{1}{\sigma}-1}.$$

The condition (1.10) satisfies if the following estimate holds:

$$C_*\left(\left(1+\sum_j h_j^{\frac{1}{\sigma}-1}\right)\left(1+\sum_j\left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}+|c_j|\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}\right)\right)\right)\leqslant \inf_{j\neq k} h_j|c_j-c_k|. \tag{1.12}$$

We see that the left hand side of (1.12) is order $M^{1-\frac{1}{2\sigma}}$ and the right hand side of (1.12) is order M^1. Hence, the condition (1.10) satisfies if we choose M large enough.

Our strategy of the proof of Theorem 1.1 is as follows. First, we define φ,ψ based on u in such a way that φ and ψ satisfy a system of nonlinear Schrödinger equations without derivatives (see (2.3)). Let R be a multi-soliton profile which satisfies the assumptions of Theorem 1.1. Then R solves (1.1) up to a small perturbation. Let (h,k) be defined in a similar way as (φ,ψ) but replace u by R. We see that (h,k) solves (2.3) up to small perturbations. Setting $\tilde\varphi=\varphi-h$ and $\tilde\psi=\psi-k$, we see that if u solves (1.1) then $(\tilde\varphi,\tilde\psi)$ solves a system and a relation between $\tilde\varphi$ and $\tilde\psi$ holds and vice versa. By using the Banach fixed point theorem, we prove that there exists a solution $(\tilde\varphi,\tilde\psi)$ of this system which exponential decays in time on $H^1(\mathbb{R})$ for t large. Combining with the assumption (1.10), we can prove a relation between $\tilde\varphi$ and $\tilde\psi$. Thus, we easily obtain the solution u of (1.1) satisfying the desired property.

This chapter is organized as follows. In Section 2, we prove the existence of multi-solitons for the equation (1.1). In Section 3, we prove some technical results which are used in the proof of the main result Theorem 1.1. More precisely, we prove the exponential decay of perturbations in the equations of h,k (Lemma 3.1) and the existence of decaying solutions for the system of equations of $\tilde\varphi,\tilde\psi$ (Lemma 3.3).

Before proving the main result, we introduce some notation used in this chapter.

Notation.

(1) *We denote the Schrödinger operator as follows*

$$L=i\partial_t+\partial_x^2.$$

(2) *Given a time $t\in\mathbb{R}$, the Strichartz space $S([t,\infty))$ is defined via the norm*

$$\|u\|_{S([t,\infty))}=\sup_{(q,r)\ admissible}\|u\|_{L_t^qL_x^r([t,\infty)\times\mathbb{R})}.$$

We denote the dual space by $N[t,\infty)=S([t,\infty))^*$. Hence for any (q,r) admissible pair we have

$$\|u\|_{N([t,\infty))}\leqslant\|u\|_{L_t^{q'}L_x^{r'}([t,\infty)\times\mathbb{R})}.$$

(3) *For $a,b\in\mathbb{R}^2$, we denote $|(a,b)|=|a|+|b|$.*

(4) *Let $a,b>0$. We denote $a\lesssim b$ if a is smaller than b up to multiplication by a positive constant and denote $a\lesssim_c b$ if a is smaller than b up to multiplication by a positive constant depending on c. Moreover, we denote $a\approx b$ if a equals to b up to multiplication by a positive constant.*

2. Proof of the main result

In this section we give the proof of Theorem 1.1. We use the Banach fixed point theorem and Strichartz estimates. We divide our proof in three steps.

Step 1. Preliminary analysis. Let $u\in C(I,H^1(\mathbb{R}))$ be a $H^1(\mathbb{R})$ solution of (1.1) on I. Consider the following transform:

$$\varphi(t,x)=\exp(i\Lambda)u(t,x), \tag{2.1}$$

$$\psi=\exp(i\Lambda)\partial_x u=\partial_x\varphi-\frac{i}{2}|\varphi|^{2\sigma}\varphi, \tag{2.2}$$

where

$$\Lambda=\frac{1}{2}\int_{-\infty}^x|u(t,y)|^{2\sigma}\,dy.$$

As in [6, section 4], using $|u|=|\varphi|$ and $\mathcal{I}m(\overline{u}\partial_x u)=\mathcal{I}m(\overline{\varphi}\psi)$, we have

$$\partial_t\Lambda=-\sigma\mathcal{I}m(|u|^{2(\sigma-1)}\overline{u}\partial_x u)+\sigma\mathcal{I}m\left[\int_{-\infty}^x\partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u\,dy\right]-\frac{1}{4}|u|^{4\sigma}.$$

Thus, using $|u| = |\varphi|$ and $\mathcal{I}m(\overline{u}\partial_x u) = \mathcal{I}m(\overline{\varphi}\psi)$, we have

$$\begin{aligned}\partial_t \Lambda &= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{I}m(\overline{\varphi}\psi) + \sigma\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)})\mathcal{I}m(\overline{u}\partial_x u)\,dx - \frac{1}{4}|\varphi|^{4\sigma}\\ &= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{I}m(\overline{\varphi}\psi) + \sigma\int_{-\infty}^{x} \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}m(\overline{\varphi}\psi)\,dx - \frac{1}{4}|\varphi|^{4\sigma}.\end{aligned}$$

Since u solves (1.1), we have

$$\begin{aligned}L\varphi &= L(\exp(i\Lambda))u + \exp(i\Lambda)Lu + 2\partial_x(\exp(i\Lambda))\partial_x u\\ &= L(\exp(i\Lambda))u + \exp(i\Lambda)(Lu + i|u|^{2\sigma}u)\\ &= L(\exp(i\Lambda))u\\ &= (i\partial_t + \partial_x^2)(\exp(i\Lambda))u,\\ &= \left[-\exp(i\Lambda)\partial_t\Lambda + \partial_x(\exp(i\Lambda)\frac{i}{2}|u|^{2\sigma})\right]u\\ &= -\varphi\partial_t\Lambda + \left[\exp(i\Lambda)\frac{-1}{4}|u|^{2\sigma} + \frac{i}{2}\exp(i\Lambda)\partial_x(|u|^{2\sigma})\right]u\\ &= -\varphi\partial_t\Lambda + \varphi\left[-\frac{1}{4}|\varphi|^{4\sigma} + \frac{i}{2}\partial_x(|\varphi|^{2\sigma})\right]\\ &= \sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{I}m(\overline{\varphi}\psi) - \sigma\varphi\int_{-\infty}^{x}\partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}m(\overline{\varphi}\psi)\,dx\\ &\quad + \frac{1}{4}|\varphi|^{4\sigma}\varphi - \frac{1}{4}\varphi|\varphi|^{4\sigma} + i\sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{R}e(\overline{\varphi}\partial_x\varphi)\\ &= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}m(\overline{\varphi}\psi) + i\mathcal{R}e(\overline{\varphi}\partial_x\varphi))\\ &\quad - \sigma\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}(\sigma-1)\partial_x(|\varphi|^2)\mathcal{I}m(\overline{\varphi}\psi)\,dx\\ &= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}m(\overline{\varphi}\psi) + i\mathcal{R}e(\overline{\varphi}\psi))\\ &\quad - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{R}e(\overline{\varphi}\psi)\mathcal{I}m(\overline{\varphi}\psi)\,dx\\ &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy.\end{aligned}$$

