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**CONSTRUCTION OF THE MULTI-SOLITON TRAINS FOR A
GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS
BY A FIXED POINT METHOD**

PHAN VAN TIN

ABSTRACT. We consider a derivative nonlinear Schrödinger equation with general nonlinearity:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0,$$

In [13], the authors prove the stability of two solitary waves in energy space for $\sigma \in (1, 2)$. As a consequence, there exists a two-soliton trains in energy space for $\sigma \in (1, 2)$. Our goal in this paper is proving the existence of multi-soliton trains in energy space for $\sigma \geq \frac{5}{2}$. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

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1. INTRODUCTION

We consider the following generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0, \tag{1.1}$$

where $\sigma \in \mathbb{R}^+$ is a given constant and $u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C}$.

1.1. Preliminaries. The equation (1.1) is studied in many works before. In special case $\sigma = 1$, there are many works on local wellposedness, global well posedness, stability of solitary waves and stability of multi-solitons before. In [11] the author give a sufficient conditions for global well posedness of (1.1) in energy space by using a Gauge transformation to remove the derivative terms. In [1], the authors show that the equation have two parameters family of solitary waves. Moreover, the authors prove the stability of these particular solutions by using variational methods. In [4], the authors give a result on stability of solitary waves when parameters in the presentation of for soliton solutions take critical values. In [7], the authors prove the stability of multi-solitons in energy space under some conditions on parameters of individual solitons.

In general case, the local well posedness and global well posedness of (1.1) is studied as in [3] for the initial data in Sobolev space $H^1(\mathbb{R})$ or $H^2(\mathbb{R})$ and [12] for small size initial data in weighted Sobolev space. In [8], the authors prove the orbital stability/instability results of solitary waves. In [2], the authors give instability results of solitary waves in the critical frequency case. In [13],

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in case $\sigma \in (1, 2)$, the authors prove the stability of the sum of two solitary waves in energy space using perturbation argument, modulational analysis and the energy argument as in [9, 10]. In this paper, we show the existence of multi-soliton trains in energy space in case $\sigma \geq \frac{5}{2}$. Before state the main result, we give some preliminaries on multi-soliton trains of (1.1).

1.2. Multi-soliton trains. As in [8] the equation (1.1) admits a two-parameters family of solitary waves solutions,

$$\psi_{\omega,c}(t, x) = \varphi_{\omega,c}(x - ct) \exp i \left(\omega t + \frac{c}{2}(x - ct) - \frac{1}{2\sigma + 2} \int_{-\infty}^{x-ct} \varphi_{\omega,c}^{2\sigma}(\eta) d\eta \right), \quad (1.2)$$

where $\omega > \frac{c^2}{4}$ and

$$\varphi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma + 1)(4\omega - c^2)}{2\sqrt{\omega} \left(\cosh(\sigma\sqrt{4\omega - c^2}y) - \frac{c}{2\sqrt{\omega}} \right)}. \quad (1.3)$$

is the positive solution of

$$-\partial_y^2 \varphi_{\omega,c} + \left(\omega - \frac{c^2}{4} \right) \varphi_{\omega,c} + \frac{c}{2} |\varphi_{\omega,c}|^{2\sigma} \varphi_{\omega,c} - \frac{2\sigma + 1}{(2\sigma + 2)^2} |\varphi_{\omega,c}|^{4\sigma} \varphi_{\omega,c} = 0. \quad (1.4)$$

Define

$$\phi_{\omega,c}(y) = \varphi_{\omega,c}(y) e^{i\theta_{\omega,c}(y)}, \quad (1.5)$$

where

$$\theta_{\omega,c} = \frac{c}{2}y - \frac{1}{2\sigma + 2} \int_{-\infty}^y \varphi_{\omega,c}^{2\sigma}(\eta) d\eta. \quad (1.6)$$

Clearly,

$$\psi_{\omega,c}(x, t) = e^{i\omega t} \phi_{\omega,c}(x - ct). \quad (1.7)$$

and $\phi_{\omega,c}$ solves

$$-\partial_y^2 \phi_{\omega,c} + \omega \phi_{\omega,c} + ic \partial_y \phi_{\omega,c} - i |\phi_{\omega,c}|^{2\sigma} \partial_y \phi_{\omega,c} = 0, \quad y \in \mathbb{R}. \quad (1.8)$$

Let $K \in \mathbb{N}$ and for each $1 \leq j \leq K$ let $(\omega_j, c_j, x_j, \theta_j) \in \mathbb{R}^4$ be given parameters such that $\omega_j > \frac{c_j^2}{4}$. Define

$$R_j(t, x) = e^{i\theta_j} \psi_{\omega_j, c_j}(t, x - x_j)$$

and define the multi-soliton profile

$$R = \sum_{j=1}^K R_j. \quad (1.9)$$

Our main result is the following.

Theorem 1.1. *Let $\sigma \geq \frac{5}{2}$, $K \in \mathbb{N}^*$ and for each $1 \leq j \leq K$, $(\theta_j, \omega_j, c_j, x_j)$ be a sequence of parameters such that $x_j = 0$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k \neq 0$, for $j \neq k$. The multi-soliton profile R is given as in (1.9). We assume that the parameters (ω_j, c_j) satisfy*

$$(1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \ll v_* = \inf_{j \neq k} h_j |c_j - c_k|. \quad (1.10)$$

Then there exists a solution u of (1.1) such that

$$\|u - R\|_{H^1} \leq C e^{-\lambda t}. \quad \forall t \geq T_0,$$

for some $C > 0$, $\lambda = \frac{1}{16} v_*$ and $T_0 \gg \max\{1, \omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|\}$ is large enough.

Remark 1.2. By Lemma 3.2, the following inequality holds for $\sigma \geq 2$:

$$(a + b)^{2(\sigma-2)} - a^{2(\sigma-2)} \lesssim b^{2(\sigma-2)} + ba^{2(\sigma-2)-1}, \quad \text{for all } a, b > 0. \quad (1.11)$$

The condition $\sigma \geq \frac{5}{2}$ ensures that the order of b on the right hand side of (1.11) is larger than 1. This is important in our analysis.

Remark 1.3. We prove there exist the parameters $(\omega_j, c_j, \theta_j, x_j)$ for $1 \leq j \leq K$ and $x_j = 0$ for all j such that the condition (1.10) satisfies. For convenience, for each j , we define $h_j = \sqrt{4\omega_j - c_j^2}$. Indeed, choosing $c_j < 0$ and $h_j \ll \min(1, |c_j|)$ for all j we have

$$\begin{aligned}\varphi_{\omega_j, c_j}^{2\sigma} &\approx \frac{h_j^2}{2\sqrt{\omega_j} \left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)} \\ \partial_x \varphi_{\omega_j, c_j} &\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)^{1+\frac{1}{2\sigma}}}.\end{aligned}$$

Using $|\sinh(x)| \leq |\cosh(x)|$ for all $x \in \mathbb{R}$ we have

$$|\partial_x \varphi_{\omega_j, c_j}| \leq \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{1}{\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}}} \lesssim |\varphi_{\omega_j, c_j}|.$$

Thus,

$$\begin{aligned}\|R_j\|_{L^\infty L^\infty} &= \|\varphi_{\omega_j, c_j}\|_{L^\infty} \lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \ll 1 \\ \|\partial_x R_j\|_{L^\infty L^\infty} &= \|\partial_x \phi_{\omega_j, c_j}\|_{L^\infty L^\infty} \\ &\lesssim \|\partial_x \varphi_{\omega_j, c_j}\|_{L^\infty} + \left\| \frac{c_j}{2} \varphi_{\omega_j, c_j} - \frac{1}{2\sigma+2} \varphi_{\omega_j, c_j}^{2\sigma+1} \right\|_{L^\infty} \\ &\lesssim \|\varphi_{\omega_j, c_j}\|_{L^\infty} + |c_j| \|\varphi_{\omega_j, c_j}\|_{L^\infty} \\ &\lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}.\end{aligned}$$

Hence,

$$\begin{aligned}\|R\|_{L^\infty L^\infty} &\lesssim \sum_j \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \lesssim 1 \\ \|\partial_x R\|_{L^\infty L^\infty} &\lesssim \sum_j \left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|R_j\|_{L^\infty H^1}^2 &= \|R_j\|_{L^\infty L^2}^2 + \|\partial_x R_j\|_{L^\infty L^2}^2 = \|\varphi_{\omega_j, c_j}\|_{L^2}^2 + \|\partial_x \varphi_{\omega_j, c_j}\|_{L^2}^2 \\ &\lesssim \|\varphi_{\omega_j, c_j}\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \left\| \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \right\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \|e^{-\frac{h_j}{2}|y|}\|_{L^2}^2 \\ &\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \frac{1}{h_j} \lesssim h_j^{\frac{1}{\sigma}} h_j^{-1} = h_j^{\frac{1}{\sigma}-1},\end{aligned}$$

where we use $h_j \leq \sqrt{2\omega_j}$. Thus,

$$\|R\|_{L^\infty H^1}^2 \lesssim \sum_j h_j^{\frac{1}{\sigma}-1}.$$

Condition (1.10) satisfies if the following estimate holds:

$$\left(1 + \sum_j h_j^{\frac{1}{\sigma}-1} \right) \left(1 + \sum_j \left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right) \right) \ll \inf_{j \neq k} h_j |c_j - c_k|. \quad (1.12)$$

Fixing h_j and replacing c_j by $M c_j$, thus $\omega_j = \frac{1}{4}(h_j^2 + M^2 c_j^2)$, we see that the left hand side of (1.12) is order $M^{1-\frac{1}{2\sigma}}$ and the right hand side of (1.12) is order M^1 . Hence, the condition (1.10) satisfies if we choose M large enough.

