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**CONSTRUCTION OF THE MULTI-SOLITON TRAINS FOR A  
GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS  
BY A FIXED POINT METHOD**

PHAN VAN TIN

ABSTRACT. We consider a derivative nonlinear Schrödinger equation with general nonlinearity:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0,$$

In [12], the authors prove the stability of two solitary waves in energy space for  $\sigma \in (1, 2)$ . As a consequence, there exists a two-soliton trains in energy space for  $\sigma \in (1, 2)$ . Our goal in this paper is proving the existence of multi-soliton trains in energy space for  $\sigma \geq \frac{5}{2}$ . Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

CONTENTS

1. Introduction	1
1.1. Preliminaries	1
1.2. Multi-soliton trains	2
2. Proof of main result	4
3. Some technical lemmas	12
3.1. Properties of solitons	12
3.2. Some useful estimates	14
3.3. Proof $G(\varphi, v) = Q(\varphi, v)$	15
3.4. Existence solution of system equations	17
Acknowledgement	20
References	20

1. INTRODUCTION

We consider the following generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma} \partial_x u = 0, \tag{1.1}$$

where  $\sigma \in \mathbb{R}^+$  is a given constant and  $u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C}$ .

**1.1. Preliminaries.** The equation (1.1) is studied in many works before. In special case  $\sigma = 1$ , there are many works on local wellposedness, global well posedness, stability of solitary waves and stability of multi-solitons before. In [11] the author give a sufficient conditions for global well posedness of (1.1) in energy space by using a Gauge transformation to remove the derivative terms. In [1], the authors show that the equation have two parameters family of solitary waves. Moreover, the authors prove the stability of these particular solutions by using variational methods. In [4], the authors give a result on stability of solitary waves when parameters in the presentation of for soliton solutions take critical values. In [7], the authors prove the stability of multi-solitons in energy space under some conditions on parameters of individual solitons.

In general case, the local well posedness and global well posedness of (1.1) is studied in [3]. In [8], the authors prove the orbital stability/instability results of solitary waves. In [2], the authors give instability results of solitary waves in the critical frequency case. In [12], in case  $\sigma \in (1, 2)$ , the authors prove the stability of the sum of two solitary waves in energy space using perturbation

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argument, modulational analysis and the energy argument as in [9, 10]. In this paper, we show the existence of multi-soliton trains in energy space in case  $\sigma \geq \frac{5}{2}$ . Before state the main result, we give some preliminaries on multi-soliton trains of (1.1).

**1.2. Multi-soliton trains.** As in [8] the equation (1.1) admits a two-parameters family of solitary waves solutions,

$$\psi_{\omega,c}(t, x) = \varphi_{\omega,c}(x - ct) \exp i \left( \omega t + \frac{c}{2}(x - ct) - \frac{1}{2\sigma + 2} \int_{-\infty}^{x-ct} \varphi_{\omega,c}^{2\sigma}(\eta) d\eta \right), \quad (1.2)$$

where  $\omega > \frac{c^2}{4}$  and

$$\varphi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma + 1)(4\omega - c^2)}{2\sqrt{\omega} \left( \cosh(\sigma\sqrt{4\omega - c^2}y) - \frac{c}{2\sqrt{\omega}} \right)}. \quad (1.3)$$

is the positive solution of

$$-\partial_y^2 \varphi_{\omega,c} + \left( \omega - \frac{c^2}{4} \right) \varphi_{\omega,c} + \frac{c}{2} |\varphi_{\omega,c}|^{2\sigma} \varphi_{\omega,c} - \frac{2\sigma + 1}{(2\sigma + 2)^2} |\varphi_{\omega,c}|^{4\sigma} \varphi_{\omega,c} = 0. \quad (1.4)$$

Define

$$\phi_{\omega,c}(y) = \varphi_{\omega,c}(y) e^{i\theta_{\omega,c}(y)}, \quad (1.5)$$

where

$$\theta_{\omega,c} = \frac{c}{2} y - \frac{1}{2\sigma + 2} \int_{-\infty}^y \varphi_{\omega,c}^{2\sigma}(\eta) d\eta. \quad (1.6)$$

Clearly,

$$\psi_{\omega,c}(x, t) = e^{i\omega t} \phi_{\omega,c}(x - ct). \quad (1.7)$$

and  $\phi_{\omega,c}$  solves

$$-\partial_y^2 \phi_{\omega,c} + \omega \phi_{\omega,c} + ic \partial_y \phi_{\omega,c} - i |\phi_{\omega,c}|^{2\sigma} \partial_y \phi_{\omega,c} = 0, \quad y \in \mathbb{R}. \quad (1.8)$$

Let  $K \in \mathbb{N}$  and for each  $1 \leq j \leq K$  let  $(\omega_j, c_j, x_j, \theta_j) \in \mathbb{R}^4$  be given parameters such that  $\omega_j > \frac{c_j^2}{4}$ . Define

$$R_j(t, x) = e^{i\theta_j} \psi_{\omega_j, c_j}(t, x - x_j)$$

and define the multi-soliton profile

$$R = \sum_{j=1}^K R_j. \quad (1.9)$$

Our main result is the following.

**Theorem 1.1.** *Let  $\sigma \geq \frac{5}{2}$ ,  $K \in \mathbb{N}^*$  and for each  $1 \leq j \leq K$ ,  $(\theta_j, \omega_j, c_j, x_j)$  be a sequence of parameters such that  $x_j = 0$ ,  $\theta_j \in \mathbb{R}$ ,  $c_j \neq c_k \neq 0$ , for  $j \neq k$ . The multi-soliton profile  $R$  is given as in (1.9). We assume that the parameters  $(\omega_j, c_j)$  satisfy*

$$(1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \ll v_* = \inf_{j \neq k} h_j |c_j - c_k|. \quad (1.10)$$

Then there exists a solution  $u$  of (1.1) such that

$$\|u - R\|_{H^1} \leq C e^{-\lambda t}. \quad \forall t \geq T_0,$$

for some  $C > 0$ ,  $\lambda = \frac{1}{8} v_*$  and  $T_0 \gg 1$ .

*Remark 1.2.* By Lemma 3.2, the following inequality holds for  $\sigma \geq 2$ :

$$(a + b)^{2(\sigma-2)} - a^{2(\sigma-2)} \lesssim b^{2(\sigma-2)} + ba^{2(\sigma-2)-1}, \quad \text{for all } a, b > 0. \quad (1.11)$$

The condition  $\sigma \geq \frac{5}{2}$  ensures that the order of  $b$  on the right hand side of (1.11) is larger than 1. This is important in our analysis.

*Remark 1.3.* We prove there exist the parameters  $(\omega_j, c_j, \theta_j, x_j)$  for  $1 \leq j \leq K$  and  $x_j = 0$  for all  $j$  such that the condition (1.10) satisfies. For convenience, for each  $j$ , we define  $h_j = \sqrt{4\omega_j - c_j^2}$ . Indeed, choosing  $c_j < 0$  and  $h_j \ll \min(1, |c_j|)$  for all  $j$  we have

$$\begin{aligned}\varphi_{\omega_j, c_j}^{2\sigma} &\approx \frac{h_j^2}{2\sqrt{\omega_j} \left( \cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)} \\ \partial_x \varphi_{\omega_j, c_j} &\approx \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left( \cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)^{1+\frac{1}{2\sigma}}}.\end{aligned}$$

Using  $|\sinh(x)| \leq |\cosh(x)|$  for all  $x \in \mathbb{R}$  we have

$$|\partial_x \varphi_{\omega_j, c_j}| \leq \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{1}{\left( \cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}}} \lesssim |\varphi_{\omega_j, c_j}|.$$

Thus,

$$\begin{aligned}\|R_j\|_{L^\infty L^\infty} &= \|\varphi_{\omega_j, c_j}\|_{L^\infty} \lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \ll 1 \\ \|\partial_x R_j\|_{L^\infty L^\infty} &= \|\partial_x \phi_{\omega_j, c_j}\|_{L^\infty L^\infty} \\ &\lesssim \|\partial_x \varphi_{\omega_j, c_j}\|_{L^\infty} + \left\| \frac{c_j}{2} \varphi_{\omega_j, c_j} - \frac{1}{2\sigma+2} \varphi_{\omega_j, c_j}^{2\sigma+1} \right\|_{L^\infty} \\ &\lesssim \|\varphi_{\omega_j, c_j}\|_{L^\infty} + |c_j| \|\varphi_{\omega_j, c_j}\|_{L^\infty} \\ &\lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}.\end{aligned}$$