As in [6, section 4], we have

$$\begin{aligned}
L\psi &= L(\exp(i\Lambda)\partial_x u)\\
&= \exp(i\Lambda)\left[-\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u + \sigma|u|^{2(\sigma-1)}\mathcal{Im}(\overline{u}\partial_x u)\partial_x u\right.\\
&\quad \left.-\sigma\int_{-\infty}^{x}\mathcal{Im}(\partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u)\,dy\partial_x u\right]\\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\mathcal{Im}(\overline{\varphi}\psi)\psi - \sigma\int_{-\infty}^{x}\partial_x(|u|^{2(\sigma-1)})\mathcal{Im}(\overline{u}\partial_x u)\,dy\psi\\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\psi\mathcal{Im}(\overline{\varphi}\psi) - \sigma\psi\int_{-\infty}^{x}\partial_x(|\varphi|^{2(\sigma-1)})\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{Im}(\overline{\varphi}\psi) - i\mathcal{Re}(\overline{\varphi}\partial_x\varphi))\\
&\quad -\sigma\psi\int_{-\infty}^{x}(\sigma-1)|\varphi|^{2(\sigma-1)}2\mathcal{Re}(\overline{\varphi}\partial\varphi)\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{Im}(\overline{\varphi}\psi) - i\mathcal{Re}(\overline{\varphi}\psi))\\
&\quad -\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{Re}(\overline{\varphi}\psi)\mathcal{Im}(\overline{\varphi}\psi)\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy.
\end{aligned}$$

Thus, if u solves (1.1) then (φ,ψ) solves

$$\begin{cases} L\varphi &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy,\\ L\psi &= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy.\end{cases} \tag{2.3}$$

For convenience, we define

$$P(\varphi,\psi) = i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2), \tag{2.4}$$

$$Q(\varphi,\psi) = -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2). \tag{2.5}$$

Let R be the multi-soliton profile satisfying the assumption of Theorem 1.1. Define h,k by

$$\begin{aligned} h(t,x) &= \exp\left(\frac{i}{2}\int_{-\infty}^{x}|R(t,x)|^{2\sigma}\,dy\right)R(t,x),\\ k &= \partial_x h - \frac{i}{2}|h|^{2\sigma}h.\end{aligned}$$

Since R_j solves (1.1) for each $1\leqslant j\leqslant K$, we have

$$LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x. \tag{2.6}$$

By Lemma 3.1 for $t\gg T_0$ large enough we have

$$\left\|-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x\right\|_{H^2} \leqslant e^{-\lambda t}. \tag{2.7}$$

Thus, we rewrite (2.6) as follows:

$$LR + i|R|^{2\sigma}R_x = e^{-\lambda t}v, \tag{2.8}$$

where

$$v = e^{\lambda t}(-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x). \tag{2.9}$$

By an elementary calculation, we have

$$\begin{cases} Lh = i\sigma|h|^{2(\sigma-1)}h^2\overline{k} - \sigma(\sigma-1)h\int_{-\infty}^{x}|h|^{2(\sigma-2)}\mathcal{Im}(k^2\overline{h}^2)\,dy + e^{-\lambda t}m(t,x),\\ Lk = -i\sigma|h|^{2(\sigma-1)}k^2\overline{h} - \sigma(\sigma-1)k\int_{-\infty}^{x}|h|^{2(\sigma-2)}\mathcal{Im}(k^2\overline{h}^2)\,dy + e^{-\lambda t}n(t,x).\end{cases} \tag{2.10}$$

where

$$m = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |R|^{2\sigma}\,dy\right) v - \sigma h \int_{-\infty}^{x} |R|^{2(\sigma-1)}\mathcal{I}m(\overline{R}v)\,dy, \tag{2.11}$$

$$n = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |R|^{2\sigma}\,dy\right) e^{-\lambda t}(\partial_x v - \sigma\partial_x R \int_{-\infty}^{x} |R|^{2(\sigma-1)}\mathcal{I}m(\overline{R}v)\,dy). \tag{2.12}$$

Since v is uniformly bounded in time in $H^2(\mathbb{R})$, we see that m, n are uniformly bounded in time in $H^1(\mathbb{R})$. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $(\tilde{\varphi}, \tilde{\psi})$ solves:

$$\begin{cases} L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-\lambda t}m(t,x), \\ L\tilde{\psi} = Q(\varphi,\psi) - Q(h,k) - e^{-\lambda t}n(t,x). \end{cases} \tag{2.13}$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h,k)$ and $f(\varphi,\psi) = (P(\varphi,\psi), Q(\varphi,\psi)$ and $H = e^{-\lambda t}(m,n)$. We find a solutions of (2.13) in Duhamel form:

$$\eta(t) = i\int_t^{\infty} [f(W+\eta) - f(W) + H](s)\,ds, \tag{2.14}$$

where $S(t)$ denote the Schrödinger group. Moreover, since $\psi = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$, we have

$$\tilde{\psi} = \partial_x\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^{2\sigma}(\tilde{\varphi}+h) - |h|^{2\sigma}h). \tag{2.15}$$

Step 2. Existence of a solution of the system
From Lemma 3.3, there exists $T_* \gg 1$ such that for $T_0 \geqslant T_*$ there exists a unique solution η of (2.13) defined on $[T_0, T_*)$ such that

$$\|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1 \quad \forall t \geqslant T_0. \tag{2.16}$$

Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}. \tag{2.17}$$

Step 3. Existence of a multi-soliton train
We prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (2.13) satisfies the relation (2.15). Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and $v = \partial_x\varphi - \frac{i}{2}|\varphi|^2\varphi$ and $\tilde{v} = v - k$. Since $(\tilde{\varphi}, \tilde{\psi})$ solves (2.13) and (h,k) solves (2.10), we have (φ,ψ) solves (2.3). Furthermore,

$$Lv = \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi). \tag{2.18}$$

Moreover,

$$\begin{aligned}
&L(|\varphi|^{2\sigma}\varphi) \\
&= (i\partial_t + \partial_x^2)(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) = i\partial_t(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) + \partial_x^2(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi} \\
&\quad + \partial_x((\sigma+1)|\varphi|^{2\sigma}\partial_x\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{\varphi}) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi} + (\sigma+1)\left[\partial_x^2\varphi|\varphi|^{2\sigma} + \partial_x\varphi\partial_x(|\varphi|^{2\sigma})\right] \\
&\quad + \sigma\left[\partial_x^2\overline{\varphi}|\varphi|^{2(\sigma-1)}\varphi^2 + (\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + (\sigma-1)|\varphi|^{2(\sigma-2)}\varphi^3(\partial_x\overline{\varphi})^2\right] \\
&= (\sigma+1)|\varphi|^{2\sigma}(i\partial_t\varphi + \partial_x^2\varphi) + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(i\partial_t\overline{\varphi} + \partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3 \\
&= (\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3.
\end{aligned}$$