Our strategy of proof of Theorem 1.1 is as follows. First, we use the transform (2.1) and (2.2). Assume u solves (1.1) then (φ, ψ) solves (2.3). Let R satisfies the assumption on Theorem 1.1. Then R solves (1.1) with a small perturbation. Let (h, k) be defined similarly as (φ, ψ) but replace u by R . We show that (h, k) solves (2.3) with small perturbations. Setting $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$, we see that if u solves (1.1) then $(\tilde{\varphi}, \tilde{\psi})$ solves a system with a relation between $\tilde{\varphi}$ and $\tilde{\psi}$ and vice versa. By using similar arguments as in [6], [5] we prove existence solution $(\tilde{\varphi}, \tilde{\psi})$ of this system which exponentially decays in time in $H^1(\mathbb{R})$ when t large. Combining with the assumption (1.10), we can prove the relation between $\tilde{\varphi}$ and $\tilde{\psi}$. Thus, we easily obtain the solution u of (1.1) satisfies the desired property.

Before proving the main result, we introduce some notations which will be used in our proof.

Notation.

(1) We denote the Schrödinger operator as follows

$$L = i\partial_t + \partial_x^2.$$

(2) Given a time $t \in \mathbb{R}$, the Strichartz space $S([t, \infty))$ is defined via the norm

$$\|u\|_{S([t, \infty))} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L^r_x([t, \infty) \times \mathbb{R})}.$$

We denote the dual space by $N([t, \infty)) = S([t, \infty))^*$. Hence for any (q, r) admissible pair we have

$$\|u\|_{N([t, \infty))} \leq \|u\|_{L^{q'} L^{r'}([t, \infty) \times \mathbb{R})}.$$

(3) For $a, b \in \mathbb{R}^2$, we denote $|(a, b)| = |a| + |b|$.

(4) Let $a, b > 0$. We denote $a \lesssim b$ if a is smaller than b up to multiply by a positive constant and denote $a \lesssim_c b$ if a is smaller than b up to multiply a positive constant depending on c . Moreover, we denote $a \approx b$ if a is equal to b up to multiply a positive constant.

2. PROOF OF MAIN RESULT

In this section we give the proof of Theorem 1.1. We use the Banach fixed point theorem and Strichartz estimates. We divide our proof into steps.

Step 1. Preliminary analysis. Let $u \in C(I, H^1(\mathbb{R}))$ be a $H^1(\mathbb{R})$ solution of (1.1) on I . Considering the following transform:

$$\varphi(t, x) = \exp(i\Lambda)u(t, x), \tag{2.1}$$

$$\psi = \exp(i\Lambda)\partial_x u = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi, \tag{2.2}$$

where

$$\Lambda = \frac{1}{2} \int_{-\infty}^x |u(t, y)|^{2\sigma} dy.$$

As in [3][section 4] we have

$$\partial_t \Lambda = -\sigma \mathcal{I}m(|u|^{2(\sigma-1)} \bar{u} \partial_x u) + \sigma \mathcal{I}m \left[\int_{-\infty}^x \partial_x (|u|^{2(\sigma-1)} \bar{u}) \partial_x u dy \right] - \frac{1}{4} |u|^{4\sigma}.$$

Thus, using $|u| = |\varphi|$ and $\mathcal{I}m(\bar{u} \partial_x u) = \mathcal{I}m(\bar{\varphi} \psi)$, we have

$$\begin{aligned} \partial_t \Lambda &= -\sigma |\varphi|^{2(\sigma-1)} \mathcal{I}m(\bar{\varphi} \psi) + \sigma \int_{-\infty}^x \partial_x (|\varphi|^{2(\sigma-1)}) \mathcal{I}m(\bar{u} \partial_x u) dx - \frac{1}{4} |\varphi|^{4\sigma} \\ &= -\sigma |\varphi|^{2(\sigma-1)} \mathcal{I}m(\bar{\varphi} \psi) + \sigma \int_{-\infty}^x \partial_x (|\varphi|^{2(\sigma-1)}) \mathcal{I}m(\bar{\varphi} \psi) dx - \frac{1}{4} |\varphi|^{4\sigma}. \end{aligned}$$

Hence, since u solves (1.1) we have

$$\begin{aligned}
L\varphi &= L(\exp(i\Lambda))u + \exp(i\Lambda)Lu + 2\partial_x(\exp(i\Lambda))\partial_x u \\
&= L(\exp(i\Lambda))u + \exp(i\Lambda)(Lu + i|u|^{2\sigma}u) \\
&= L(\exp(i\Lambda))u \\
&= (i\partial_t + \partial_x^2)(\exp(i\Lambda))u, \\
&= \left[-\exp(i\Lambda)\partial_t\Lambda + \partial_x(\exp(i\Lambda)\frac{i}{2}|u|^{2\sigma}) \right] u \\
&= -\varphi\partial_t\Lambda + \left[\exp(i\Lambda)\frac{-1}{4}|u|^{2\sigma} + \frac{i}{2}\exp(i\Lambda)\partial_x(|u|^{2\sigma}) \right] u \\
&= -\varphi\partial_t\Lambda + \varphi \left[-\frac{1}{4}|\varphi|^{4\sigma} + \frac{i}{2}\partial_x(|\varphi|^{2\sigma}) \right] \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - \sigma\varphi \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dx + \frac{1}{4}|\varphi|^{4\sigma}\varphi - \frac{1}{4}\varphi|\varphi|^{4\sigma} + i\sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi) \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) + i\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)) - \sigma\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}(\sigma-1)\partial_x(|\varphi|^2)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dx \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) + i\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)) - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy.
\end{aligned}$$

As in [3][section 4], we have

$$\begin{aligned}
L\psi &= L(\exp(i\Lambda)\partial_x u) \\
&= \exp(i\Lambda) \left[-\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u + \sigma|u|^{2(\sigma-1)}\mathcal{I}\mathfrak{m}(\bar{u}\partial_x u)\partial_x u - \sigma \int_{-\infty}^x \mathcal{I}\mathfrak{m}(\partial_x(|u|^{2(\sigma-1)}\bar{u}))\partial_x u dy \partial_x u \right] \\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi)\psi - \sigma \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{u}\partial_x u) dy \psi \\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\psi\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - \sigma\psi \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - i\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)) - \sigma\psi \int_{-\infty}^x (\sigma-1)|\varphi|^{2(\sigma-1)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - i\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)) - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy.
\end{aligned}$$

Thus, if u solves (1.1) then (φ, ψ) solves

$$\begin{cases} L\varphi &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy, \\ L\psi &= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy. \end{cases} \quad (2.3)$$

For convenience, we define

$$P(\varphi, \psi) = i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2), \quad (2.4)$$

$$Q(\varphi, \psi) = -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2). \quad (2.5)$$

Let R be multi-soliton profile defined as in Section 1.2. Define

$$h = \exp\left(\frac{i}{2} \int_{-\infty}^x |R(t, x)|^{2\sigma} dy\right) R(t, x),$$

$$k = \partial_x h - \frac{i}{2}|h|^{2\sigma}h.$$

Since for each $1 \leq j \leq K$, R_j solves (1.1) we have

$$LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x, \quad (2.6)$$

Set $\lambda = \frac{1}{16}v_*$. By Lemma 3.1 for $t \gg T_0$ large enough we have

$$\|-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x\|_{H^2} \leq e^{-\lambda t}. \quad (2.7)$$