Hence,

$$\begin{aligned}\|R\|_{L^\infty L^\infty} &\lesssim \sum_j \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \lesssim 1 \\ \|\partial_x R\|_{L^\infty L^\infty} &\lesssim \sum_j \left( \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|R_j\|_{L^\infty H^1}^2 &= \|R_j\|_{L^\infty L^2}^2 + \|\partial_x R_j\|_{L^\infty L^2}^2 = \|\varphi_{\omega_j, c_j}\|_{L^2}^2 + \|\partial_x \varphi_{\omega_j, c_j}\|_{L^2}^2 \\ &\lesssim \|\varphi_{\omega_j, c_j}\|_{L^2}^2 \lesssim \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \left\| \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \right\|_{L^2}^2 \lesssim \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \|e^{-\frac{h_j}{2}|y|}\|_{L^2}^2 \\ &\approx \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{\sigma}} \frac{1}{h_j} \lesssim h_j^{\frac{1}{\sigma}} h_j^{-1} = h_j^{\frac{1}{\sigma}-1},\end{aligned}$$

where we use  $h_j \leq \sqrt{2\omega_j}$ . Thus,

$$\|R\|_{L^\infty H^1}^2 \lesssim \sum_j h_j^{\frac{1}{\sigma}-1}.$$

Condition (1.10) satisfies if the following estimate holds:

$$\left( 1 + \sum_j h_j^{\frac{1}{\sigma}-1} \right) \left( 1 + \sum_j \left( \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right) \right) \ll \inf_{j \neq k} h_j |c_j - c_k|. \quad (1.12)$$

Fixing  $h_j$  and replacing  $c_j$  by  $M c_j$ , thus  $\omega_j = \frac{1}{4}(h_j^2 + M^2 c_j^2)$ , we see that the left hand side of (1.12) is order  $M^{1-\frac{1}{2\sigma}}$  and the right hand side of (1.12) is order  $M^1$ . Hence, the condition (1.10) satisfies if we choose  $M$  large enough.

Before proving the main result, we introduce some notations which will use in our proof.

**Notation.**

(1) We denote the Schrödinger operator as follows

$$L = i\partial_t + \partial_x^2.$$

(2) Given a time  $t \in \mathbb{R}$ , the Strichartz space  $S([t, \infty))$  is defined via the norm

$$\|u\|_{S([t, \infty))} = \sup_{(q,r) \text{ admissible}} \|u\|_{L_t^q L^r_x([t, \infty) \times \mathbb{R})}.$$

We denote the dual space by  $N([t, \infty)) = S([t, \infty))^*$ . Hence for any  $(q, r)$  admissible pair we have

$$\|u\|_{N([t, \infty))} \leq \|u\|_{L^{q'} L^{r'}([t, \infty) \times \mathbb{R})}.$$

(3) For  $a, b \in \mathbb{R}^2$ , we denote  $|(a, b)| = |a| + |b|$ .

(4) Let  $a, b > 0$ . We denote  $a \lesssim b$  if  $a$  is smaller than  $b$  up to multiply by a positive constant and denote  $a \lesssim_c b$  if  $a$  is smaller than  $b$  up to multiply a positive constant depending on  $c$ . Moreover, we denote  $a \approx b$  if  $a$  equal to  $b$  up to multiply a positive constant.

## 2. PROOF OF MAIN RESULT

In this section we give the proof of Theorem 1.1. We use the Banach fixed point theorem and Strichartz estimates. We divide our proof into steps.

**Step 1. Preliminary analysis.** Let  $u \in C(I, H^1(\mathbb{R}))$  be a  $H^1(\mathbb{R})$  solution of (1.1) on  $I$ . Considering the following transform:

$$\varphi(t, x) = \exp(i\Lambda)u(t, x), \quad (2.1)$$

$$\psi = \exp(i\Lambda)\partial_x u = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi, \quad (2.2)$$

where

$$\Lambda = \frac{1}{2} \int_{-\infty}^x |u(t, y)|^{2\sigma} dy.$$

As in [3][section 4] we have

$$\partial_t \Lambda = -\sigma \mathcal{I}m(|u|^{2(\sigma-1)} \bar{u} \partial_x u) + \sigma \mathcal{I}m \left[ \int_{-\infty}^x \partial_x (|u|^{2(\sigma-1)} \bar{u}) \partial_x u dy \right] - \frac{1}{4} |u|^{4\sigma}.$$

Thus, using  $|u| = |\varphi|$  and  $\mathcal{I}m(\bar{u} \partial_x u) = \mathcal{I}m(\bar{\varphi} \psi)$ , we have

$$\begin{aligned} \partial_t \Lambda &= -\sigma |\varphi|^{2(\sigma-1)} \mathcal{I}m(\bar{\varphi} \psi) + \sigma \int_{-\infty}^x \partial_x (|\varphi|^{2(\sigma-1)}) \mathcal{I}m(\bar{u} \partial_x u) dx - \frac{1}{4} |\varphi|^{4\sigma} \\ &= -\sigma |\varphi|^{2(\sigma-1)} \mathcal{I}m(\bar{\varphi} \psi) + \sigma \int_{-\infty}^x \partial_x (|\varphi|^{2(\sigma-1)}) \mathcal{I}m(\bar{\varphi} \psi) dx - \frac{1}{4} |\varphi|^{4\sigma}. \end{aligned}$$

Hence, since  $u$  solves (1.1) we have

$$\begin{aligned}
L\varphi &= L(\exp(i\Lambda))u + \exp(i\Lambda)Lu + 2\partial_x(\exp(i\Lambda))\partial_x u \\
&= L(\exp(i\Lambda))u + \exp(i\Lambda)(Lu + i|u|^{2\sigma}u) \\
&= L(\exp(i\Lambda))u \\
&= (i\partial_t + \partial_x^2)(\exp(i\Lambda))u, \\
&= \left[ -\exp(i\Lambda)\partial_t\Lambda + \partial_x(\exp(i\Lambda)\frac{i}{2}|u|^{2\sigma}) \right] u \\
&= -\varphi\partial_t\Lambda + \left[ \exp(i\Lambda)\frac{-1}{4}|u|^{2\sigma} + \frac{i}{2}\exp(i\Lambda)\partial_x(|u|^{2\sigma}) \right] u \\
&= -\varphi\partial_t\Lambda + \varphi \left[ -\frac{1}{4}|\varphi|^{4\sigma} + \frac{i}{2}\partial_x(|\varphi|^{2\sigma}) \right] \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - \sigma\varphi \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dx + \frac{1}{4}|\varphi|^{4\sigma}\varphi - \frac{1}{4}\varphi|\varphi|^{4\sigma} + i\sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi) \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) + i\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)) - \sigma\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}(\sigma-1)\partial_x(|\varphi|^2)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dx \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) + i\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)) - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dx \\
&= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy.
\end{aligned}$$

As in [3][section 4], we have

$$\begin{aligned}
L\psi &= L(\exp(i\Lambda)\partial_x u) \\
&= \exp(i\Lambda) \left[ -\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u + \sigma|u|^{2(\sigma-1)}\mathcal{I}\mathfrak{m}(\bar{u}\partial_x u)\partial_x u - \sigma \int_{-\infty}^x \mathcal{I}\mathfrak{m}(\partial_x(|u|^{2(\sigma-1)}\bar{u}))\partial_x u dy \partial_x u \right] \\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi)\psi - \sigma \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{u}\partial_x u) dy \psi \\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\psi\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - \sigma\psi \int_{-\infty}^x \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - i\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)) - \sigma\psi \int_{-\infty}^x (\sigma-1)|\varphi|^{2(\sigma-1)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\partial_x\varphi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) - i\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)) - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}2\mathcal{R}\mathfrak{e}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi)\mathcal{I}\mathfrak{m}(\bar{\varphi}\psi) dy \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy.
\end{aligned}$$

Thus, if  $u$  solves (1.1) then  $(\varphi, \psi)$  solves

$$\begin{cases} L\varphi &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy, \\ L\psi &= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2) dy. \end{cases} \quad (2.3)$$

For convenience, we define

$$P(\varphi, \psi) = i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{\psi} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2), \quad (2.4)$$

$$Q(\varphi, \psi) = -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\bar{\varphi} - \sigma(\sigma-1)\psi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}\mathfrak{m}(\psi^2\bar{\varphi}^2). \quad (2.5)$$

Let  $R$  be multi-soliton profile defined as in Section 1.2. Define

$$h = \exp\left(\frac{i}{2} \int_{-\infty}^x |R(t, x)|^{2\sigma} dy\right) R(t, x),$$

$$k = \partial_x h - \frac{i}{2}|h|^{2\sigma}h.$$

Since for each  $1 \leq j \leq K$ ,  $R_j$  solves (1.1) we have

$$LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x, \quad (2.6)$$

By Lemma 3.1 for  $t$  large enough we have

$$\|-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x\|_{H^2} \leq e^{-\lambda t}. \quad (2.7)$$