Combining with (2.18) and using (2.3), we have

$$
\begin{aligned}
Lv &= \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi)\\
&= \partial_x L\varphi - \frac{i}{2}\Big[(\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi})\\
&\quad +(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) + \partial_x P(\varphi,v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\left(P(\varphi,\psi) - P(\varphi,v)\right)\\
&\quad - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi,v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)})\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi,v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&\quad - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad +\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\left(P(\varphi,\psi) - P(\varphi,v)\right)\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)}) + G(\varphi,v),
\end{aligned}
$$

where $G(\varphi,v)$ contains the remaining ingredients and $G(\varphi,v)$ only depends on φ and v:

$$
\begin{aligned}
&G(\varphi,v)\\
&= \partial_x P(\varphi,v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi,v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi,v)}\\
&\quad - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad +\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big].
\end{aligned}
\tag{2.19}
$$

As the calculations of $L\psi$ in the step 1, noting that the role of v is similar to the role of ψ in the process of calculation, we have $G(\varphi,v) = Q(\varphi,v)$ (see Lemma 3.2 for a detailed proof). Hence,

$$
\begin{aligned}
L\psi - Lv = Q(\varphi,\psi) - Q(\varphi,v) &- \partial_x(P(\varphi,\psi) - P(\varphi,v))\\
&+ \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\left(P(\varphi,\psi) - P(\varphi,v)\right)\\
&- \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
L\tilde{\psi} - L\tilde{v} &= L\psi - Lv\\
&= Q(\varphi,\tilde{\psi}+k) - Q(\varphi,\tilde{v}+k) - \partial_x(P(\varphi,\tilde{\psi}+k) - P(\varphi,\tilde{v}+k)\\
&\quad + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi,\tilde{\psi}+k) - P(\varphi,\tilde{v}+k))\\
&\quad - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\tilde{\psi}+k)} - \overline{P(\varphi,\tilde{v}+k)}).
\end{aligned}
\tag{2.20}
$$

Multiplying both side of (2.20) by $\overline{\tilde{\psi}-\tilde{v}}$, taking imaginary part and integrating over space with integration by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\\
&=\mathcal{I}m\int_{\mathbb{R}}(Q(\varphi,\tilde{\psi}+k)-Q(\varphi,\tilde{v}+k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx && (2.21)\\
&\quad-\mathcal{I}m\int_{\mathbb{R}}\partial_x(P(\varphi,\tilde{\psi}+k)-P(\varphi,\tilde{v}+k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx && (2.22)\\
&\quad+(\sigma+1)\mathcal{I}m\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2\sigma}(P(\varphi,\tilde{\psi}+k)-P(\varphi,\tilde{v}+k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx && (2.23)\\
&\quad-\sigma\mathcal{I}m\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\tilde{\psi}+k)}-\overline{P(\varphi,\tilde{v}+k)})(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx. && (2.24)
\end{aligned}
$$

We denote by A,B,C,D the terms (2.21), (2.22), (2.23) and (2.24) respectively. First, we try to estimate A,B,C,D in term of R. We have

$$
\begin{aligned}
|A|&\lesssim\left|\int_{\mathbb{R}}(Q(\varphi,\tilde{\psi}+k)-Q(\varphi,\tilde{v}+k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim\left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\overline{\varphi}((\tilde{\psi}+k)^2-(\tilde{v}+k)^2)(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}\left[(\tilde{\psi}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right.\right.\\
&\quad\left.\left.-(\tilde{v}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{v}+k)^2\overline{\varphi}^2)\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim\left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\overline{\varphi}((\tilde{\psi}+k)^2-(\tilde{v}+k)^2)(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}\left[(\tilde{\psi}-\tilde{v})\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}\left[(\tilde{v}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right\|_{L_x^\infty}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\tilde{v}+k\|_{L^2}\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right\|_{L_x^\infty}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L_x^1}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}((\tilde{\psi}+k)^2-(\tilde{v}+k)^2)\|_{L^1}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_1, && (2.25)
\end{aligned}
$$

where,

$$K_1:=\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1}+\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.$$

Furthermore,

$$
\begin{aligned}
|B| &\lesssim \left|\int_{\mathbb{R}} \partial_x(|\varphi|^{2(\sigma-1)}\varphi^2(\overline{\tilde{\psi}}-\overline{\tilde{v}}))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}} \partial_x\left(\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right)(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right| \\
&\lesssim \left|\int_{\mathbb{R}} \partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)(\overline{\tilde{\psi}}-\overline{\tilde{v}})^2\,dx\right| + \left||\varphi|^{2(\sigma-1)}\varphi^2\frac{1}{2}\partial_x((\tilde{\psi}-\tilde{v})^2)\,dx\right| \qquad (2.26)\\
&\quad + \left|\int_{\mathbb{R}} \partial_x\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}} \varphi|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|.
\end{aligned}
$$

By using integration by parts for the second term of (2.26) and using Hölder inequality we have

$$
\begin{aligned}
|B| &\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x\varphi\|_{L^2}\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\|_{L_x^\infty}\|\tilde{\psi}-\tilde{v}\|_{L^2} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L_x^1} \qquad (2.27)\\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty} \\
&= \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 K_2, \qquad (2.28)
\end{aligned}
$$

where

$$
K_2 := \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x\varphi\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} + \|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}.
$$

Using (2.4), we have

$$
\begin{aligned}
|C| &\lesssim \left|\int_{\mathbb{R}} |\varphi|^{2\sigma}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{\tilde{\psi}}-\overline{\tilde{v}})^2\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}} |\varphi|^{2\sigma}\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right| \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\|_{L_x^\infty} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} \\
&= \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 K_3, \qquad (2.29)
\end{aligned}
$$

where

$$
K_3 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.
$$

Now, we give an estimate for D. We have

$$
\begin{aligned}
|D| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)}\varphi^2|\varphi|^{2(\sigma-1)}\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)}\varphi^2\varphi \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx \right| \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\Big\|\int_{-\infty}^{x} |\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\Big\|_{L^\infty_x} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} \\
&= \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 K_4,
\end{aligned} \tag{2.30}
$$

where

$$
K_4 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.
$$

Combining (2.25), (2.28), (2.29) and (2.30), we have

$$
\left|\partial_t\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\right| \lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2(K_1+K_2+K_3+K_4).
$$

Using the Grönwall inequality, we have

$$
\begin{aligned}
\|\tilde{\psi}(t)-\tilde{v}(t)\|_{L^2}^2 &\lesssim \|\tilde{\psi}(N)-\tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (K_1+K_2+K_3+K_4)\,ds\right) \\
&\leqslant e^{-2\lambda N}\exp\left(\int_t^N (K_1+K_2+K_3+K_4)\,ds\right).
\end{aligned} \tag{2.31}
$$

Now, we try to estimate $K_1+K_2+K_3+K_4$ in term of R. When we have this kind of estimate, we will use the assumption (1.10) to obtain that $\tilde{\psi}=\tilde{v}$. We have

$$
\begin{aligned}
&\int_t^N (K_1+K_2+K_3+K_4)\,ds \\
&= \int_t^N \|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1} \\
&\quad + \|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\,ds && (2.32) \\
&\quad + \int_t^N \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x\varphi\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} \\
&\quad + \|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\,ds && (2.33) \\
&\quad + \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\,ds && (2.34) \\
&\quad + \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\,ds && (2.35)
\end{aligned}
$$