Thus, we rewrite (2.6) as follows:

$$LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x, \quad (2.8)$$

where

$$v = e^{\lambda t}(-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x). \quad (2.9)$$

By elementary calculation, we have

$$\begin{cases} Lh = i\sigma|h|^{2(\sigma-1)}h^2\bar{k} - \sigma(\sigma-1)h \int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\bar{h}^2) dy + e^{-\lambda t}m(t, x), \\ Lk = -i\sigma|h|^{2(\sigma-1)}k^2\bar{h} - \sigma(\sigma-1)k \int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\bar{h}^2) dy + e^{-\lambda t}n(t, x). \end{cases} \quad (2.10)$$

where

$$m = \exp\left(\frac{i}{2} \int_{-\infty}^x |R|^{2\sigma} dy\right) v - \sigma h \int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\bar{R}v) dy, \quad (2.11)$$

$$n = \exp\left(\frac{i}{2} \int_{-\infty}^x |R|^{2\sigma} dy\right) e^{-\lambda t}(\partial_x v - \sigma \partial_x R \int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\bar{R}v) dy). \quad (2.12)$$

Since v is uniformly bounded in $H^2(\mathbb{R})$, we see that m, n are uniformly bounded in $H^1(\mathbb{R})$. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $(\tilde{\varphi}, \tilde{\psi})$ solves:

$$\begin{cases} L\tilde{\varphi} = P(\varphi, \psi) - P(h, k) - e^{-\lambda t}m(t, x), \\ L\tilde{\psi} = Q(\varphi, \psi) - Q(h, k) - e^{-\lambda t}n(t, x). \end{cases} \quad (2.13)$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h, k)$ and $f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi))$ and $H = e^{-\lambda t}(m, n)$. We find solution of (2.13) in Duhamel form:

$$\eta(t) = i \int_t^\infty [f(W + \eta) - f(W) + H](s) ds, \quad (2.14)$$

where $S(t)$ denote the Schrödinger group. Moreover, since $\psi = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$ we have

$$\tilde{\psi} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^{2\sigma}(\tilde{\varphi} + h) - |h|^{2\sigma}h). \quad (2.15)$$

Step 2. Existence solution of system equations

Since Lemma 3.4 there exists $T_* \gg 1$ such that for $T_0 \geq T_*$ there exists unique solution η of (2.13) define on $[T_0, T_*)$ such that

$$\|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial_x \eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1 \quad \forall t \geq T_0. \quad (2.16)$$

for the constant λ defined as in step 1. Thus, for all $t \geq T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}. \quad (2.17)$$

Step 3. Existence of multi-soliton trains of (1.1) We prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (2.13) satisfies relation (2.15). Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and $v = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$ and $\tilde{v} = v - k$. Since $(\tilde{\varphi}, \tilde{\psi})$ solves (2.13) and (h, k) solves (2.10) we have (φ, ψ) solves (2.3). Furthermore,

$$Lv = \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi). \quad (2.18)$$

Moreover,

$$\begin{aligned}
L(|\varphi|^{2\sigma}\varphi) &= (i\partial_t + \partial_x^2)(\varphi^{\sigma+1}\bar{\varphi}^\sigma) = i\partial_t(\varphi^{\sigma+1}\bar{\varphi}^\sigma) + \partial_x^2(\varphi^{\sigma+1}\bar{\varphi}^\sigma) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\bar{\varphi} + \partial_x((\sigma+1)|\varphi|^{2\sigma}\partial_x\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\bar{\varphi}) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\bar{\varphi} + (\sigma+1)[\partial_x^2\varphi|\varphi|^{2\sigma} + \partial_x\varphi\partial_x(|\varphi|^{2\sigma})] \\
&\quad + \sigma\left[\partial_x^2\bar{\varphi}|\varphi|^{2(\sigma-1)}\varphi^2 + (\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + (\sigma-1)|\varphi|^{2(\sigma-2)}\varphi^3(\partial_x\bar{\varphi})^2\right] \\
&= (\sigma+1)|\varphi|^{2\sigma}(i\partial_t\varphi + \partial_x^2\varphi) + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(i\partial_t\bar{\varphi} + \partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3 \\
&= (\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3.
\end{aligned}$$

Combining with (2.18), using (2.3) we have

$$\begin{aligned}
Lv &= \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi) \\
&= \partial_x L\varphi - \frac{i}{2}\left[(\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\right. \\
&\quad \left.+ \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right] \\
&= \partial_x(P(\varphi, \psi) - P(\varphi, v)) + \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) \\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right] \\
&= \partial_x(P(\varphi, \psi) - P(\varphi, v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}) + G(\varphi, v),
\end{aligned}$$

where $G(\varphi, v)$ is the remaining ingredients and $G(\varphi, v)$ depend only on φ and v :

$$\begin{aligned}
G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right].
\end{aligned} \tag{2.19}$$

As the calculations of $L\psi$ in step 1, noting that the role of v is similar the one of ψ in step 1, we have $G(\varphi, v) = Q(\varphi, v)$ (see Lemma 3.3 for a detail proof of $G(\varphi, v) = Q(\varphi, v)$). Hence,

$$\begin{aligned}
L\psi - Lv &= Q(\varphi, \psi) - Q(\varphi, v) - \partial_x(P(\varphi, \psi) - P(\varphi, v)) + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) \\
&\quad - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}).
\end{aligned}$$

Thus,

$$\begin{aligned}
L\tilde{\psi} - L\tilde{v} &= L\psi - Lv \\
&= Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k) - \partial_x(P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) \\
&\quad + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi} + k)} - \overline{P(\varphi, \tilde{v} + k)}).
\end{aligned} \tag{2.20}$$

Multiplying both side of (2.20) by $\overline{\tilde{\psi} - \tilde{v}}$, taking imaginary part and integrating over space with integration by parts we obtain

$$\begin{aligned} & \frac{1}{2} \partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \\ &= \mathcal{I}m \int_{\mathbb{R}} (Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \end{aligned} \quad (2.21)$$

$$- \mathcal{I}m \int_{\mathbb{R}} \partial_x (P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \quad (2.22)$$

$$+ (\sigma + 1) \mathcal{I}m \int_{\mathbb{R}} \frac{i}{2} |\varphi|^{2\sigma} (P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \quad (2.23)$$

$$- \sigma \mathcal{I}m \int_{\mathbb{R}} \frac{i}{2} |\varphi|^{2(\sigma-1)} \varphi^2 (\overline{P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)}) (\overline{\tilde{\psi} - \tilde{v}}) dx. \quad (2.24)$$

We define A, B, C, D are (2.21), (2.22), (2.23) and (2.24) respectively. Firstly, we try to estimate A, B, C, D . We have

$$\begin{aligned} |A| &\lesssim \left| \int_{\mathbb{R}} (Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[(\tilde{\psi} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy - (\tilde{v} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{v} + k)^2 \overline{\varphi}^2) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[(\tilde{\psi} - \tilde{v}) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[(\tilde{v} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy \right\|_{L_x^\infty} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\tilde{v} + k\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right\|_{L_x^\infty} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L_x^1} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)\|_{L^1} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L^1} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_1, \end{aligned} \quad (2.25)$$

where,

$$K_1 := \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L^1} + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Furthermore,

$$\begin{aligned}
|B| &\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} - \tilde{v}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \left(\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right) (\tilde{\psi} - \tilde{v}) dx \right| \\
&\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2) (\tilde{\psi} - \tilde{v})^2 dx \right| + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 \frac{1}{2} \partial_x ((\tilde{\psi} - \tilde{v})^2) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy (\tilde{\psi} - \tilde{v}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \varphi |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) (\tilde{\psi} - \tilde{v}) dx \right|.
\end{aligned} \tag{2.26}$$

By using integration by parts for second term of (2.26) and using Hölder inequality we have

$$\begin{aligned}
|B| &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy \right\|_{L^\infty} \|\tilde{\psi} - \tilde{v}\|_{L^2} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} - \tilde{v}) (\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_2,
\end{aligned} \tag{2.27}$$

where

$$K_2 := \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\partial_x \varphi\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty}.$$