Thus, we rewrite (2.6) as follows:

$$LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x, \quad (2.8)$$

where

$$v = e^{\lambda t}(-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x). \quad (2.9)$$

By elementary calculation, we have

$$\begin{cases} Lh = i\sigma|h|^{2(\sigma-1)}h^2\bar{k} - \sigma(\sigma-1)h \int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\bar{h}^2) dy + e^{-\lambda t}m(t, x), \\ Lk = -i\sigma|h|^{2(\sigma-1)}k^2\bar{h} - \sigma(\sigma-1)k \int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\bar{h}^2) dy + e^{-\lambda t}n(t, x). \end{cases} \quad (2.10)$$

where

$$m = \exp\left(\frac{i}{2} \int_{-\infty}^x |R|^{2\sigma} dy\right) v - \sigma h \int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\bar{R}v) dy, \quad (2.11)$$

$$n = \exp\left(\frac{i}{2} \int_{-\infty}^x |R|^{2\sigma} dy\right) e^{-\lambda t}(\partial_x v - \sigma \partial_x R \int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\bar{R}v) dy). \quad (2.12)$$

Since  $v$  is uniformly bounded in  $H^2(\mathbb{R})$ , we see that  $m, n$  are uniformly bounded in  $H^1(\mathbb{R})$ . Let  $\tilde{\varphi} = \varphi - h$  and  $\tilde{\psi} = \psi - k$ . Then  $(\tilde{\varphi}, \tilde{\psi})$  solves:

$$\begin{cases} L\tilde{\varphi} = P(\varphi, \psi) - P(h, k) - e^{-\lambda t}m(t, x), \\ L\tilde{\psi} = Q(\varphi, \psi) - Q(h, k) - e^{-\lambda t}n(t, x). \end{cases} \quad (2.13)$$

Set  $\eta = (\tilde{\varphi}, \tilde{\psi})$ ,  $W = (h, k)$  and  $f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi))$  and  $H = e^{-\lambda t}(m, n)$ . We find solution of (2.13) in Duhamel form:

$$\eta(t) = i \int_t^\infty [f(W + \eta) - f(W) + H](s) ds, \quad (2.14)$$

where  $S(t)$  denote the Schrödinger group. Moreover, since  $\psi = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$  we have

$$\tilde{\psi} = \partial_x \tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^{2\sigma}(\tilde{\varphi} + h) - |h|^{2\sigma}h). \quad (2.15)$$

### Step 2. Existence solution of system equations

Since Lemma 3.4 there exists  $T_* \gg 1$  such that for  $T_0 \geq T_*$  there exists unique solution  $\eta$  of (2.13) define on  $[T_0, T_*)$  such that

$$\|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial_x \eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1 \quad \forall t \geq T_0. \quad (2.16)$$

for the constant  $\lambda$  defined as in step 1. Thus, for all  $t \geq T_0$ , we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}. \quad (2.17)$$

**Step 3. Existence of multi-soliton trains of (1.1)** We prove that the solution  $\eta = (\tilde{\varphi}, \tilde{\psi})$  of (2.13) satisfies relation (2.15). Set  $\varphi = \tilde{\varphi} + h$ ,  $\psi = \tilde{\psi} + k$  and  $v = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$  and  $\tilde{v} = v - k$ . Since  $(\tilde{\varphi}, \tilde{\psi})$  solves (2.13) and  $(h, k)$  solves (2.10) we have  $(\varphi, \psi)$  solves (2.3). Furthermore,

$$Lv = \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi). \quad (2.18)$$

Moreover,

$$\begin{aligned}
L(|\varphi|^{2\sigma}\varphi) &= (i\partial_t + \partial_x^2)(\varphi^{\sigma+1}\bar{\varphi}^\sigma) = i\partial_t(\varphi^{\sigma+1}\bar{\varphi}^\sigma) + \partial_x^2(\varphi^{\sigma+1}\bar{\varphi}^\sigma) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\bar{\varphi} + \partial_x((\sigma+1)|\varphi|^{2\sigma}\partial_x\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\bar{\varphi}) \\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\bar{\varphi} + (\sigma+1)[\partial_x^2\varphi|\varphi|^{2\sigma} + \partial_x\varphi\partial_x(|\varphi|^{2\sigma})] \\
&\quad + \sigma\left[\partial_x^2\bar{\varphi}|\varphi|^{2(\sigma-1)}\varphi^2 + (\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + (\sigma-1)|\varphi|^{2(\sigma-2)}\varphi^3(\partial_x\bar{\varphi})^2\right] \\
&= (\sigma+1)|\varphi|^{2\sigma}(i\partial_t\varphi + \partial_x^2\varphi) + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(i\partial_t\bar{\varphi} + \partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3 \\
&= (\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) \\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3.
\end{aligned}$$

Combining with (2.18), using (2.3) we have

$$\begin{aligned}
Lv &= \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi) \\
&= \partial_x L\varphi - \frac{i}{2}\left[(\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\bar{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\right. \\
&\quad \left.+ \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right] \\
&= \partial_x(P(\varphi, \psi) - P(\varphi, v)) + \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) \\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right] \\
&= \partial_x(P(\varphi, \psi) - P(\varphi, v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}) + G(\varphi, v),
\end{aligned}$$

where  $G(\varphi, v)$  is the remaining ingredients and  $G(\varphi, v)$  depend only on  $\varphi$  and  $v$ :

$$\begin{aligned}
G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right].
\end{aligned} \tag{2.19}$$

As the calculations of  $L\psi$  in step 1, noting that the role of  $v$  is similar the one of  $\psi$  in step 1, we have  $G(\varphi, v) = Q(\varphi, v)$  (see Lemma 3.3 for a detail proof of  $G(\varphi, v) = Q(\varphi, v)$ ). Hence,

$$\begin{aligned}
L\psi - Lv &= Q(\varphi, \psi) - Q(\varphi, v) - \partial_x(P(\varphi, \psi) - P(\varphi, v)) + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \psi) - P(\varphi, v)) \\
&\quad - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}).
\end{aligned}$$

Thus,

$$\begin{aligned}
L\tilde{\psi} - L\tilde{v} &= L\psi - Lv \\
&= Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k) - \partial_x(P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) \\
&\quad + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi} + k)} - \overline{P(\varphi, \tilde{v} + k)}).
\end{aligned} \tag{2.20}$$



Multiplying both side of (2.20) by  $\overline{\tilde{\psi} - \tilde{v}}$ , taking imaginary part and integrating over space with integration by parts we obtain

$$\begin{aligned} & \frac{1}{2} \partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \\ &= \mathcal{I}m \int_{\mathbb{R}} (Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \end{aligned} \quad (2.21)$$

$$- \mathcal{I}m \int_{\mathbb{R}} \partial_x (P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \quad (2.22)$$

$$+ (\sigma + 1) \mathcal{I}m \int_{\mathbb{R}} \frac{i}{2} |\varphi|^{2\sigma} (P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \quad (2.23)$$

$$- \sigma \mathcal{I}m \int_{\mathbb{R}} \frac{i}{2} |\varphi|^{2(\sigma-1)} \varphi^2 \overline{(P(\varphi, \tilde{\psi} + k) - P(\varphi, \tilde{v} + k))} (\overline{\tilde{\psi} - \tilde{v}}) dx. \quad (2.24)$$

We define  $A, B, C, D$  are (2.21), (2.22), (2.23) and (2.24) respectively. We have

$$\begin{aligned} |A| &\lesssim \left| \int_{\mathbb{R}} (Q(\varphi, \tilde{\psi} + k) - Q(\varphi, \tilde{v} + k)) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[ (\tilde{\psi} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy - (\tilde{v} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{v} + k)^2 \overline{\varphi}^2) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2) (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[ (\tilde{\psi} - \tilde{v}) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\quad + \left| \int_{\mathbb{R}} \left[ (\tilde{v} + k) \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right] (\overline{\tilde{\psi} - \tilde{v}}) dx \right| \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m((\tilde{\psi} + k)^2 \overline{\varphi}^2) dy \right\|_{L^\infty} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\tilde{v} + k\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right\|_{L^\infty} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L^1_x} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)\|_{L^1} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L^1} \\ &\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\ &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_1, \end{aligned} \quad (2.25)$$

where,

$$K_1 := \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)} (\tilde{\psi} + k)^2\|_{L^1} + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Furthermore,

$$\begin{aligned}
|B| &\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} - \tilde{v}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \left( \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right) (\tilde{\psi} - \tilde{v}) dx \right| \\
&\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2) (\tilde{\psi} - \tilde{v})^2 dx \right| + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 \frac{1}{2} \partial_x ((\tilde{\psi} - \tilde{v})^2) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy (\tilde{\psi} - \tilde{v}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \varphi |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) (\tilde{\psi} - \tilde{v}) dx \right|. \tag{2.26}
\end{aligned}$$