Using (2.16) and (2.17), we have

$$
\|\varphi\|_{L^\infty} \leqslant \|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty} \tag{2.36}
$$

$$
\|\varphi\|_{L^2} \leqslant \|\tilde{\varphi}\|_{L^2} + \|h\|_{L^2} \lesssim 1 + \|h\|_{L^2} \tag{2.37}
$$

$$
\|\psi\|_{L^\infty} \lesssim 1 \tag{2.38}
$$

We denote by Z_1, Z_2, Z_3, Z_4 the terms (2.32), (2.33), (2.34) and (2.35) respectively. Using (2.36), (2.37), (2.38), (2.16) and (2.17), for $N \gg t$, we have

$$\begin{aligned}
|Z_1| &\lesssim \|\varphi\|^3_{L^4(t,N)L^\infty}\|\varphi\|^{2(\sigma-2)}_{L^\infty L^\infty}\|\tilde{\psi}+\tilde{v}+2k\|_{L^4(t,N)L^\infty}\\
&\quad + (N-t)\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}\|_{L^\infty L^2}+\|k\|_{L^\infty L^2})^2\\
&\quad + \|\tilde{v}+k\|_{L^{\frac{4}{3}}(t,N)L^2}\|\varphi\|_{L^\infty L^2}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^4(t,N)L^\infty}+\|k\|_{L^4(t,N)L^\infty})\\
&\lesssim (N-t)^{\frac{3}{4}}\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}(1+\|k\|_{L^\infty L^\infty}(N-t)^{\frac{1}{4}})\\
&\quad + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|^2_{L^\infty L^2})\\
&\quad + (N-t)^{\frac{3}{4}}(1+\|k\|_{L^\infty L^2})(1+\|h\|_{L^\infty L^2})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+(N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty})\\
&\lesssim (N-t)\|k\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})+(N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|^2_{L^\infty L^2})\\
&\quad + (N-t)\|k\|_{L^\infty L^\infty}(1+\|k\|_{L^\infty L^2})(1+\|h\|_{L^\infty L^2})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})\\
&:= (N-t)W_1(h,k).
\end{aligned}$$

Similarly, for $N \gg t$, we have

$$\begin{aligned}
|Z_2| &\lesssim \|\partial_x\varphi\varphi^{2\sigma-1}\|_{L^1(t,N)L^\infty}+(N-t)\|\partial_x\varphi\|_{L^\infty(t,N)L^2}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}\|\tilde{\psi}+\tilde{v}+k\|_{L^\infty(t,N)L^2}\\
&\quad + (N-t)^{\frac{3}{4}}\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^4(t,N)L^\infty}+\|k\|_{L^4(t,N)L^\infty})\\
&\lesssim (N-t)^{\frac{3}{4}}(\|\partial_x\tilde{\varphi}\|_{L^4(t,N)L^\infty}+\|\partial_x h\|_{L^4(t,N)L^\infty})\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}\\
&\quad + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2})\\
&\quad + (N-t)^{\frac{3}{4}}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})(1+(N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty})\\
&\lesssim (N-t)\|\partial_x h\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})+(N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2})\\
&\quad + (N-t)\|k\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})\\
&:= (N-t)W_2(h,k),
\end{aligned}$$

and

$$\begin{aligned}
&|Z_3| = |Z_4|\\
&\lesssim (N-t)(\|\tilde{\varphi}\|_{L^\infty L^\infty}+\|h\|_{L^\infty L^\infty})^{4\sigma}\\
&\quad + (N-t)\|\varphi\|_{L^\infty L^2}\|\varphi\|^{2\sigma}_{L^\infty L^\infty}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^\infty L^2}+\|k\|_{L^\infty L^2})\\
&\lesssim (N-t)(1+\|h\|^{4\sigma}_{L^\infty L^\infty})+(N-t)(1+\|h\|_{L^\infty L^2})(1+\|h\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2})\\
&:= (N-t)W_3(h,k).
\end{aligned}$$

Hence, from (2.31), we have

$$\begin{aligned}
\|\tilde{\psi(t)}-\tilde{v(t)}\|^2_{L^2} &\lesssim e^{-2\lambda N}\exp\left(\int_t^N (K_1+K_2+K_3+K_4)\,ds\right)\\
&\lesssim e^{-2\lambda N}\exp((N-t)(W_1(h,k)+W_2(h,k)+W_3(h,k))) \qquad (2.39)
\end{aligned}$$

The above estimate is not enough explicit. As said above, we would like to estimate the right hand side of (2.39) in terms of R. Noting that $|h| = |R|$ and $|k| = |\partial_x R|$, we have

$$\begin{aligned}
W_1(h,k) &= \|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^{2\sigma-1}_{L^\infty L^\infty})+(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|\partial_x R\|^2_{L^\infty L^2})\\
&\quad + \|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\,[\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty L^\infty})+(1+\|\partial_x R\|_{L^\infty L^2})\\
&\quad +\|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})]\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times\\
&\quad \times\left[\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty H^1})+(1+\|R\|^2_{L^\infty H^1})+\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^2_{L^\infty H^1})\right]\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}).
\end{aligned}$$

Similarly, by noting that $|\partial_x h| \leqslant |k| + |h|^{2\sigma+1}$, we have

$$\begin{aligned}
W_2(h,k) &\lesssim (\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\quad + (1+\|h\|^{2(\sigma-1)})(1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times \\
&\quad \times \big[(\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\quad +(1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|_{L^\infty L^\infty})\big] \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times \\
&\quad \times \big[(1+\|h\|_{L^\infty L^\infty})(\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty}) + (1+\|k\|_{L^\infty L^2})\big] \\
&= (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times \\
&\quad \times \big[(1+\|R\|_{L^\infty L^\infty})(\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) + (1+\|\partial_x R\|_{L^\infty L^2})\big] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}),
\end{aligned}$$

and

$$\begin{aligned}
W_3(h,k) &= (1+\|R\|^{4\sigma}_{L^\infty L^\infty}) + (1+\|R\|_{L^\infty L^2})(1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|\partial_x R\|_{L^\infty L^2}) \\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})\left[(1+\|R\|^2_{L^\infty L^\infty}) + (1+\|R\|_{L^\infty L^2})(1+\|\partial_x R\|_{L^\infty L^2})\right] \\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}).
\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}
&W_1(h,k) + W_2(h,k) + W_3(h,k) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\quad + (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\quad + (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^{2\sigma}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}).
\end{aligned}$$

Thus, there exists a positive constant C_0 such that

$$\begin{aligned}
&W_1(h,k) + W_2(h,k) + W_3(h,k) \\
&\leqslant C_0\left((1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty})\right).
\end{aligned}$$