Using (2.4), we have

$$\begin{aligned}
|C| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2\sigma} |\varphi|^{2(\sigma-1)} \varphi^2 (\tilde{\psi} - \tilde{v})^2 dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2\sigma} \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy (\tilde{\psi} - \tilde{v}) dx \right| \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy \right\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} - \tilde{v}) (\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_3,
\end{aligned} \tag{2.28}$$

where

$$K_3 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Now, we give an estimate for D . We have

$$\begin{aligned}
|D| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 |\varphi|^{2(\sigma-1)} \bar{\varphi}^2 (\tilde{\psi} - \tilde{v})(\bar{\tilde{\psi}} - \bar{\tilde{v}}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\bar{\varphi}^2((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy (\bar{\tilde{\psi}} - \bar{\tilde{v}}) dx \right| \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\bar{\varphi}^2((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} - \tilde{v})(\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_4, \tag{2.29}
\end{aligned}$$

where

$$K_4 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Combining (2.25), (2.27), (2.28) and (2.29) we have

$$\left| \partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \right| \lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 (K_1 + K_2 + K_3 + K_4).$$

Let $N > t \gg 1$. Integrating over time from t to N we have

$$\int_t^N \left| \frac{\partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2}{\|\tilde{\psi} - \tilde{v}\|_{L^2}^2} ds \right| \lesssim \int_t^N (K_1 + K_2 + K_3 + K_4) ds,$$

This implies that

$$\text{Ln}(\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2) - \text{Ln}(\|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2) \lesssim \int_t^N \left| \frac{\partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2}{\|\tilde{\psi} - \tilde{v}\|_{L^2}^2} ds \right| \lesssim \int_t^N (K_1 + K_2 + K_3 + K_4) ds,$$

Hence, using (2.16) and (2.17) we have

$$\begin{aligned}
\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right) \\
&\leq e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right). \tag{2.30}
\end{aligned}$$

Now, we try to estimate $K_1 + K_2 + K_3 + K_4$ in term of R . When we have this kind of estimate, we will use the assumption (1.10) to obtain that $\tilde{\psi} = \tilde{v}$. We have

$$\begin{aligned}
&\int_t^N (K_1 + K_2 + K_3 + K_4) ds \\
&= \int_t^N \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi} + k)^2\|_{L^1} + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.31}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x\varphi\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\varphi^{2\sigma-1}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} ds \tag{2.32}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.33}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.34}
\end{aligned}$$

Using (2.16) and (2.17), we have

$$\|\varphi\|_{L^\infty} \leq \|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty} \quad (2.35)$$

$$\|\varphi\|_{L^2} \leq \|\tilde{\varphi}\|_{L^2} + \|h\|_{L^2} \lesssim 1 + \|h\|_{L^2} \quad (2.36)$$

$$\|\psi\|_{L^\infty} \lesssim 1 \quad (2.37)$$

We denote Z_1, Z_2, Z_3, Z_4 as (2.31), (2.32), (2.33) and (2.34) respectively. Using (2.35), (2.36), (2.37), (2.16) and (2.17), for $N \gg t$, we have

$$\begin{aligned} |Z_1| &\lesssim \|\varphi\|_{L^4(t,N)L^\infty}^3 \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-2)} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^4(t,N)L^\infty} + (N-t) \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2})^2 \\ &\quad + \|\tilde{v} + k\|_{L^{\frac{4}{3}}(t,N)L^2} \|\varphi\|_{L^\infty L^2} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi} + \tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\ &\lesssim (N-t)^{\frac{3}{4}} \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} (1 + \|k\|_{L^\infty L^\infty} (N-t)^{\frac{1}{4}}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}^2) \\ &\quad + (N-t)^{\frac{3}{4}} (1 + \|k\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + (N-t)^{\frac{1}{4}} \|k\|_{L^\infty L^\infty}) \\ &\lesssim (N-t) \|k\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}^2) \\ &\quad + (N-t) \|k\|_{L^\infty L^\infty} (1 + \|k\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) \\ &:= (N-t)W_1(h, k). \end{aligned}$$

Similarly, for $N \gg t$, we have

$$\begin{aligned} |Z_2| &\lesssim \|\partial_x \varphi \varphi^{2\sigma-1}\|_{L^1(t,N)L^\infty} + (N-t) \|\partial_x \varphi\|_{L^\infty(t,N)L^2} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} \|\tilde{\psi} + \tilde{v} + k\|_{L^\infty(t,N)L^2} \\ &\quad + (N-t)^{\frac{3}{4}} \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} (\|\tilde{\psi} + \tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\ &\lesssim (N-t)^{\frac{3}{4}} (\|\partial_x \tilde{\varphi}\|_{L^4(t,N)L^\infty} + \|\partial_x h\|_{L^4(t,N)L^\infty}) \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}) \\ &\quad + (N-t)^{\frac{3}{4}} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) (1 + (N-t)^{\frac{1}{4}} \|k\|_{L^\infty L^\infty}) \\ &\lesssim (N-t) \|\partial_x h\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}) \\ &\quad + (N-t) \|k\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) \\ &:= (N-t)W_2(h, k), \end{aligned}$$

and

$$\begin{aligned} |Z_3| = |Z_4| &\lesssim (N-t) (\|\tilde{\varphi}\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty})^{4\sigma} + (N-t) \|\varphi\|_{L^\infty L^2} \|\varphi\|_{L^\infty L^\infty}^{2\sigma} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi} + \tilde{v}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2}) \\ &\lesssim (N-t) (1 + \|h\|_{L^\infty L^\infty}^{4\sigma}) + (N-t) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{4\sigma-2}) (1 + \|k\|_{L^\infty L^2}) \\ &:= (N-t)W_3(h, k). \end{aligned}$$

Hence, since (2.30), we obtain

$$\begin{aligned} \|\psi(\tilde{t}) - v(\tilde{t})\|_{L^2}^2 &\lesssim e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right) \\ &\lesssim e^{-2\lambda N} \exp((N-t)(W_1(h, k) + W_2(h, k) + W_3(h, k))) \end{aligned} \quad (2.38)$$

The above estimate is not enough explicit. As we mention above, we will estimate the right hand side of (2.38) in term of R . Noting that $|h| = |R|$ and $|k| = |\partial_x R|$, we have

$$\begin{aligned} W_1(h, k) &= \|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty L^\infty}^{2\sigma-1}) + (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|\partial_x R\|_{L^\infty L^2}^2) \\ &\quad + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|\partial_x R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) \\ &\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [\|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty L^\infty}) + (1 + \|\partial_x R\|_{L^\infty L^2}) \\ &\quad + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|\partial_x R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^2})] \\ &\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [\|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty H^1}) + (1 + \|R\|_{L^\infty H^1}^2) + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty H^1}^2)] \\ &\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|R\|_{L^\infty H^1}^2) (1 + \|\partial_x R\|_{L^\infty L^\infty}). \end{aligned}$$

Similarly, by noting that $|\partial_x h| \leq |k| + |h|^{2\sigma+1}$ we have

$$\begin{aligned}
W_2(h, k) &\lesssim (\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1})(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|h\|_{L^\infty L^\infty}) + (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|k\|_{L^\infty L^2}) \\
&\quad + \|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|h\|_{L^\infty L^\infty}) \\
&\lesssim (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1})(1 + \|h\|_{L^\infty L^\infty}) + (1 + \|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty})] \\
&\lesssim (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(1 + \|h\|_{L^\infty L^\infty})(\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|k\|_{L^\infty L^2})] \\
&= (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(1 + \|R\|_{L^\infty L^\infty})(\|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|\partial_x R\|_{L^\infty L^2})] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1})(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}),
\end{aligned}$$

and

$$\begin{aligned}
W_3(h, k) &= (1 + \|R\|_{L^\infty L^\infty}^{4\sigma}) + (1 + \|R\|_{L^\infty L^2})(1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|\partial_x R\|_{L^\infty L^2}) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2}) [(1 + \|R\|_{L^\infty L^\infty}^2) + (1 + \|R\|_{L^\infty L^2})(1 + \|\partial_x R\|_{L^\infty L^2})] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|R\|_{L^\infty H^1}^2).
\end{aligned}$$

By combining the above estimates we have

$$\begin{aligned}
W_1(h, k) + W_2(h, k) + W_3(h, k) &\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|R\|_{L^\infty H^1}^2) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \\
&\quad + (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty L^\infty}^{2\sigma})(1 + \|R\|_{L^\infty H^1}^2) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}).
\end{aligned}$$