By using integration by parts for second term of (2.26) and using Hölder inequality we have

$$\begin{aligned}
|B| &\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy \right\|_{L^\infty} \|\tilde{\psi} - \tilde{v}\|_{L^2} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} - \tilde{v}) (\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} \\
&\quad + \|\partial_x \varphi\|_{L^2} \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_2, \tag{2.27}
\end{aligned}$$

where

$$K_2 := \|\partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\partial_x \varphi\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\varphi^{2\sigma-1} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty}.$$

Using (2.4), we have

$$\begin{aligned}
|C| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2\sigma} |\varphi|^{2(\sigma-1)} \varphi^2 (\tilde{\psi} - \tilde{v})^2 dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2\sigma} \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 ((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy (\tilde{\psi} - \tilde{v}) dx \right| \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\overline{\varphi}^2 (\tilde{\psi} - \tilde{v})) (\tilde{\psi} + \tilde{v} + 2k) dy \right\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} - \tilde{v}) (\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_3, \tag{2.28}
\end{aligned}$$

where

$$K_3 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)} (\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Now, we give an estimate for  $D$ . We have

$$\begin{aligned}
|D| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 |\varphi|^{2(\sigma-1)} \bar{\varphi}^2 (\tilde{\psi} - \tilde{v})(\bar{\tilde{\psi}} - \bar{\tilde{v}}) dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 \varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\bar{\varphi}^2((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy (\bar{\tilde{\psi}} - \bar{\tilde{v}}) dx \right| \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \left\| \int_{-\infty}^x |\varphi|^{2(\sigma-2)} \mathcal{I}m(\bar{\varphi}^2((\tilde{\psi} + k)^2 - (\tilde{v} + k)^2)) dy \right\|_{L^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} - \tilde{v})(\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_4, \tag{2.29}
\end{aligned}$$

where

$$K_4 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.$$

Combining (2.25), (2.27), (2.28) and (2.29) we have

$$\left| \partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \right| \lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 (K_1 + K_2 + K_3 + K_4).$$

Let  $N > t \gg 1$ . Integrating over time from  $t$  to  $N$  we have

$$\int_t^N \left| \frac{\partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2}{\|\tilde{\psi} - \tilde{v}\|_{L^2}^2} ds \right| \lesssim \int_t^N (K_1 + K_2 + K_3 + K_4) ds,$$

This implies that

$$\text{Ln}(\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2) - \text{Ln}(\|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2) \lesssim \int_t^N \left| \frac{\partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2}{\|\tilde{\psi} - \tilde{v}\|_{L^2}^2} ds \right| \lesssim \int_t^N (K_1 + K_2 + K_3 + K_4) ds,$$

Hence, using (2.16) and (2.17) we have

$$\begin{aligned}
\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right) \\
&\leq e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right). \tag{2.30}
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_t^N (K_1 + K_2 + K_3 + K_4) ds \\
&= \int_t^N \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi} + k)^2\|_{L^1} + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.31}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\partial_x(|\varphi|^{2(\sigma-1)} \varphi^2)\|_{L^\infty} + \|\partial_x \varphi\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} + \|\varphi^{2\sigma-1}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} ds \tag{2.32}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.33}
\end{aligned}$$

$$\begin{aligned}
&+ \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} ds \tag{2.34}
\end{aligned}$$

Using (2.16) and (2.17), we have

$$\|\varphi\|_{L^\infty} \leq \|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty} \tag{2.35}$$

$$\|\varphi\|_{L^2} \leq \|\tilde{\varphi}\|_{L^2} + \|h\|_{L^2} \lesssim 1 + \|h\|_{L^2} \tag{2.36}$$

$$\|\psi\|_{L^\infty} \lesssim 1 \tag{2.37}$$

We denote  $Z_1, Z_2, Z_3, Z_4$  as (2.31), (2.32), (2.33) and (2.34) respectively. Using (2.35), (2.36), (2.37), (2.16) and (2.17), for  $N \gg t$ , we have

$$\begin{aligned}
|Z_1| &\lesssim \|\varphi\|_{L^4(t,N)L^\infty}^3 \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-2)} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^4(t,N)L^\infty} + (N-t) \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2})^2 \\
&\quad + \|\tilde{v} + k\|_{L^{\frac{4}{3}}(t,N)L^2} \|\varphi\|_{L^\infty L^2} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi} + \tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\
&\lesssim (N-t)^{\frac{3}{4}} \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} (1 + \|k\|_{L^\infty L^\infty} (N-t)^{\frac{1}{4}}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}^2) \\
&\quad + (N-t)^{\frac{3}{4}} (1 + \|k\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + (N-t)^{\frac{1}{4}} \|k\|_{L^\infty L^\infty}) \\
&\lesssim (N-t) \|k\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}^2) \\
&\quad + (N-t) \|k\|_{L^\infty L^\infty} (1 + \|k\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) \\
&:= (N-t) W_1(h, k).
\end{aligned}$$

Similarly, for  $N \gg t$ , we have

$$\begin{aligned}
|Z_2| &\lesssim \|\partial_x \varphi \varphi^{2\sigma-1}\|_{L^1(t,N)L^\infty} + (N-t) \|\partial_x \varphi\|_{L^\infty(t,N)L^2} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} \|\tilde{\psi} + \tilde{v} + k\|_{L^\infty(t,N)L^2} \\
&\quad + (N-t)^{\frac{3}{4}} \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} (\|\tilde{\psi} + \tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\
&\lesssim (N-t)^{\frac{3}{4}} (\|\partial_x \tilde{\varphi}\|_{L^4(t,N)L^\infty} + \|\partial_x h\|_{L^4(t,N)L^\infty}) \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}) \\
&\quad + (N-t)^{\frac{3}{4}} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) (1 + (N-t)^{\frac{1}{4}} \|k\|_{L^\infty L^\infty}) \\
&\lesssim (N-t) \|\partial_x h\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) + (N-t) (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|k\|_{L^\infty L^2}) \\
&\quad + (N-t) \|k\|_{L^\infty L^\infty} (1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) \\
&:= (N-t) W_2(h, k),
\end{aligned}$$

and

$$\begin{aligned}
|Z_3| = |Z_4| &\lesssim (N-t) (\|\tilde{\varphi}\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty})^{4\sigma} + (N-t) \|\varphi\|_{L^\infty L^2} \|\varphi\|_{L^\infty L^\infty}^{2\sigma} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi} + \tilde{v}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2}) \\
&\lesssim (N-t) (1 + \|h\|_{L^\infty L^\infty}^{4\sigma}) + (N-t) (1 + \|h\|_{L^\infty L^2}) (1 + \|h\|_{L^\infty L^\infty}^{4\sigma-2}) (1 + \|k\|_{L^\infty L^2}) \\
&:= (N-t) W_3(h, k).
\end{aligned}$$

Hence, since (2.30), we obtain

$$\begin{aligned}
\|\psi(\tilde{t}) - v(\tilde{t})\|_{L^2}^2 &\lesssim e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) ds\right) \\
&\lesssim e^{-2\lambda N} \exp((N-t)(W_1(h, k) + W_2(h, k) + W_3(h, k))) \quad (2.38)
\end{aligned}$$

Noting that  $|h| = |R|$  and  $|k| = |\partial_x R|$ , we have

$$\begin{aligned}
W_1(h, k) &= \|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty L^\infty}^{2\sigma-1}) + (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|\partial_x R\|_{L^\infty L^2}^2) \\
&\quad + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|\partial_x R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [\|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty L^\infty}) + (1 + \|\partial_x R\|_{L^\infty L^2}) \\
&\quad + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|\partial_x R\|_{L^\infty L^2}) (1 + \|R\|_{L^\infty L^2})] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [\|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty H^1}) + (1 + \|R\|_{L^\infty H^1}^2) + \|\partial_x R\|_{L^\infty L^\infty} (1 + \|R\|_{L^\infty H^1}^2)] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) (1 + \|R\|_{L^\infty H^1}^2) (1 + \|\partial_x R\|_{L^\infty L^\infty}).
\end{aligned}$$