Let $C_* = 16C_0$. Using the assumption (1.10), we have

$$W_1(h,k) + W_2(h,k) + W_3(h,k) \leqslant \frac{v_*}{16} = \lambda,$$

for t large enough. Thus, by (2.39), we have

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|^2_{L^2} \leqslant e^{-2\lambda N + (N-t)\lambda},$$

for t large enough. Letting $N \to \infty$ in the above estimate, we obtain

$$\|\tilde{\psi}(t) - \tilde{v}\|^2_{L^2} = 0,$$

for all t large enough. This implies that

$$\tilde{\psi} = \partial_x \varphi - \frac{i}{2}|\varphi|^2\varphi - k, \tag{2.40}$$

and then

$$\psi = \partial_x \varphi - \frac{i}{2}|\varphi|^2\varphi.$$

Moreover, since $(\tilde{\psi}, \tilde{\varphi})$ solves (2.13) we have (ψ, φ) solves (2.3). Combining with (2.40), if we set

$$u = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |\varphi|^{2\sigma}\, dy\right)\varphi$$

then u solves (1.1). Furthermore,

$$\|u-R\|_{H^1} = \left\|\exp\left(-\frac{i}{2}|\varphi|^{2\sigma}\,dy\right)\varphi - \exp\left(\frac{i}{2}|h|^{2\sigma}\,dy\right)h\right\|_{H^1}$$
$$\lesssim C(\|\varphi\|_{H^1},\|h\|_{H^1})\|\varphi-h\|_{H^1} \lesssim \|\tilde{\varphi}\|_{H^1} \lesssim e^{-\lambda t},$$

Thus for t large enough, we have

$$\|u-R\|_{H^1} \leqslant Ce^{-\lambda t}, \tag{2.41}$$

for $\lambda = \frac{1}{16}v_*$ and $C = C(\omega_1,...,\omega_K,c_1,...,c_K)$. This completes the proof of Theorem 1.1.

Remark 2.1. In the case $\sigma = 1$, the integrals in (2.3) disappear. In the case, $\sigma = 2$, the integrals (2.3) reduce into $\int_{-\infty}^{x} \mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy$, we do not need to use the inequality (1.11). Thus, by similar arguments as in the proof of Theorem 1.1 we may prove that there exist multi-solitons solutions of (1.1) when $\sigma = 1$ or $\sigma = 2$.

3. Some technical lemmas

3.1. **Properties of solitons.** In this section, we give the proof of (2.7). We have the following result.

Lemma 3.1. *There exist $C > 0$ and a constant $\lambda > 0$ such that for $t > 0$ large enough, the estimate (2.7) uniformly holds in time.*

Proof. First, we need some estimates on the profile. We have

$$|R_j(t,x)| = |\psi_{\omega_j,c_j}(t,x)| = |\phi_{\omega_j,c_j}(x-c_jt)| = |\varphi_{\omega_j,c_j}(x-c_jt)|$$
$$\approx \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x-c_jt)) - \frac{c_j}{2\sqrt{\omega_j}}\right)}\right)^{\frac{1}{2\sigma}}$$
$$\lesssim \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x-c_jt)) - \frac{|c_j|}{2\sqrt{\omega_j}}\cosh(\sigma h_j(x-c_jt))\right)}\right)^{\frac{1}{2\sigma}}$$
$$\lesssim \left(\frac{4\omega_j - c_j^2}{(2\sqrt{\omega_j}-|c_j|)\cosh(\sigma h_j(x-c_jt))}\right)^{\frac{1}{2\sigma}} \lesssim \left(\frac{2\sqrt{\omega_j}+|c_j|}{\cosh(\sigma h_j(x-c_jt))}\right)^{\frac{1}{2\sigma}}$$
$$\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x-c_jt|},$$

Furthermore,

$$\partial_x\varphi_{\omega_j,c_j}(y) \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.$$

Thus,

$$|\partial_x\varphi_{\omega_j,c_j}(y)| \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{|\sinh(\sigma h_j y)|}{\left(1-\frac{|c_j|}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}\cosh(\sigma h_j y)^{1+\frac{1}{2\sigma}}}$$
$$\lesssim_{\omega_j,|c_j|} \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|y|},$$

Using the above estimates, we have

$$|\partial_x R_j(t,x)| = |\partial_x\psi_{\omega_j,c_j}(t,x)| = |\partial_x\phi_{\omega_j,c_j}(x-c_jt)|$$
$$= |\partial_x\varphi_{\omega_j,c_j}(x-c_jt) + i\varphi_{\omega_j,c_j}(x-c_jt)\partial_x\theta_{\omega_j,c_j}(x-c_jt)|$$
$$\lesssim |\partial_x\varphi_{\omega_j,c_j}(x-c_jt)| + |\varphi_{\omega_j,c_j}(x-c_jt)||\partial_x\theta_{\omega_j,c_j}(x-c_jt)|$$
$$\lesssim_{\omega_j,|c_j|} |\partial_x\varphi_{\omega_j,c_j}(x-c_jt)| + e^{\frac{-h_j}{2}|x-c_jt|}$$
$$\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x-c_jt|}.$$

By similar arguments, we have

$$|\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)| \lesssim_{\omega_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|},$$

For convenience, we set

$$\begin{aligned}\chi &= -i|R|^{2\sigma}\partial_x R + i\Sigma_j |R_j|^{2\sigma}\partial_x R_j,\\ f(R,\overline{R},\partial_x R) &= i|R|^{2\sigma}\partial_x R,\\ g(R,\overline{R},\partial_x R,\partial_x\overline{R},\partial_x^2 R) &= i\partial_x(|R|^{2\sigma}\partial_x R),\\ r(R,\partial_x R,..,\partial_x^3 R,\partial_x\overline{R},\partial_x^2\overline{R}) &= i\partial_x^2(|R|^{2\sigma}\partial_x R).\end{aligned}$$

Fix $t>0$, for each $x\in\mathbb{R}$, choose $m=m(x)\in\{1,2,...,K\}$ so that

$$|x-c_mt| = \min_j |x-c_jt|.$$

For $j\neq m$ we have

$$|x-c_jt| \geqslant \frac{1}{2}(|x-c_jt|+|x-c_mt|) \geqslant \frac{1}{2}|c_jt-c_mt| = \frac{t}{2}|c_j-c_m|.$$

Thus, we have

$$\begin{aligned}&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)|\\ &\leqslant \sum_{j\neq m}(|R_j(t,x)| + |\partial_x R_j(t,x)| + |\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)|)\\ &\lesssim_{\omega_1,..,\omega_K,|c_1|,..,|c_K|} \delta_m(t,x) := \sum_{j\neq m} e^{\frac{-h_j}{2}|x-c_jt|}.\end{aligned}$$

Recall that

$$v_* = \inf_{j\neq k} h_j|c_j-c_k|.$$

We have

$$\begin{aligned}&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)| \lesssim \delta_m(t,x)\\ &\lesssim e^{-\frac{1}{4}v_*t}.\end{aligned}$$

We see that f,g,r are polynomials in R, $\partial_x R$, $\partial_x^2 R$, $\partial_x^3 R$, $\partial_x\overline{R}$ and $\partial_x^2\overline{R}$. Denote

$$A = \sup_{|u|+|\partial_x u|+|\partial_x^2 u|+|\partial_x^3 u|\leqslant\sum_j\|R_j\|_{H^4}} (|df|+|dg|+|dr|).$$