Noting that the notation " \lesssim " in the above estimate is completely independent on R and parameters $\omega_1, \dots, \omega_K, c_1, \dots, c_K$. Hence, using the assumption (1.10), we obtain that

$$W_1(h, k) + W_2(h, k) + W_3(h, k) \leq \lambda,$$

for t large enough. Thus, (2.38) is rewritten as

$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 \leq e^{-2\lambda N + (N-t)\lambda},$$

for t large enough. Letting $N \rightarrow \infty$ in two sides of the above estimate, we obtain

$$\|\tilde{\psi}(t) - \tilde{v}\|_{L^2}^2 = 0,$$

for all t enough large. Hence, we have proved that

$$\tilde{\psi} = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi - k, \quad (2.39)$$

and then

$$\psi = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi.$$

Moreover, since $(\tilde{\psi}, \tilde{\varphi})$ solves (2.13) we have (ψ, φ) solves (2.3). Combining with (2.39) if we set $u = \exp\left(-\frac{i}{2} \int_{-\infty}^x |\varphi|^{2\sigma} dy\right) \varphi$ then u solves (1.1). Furthermore,

$$\begin{aligned}
\|u - R\|_{H^1} &= \left\| \exp\left(-\frac{i}{2} \int_{-\infty}^x |\varphi|^{2\sigma} dy\right) \varphi - \exp\left(\frac{i}{2} \int_{-\infty}^x |h|^{2\sigma} dy\right) h \right\|_{H^1} \lesssim C(\|\varphi\|_{H^1}, \|h\|_{H^1}) \|\varphi - h\|_{H^1} \\
&\lesssim \|\tilde{\varphi}\|_{H^1} \lesssim e^{-\lambda t},
\end{aligned}$$

Thus for t large enough, we have

$$\|u - R\|_{H^1} \leq C e^{-\lambda t}, \quad (2.40)$$

for $\lambda = \frac{1}{16} v_*$ and $C = C(\omega_1, \dots, \omega_K, c_1, \dots, c_K)$ depend on parameters. Moreover, choosing $t \gg T_0$ to be large enough depending parameters $\omega_1, \dots, \omega_K, c_1, \dots, c_K$, we may reduce the constant C in the right hand side of (2.40) independent on any parameter. This completes the proof of Theorem 1.1.

Remark 2.1. In the case $\sigma = 1$, the integrals in (2.3) disappear. In the case, $\sigma = 2$, the integrals (2.3) reduce into $\int_{-\infty}^x \mathcal{I}m(\psi^2 \bar{\varphi}^2) dy$, we do not need use the inequality (3.2). Thus, in these cases, by a similar arguments in proof of Theorem 1.1 we may obtain similar existence results of multi-solitons of (1.1) when $\sigma = 1$ or $\sigma = 2$.

3. SOME TECHNICAL LEMMAS

3.1. Properties of solitons. In this section, we give the proof of (2.7). We have the following result.

Lemma 3.1. *There exist $C > 0$ and a constant $\lambda > 0$ such that for $t > 0$ large enough, the estimate (2.7) holds uniformly in t .*

Proof. First, we need some estimates on profile. We have

$$\begin{aligned} |R_j(t, x)| &= |\psi_{\omega_j, c_j}(t, x)| = |\phi_{\omega_j, c_j}(x - c_j t)| = |\varphi_{\omega_j, c_j}(x - c_j t)| \approx \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j} \left(\cosh(\sigma h_j(x - c_j t)) - \frac{c_j}{2\sqrt{\omega_j}} \right)} \right)^{\frac{1}{2\sigma}} \\ &\lesssim \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j} \left(\cosh(\sigma h_j(x - c_j t)) - \frac{|c_j|}{2\sqrt{\omega_j}} \cosh(\sigma h_j(x - c_j t)) \right)} \right)^{\frac{1}{2\sigma}} \\ &\lesssim \left(\frac{4\omega_j - c_j^2}{(2\sqrt{\omega_j} - |c_j|) \cosh(\sigma h_j(x - c_j t))} \right)^{\frac{1}{2\sigma}} \lesssim \left(\frac{2\sqrt{\omega_j} + |c_j|}{\cosh(\sigma h_j(x - c_j t))} \right)^{\frac{1}{2\sigma}} \\ &\lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|}, \end{aligned}$$

Furthermore,

$$\partial_x \varphi_{\omega_j, c_j}(y) \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{\sqrt{\omega_j}} \right)^{1 + \frac{1}{2\sigma}}}.$$

Thus,

$$\begin{aligned} |\partial_x \varphi_{\omega_j, c_j}(y)| &\lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{|\sinh(\sigma h_j y)|}{\left(1 - \frac{|c_j|}{\sqrt{\omega_j}} \right)^{1 + \frac{1}{2\sigma}} \cosh(\sigma h_j y)^{1 + \frac{1}{2\sigma}}} \\ &\lesssim_{\omega_j, |c_j|} \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|y|}, \end{aligned}$$

Using the above estimates, we have

$$\begin{aligned} |\partial_x R_j(t, x)| &= |\partial_x \psi_{\omega_j, c_j}(t, x)| = |\partial_x \phi_{\omega_j, c_j}(x - c_j t)| = |\partial_x \varphi_{\omega_j, c_j}(x - c_j t) + i\varphi_{\omega_j, c_j}(x - c_j t) \partial_x \theta_{\omega_j, c_j}(x - c_j t)| \\ &\lesssim |\partial_x \varphi_{\omega_j, c_j}(x - c_j t)| + |\varphi_{\omega_j, c_j}(x - c_j t)| |\partial_x \theta_{\omega_j, c_j}(x - c_j t)| \\ &\lesssim_{\omega_j, |c_j|} |\partial_x \varphi_{\omega_j, c_j}(x - c_j t)| + e^{-\frac{h_j}{2}|x - c_j t|} \\ &\lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|}. \end{aligned}$$

By similar arguments we have

$$|\partial_x^2 R_j(t, x)| + |\partial_x^3 R_j(t, x)| \lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|},$$

Now, we comeback to prove Lemma 3.1. For convenience, we set

$$\begin{aligned} \chi &= -i|R|^{2\sigma} \partial_x R + i\Sigma_j |R_j|^{2\sigma} \partial_x R_j, \\ f(R, \bar{R}, \partial_x R) &= i|R|^{2\sigma} \partial_x R, \\ g(R, \bar{R}, \partial_x R, \partial_x \bar{R}, \partial_x^2 R) &= i\partial_x (|R|^{2\sigma} \partial_x R), \\ r(R, \partial_x R, \dots, \partial_x^3 R, \partial_x \bar{R}, \partial_x^2 \bar{R}) &= i\partial_x^2 (|R|^{2\sigma} \partial_x R). \end{aligned}$$

Fix $t > 0$, for each $x \in \mathbb{R}$, choose $m = m(x) \in \{1, 2, \dots, K\}$ so that

$$|x - c_m t| = \min_j |x - c_j t|.$$

For $j \neq m$ we have

$$|x - c_j t| \geq \frac{1}{2}(|x - c_j t| + |x - c_m t|) \geq \frac{1}{2}|c_j t - c_m t| = \frac{t}{2}|c_j - c_m|.$$

Thus, we have

$$\begin{aligned} & |(R - R_m)(t, x)| + |\partial_x(R - R_m)(t, x)| + |\partial_x^2(R - R_m)(t, x)| + |\partial_x^3(R - R_m)(t, x)| \\ & \leq \sum_{j \neq m} (|R_j(t, x)| + |\partial_x R_j(t, x)| + |\partial_x^2 R_j(t, x)| + |\partial_x^3 R_j(t, x)|) \\ & \lesssim_{\omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|} \delta_m(t, x) := \sum_{j \neq m} e^{-\frac{h_j}{2}|x - c_j t|}. \end{aligned}$$

Recall that

$$v_* = \inf_{j \neq k} h_j |c_j - c_k|.$$

We have

$$|(R - R_m)(t, x)| + |\partial_x(R - R_m)(t, x)| + |\partial_x^2(R - R_m)(t, x)| + |\partial_x^3(R - R_m)(t, x)| \lesssim \delta_m(t, x) \lesssim e^{-\frac{1}{4}v_* t}.$$