Similarly, by noting that  $|\partial_x h| \leq |k| + |h|^{2\sigma+1}$  we have

$$\begin{aligned}
W_2(h, k) &\lesssim (\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1})(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|h\|_{L^\infty L^\infty}) + (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|k\|_{L^\infty L^2}) \\
&\quad + \|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|h\|_{L^\infty L^\infty}) \\
&\lesssim (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1})(1 + \|h\|_{L^\infty L^\infty}) + (1 + \|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty})] \\
&\lesssim (1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(1 + \|h\|_{L^\infty L^\infty})(\|k\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|k\|_{L^\infty L^2})] \\
&= (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)}) [(1 + \|R\|_{L^\infty L^\infty})(\|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|\partial_x R\|_{L^\infty L^2})] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1})(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}),
\end{aligned}$$

and

$$\begin{aligned}
W_3(h, k) &= (1 + \|R\|_{L^\infty L^\infty}^{4\sigma}) + (1 + \|R\|_{L^\infty L^2})(1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|\partial_x R\|_{L^\infty L^2}) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2}) [(1 + \|R\|_{L^\infty L^\infty}^2) + (1 + \|R\|_{L^\infty L^2})(1 + \|\partial_x R\|_{L^\infty L^2})] \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|R\|_{L^\infty H^1}^2).
\end{aligned}$$

By combining the above estimates we have

$$\begin{aligned}
W_1(h, k) + W_2(h, k) + W_3(h, k) &\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) + (1 + \|R\|_{L^\infty L^\infty}^{4\sigma-2})(1 + \|R\|_{L^\infty H^1}^2) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \\
&\quad + (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty L^\infty}^{2\sigma})(1 + \|R\|_{L^\infty H^1}^2) \\
&\lesssim (1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \ll \lambda,
\end{aligned}$$

by the assumption (1.10). Thus, by let  $N \rightarrow \infty$  in (2.38) we obtain

$$\|\tilde{\psi}(t) - \tilde{v}\|_{L^2}^2 = 0,$$

for all  $t$  enough large. Hence, we have proved that

$$\tilde{\psi} = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi - k, \quad (2.39)$$

and then

$$\psi = \partial_x \varphi - \frac{i}{2} |\varphi|^2 \varphi.$$

Moreover, since  $(\tilde{\psi}, \tilde{\varphi})$  solves (2.13) we have  $(\psi, \varphi)$  solves (2.3). Combining with (2.39) if we set  $u = \exp\left(-\frac{i}{2} \int_{-\infty}^x |\varphi|^{2\sigma} dy\right) \varphi$  then  $u$  solves (1.1). Furthermore,

$$\begin{aligned}
\|u - R\|_{H^1} &= \left\| \exp\left(-\frac{i}{2} \int_{-\infty}^x |\varphi|^{2\sigma} dy\right) \varphi - \exp\left(\frac{i}{2} \int_{-\infty}^x |h|^{2\sigma} dy\right) h \right\|_{H^1} \lesssim C(\|\varphi\|_{H^1}, \|h\|_{H^1}) \|\varphi - h\|_{H^1} \\
&\lesssim \|\tilde{\varphi}\|_{H^1} \lesssim e^{-\lambda t},
\end{aligned}$$

Thus for  $t$  large enough, we have

$$\|u - R\|_{H^1} \leq C e^{-\lambda t},$$

for  $\lambda = \frac{1}{8} v_*$  and  $C = C(\omega_1, \dots, \omega_K, c_1, \dots, c_K)$  depend on parameters, which implies the desired result.

### 3. SOME TECHNICAL LEMMAS

**3.1. Properties of solitons.** In this section, we give the proof of (2.7). We have the following result.

**Lemma 3.1.** *There exist  $C > 0$  and a constant  $\lambda > 0$  such that for  $t > 0$  large enough, the estimate (2.7) holds uniformly in  $t$ .*

*Proof.* First, we need some estimates on profile. We have

$$\begin{aligned}
|R_j(t, x)| &= |\psi_{\omega_j, c_j}(t, x)| = |\phi_{\omega_j, c_j}(x - c_j t)| = |\varphi_{\omega_j, c_j}(x - c_j t)| \approx \left( \frac{4\omega_j - c_j^2}{2\sqrt{\omega_j} \left( \cosh(\sigma h_j(x - c_j t)) - \frac{c_j}{2\sqrt{\omega_j}} \right)} \right)^{\frac{1}{2\sigma}} \\
&\lesssim \left( \frac{4\omega_j - c_j^2}{2\sqrt{\omega_j} \left( \cosh(\sigma h_j(x - c_j t)) - \frac{|c_j|}{2\sqrt{\omega_j}} \cosh(\sigma h_j(x - c_j t)) \right)} \right)^{\frac{1}{2\sigma}} \\
&\lesssim \left( \frac{4\omega_j - c_j^2}{(2\sqrt{\omega_j} - |c_j|) \cosh(\sigma h_j(x - c_j t))} \right)^{\frac{1}{2\sigma}} \lesssim \left( \frac{2\sqrt{\omega_j} + |c_j|}{\cosh(\sigma h_j(x - c_j t))} \right)^{\frac{1}{2\sigma}} \\
&\lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|},
\end{aligned}$$

Furthermore,

$$\partial_x \varphi_{\omega_j, c_j}(y) \approx \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left( \cosh(\sigma h_j y) - \frac{c_j}{\sqrt{\omega_j}} \right)^{1 + \frac{1}{2\sigma}}}.$$

Thus,

$$\begin{aligned}
|\partial_x \varphi_{\omega_j, c_j}(y)| &\lesssim \left( \frac{h_j^2}{2\sqrt{\omega_j}} \right)^{\frac{1}{2\sigma}} \frac{|\sinh(\sigma h_j y)|}{\left( 1 - \frac{|c_j|}{\sqrt{\omega_j}} \right)^{1 + \frac{1}{2\sigma}} \cosh(\sigma h_j y)^{1 + \frac{1}{2\sigma}}} \\
&\lesssim_{\omega_j, |c_j|} \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|y|},
\end{aligned}$$

Using the above estimates, we have

$$\begin{aligned}
|\partial_x R_j(t, x)| &= |\partial_x \psi_{\omega_j, c_j}(t, x)| = |\partial_x \phi_{\omega_j, c_j}(x - c_j t)| = |\partial_x \varphi_{\omega_j, c_j}(x - c_j t) + i\varphi_{\omega_j, c_j}(x - c_j t) \partial_x \theta_{\omega_j, c_j}(x - c_j t)| \\
&\lesssim |\partial_x \varphi_{\omega_j, c_j}(x - c_j t)| + |\varphi_{\omega_j, c_j}(x - c_j t)| |\partial_x \theta_{\omega_j, c_j}(x - c_j t)| \\
&\lesssim_{\omega_j, |c_j|} |\partial_x \varphi_{\omega_j, c_j}(x - c_j t)| + e^{-\frac{h_j}{2}|x - c_j t|} \\
&\lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|}.
\end{aligned}$$

By similar arguments we have

$$|\partial_x^2 R_j(t, x)| + |\partial_x^3 R_j(t, x)| \lesssim_{\omega_j, |c_j|} e^{-\frac{h_j}{2}|x - c_j t|},$$

Now, we come back to prove Lemma 3.1. For convenience, we set

$$\begin{aligned}
\chi &= -i|R|^{2\sigma} \partial_x R + i\Sigma_j |R_j|^{2\sigma} \partial_x R_j, \\
f(R, \bar{R}, \partial_x R) &= i|R|^{2\sigma} \partial_x R, \\
g(R, \bar{R}, \partial_x R, \partial_x \bar{R}, \partial_x^2 R) &= i\partial_x (|R|^{2\sigma} \partial_x R), \\
r(R, \partial_x R, \dots, \partial_x^3 R, \partial_x \bar{R}, \partial_x^2 \bar{R}) &= i\partial_x^2 (|R|^{2\sigma} \partial_x R).
\end{aligned}$$

Fix  $t > 0$ , for each  $x \in \mathbb{R}$ , choose  $m = m(x) \in \{1, 2, \dots, K\}$  so that

$$|x - c_m t| = \min_j |x - c_j t|.$$

For  $j \neq m$  we have

$$|x - c_j t| \geq \frac{1}{2}(|x - c_j t| + |x - c_m t|) \geq \frac{1}{2}|c_j t - c_m t| = \frac{t}{2}|c_j - c_m|.$$

Thus, we have

$$\begin{aligned}
&|(R - R_m)(t, x)| + |\partial_x(R - R_m)(t, x)| + |\partial_x^2(R - R_m)(t, x)| + |\partial_x^3(R - R_m)(t, x)| \\
&\leq \Sigma_{j \neq m} (|R_j(t, x)| + |\partial_x R_j(t, x)| + |\partial_x^2 R_j(t, x)| + |\partial_x^3 R_j(t, x)|) \\
&\lesssim_{\omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|} \delta_m(t, x) := \Sigma_{j \neq m} e^{-\frac{h_j}{2}|x - c_j t|}.
\end{aligned}$$