We have

$$\begin{aligned}&|\chi| + |\partial_x\chi| + |\partial_x^2\chi|\\ &\leqslant |f(R,\overline{R},\partial_x R) - f_{R_m,\partial_x\overline{R}_m,R_m}| + |g(R,\overline{R},\partial_x R,..) - g(R_m,\overline{R}_m,\partial_x R_m,..)|\\ &\quad + |r(R,\partial_x R,..,\partial_x^3 R,\overline{R},..) - r(R_m,\partial_x R_m,..,\partial_x^3 R_m,\overline{R}_m,..)|\\ &+ \Sigma_{j\neq m}(f(R_j,\overline{R}_j,\partial_x R_j) + g(R_j,\partial_x R_j,\partial_x^2 R_j,\overline{R}_j,\partial_x\overline{R}_j) + r(R_j,...,\partial_x^3 R_j,\overline{R}_j,...,\partial_x^2\overline{R}_j))\\ &\lesssim A(|R-R_m| + |\partial_x(R-R_m)| + |\partial_x^2(R-R_m)| + |\partial_x^3(R-R_m)|)\\ &\quad + A\Sigma_{j\neq m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|)\\ &\lesssim 2A\Sigma_{j\neq m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|)\\ &\lesssim 2A\delta_m(t,x).\end{aligned}$$

In particular,

$$\|\chi\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_*t}. \tag{3.1}$$

Moreover,

$$\begin{aligned}\|\chi\|_{W^{2,1}} &\lesssim \Sigma_j(\||R_j|^{2\sigma}\partial_x R_j\|_{L^1} + \|\partial_x(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1} + \|\partial_x^2(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1})\\ &\lesssim \Sigma_j(\|R_j\|_{H^1}^{(}2\sigma+1) + \|R_j\|_{H^2}^{2\sigma+1} + \|R_j\|_{H^3}^{2\sigma+1}) < \infty.\end{aligned}$$

Thus, using Hölder inequality we obtain

$$\|\chi\|_{H^2} \lesssim_{\omega_1,..,\omega_K,|c_1|,..,|c_K|} e^{-\frac{1}{8}v_*t}.$$

It follows that if $t \gg \max\{\omega_1, ..., \omega_K, |c_1|, ..., |c_K|\}$ is large enough then

$$\|\chi\|_{H^2} \leqslant e^{-\frac{1}{16}v_* t}.$$

Setting $\lambda = \frac{1}{16}v_*$, we obtain the desired result. □

3.2. **Proof $G(\varphi, v) = Q(\varphi, v)$.** Let $G(\varphi, v)$ be defined as in (2.19) and Q be defined as in (2.5). Then we have the following result.

Lemma 3.2. *Let $v = \partial_x\varphi - \frac{i}{2}|\varphi|^2\varphi$. Then the following equality holds:*

$$G(\varphi, v) = Q(\varphi, v).$$

Proof. We have

$$\begin{aligned}
P(\varphi, v) &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy,\\
Q(\varphi, v) &= -i\sigma|\varphi|^{2(\sigma-1)}v^2\overline{\varphi} - \sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v)\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&\quad - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big].
\end{aligned}$$

The term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $G(\varphi, v)$ is the following.

$$\begin{aligned}
&- \sigma(\sigma-1)\partial_x\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&\quad - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(-1)\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-1)\sigma(\sigma-1)\overline{\varphi}\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\varphi + \frac{i}{2}\sigma|\varphi|^{2\sigma}\varphi\right)\\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)\\
&= -\sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy,
\end{aligned}$$

which equals to the term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $Q(\varphi, v)$. We only need to check the equality of the remaining terms. The remaining terms of $G(\varphi, v)$ is the following.

$$\begin{aligned}
&i\sigma\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2\overline{v}) - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{I}m(v^2\overline{\varphi}^2)\\
&- \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v})\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}^2 v) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} \qquad (3.2)\\
&\quad - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]. \qquad (3.3)
\end{aligned}$$

Noting that $\partial_x(|\varphi|^2) = 2\mathcal{R}e(v\overline{\varphi})$ and $v = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$, we have

$$
\begin{aligned}
&\text{the term (3.2)}\\
&= i\sigma\partial_x(|\varphi|^{2(\sigma-1)})\varphi^2\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}2\varphi\partial_x\varphi\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{v}\\
&\quad - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi 2\mathcal{R}e(v\overline{\varphi})\mathcal{I}m(v\overline{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}\\
&\quad + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v}) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi^2\overline{v} + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x(\overline{v} - \overline{\partial_x\varphi})\\
&\quad - 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{R}e(v\overline{\varphi})\mathcal{I}m(v\overline{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&= 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi(i\varphi\overline{v} - \mathcal{I}m(v\overline{\varphi})) + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v}\\
&\quad + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\left(\frac{i}{2}|\varphi|^{2\sigma}\overline{\varphi}\right) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}\\
&\quad - \frac{1}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(2\sigma|\varphi|^{2(\sigma-1)}\mathcal{R}e(v\overline{\varphi}) + |\varphi|^{2\sigma}\partial_x\overline{\varphi})\\
&\quad + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}\\
&\quad - \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\partial_x\overline{\varphi} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}\\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2(\overline{v} - \partial_x\overline{\varphi})\\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi.
\end{aligned}
$$

Moreover, using $\mathcal{R}e(\partial_x\varphi\overline{\varphi}) = \mathcal{R}e(v\overline{\varphi})$ we have

$$
\begin{aligned}
&\text{the term (3.3)}\\
&= \frac{-i}{2}\Big[\sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi(\partial_x\overline{\varphi}\varphi + \partial_x\varphi\overline{\varphi})\\
&\quad + \sigma(\sigma-1)(\partial\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&= \frac{-i}{2}\Big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2(\partial_x\varphi\overline{\varphi} + \partial_x\overline{\varphi}\varphi)\\
&\quad + 2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\Big]\\
&= \frac{-i}{2}\Big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})\\
&\quad + 2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\Big]\\
&= -i\Big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})\\
&\quad + \sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\Big]\\
&= -i\Big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi(\partial_x\overline{\varphi}\varphi + \partial_x\varphi\overline{\varphi})\\
&\quad + 2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\Big]\\
&= -i\Big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}(\mathcal{R}e(v\overline{\varphi}))^2\varphi\Big]\\
&= -2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2\\
&\quad - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi}).
\end{aligned}
$$

Combining the above expressions we obtain

$$
\begin{aligned}
&\text{the remaining term of } G(\varphi, v)\\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{Re}(v\overline{\varphi})\\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(\varphi\overline{v} - \mathcal{Re}(v\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&= -2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{Im}(\varphi\overline{v}) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{Im}(\varphi\overline{v}) + i\partial_x\overline{\varphi}\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{Im}(\varphi\partial_x\overline{\varphi}) + |\varphi|^{2\sigma+2} + i\mathcal{Re}(\varphi\partial_x\overline{\varphi}) - \mathcal{Im}(\varphi\partial_x\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(|\varphi|^{2\sigma+2} + i\overline{\varphi}\partial_x\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}(\partial_x\varphi)^2 - \sigma|\varphi|^{4\sigma}\partial_x\varphi + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)^2 - \sigma|\varphi|^{4\sigma}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}v^2.
\end{aligned}
$$

This is exactly the remaining terms of $Q(\varphi, v)$. Thus, $G(\varphi, v) = Q(\varphi, v)$. □

3.3. **Existence of a solution of the system.** In this section, using similar arguments as in [9, 10], we prove the existence of a solution of (2.13). For convenience, we recall the equation:

$$\eta(t) = i\int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{3.4}$$

where

$$
\begin{aligned}
W &= (h, k),\\
H &= e^{-\lambda t}(m, n),\\
f(\varphi, \psi) &= (P(\varphi, \psi), Q(\varphi, \psi)).
\end{aligned}
$$

We have the following lemma.