□

We see that f, g, r are polynomials of $R, \partial_x R, \partial_x^2 R, \partial_x^3 R, \partial_x \bar{R}$ and $\partial_x^2 \bar{R}$. Denote

$$A = \sup_{|z| + |\partial_x z| + |\partial_x^2 z| + |\partial_x^3 z| \leq \sum_j \|R_j\|_{H^4}} (|df| + |dg| + |dr|).$$

where we denote $|df(x, y, z, \dots)| = |\partial_x f| + |\partial_y f| + |\partial_z f| + \dots$ as norm of differential of f . We have

$$\begin{aligned} & |\chi| + |\partial_x \chi| + |\partial_x^2 \chi| \\ & \leq |f(R, \bar{R}, \partial_x R) - f_{R_m, \partial_x \bar{R}_m, R_m}| + |g(R, \bar{R}, \partial_x R, \dots) - g(R_m, \bar{R}_m, \partial_x R_m, \dots)| \\ & \quad + |r(R, \partial_x R, \dots, \partial_x^3 R, \bar{R}, \dots) - r(R_m, \partial_x R_m, \dots, \partial_x^3 R_m, \bar{R}_m, \dots)| \\ & + \sum_{j \neq m} (f(R_j, \bar{R}_j, \partial_x R_j) + g(R_j, \partial_x R_j, \partial_x^2 R_j, \bar{R}_j, \partial_x \bar{R}_j) + r(R_j, \dots, \partial_x^3 R_j, \bar{R}_j, \dots, \partial_x^2 \bar{R}_j)) \\ & \lesssim A(|R - R_m| + |\partial_x(R - R_m)| + |\partial_x^2(R - R_m)| + |\partial_x^3(R - R_m)|) + A \sum_{j \neq m} (|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\ & \lesssim 2A \sum_{j \neq m} (|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\ & \lesssim 2A \delta_m(t, x). \end{aligned}$$

In particular,

$$\|\chi\|_{W^{2, \infty}} \lesssim e^{-\frac{1}{4}v_* t}. \quad (3.1)$$

Moreover,

$$\begin{aligned} \|\chi\|_{W^{2, 1}} & \lesssim \sum_j (\| |R_j|^{2\sigma} \partial_x R_j \|_{L^1} + \|\partial_x (|R_j|^{2\sigma} \partial_x R_j)\|_{L^1} + \|\partial_x^2 (|R_j|^{2\sigma} \partial_x R_j)\|_{L^1}) \\ & \lesssim \sum_j (\|R_j\|_{H^1}^{2\sigma+1} + \|R_j\|_{H^2}^{2\sigma+1} + \|R_j\|_{H^3}^{2\sigma+1}) < \infty. \end{aligned}$$

Thus, using Hölder inequality we obtain

$$\|\chi\|_{H^2} \lesssim_{\omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|} e^{-\frac{1}{8}v_* t}.$$

It follows that if $t \gg \max\{\omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|\}$ is large enough then

$$\|\chi\|_{H^2} \leq e^{-\frac{1}{16}v_* t}.$$

Set $\lambda = \frac{1}{16}v_*$ we obtain the desired result.

3.2. Some useful estimates.

Lemma 3.2. *Let $x \geq 0$. Then there exists $C = C(x)$ such that*

$$(a+b)^x - a^x \leq C(x)(b^x + ba^{x-1}). \quad (3.2)$$

for all $a, b \geq 0$.

Proof. If $x = 0$ or $x = 1$ or $b = 0$ or $a = 0$ then (3.2) is true for $C(x) = 1$. Consider $a, b > 0$. If $0 < x < 1$ then using $m^x > m$ for $m < 1$ and $0 < x < 1$ we have

$$\left(\frac{a}{a+b}\right)^x + \left(\frac{b}{a+b}\right)^x > \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Hence,

$$(a+b)^x < a^x + b^x,$$

if we choose $C(x) = 1$ then (3.2) holds. Now, considering $a, b > 0$ and $x > 1$, we set

$$g(z) = z^x, \quad \forall z \in \mathbb{R}.$$

We have g is class C^1 . Thus, there exists $\xi \in (a, a+b)$ such that

$$|(a+b)^x - a^x| = |g(a+b) - g(a)| = |bg'(\xi)| = bx\xi^{x-1} < xb(a+b)^{x-1}.$$

If $x - 1 \leq 1$ then $(a+b)^{x-1} \leq a^{x-1} + b^{x-1}$ and hence we choose $C(x) = x$. If $x - 1 > 1$ then by Jensen inequality for convex function $f(z) = z^{x-1}$ we have

$$\left(\frac{a+b}{2}\right)^{x-1} \leq \frac{a^{x-1} + b^{x-1}}{2}.$$

We obtain

$$(a+b)^x - a^x < xb(a+b)^{x-1} \leq 2^{x-2}xb(a^{x-1} + b^{x-1}).$$

Choosing $C(x) = 2^{x-2}x$ we obtain the desired result. \square

3.3. Proof $G(\varphi, v) = Q(\varphi, v)$. Let $G(\varphi, v)$ be defined as in (2.19) and Q be defined as in (2.5). Then we have the following result.

Lemma 3.3. *Let $v = \partial_x \varphi - \frac{i}{2}|\varphi|^2 \varphi$. Then the following equality holds:*

$$G(\varphi, v) = Q(\varphi, v).$$

Proof. We have

$$\begin{aligned} P(\varphi, v) &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{v} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy, \\ Q(\varphi, v) &= -i\sigma|\varphi|^{2(\sigma-1)}v^2\bar{\varphi} - \sigma(\sigma-1)v \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\ &\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]. \end{aligned}$$

The term contains $\int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy$ in the expression of $G(\varphi, v)$ is as follows:

$$\begin{aligned} & -\sigma(\sigma-1)\partial_x\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(-1)\sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ & + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-1)\sigma(\sigma-1)\bar{\varphi} \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ & = -\sigma(\sigma-1) \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \left(\partial_x\varphi - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\varphi + \frac{i}{2}\sigma|\varphi|^{2\sigma}\varphi\right) \\ & = -\sigma(\sigma-1) \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \left(\partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) \\ & = -\sigma(\sigma-1)v \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy, \end{aligned}$$

which is exactly the term contains $\int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{Im}(v^2 \bar{\varphi}^2) dy$ in the expression of $Q(\varphi, v)$. We only need to check the equality of the remaining terms. The remaining terms of $G(\varphi, v)$ is as follows:

$$\begin{aligned} & i\sigma \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2 \bar{v}) - \sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi \mathcal{Im}(v^2 \bar{\varphi}^2) - \frac{i}{2} (\sigma+1) |\varphi|^{2\sigma} (i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \bar{v}) \\ & + \frac{i}{2} \sigma |\varphi|^{2(\sigma-1)} \varphi^2 (-i\sigma |\varphi|^{2(\sigma-1)} \bar{\varphi}^2 v) - i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \partial_x^2 \bar{\varphi} \end{aligned} \quad (3.3)$$

$$- \frac{i}{2} \left[(\sigma+1) \partial_x \varphi \partial_x (|\varphi|^{2\sigma}) + \sigma(\sigma+1) |\partial_x \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + \sigma(\sigma-1) (\partial_x \bar{\varphi})^2 |\varphi|^{2(\sigma-2)} \varphi^3 \right]. \quad (3.4)$$

Noting that $\partial_x (|\varphi|^2) = 2\mathcal{Re}(v\bar{\varphi})$ and $v = \partial_x \varphi - \frac{i}{2} |\varphi|^{2\sigma} \varphi$, we have

the term (3.3)

$$\begin{aligned} & = i\sigma \partial_x (|\varphi|^{2(\sigma-1)}) \varphi^2 \bar{v} + i\sigma |\varphi|^{2(\sigma-1)} 2\varphi \partial_x \varphi \bar{v} + i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \partial_x \bar{v} - \sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi 2\mathcal{Re}(v\bar{\varphi}) \mathcal{Im}(v\bar{\varphi}) \\ & + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \bar{v} + \sigma^2 |\varphi|^{4\sigma-2} \varphi \mathcal{Re}(\varphi \bar{v}) - i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \partial_x^2 \bar{\varphi} \\ & = 2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \mathcal{Re}(v\bar{\varphi}) \varphi^2 \bar{v} + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \bar{v} + i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \partial_x (\bar{v} - \overline{\partial_x \varphi}) \\ & - 2\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi \mathcal{Re}(v\bar{\varphi}) \mathcal{Im}(v\bar{\varphi}) + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \bar{v} + \sigma^2 |\varphi|^{4\sigma-2} \varphi \mathcal{Re}(\varphi \bar{v}) \\ & = 2\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \mathcal{Re}(v\bar{\varphi}) \varphi (i\varphi \bar{v} - \mathcal{Im}(v\bar{\varphi})) + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \bar{v} + i\sigma |\varphi|^{2(\sigma-1)} \varphi^2 \partial_x \left(\frac{i}{2} |\varphi|^{2\sigma} \bar{\varphi} \right) \\ & + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \bar{v} + \sigma^2 |\varphi|^{4\sigma-2} \varphi \mathcal{Re}(\varphi \bar{v}) \\ & = 2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi (\mathcal{Re}(v\bar{\varphi}))^2 + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \varphi \bar{v} \\ & - \frac{1}{2} \sigma |\varphi|^{2(\sigma-1)} \varphi^2 (2\sigma |\varphi|^{2(\sigma-1)} \mathcal{Re}(v\bar{\varphi}) + |\varphi|^{2\sigma} \partial_x \bar{\varphi}) + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \bar{v} + \sigma^2 |\varphi|^{4\sigma-2} \varphi \mathcal{Re}(\varphi \bar{v}) \\ & = 2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi (\mathcal{Re}(v\bar{\varphi}))^2 + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \varphi \bar{v} \\ & - \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \partial_x \bar{\varphi} + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 \bar{v} \\ & = 2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi (\mathcal{Re}(v\bar{\varphi}))^2 + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \varphi \bar{v} + \frac{1}{2} \sigma |\varphi|^{4\sigma-2} \varphi^2 (\bar{v} - \partial_x \bar{\varphi}) \\ & = 2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi (\mathcal{Re}(v\bar{\varphi}))^2 + 2i\sigma |\varphi|^{2(\sigma-1)} \varphi \partial_x \varphi \bar{v} + \frac{i}{4} \sigma |\varphi|^{6\sigma} \varphi. \end{aligned}$$