Define

$$v_* = \inf_{j \neq k} h_j |c_j - c_k|.$$

We have

$$|(R - R_m)(t, x)| + |\partial_x(R - R_m)(t, x)| + |\partial_x^2(R - R_m)(t, x)| + |\partial_x^3(R - R_m)(t, x)| \lesssim \delta_m(t, x) \lesssim e^{-\frac{1}{4}v_*t}.$$

□

We see that  $f, g, r$  are polynomials of  $R, \partial_x R, \partial_x^2 R, \partial_x^3 R, \partial_x \bar{R}$  and  $\partial_x^2 \bar{R}$ . Denote

$$A = \sup_{|z| + |\partial_x z| + |\partial_x^2 z| + |\partial_x^3 z| \leq \Sigma_j \|R_j\|_{H^4}} (|df| + |dg| + |dr|).$$

where we denote  $|df(x, y, z, \dots)| = |\partial_x f| + |\partial_y f| + |\partial_z f| + \dots$  as norm of differential of  $f$ . We have

$$\begin{aligned} & |\chi| + |\partial_x \chi| + |\partial_x^2 \chi| \\ & \leq |f(R, \bar{R}, \partial_x R) - f_{R_m, \partial_x \bar{R}_m, R_m}| + |g(R, \bar{R}, \partial_x R, \dots) - g(R_m, \bar{R}_m, \partial_x R_m, \dots)| \\ & \quad + |r(R, \partial_x R, \dots, \partial_x^3 R, \bar{R}, \dots) - r(R_m, \partial_x R_m, \dots, \partial_x^3 R_m, \bar{R}_m, \dots)| \\ & \lesssim A(|R - R_m| + |\partial_x(R - R_m)| + |\partial_x^2(R - R_m)| + |\partial_x^3(R - R_m)|) + A \Sigma_{j \neq m} (|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\ & \lesssim 2A \Sigma_{j \neq m} (|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\ & \lesssim 2A \delta_m(t, x). \end{aligned} \quad + \Sigma_{j \neq m} (f)$$

In particular,

$$\|\chi\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_*t}. \quad (3.1)$$

Moreover,

$$\begin{aligned} \|\chi\|_{W^{2,1}} & \lesssim \Sigma_j (\| |R_j|^{2\sigma} \partial_x R_j \|_{L^1} + \|\partial_x (|R_j|^{2\sigma} \partial_x R_j)\|_{L^1} + \|\partial_x^2 (|R_j|^{2\sigma} \partial_x R_j)\|_{L^1}) \\ & \lesssim \Sigma_j (\|R_j\|_{H^1}^{2\sigma+1} + \|R_j\|_{H^2}^{2\sigma+1} + \|R_j\|_{H^3}^{2\sigma+1}) < \infty. \end{aligned}$$

Thus, using Hölder inequality we obtain

$$\|\chi\|_{H^2} \lesssim_{\omega_1, \dots, \omega_K, |c_1|, \dots, |c_K|} e^{-\frac{1}{8}v_*t}.$$

Set  $\lambda = \frac{1}{8}v_*$  we obtain the desired result.

### 3.2. Some useful estimates.

**Lemma 3.2.** *Let  $x \geq 0$ . Then there exists  $C = C(x)$  such that*

$$(a + b)^x - a^x \leq C(x)(b^x + ba^{x-1}). \quad (3.2)$$

for all  $a, b \geq 0$ .

*Proof.* If  $x = 0$  or  $x = 1$  or  $b = 0$  or  $a = 0$  then (3.2) is true for  $C(x) = 1$ . Consider  $a, b > 0$ . If  $0 < x < 1$  then using  $m^x > m$  for  $m < 1$  and  $0 < x < 1$  we have

$$\left(\frac{a}{a+b}\right)^x + \left(\frac{b}{a+b}\right)^x > \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Hence,

$$(a + b)^x < a^x + b^x,$$

if we choose  $C(x) = 1$  then (3.2) holds. Now, considering  $a, b > 0$  and  $x > 1$ , we set

$$g(z) = z^x, \quad \forall z \in \mathbb{R}.$$

We have  $g$  is class  $C^1$ . Thus, there exists  $\xi \in (a, a + b)$  such that

$$|(a + b)^x - a^x| = |g(a + b) - g(a)| = |bg'(\xi)| = bx\xi^{x-1} < xb(a + b)^{x-1}.$$

If  $x - 1 \leq 1$  then  $(a + b)^{x-1} \leq a^{x-1} + b^{x-1}$  and hence we choose  $C(x) = x$ . If  $x - 1 > 1$  then by Jensen inequality for convex function  $f(z) = z^{x-1}$  we have

$$\left(\frac{a + b}{2}\right)^{x-1} \leq \frac{a^{x-1} + b^{x-1}}{2}.$$

We obtain

$$(a+b)^x - a^x < xb(a+b)^{x-1} \leq 2^{x-2}xb(a^{x-1} + b^{x-1}).$$

Choosing  $C(x) = 2^{x-2}x$  we obtain the desired result.  $\square$

**3.3. Proof**  $G(\varphi, v) = Q(\varphi, v)$ . Let  $G(\varphi, v)$  be defined as in (2.19) and  $Q$  be defined as in (2.5). Then we have the following result.

**Lemma 3.3.** *Let  $v = \partial_x \varphi - \frac{i}{2}|\varphi|^2 \varphi$ . Then the following equality holds:*

$$G(\varphi, v) = Q(\varphi, v).$$

*Proof.* We have

$$\begin{aligned} P(\varphi, v) &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{v} - \sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy, \\ Q(\varphi, v) &= -i\sigma|\varphi|^{2(\sigma-1)}v^2\bar{\varphi} - \sigma(\sigma-1)v \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\ &\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]. \end{aligned}$$

The term contains  $\int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy$  in the expression of  $G(\varphi, v)$  is as follows:

$$\begin{aligned} & -\sigma(\sigma-1)\partial_x\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(-1)\sigma(\sigma-1)\varphi \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ & + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-1)\sigma(\sigma-1)\bar{\varphi} \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \\ & = -\sigma(\sigma-1) \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \left(\partial_x\varphi - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\varphi + \frac{i}{2}\sigma|\varphi|^{2\sigma}\varphi\right) \\ & = -\sigma(\sigma-1) \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy \left(\partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) \\ & = -\sigma(\sigma-1)v \int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy, \end{aligned}$$

which is exactly the term contains  $\int_{-\infty}^x |\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\bar{\varphi}^2) dy$  in the expression of  $Q(\varphi, v)$ . We only need to check the equality of the remaining terms. The remaining terms of  $G(\varphi, v)$  is as follows:

$$\begin{aligned} & i\sigma\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2\bar{v}) - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{I}m(v^2\bar{\varphi}^2) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\bar{v}) \\ & + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}^2v) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \end{aligned} \quad (3.3)$$

$$- \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]. \quad (3.4)$$



Noting that  $\partial_x(|\varphi|^2) = 2\operatorname{Re}(v\bar{\varphi})$  and  $v = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$ , we have

the term (3.3)

$$\begin{aligned}
&= i\sigma\partial_x(|\varphi|^{2(\sigma-1)})\varphi^2\bar{v} + i\sigma|\varphi|^{2(\sigma-1)}2\varphi\partial_x\varphi\bar{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\bar{v} - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi 2\operatorname{Re}(v\bar{\varphi})\operatorname{Im}(v\bar{\varphi}) \\
&\quad + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\bar{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\operatorname{Re}(\varphi\bar{v}) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\bar{\varphi} \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\operatorname{Re}(v\bar{\varphi})\varphi^2\bar{v} + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\bar{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x(\bar{v} - \overline{\partial_x\varphi}) \\
&\quad - 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\operatorname{Re}(v\bar{\varphi})\operatorname{Im}(v\bar{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\bar{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\operatorname{Re}(\varphi\bar{v}) \\
&= 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\operatorname{Re}(v\bar{\varphi})\varphi(i\varphi\bar{v} - \operatorname{Im}(v\bar{\varphi})) + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\bar{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\left(\frac{i}{2}|\varphi|^{2\sigma}\bar{\varphi}\right) \\
&\quad + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\bar{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\operatorname{Re}(\varphi\bar{v}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\operatorname{Re}(v\bar{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} \\
&\quad - \frac{1}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(2\sigma|\varphi|^{2(\sigma-1)}\operatorname{Re}(v\bar{\varphi}) + |\varphi|^{2\sigma}\partial_x\bar{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\bar{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\operatorname{Re}(\varphi\bar{v}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\operatorname{Re}(v\bar{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} \\
&\quad - \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\partial_x\bar{\varphi} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\bar{v} \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\operatorname{Re}(v\bar{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2(\bar{v} - \partial_x\bar{\varphi}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\operatorname{Re}(v\bar{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi.
\end{aligned}$$