Lemma 3.3. *Let $H = H(t,x) : [0,\infty)\times\mathbb{R} \to \mathbb{C}^2$, $W = W(t,x) : [0,\infty)\times\mathbb{R} \to \mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2\times L^2} \leqslant C_1, \quad \forall t \geqslant T_0, \tag{3.5}$$

$$\|\partial W(t)\|_{L^2\times L^2} + \|\partial W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2\times L^2} \leqslant C_2, \quad \forall t \geqslant T_0. \tag{3.6}$$

Consider equation (3.4). There exists a constant λ_* independent of C_2 such that if $\lambda \geqslant \lambda_*$ then there exists a unique solution η of (3.4) on $[T_0,\infty)\times\mathbb{R}$ satisfying

$$e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0.$$

Proof. We rewrite (3.4) by $\eta = \Phi\eta$. We show that, for λ large enough, Φ is a contraction map in the following ball

$$B = \left\{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1\right\}.$$

We will use condition $\lambda \gg 1$ in the proof without specifying it.

Step 1. Proof Φ maps B into B

Let $t \geqslant T_0$, $\eta = (\eta_1, \eta_2) \in B$, $W = (w_1, w_2)$ and $H = (h_1, h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))\times N([t,\infty))}, \tag{3.7}$$

$$+ \|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{3.8}$$

For (3.8), using (3.5), we have

$$\begin{aligned}\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} &= \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \\ &\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau \leqslant \frac{1}{\lambda}e^{-\lambda t} < \frac{1}{10}e^{-\lambda t}. \end{aligned} \tag{3.9}$$

For (3.7), we have

$$\begin{aligned}&|P(W+\eta)-P(W)| \\ &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1,w_2)| \\ &\lesssim \left||w_1+\eta_1|^{2\sigma-1)}(w_1+\eta_1)^2\overline{w_2+\eta_2} - |w_1|^{2(\sigma-1)}w_1^2\overline{w_2}\right| \qquad (3.10)\\ &+ \Bigg|(w_1+\eta_1)\int_{-\infty}^x |w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w_1}+\overline{\eta_1})^2) \\ &\quad -w_1\int_{-\infty}^x |w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{\eta_1}^2)\Bigg|. \qquad (3.11)\end{aligned}$$

Using the assumption $\sigma \geqslant \frac{5}{2}$ and the inequality (1.11) we have

$$\begin{aligned}&\text{the term (3.10)} \\ &\lesssim \left||w_1+\eta_1|^{2(\sigma-1)} - |w_1|^{2(\sigma-1)}||w_1+\eta_1|^2|w_2+\eta_2|\right| \\ &\quad + \left||w_1|^{2(\sigma-1)}|(w_1+\eta_1)^2 - w_1^2||w_2+\eta_2|\right| + \left||w_1|^{2(\sigma-1)}|w_1|^2|\eta_2|\right| \\ &\lesssim (|\eta_1|^{2(\sigma-1)} + |\eta_1||w_1|^{2(\sigma-1)-1})(|W|+|\eta|)^3 \\ &\quad + |w_1|^{2(\sigma-1)}(|w_1||\eta_1| + |\eta_1|^2)|w_2+\eta_2| + |w_1|^{2\sigma}|\eta_2| \\ &\lesssim (|\eta|^{2(\sigma-1)} + |\eta||W|^{2(\sigma-1)-1})(|W|^3+|\eta|^3) \\ &\quad + |W|^{2(\sigma-1)}(|W||\eta| + |\eta|^2)(|W|+|\eta|) + |W|^{2\sigma}|\eta| \\ &\lesssim |\eta|(|\eta|^{2\sigma-3} + |W|^{2\sigma-3})(|\eta|^3+|W|^3) + |\eta||W|^{2(\sigma-1)}(|W|^2+|\eta|^2) + |W|^{2\sigma}|\eta| \\ &\lesssim |\eta|(|\eta|^{2\sigma}+|W|^{2\sigma}) + |\eta||W|^{2\sigma} + |\eta|^3|W|^{2(\sigma-1)} + |W|^{2\sigma}|\eta| \\ &\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma}.\end{aligned}$$

Moreover,

$$
\begin{aligned}
&\text{the term (3.11)}\\
&\lesssim |\eta_1| \int_{-\infty}^{x} |w_1+\eta_1|^{2(\sigma-2)}|w_2+\eta_2|^2|w_1+\eta_1|^2\,dy\\
&\quad + |w_1| \int_{-\infty}^{x} (|w_1+\eta_1|^{2(\sigma-2)} - |w_1|^{2(\sigma-2)})|w_2+\eta_2|^2|w_1+\eta_1|^2\,dy\\
&\quad + |w_1| \int_{-\infty}^{x} |w_1|^{2(\sigma-2)}|\mathcal{I}m((w_2+\eta_2)^2 - w_2^2)(\overline{w_1}+\overline{\eta_1})^2|\,dy\\
&\quad + |w_1| \int_{-\infty}^{x} |w_1|^{2(\sigma-2)}|\mathcal{I}m(w_2^2((\overline{w_1}+\eta_1)^2 - \overline{\eta_1}^2))|\,dy\\
&\lesssim |\eta| \int_{-\infty}^{x} |W|^{2\sigma}+|\eta|^{2\sigma}\,dy + |W| \int_{-\infty}^{x} (|\eta_1|^{2(\sigma-2)} + |\eta_1||w_1|^{2\sigma-5})(|W|^4+|\eta|^4)\,dy\\
&\quad + |W| \int_{-\infty}^{x} |W|^{2(\sigma-2)}(|\eta_2|^2+|w_2||\eta_2|)(|W|^2+|\eta|^2)\,dy\\
&\quad + |W| \int_{-\infty}^{x} |W|^{2(\sigma-2)}|w_2|^2(|\eta_1|^2+|\eta_1||w_1|)\,dy\\
&\lesssim |\eta| \int_{-\infty}^{x} |W|^{2\sigma}+|\eta|^{2\sigma}\,dy + |W| \int_{-\infty}^{x} |\eta|(|W|^{2\sigma}+|\eta|^{2\sigma})\,dy\\
&+ |W| \int_{-\infty}^{x} |W|^{2(\sigma-2)}|\eta|(|W|^3+|\eta|^3)\,dy + |W| \int_{-\infty}^{x} |W|^{2(\sigma-2)}|W|^2|\eta|(|W|+|\eta|)\,dy\\
&\lesssim |\eta| \int_{-\infty}^{x} |W|^{2\sigma}+|\eta|^{2\sigma}\,dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|P(W+\eta)-P(W)|\\
&\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^{x} |W|^{2\sigma}+|\eta|^{2\sigma}\,dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|Q(W+\eta)-Q(W)|\\
&\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^{x} |W|^{2\sigma}+|\eta|^{2\sigma}\,dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy.
\end{aligned}
$$