Moreover, using $\mathcal{Re}(\partial_x \varphi \bar{\varphi}) = \mathcal{Re}(v\bar{\varphi})$ we have

the term (3.4)

$$\begin{aligned} & = \frac{-i}{2} \left[\sigma(\sigma+1) |\partial_x \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + \sigma(\sigma+1) |\varphi|^{2(\sigma-1)} \partial_x \varphi (\partial_x \bar{\varphi} \varphi + \partial_x \varphi \bar{\varphi}) + \sigma(\sigma-1) (\partial_x \bar{\varphi})^2 |\varphi|^{2(\sigma-2)} \varphi^3 \right] \\ & = \frac{-i}{2} \left[2\sigma |\partial \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + \sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \partial_x \bar{\varphi} \varphi^2 (\partial_x \varphi \bar{\varphi} + \partial_x \bar{\varphi} \varphi) + 2\sigma(\sigma+1) |\varphi|^{2(\sigma-1)} \partial_x \varphi \mathcal{Re}(v\bar{\varphi}) \right] \\ & = \frac{-i}{2} \left[2\sigma |\partial \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + 2\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \partial_x \bar{\varphi} \varphi^2 \mathcal{Re}(v\bar{\varphi}) + 2\sigma(\sigma+1) |\varphi|^{2(\sigma-1)} \partial_x \varphi \mathcal{Re}(v\bar{\varphi}) \right] \\ & = -i \left[\sigma |\partial \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + \sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \partial_x \bar{\varphi} \varphi^2 \mathcal{Re}(v\bar{\varphi}) + \sigma(\sigma+1) |\varphi|^{2(\sigma-1)} \partial_x \varphi \mathcal{Re}(v\bar{\varphi}) \right] \\ & = -i \left[\sigma |\partial \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + \sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \mathcal{Re}(v\bar{\varphi}) \varphi (\partial_x \bar{\varphi} \varphi + \partial_x \varphi \bar{\varphi}) + 2\sigma |\varphi|^{2(\sigma-1)} \partial_x \varphi \mathcal{Re}(v\bar{\varphi}) \right] \\ & = -i \left[\sigma |\partial \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi + 2\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} (\mathcal{Re}(v\bar{\varphi}))^2 \varphi \right] \\ & = -2i\sigma(\sigma-1) |\varphi|^{2(\sigma-2)} \varphi (\mathcal{Re}(v\bar{\varphi}))^2 - i\sigma |\partial_x \varphi|^2 |\varphi|^{2(\sigma-1)} \varphi - 2i\sigma |\varphi|^{2(\sigma-1)} \partial_x \varphi \mathcal{Re}(v\bar{\varphi}). \end{aligned}$$

Combining the above expressions we obtain

$$\begin{aligned}
& \text{the remaining term of } G(\varphi, v) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\bar{\varphi}) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(\varphi\bar{v} - \mathcal{R}e(v\bar{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{I}m(\varphi\bar{v}) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\bar{v}) + i\partial_x\bar{\varphi}\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\partial_x\bar{\varphi}) + |\varphi|^{2\sigma+2} + i\mathcal{R}e(\varphi\partial_x\bar{\varphi}) - \mathcal{I}m(\varphi\partial_x\bar{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(|\varphi|^{2\sigma+2} + i\bar{\varphi}\partial_x\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}(\partial_x\varphi)^2 - \sigma|\varphi|^{4\sigma}\partial_x\varphi + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)^2 - \sigma|\varphi|^{4\sigma}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}v^2.
\end{aligned}$$

This is exactly the remaining terms of $Q(\varphi, v)$. Thus, $G(\varphi, v) = Q(\varphi, v)$. \square

3.4. Existence solution of system equations. In this section, using similar arguments as in [5] and [6] we prove existence of solution of (2.13):

$$\eta(t) = i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s) ds, \quad (3.5)$$

where

$$W = (h, k), \quad (3.6)$$

$$H = e^{-\lambda t}(m, n), \quad (3.7)$$

$$f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi)). \quad (3.8)$$

We have the following lemma:

Lemma 3.4. *Let $H = H(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$, $W = W(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geq 0$:*

$$\|W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2 \times L^2} \leq C_1 \quad \forall t \geq T_0, \quad (3.9)$$

$$\|\partial W(t)\|_{L^2 \times L^2} + \|\partial W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2 \times L^2} \leq C_2, \quad \forall t \geq T_0. \quad (3.10)$$

Consider equation (3.5). There exists a constant λ_ independent of C_2 such that if $\lambda \geq \lambda_*$ then there exists a unique solution η to (3.5) on $[T_0, \infty) \times \mathbb{R}$ satisfying*

$$e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial\eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1, \quad \forall t \geq T_0.$$

Proof. We write (3.5) as $\eta = \Phi\eta$. We shall show that, for λ sufficiently large, Φ is contraction in the ball

$$B = \{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1\}.$$

We will use condition $\lambda \gg 1$ in our analysis without notation.

Step 1. Prove Φ map B into B

Let $t \geq T_0$, $\eta = (\eta_1, \eta_2) \in B$, $W = (w_1, w_2)$ and $H = (h_1, h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t, \infty)) \times N([t, \infty))}, \quad (3.11)$$

$$+ \|H\|_{L^1_t L^2_x([t, \infty)) \times L^1_t L^2_x([t, \infty))}. \quad (3.12)$$

For (3.12), using (3.9), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty)) \times L^1_\tau L^2_x([t,\infty))} = \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau} d\tau \leq \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \quad (3.13)$$

For (3.11), we have

$$\begin{aligned} |P(W + \eta) - P(W)| &= |P(w_1 + \eta_1, w_2 + \eta_2) - P(w_1, w_2)| \\ &\lesssim \left| |w_1 + \eta_1|^{2(\sigma-1)} (w_1 + \eta_1)^2 \overline{w_2 + \eta_2} - |w_1|^{2(\sigma-1)} w_1^2 \overline{w_2} \right| \end{aligned} \quad (3.14)$$

$$+ \left| (w_1 + \eta_1) \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\overline{w_1} + \overline{\eta_1})^2) - w_1 \int_{-\infty}^x |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \overline{\eta_1}^2) \right|. \quad (3.15)$$

Denote, using the assumption $\sigma \geq \frac{5}{2}$ and Lemma 3.2 we have

the term (3.14)

$$\begin{aligned} &\lesssim \left| |w_1 + \eta_1|^{2(\sigma-1)} - |w_1|^{2(\sigma-1)} \right| |w_1 + \eta_1|^2 |w_2 + \eta_2| + \left| |w_1|^{2(\sigma-1)} (w_1 + \eta_1)^2 - w_1^2 |w_2 + \eta_2| \right| \\ &\quad + \left| |w_1|^{2(\sigma-1)} |w_1|^2 |\eta_2| \right| \\ &\lesssim (|\eta_1|^{2(\sigma-1)} + |\eta_1| |w_1|^{2(\sigma-1)-1}) (|W| + |\eta|)^3 + |w_1|^{2(\sigma-1)} (|w_1| |\eta_1| + |\eta_1|^2) |w_2 + \eta_2| + |w_1|^{2\sigma} |\eta_2| \\ &\lesssim (|\eta|^{2(\sigma-1)} + |\eta| |W|^{2(\sigma-1)-1}) (|W|^3 + |\eta|^3) + |W|^{2(\sigma-1)} (|W| |\eta| + |\eta|^2) (|W| + |\eta|) + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta| (|\eta|^{2\sigma-3} + |W|^{2\sigma-3}) (|\eta|^3 + |W|^3) + |\eta| |W|^{2(\sigma-1)} (|W|^2 + |\eta|^2) + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta| (|\eta|^{2\sigma} + |W|^{2\sigma}) + |\eta| |W|^{2\sigma} + |\eta|^3 |W|^{2(\sigma-1)} + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta|^{2\sigma+1} + |\eta| |W|^{2\sigma}. \end{aligned}$$