Moreover, using  $\operatorname{Re}(\partial_x\varphi\bar{\varphi}) = \operatorname{Re}(v\bar{\varphi})$  we have

the term (3.4)

$$\begin{aligned}
&= \frac{-i}{2}\left[\sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi(\partial_x\bar{\varphi}\varphi + \partial_x\varphi\bar{\varphi}) + \sigma(\sigma-1)(\partial\bar{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right] \\
&= \frac{-i}{2}\left[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\bar{\varphi}\varphi^2(\partial_x\varphi\bar{\varphi} + \partial_x\bar{\varphi}\varphi) + 2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\operatorname{Re}(v\bar{\varphi})\right] \\
&= \frac{-i}{2}\left[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\bar{\varphi}\varphi^2\operatorname{Re}(v\bar{\varphi}) + 2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\operatorname{Re}(v\bar{\varphi})\right] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\bar{\varphi}\varphi^2\operatorname{Re}(v\bar{\varphi}) + \sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\operatorname{Re}(v\bar{\varphi})\right] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\operatorname{Re}(v\bar{\varphi})\varphi(\partial_x\bar{\varphi}\varphi + \partial_x\varphi\bar{\varphi}) + 2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\operatorname{Re}(v\bar{\varphi})\right] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}(\operatorname{Re}(v\bar{\varphi}))^2\varphi\right] \\
&= -2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\operatorname{Re}(v\bar{\varphi}))^2 - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\operatorname{Re}(v\bar{\varphi}).
\end{aligned}$$

Combining the above expressions we obtain

$$\begin{aligned}
& \text{the remaining term of } G(\varphi, v) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\bar{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\bar{\varphi}) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(\varphi\bar{v} - \mathcal{R}e(v\bar{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{I}m(\varphi\bar{v}) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\bar{v}) + i\partial_x\bar{\varphi}\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\partial_x\bar{\varphi}) + |\varphi|^{2\sigma+2} + i\mathcal{R}e(\varphi\partial_x\bar{\varphi}) - \mathcal{I}m(\varphi\partial_x\bar{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(|\varphi|^{2\sigma+2} + i\bar{\varphi}\partial_x\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}(\partial_x\varphi)^2 - \sigma|\varphi|^{4\sigma}\partial_x\varphi + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)^2 - \sigma|\varphi|^{4\sigma}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\bar{\varphi}v^2.
\end{aligned}$$

This is exactly the remaining terms of  $Q(\varphi, v)$ . Thus,  $G(\varphi, v) = Q(\varphi, v)$ .  $\square$

**3.4. Existence solution of system equations.** In this section, using similar arguments as in [5] and [6] we prove existence of solution of (2.13):

$$\eta(t) = i \int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s) ds, \quad (3.5)$$

where

$$W = (h, k), \quad (3.6)$$

$$H = e^{-\lambda t}(m, n), \quad (3.7)$$

$$f(\varphi, \psi) = (P(\varphi, \psi), Q(\varphi, \psi)). \quad (3.8)$$

As in [13, Lemma 4.3] we have the following:

**Lemma 3.4.** *Let  $H = H(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$ ,  $W = W(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{C}^2$  be given vector functions which satisfy for some  $C_1 > 0$ ,  $C_2 > 0$ ,  $\lambda > 0$ ,  $T_0 \geq 0$ :*

$$\|W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2 \times L^2} \leq C_1 \quad \forall t \geq T_0, \quad (3.9)$$

$$\|\partial W(t)\|_{L^2 \times L^2} + \|\partial W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2 \times L^2} \leq C_2, \quad \forall t \geq T_0. \quad (3.10)$$

*Consider equation (3.5). There exists a constant  $\lambda_*$  independent of  $C_2$  such that if  $\lambda \geq \lambda_*$  then there exists a unique solution  $\eta$  to (3.5) on  $[T_0, \infty) \times \mathbb{R}$  satisfying*

$$e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial\eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1, \quad \forall t \geq T_0.$$

*Proof.* We write (3.5) as  $\eta = \Phi\eta$ . We shall show that, for  $\lambda$  sufficiently large,  $\Phi$  is contraction in the ball

$$B = \{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t, \infty)) \times S([t, \infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t, \infty)) \times S([t, \infty))} \leq 1\}.$$

We will use condition  $\lambda \gg 1$  in our analysis without notation.

**Step 1. Prove  $\Phi$  map  $B$  into  $B$**

Let  $t \geq T_0$ ,  $\eta = (\eta_1, \eta_2) \in B$ ,  $W = (w_1, w_2)$  and  $H = (h_1, h_2)$ . By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t, \infty)) \times N([t, \infty))}, \quad (3.11)$$

$$+ \|H\|_{L^1_t L^2_x([t, \infty)) \times L^1_t L^2_x([t, \infty))}. \quad (3.12)$$

For (3.12), using (3.9), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty)) \times L^1_\tau L^2_x([t,\infty))} = \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau} d\tau \leq \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \quad (3.13)$$

For (3.11), we have

$$\begin{aligned} |P(W + \eta) - P(W)| &= |P(w_1 + \eta_1, w_2 + \eta_2) - P(w_1, w_2)| \\ &\lesssim \left| |w_1 + \eta_1|^{2(\sigma-1)} (w_1 + \eta_1)^2 \overline{w_2 + \eta_2} - |w_1|^{2(\sigma-1)} w_1^2 \overline{w_2} \right| \end{aligned} \quad (3.14)$$

$$+ \left| (w_1 + \eta_1) \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\overline{w_1} + \overline{\eta_1})^2) - w_1 \int_{-\infty}^x |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \overline{\eta_1}^2) \right|. \quad (3.15)$$

Denote, using the assumption  $\sigma \geq \frac{5}{2}$  and Lemma 3.2 we have

the term (3.14)

$$\begin{aligned} &\lesssim \left| |w_1 + \eta_1|^{2(\sigma-1)} - |w_1|^{2(\sigma-1)} \right| |w_1 + \eta_1|^2 |w_2 + \eta_2| + \left| |w_1|^{2(\sigma-1)} (w_1 + \eta_1)^2 - w_1^2 |w_2 + \eta_2| \right| \\ &\quad + \left| |w_1|^{2(\sigma-1)} |w_1|^2 |\eta_2| \right| \\ &\lesssim (|\eta_1|^{2(\sigma-1)} + |\eta_1| |w_1|^{2(\sigma-1)-1}) (|W| + |\eta|)^3 + |w_1|^{2(\sigma-1)} (|w_1| |\eta_1| + |\eta_1|^2) |w_2 + \eta_2| + |w_1|^{2\sigma} |\eta_2| \\ &\lesssim (|\eta|^{2(\sigma-1)} + |\eta| |W|^{2(\sigma-1)-1}) (|W|^3 + |\eta|^3) + |W|^{2(\sigma-1)} (|W| |\eta| + |\eta|^2) (|W| + |\eta|) + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta| (|\eta|^{2\sigma-3} + |W|^{2\sigma-3}) (|\eta|^3 + |W|^3) + |\eta| |W|^{2(\sigma-1)} (|W|^2 + |\eta|^2) + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta| (|\eta|^{2\sigma} + |W|^{2\sigma}) + |\eta| |W|^{2\sigma} + |\eta|^3 |W|^{2(\sigma-1)} + |W|^{2\sigma} |\eta| \\ &\lesssim |\eta|^{2\sigma+1} + |\eta| |W|^{2\sigma}. \end{aligned}$$

Moreover,

the term (3.15)

$$\begin{aligned} &\lesssim |\eta_1| \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} |w_2 + \eta_2|^2 |w_1 + \eta_1|^2 dy \\ &\quad + |w_1| \int_{-\infty}^x (|w_1 + \eta_1|^{2(\sigma-2)} - |w_1|^{2(\sigma-2)}) |w_2 + \eta_2|^2 |w_1 + \eta_1|^2 dy \\ &\quad + |w_1| \int_{-\infty}^x |w_1|^{2(\sigma-2)} |\mathcal{I}m((w_2 + \eta_2)^2 - w_2^2) (\overline{w_1} + \overline{\eta_1})^2| dy \\ &\quad + |w_1| \int_{-\infty}^x |w_1|^{2(\sigma-2)} |\mathcal{I}m(w_2^2 ((\overline{w_1} + \eta_1)^2 - \overline{\eta_1}^2))| dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x (|\eta_1|^{2(\sigma-2)} + |\eta_1| |w_1|^{2\sigma-5}) (|W|^4 + |\eta|^4) dy \\ &\quad + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} (|\eta_2|^2 + |w_2| |\eta_2|) (|W|^2 + |\eta|^2) dy + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |w_2|^2 (|\eta_1|^2 + |\eta_1| |w_1|) dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| (|W|^{2\sigma} + |\eta|^{2\sigma}) dy \\ &\quad + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |\eta| (|W|^3 + |\eta|^3) dy + |W| \int_{-\infty}^x |W|^{2(\sigma-2)} |W|^2 |\eta| (|W| + |\eta|) dy \\ &\lesssim |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| |W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &|P(W + \eta) - P(W)| \\ &\lesssim |\eta|^{2\sigma+1} + |\eta| |W|^{2\sigma} + |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta| |W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Similarly,

$$\begin{aligned} & |Q(W + \eta) - Q(W)| \\ & \lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy + |W| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy. \end{aligned}$$