Hence, using $\sigma \geqslant \frac{5}{2}$, we have:

$$\begin{aligned}
&\|f(W+\eta)-f(W)\|_{N([t,\infty))\times N([t,\infty))}\\
&\lesssim \|P(W+\eta)-P(W)\|_{L^1_\tau L^2_x([t,\infty))}+\|Q(W+\eta)-Q(W)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\eta|^{2\sigma+1}\|_{L^1_\tau L^2_x([t,\infty))}+\||\eta|\int_{-\infty}^x |W|^{2\sigma}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^2_x([t,\infty))}\\
&\quad+\||W|\int_{-\infty}^x |\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\eta|\|_{L^\infty L^2_x([t,\infty))}\||\eta|\|^4_{L^4_\tau L^\infty_x([t,\infty))}\\
&\quad+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\left\|\int_{-\infty}^x |W|^{2\sigma}+|\eta|^{2\sigma}\,dy\right\|_{L^\infty_\tau L^\infty_x([t,\infty))}\\
&\quad+\||W|\|_{L^\infty_\tau L^2_x([t,\infty))}\|\int_{-\infty}^x |\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^\infty_x([t,\infty))}\\
&\lesssim e^{-5\lambda t}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\||W|^{2\sigma}+|\eta|^{2\sigma}\|_{L^\infty_\tau L^1_x}\\
&\quad+\|W\|_{L^\infty_t L^2_x}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\||W|^{2\sigma-1}+|\eta|^{2\sigma-1}\|_{L^\infty_\tau L^2_x([t,\infty))}\\
&\lesssim e^{-5\lambda t}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}=e^{-5\lambda t}+\int_t^\infty e^{-\lambda\tau}d\tau\\
&\lesssim e^{-5\lambda t}+\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t},
\end{aligned}$$

Combining with (3.9) and (3.7), (3.8) we obtain

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}<\frac{1}{5}e^{-\lambda t}. \tag{3.12}$$

We have

$$\|\partial_x\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}\lesssim \|\partial_x(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} \tag{3.13}$$

$$+\|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{3.14}$$

For (3.14), using (3.6) we have

$$\|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}\lesssim\int_t^\infty e^{-\lambda\tau}\,d\tau=\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t}, \tag{3.15}$$

For (3.13), we have

$$\begin{aligned}
\|\partial_x(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))}&=\|\partial_x(P(W+\eta)-P(W))\|_{N([t,\infty))}\\
&\quad+\|\partial_x(Q(W+\eta)-Q(W))\|_{N([t,\infty))}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&|\partial_x(P(W+\eta)-P(W))|\\
&\lesssim |\partial_x(|w_1+\eta_1|^{2(\sigma-1)}(w_1+\eta_1)^2(\overline{w}_2+\overline{\eta}_2)-|w_1|^{2(\sigma-1)}w_1^2\overline{w}_2)| && (3.16)\\
&\quad+\Big|\partial_x(w_1+\eta_1)\int_{-\infty}^x |w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\,dy\\
&\quad-\partial_x w_1\int_{-\infty}^x |w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{w}_1^2)\,dy\Big| && (3.17)\\
&\quad+\Big|(w_1+\eta_1)|w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\\
&\quad-w_1|w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{w}_1)\Big|. && (3.18)
\end{aligned}$$

For (3.16), we have

$$\begin{aligned}
&\text{the term (3.16)}\\
&\lesssim(|\eta|+|\eta|^{2\sigma}+|\partial_x\eta|)(|W|+|W|^{2\sigma}+|\eta|+|\eta|^{2\sigma}+|\partial_x\eta|)
\end{aligned}$$

Thus,

$$\|\text{the term (3.16)}\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \||\eta|+|\partial\eta|\|_{L^1_\tau L^2_x} \lesssim \frac{1}{\lambda}e^{-\lambda t} < \frac{1}{10}e^{-\lambda t}.$$

For (3.17), using the inequality (1.11), we have

$$\begin{aligned}
&\|\text{ the term (3.17)}\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}\Big\|\int_{-\infty}^x |w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\,dy\Big\|_{L^\infty_t L^\infty_x}\\
&\quad+\|\partial_x w_1\|_{L^\infty_t L^2_x}\times\\
&\quad\times\Big\|\int_{-\infty}^x (|w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)-|w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{w}_1^2))\,dy\Big\|_{L^{1\tau}L^\infty_x}\\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}\||w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\|_{L^\infty_t L^1_x}\\
&\quad+\||w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)-|w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{w}_1^2)\|_{L^1_\tau L^1_x}\\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))} \leqslant \int_t^\infty e^{-\lambda\tau}\,d\tau \lesssim \frac{1}{\lambda}e^{-\lambda t} < \frac{1}{10}e^{-\lambda t},
\end{aligned}$$

For (3.18), using the inequality (1.11), we have

$$\begin{aligned}
&\|\text{the term (3.18)}\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\leqslant \int_t^\infty e^{-\lambda\tau}\,d\tau \lesssim \frac{1}{\lambda}e^{-\lambda t} < \frac{1}{10}e^{-\lambda t},
\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}
&\|\partial_x(P(W+\eta)-P(W))\|_{N([t,\infty))}\\
&\leqslant \|\partial_x(P(W+\eta)-P(W))\|_{L^1_\tau L^2_x([t,\infty))} \leqslant \frac{3}{10}e^{-\lambda t},
\end{aligned}\tag{3.19}$$

Similarly,

$$\|\partial_x(Q(W+\eta)-Q(W))\|_{N([t,\infty))} \leqslant \frac{3}{10}e^{-\lambda t},\tag{3.20}$$

Combining the estimates (3.13), (3.14), (3.15), (3.19) and (3.20), we have

$$\|\partial_x\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant \frac{7}{10}e^{-\lambda t}.\tag{3.21}$$

Combining (3.12) with (3.21), we obtain

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}+\|\partial_x\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant \frac{9}{10}e^{-\lambda t},\tag{3.22}$$

Thus, for λ large enough

$$\|\Phi\eta\|_X < 1.$$

This implies that Φ maps B into B.

Step 2. Φ is a contraction map on B

By using (3.5), (3.6) and a similar estimate of (3.22), we can show that, for any $\eta\in B$ and $\kappa\in B$ we have

$$\|\Phi\eta-\Phi\kappa\|_X \leqslant \frac{1}{2}\|\eta-\kappa\|_X.$$

for λ large enough. From Banach fixed point theorem, there exists a unique solution in B of (3.4) and thus a solution of (2.13). This completes the proof of Lemma 3.3. □

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