Moreover,

the term (3.15)

$$\begin{aligned} &\lesssim |\eta_1| \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} |w_2 + \eta_2|^2 |w_1 + \eta_1|^2 dy \\ &\quad + |w_1| \int_{-\infty}^x (|w_1 + \eta_1|^{2(\sigma-2)} - |w_1|^{2(\sigma-2)}) |w_2 + \eta_2|^2 |w_1 + \eta_1|^2 dy \\ &\quad + |w_1| \int_{-\infty}^x |w_1|^{2(\sigma-2)} |\mathcal{I}m((w_2 + \eta_2)^2 - w_2^2) (\overline{w_1} + \overline{\eta_1})^2| dy \\ &\quad + |w_1| \int_{-\infty}^x |w_1|^{2(\sigma-2)} |\mathcal{I}m(w_2^2 ((\overline{w_1} + \eta_1)^2 - \overline{\eta_1}^2))| dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x (|\eta_1|^{2(\sigma-2)} + |\eta_1| |w_1|^{2\sigma-5}) (|W|^4 + |\eta|^4) dy \\ &\quad + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} (|\eta_2|^2 + |w_2| |\eta_2|) (|W|^2 + |\eta|^2) dy + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |w_2|^2 (|\eta_1|^2 + |\eta_1| |w_1|) dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| (|W|^{2\sigma} + |\eta|^{2\sigma}) dy \\ &\quad + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |\eta| (|W|^3 + |\eta|^3) dy + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |W|^2 |\eta| (|W| + |\eta|) dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| |W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &|P(W + \eta) - P(W)| \\ &\lesssim |\eta|^{2\sigma+1} + |\eta| |W|^{2\sigma} + |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| |W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Similarly,

$$\begin{aligned} & |Q(W + \eta) - Q(W)| \\ & \lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Hence, using $\sigma \geq \frac{5}{2}$, we have:

$$\begin{aligned} & \|f(W + \eta) - f(W)\|_{N([t, \infty)) \times N([t, \infty))} \\ & \lesssim \|P(W + \eta) - P(W)\|_{L_\tau^1 L_x^2([t, \infty))} + \|Q(W + \eta) - Q(W)\|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta|^{2\sigma+1} \|_{L_\tau^1 L_x^2([t, \infty))} + \| |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^2([t, \infty))} \\ & \quad + \| |W| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta| \|_{L^\infty L_x^2([t, \infty))} \| |\eta| \|_{L_\tau^4 L_x^\infty([t, \infty))} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy \|_{L_\tau^\infty L_x^\infty([t, \infty))} \\ & \quad + \| |W| \|_{L_\tau^\infty L_x^2([t, \infty))} \| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^\infty([t, \infty))} \\ & \lesssim e^{-5\lambda t} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| |W|^{2\sigma} + |\eta|^{2\sigma} \|_{L_\tau^\infty L_x^1} + \| W \|_{L_t^\infty L_x^2} \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| |W|^{2\sigma-1} + |\eta|^{2\sigma-1} \|_{L_\tau^\infty L_x^2([t, \infty))} \\ & \lesssim e^{-5\lambda t} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} = e^{-5\lambda t} + \int_t^\infty e^{-\lambda \tau} d\tau \lesssim e^{-5\lambda t} + \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \end{aligned}$$

Combining with (3.13) and (3.11), (3.12) we obtain

$$\|\Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} < \frac{1}{5} e^{-\lambda t}. \quad (3.16)$$

We have

$$\|\partial_x \Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} \lesssim \|\partial_x(f(W + \eta) - f(W))\|_{N([t, \infty)) \times N([t, \infty))} \quad (3.17)$$

$$+ \|\partial_x H\|_{L_\tau^1 L_x^2([t, \infty)) \times L_\tau^1 L_x^2([t, \infty))}. \quad (3.18)$$

For (3.18), using (3.10) we have

$$\|\partial_x H\|_{L_\tau^1 L_x^2([t, \infty)) \times L_\tau^1 L_x^2([t, \infty))} \lesssim \int_t^\infty e^{-\lambda \tau} d\tau = \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \quad (3.19)$$

For (3.17), we have

$$\|\partial_x(f(W + \eta) - f(W))\|_{N([t, \infty)) \times N([t, \infty))} = \|\partial_x(P(W + \eta) - P(W))\|_{N([t, \infty))} + \|\partial_x(Q(W + \eta) - Q(W))\|_{N([t, \infty))}.$$

Furthermore, using the notation 1.2 (3), we have

$$\begin{aligned} & |\partial_x(P(W + \eta) - P(W))| \\ & \lesssim |\partial_x(|w_1 + \eta_1|^{2(\sigma-1)}(w_1 + \eta_1)^2(\bar{w}_2 + \bar{\eta}_2) - |w_1|^{2(\sigma-1)}w_1^2\bar{w}_2)| \quad (3.20) \end{aligned}$$

$$+ \left| \partial_x(w_1 + \eta_1) \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2(\bar{w}_1 + \bar{\eta}_1)^2) dy - \partial_x w_1 \int_{-\infty}^x |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) dy \right| \quad (3.21)$$

$$+ \left| (w_1 + \eta_1) |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2(\bar{w}_1 + \bar{\eta}_1)^2) - w_1 |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) \right|. \quad (3.22)$$

For (3.20), we have

the term (3.20)

$$\lesssim (|\eta| + |\eta|^{2\sigma} + |\partial_x \eta|)(|W| + |W|^{2\sigma} + |\eta| + |\eta|^{2\sigma} + |\partial_x \eta|)$$

Thus,

$$\begin{aligned} & \|\text{the term (3.20)}\|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta| + |\partial_x \eta| \|_{L_\tau^1 L_x^2} \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \end{aligned}$$

For (3.21), using Lemma 3.2 we have

$$\begin{aligned}
& \| \text{the term (3.21)} \|_{L^1_\tau L^2_x(t, \infty)} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} \left\| \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) dy \right\|_{L^\infty_\tau L^\infty_x} \\
& \quad + \| \partial_x w_1 \|_{L^\infty_\tau L^2_x} \left\| \int_{-\infty}^x (|w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) - |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2)) dy \right\|_{L^1_\tau L^\infty_x} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} \| |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) \|_{L^\infty_t L^1_x} \\
& \quad + \| |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) - |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) \|_{L^1_\tau L^1_x} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} + \| |\eta| \|_{L^1_\tau L^2_x(t, \infty)} \leq \int_t^\infty e^{-\lambda\tau} d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},
\end{aligned}$$

For (3.22), using Lemma 3.2 we have

$$\| \text{the term (3.22)} \|_{L^1_\tau L^2_x(t, \infty)} \lesssim \| |\eta| \|_{L^1_\tau L^2_x(t, \infty)} \leq \int_t^\infty e^{-\lambda\tau} d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},$$

Combining the above we obtain

$$\| \partial_x (P(W + \eta) - P(W)) \|_{N(t, \infty)} \leq \| \partial_x (P(W + \eta) - P(W)) \|_{L^1_\tau L^2_x(t, \infty)} \leq \frac{3}{10} e^{-\lambda t}, \quad (3.23)$$

Similarly,

$$\| \partial_x (Q(W + \eta) - Q(W)) \|_{N(t, \infty)} \leq \frac{3}{10} e^{-\lambda t}, \quad (3.24)$$

Combining (3.17), (3.18), (3.19), (3.23), (3.24) we have

$$\| \partial_x \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} \leq \frac{7}{10} e^{-\lambda t}. \quad (3.25)$$

Combining (3.16) and (3.25) we obtain

$$\| \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} + \| \partial_x \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} \leq \frac{9}{10} e^{-\lambda t}, \quad (3.26)$$

Thus, for λ large enough

$$\| \Phi \eta \|_X < 1.$$

Which implies that Φ map B onto B .

Step 2. Φ is a contraction map on B

By using (3.9), (3.10) and similar estimate of (3.26), we can show that, for any $\eta \in B$ and $\kappa \in B$ we have

$$\| \Phi \eta - \Phi \kappa \|_X \leq \frac{1}{2} \| \eta - \kappa \|_X.$$

for λ large enough. By Banach fixed point theorem there exists unique solution on B of (3.5) and then solution of (2.13). \square

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