Hence, using  $\sigma \geq \frac{5}{2}$ , we have:

$$\begin{aligned} & \|f(W + \eta) - f(W)\|_{N([t, \infty)) \times N([t, \infty))} \\ & \lesssim \|P(W + \eta) - P(W)\|_{L_\tau^1 L_x^2([t, \infty))} + \|Q(W + \eta) - Q(W)\|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta|^{2\sigma+1} \|_{L_\tau^1 L_x^2([t, \infty))} + \| |\eta| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^2([t, \infty))} \\ & \quad + \| |W| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta| \|_{L^\infty L_x^2([t, \infty))} \| |\eta| \|_{L_\tau^4 L_x^\infty([t, \infty))} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| \int_{-\infty}^x |W|^{2\sigma} + |\eta|^{2\sigma} dy \|_{L_\tau^\infty L_x^\infty([t, \infty))} \\ & \quad + \| |W| \|_{L_\tau^\infty L_x^2([t, \infty))} \| \int_{-\infty}^x |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma} dy \|_{L_\tau^1 L_x^\infty([t, \infty))} \\ & \lesssim e^{-5\lambda t} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| |W|^{2\sigma} + |\eta|^{2\sigma} \|_{L_\tau^\infty L_x^1} + \| |W| \|_{L_\tau^\infty L_x^2} \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} \| |W|^{2\sigma-1} + |\eta|^{2\sigma-1} \|_{L_\tau^\infty L_x^2([t, \infty))} \\ & \lesssim e^{-5\lambda t} + \| |\eta| \|_{L_\tau^1 L_x^2([t, \infty))} = e^{-5\lambda t} + \int_t^\infty e^{-\lambda \tau} d\tau \lesssim e^{-5\lambda t} + \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \end{aligned}$$

Combining with (3.13) and (3.11), (3.12) we obtain

$$\|\Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} < \frac{1}{5} e^{-\lambda t}. \quad (3.16)$$

We have

$$\|\partial_x \Phi\eta\|_{S([t, \infty)) \times S([t, \infty))} \lesssim \|\partial_x(f(W + \eta) - f(W))\|_{N([t, \infty)) \times N([t, \infty))} \quad (3.17)$$

$$+ \|\partial_x H\|_{L_\tau^1 L_x^2([t, \infty)) \times L_\tau^1 L_x^2([t, \infty))}. \quad (3.18)$$

For (3.18), using (3.10) we have

$$\|\partial_x H\|_{L_\tau^1 L_x^2([t, \infty)) \times L_\tau^1 L_x^2([t, \infty))} \lesssim \int_t^\infty e^{-\lambda \tau} d\tau = \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \quad (3.19)$$

For (3.17), we have

$$\|\partial_x(f(W + \eta) - f(W))\|_{N([t, \infty)) \times N([t, \infty))} = \|\partial_x(P(W + \eta) - P(W))\|_{N([t, \infty))} + \|\partial_x(Q(W + \eta) - Q(W))\|_{N([t, \infty))}.$$

Furthermore, using the notation 1.2 (3), we have

$$\begin{aligned} & |\partial_x(P(W + \eta) - P(W))| \\ & \lesssim |\partial_x(|w_1 + \eta_1|^{2(\sigma-1)}(w_1 + \eta_1)^2(\bar{w}_2 + \bar{\eta}_2) - |w_1|^{2(\sigma-1)}w_1^2\bar{w}_2)| \quad (3.20) \end{aligned}$$

$$+ \left| \partial_x(w_1 + \eta_1) \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2(\bar{w}_1 + \bar{\eta}_1)^2) dy - \partial_x w_1 \int_{-\infty}^x |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) dy \right| \quad (3.21)$$

$$+ \left| (w_1 + \eta_1) |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2(\bar{w}_1 + \bar{\eta}_1)^2) - w_1 |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) \right|. \quad (3.22)$$

For (3.20), we have

the term (3.20)

$$\lesssim (|\eta| + |\eta|^{2\sigma} + |\partial_x \eta|)(|W| + |W|^{2\sigma} + |\eta| + |\eta|^{2\sigma} + |\partial_x \eta|)$$

Thus,

$$\begin{aligned} & \|\text{the term (3.20)}\|_{L_\tau^1 L_x^2([t, \infty))} \\ & \lesssim \| |\eta| + |\partial_x \eta| \|_{L_\tau^1 L_x^2} \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \end{aligned}$$

For (3.21), using Lemma 3.2 we have

$$\begin{aligned}
& \| \text{the term (3.21)} \|_{L^1_\tau L^2_x(t, \infty)} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} \left\| \int_{-\infty}^x |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) dy \right\|_{L^\infty_\tau L^\infty_x} \\
& \quad + \| \partial_x w_1 \|_{L^\infty_\tau L^2_x} \left\| \int_{-\infty}^x (|w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) - |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2)) dy \right\|_{L^1_\tau L^\infty_x} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} \| |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) \|_{L^\infty_t L^1_x} \\
& \quad + \| |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\bar{w}_1 + \bar{\eta}_1)^2) - |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \bar{w}_1^2) \|_{L^1_\tau L^1_x} \\
& \lesssim \| \partial \eta_1 \|_{L^1_\tau L^2_x(t, \infty)} + \| |\eta| \|_{L^1_\tau L^2_x(t, \infty)} \leq \int_t^\infty e^{-\lambda\tau} d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},
\end{aligned}$$

For (3.22), using Lemma 3.2 we have

$$\| \text{the term (3.22)} \|_{L^1_\tau L^2_x(t, \infty)} \lesssim \| |\eta| \|_{L^1_\tau L^2_x(t, \infty)} \leq \int_t^\infty e^{-\lambda\tau} d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},$$

Combining the above we obtain

$$\| \partial_x (P(W + \eta) - P(W)) \|_{N(t, \infty)} \leq \| \partial_x (P(W + \eta) - P(W)) \|_{L^1_\tau L^2_x(t, \infty)} \leq \frac{3}{10} e^{-\lambda t}, \quad (3.23)$$

Similarly,

$$\| \partial_x (Q(W + \eta) - Q(W)) \|_{N(t, \infty)} \leq \frac{3}{10} e^{-\lambda t}, \quad (3.24)$$

Combining (3.17), (3.18), (3.19), (3.23), (3.24) we have

$$\| \partial_x \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} \leq \frac{7}{10} e^{-\lambda t}. \quad (3.25)$$

Combining (3.16) and (3.25) we obtain

$$\| \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} + \| \partial_x \Phi \eta \|_{S(t, \infty) \times S(t, \infty)} \leq \frac{9}{10} e^{-\lambda t}, \quad (3.26)$$

Thus, for  $\lambda$  large enough

$$\| \Phi \eta \|_X < 1.$$

Which implies that  $\Phi$  map  $B$  onto  $B$ .

**Step 2.  $\Phi$  is a contraction map on  $B$**

By using (3.9), (3.10) and similar estimate of (3.26), we can show that, for any  $\eta \in B$  and  $\kappa \in B$  we have

$$\| \Phi \eta - \Phi \kappa \|_X \leq \frac{1}{2} \| \eta - \kappa \|_X.$$

for  $\lambda$  large enough. By Banach fixed point theorem there exists unique solution on  $B$  of (3.5) and then solution of (2.13).  $\square$

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#### REFERENCES

- [1] M. Colin and M. Ohta. Stability of solitary waves for derivative nonlinear Schrödinger equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(5):753–764, 2006.
- [2] Z. Guo, C. Ning, and Y. Wu. Instability of the solitary wave solutions for the generalized derivative nonlinear Schrödinger equation in the critical frequency case. *Math. Res. Lett.*, 27(2):339–375, 2020.
- [3] M. Hayashi and T. Ozawa. Well-posedness for a generalized derivative nonlinear Schrödinger equation. *J. Differential Equations*, 261(10):5424–5445, 2016.
- [4] S. Kwon and Y. Wu. Orbital stability of solitary waves for derivative nonlinear Schrödinger equation. *J. Anal. Math.*, 135(2):473–486, 2018.
- [5] S. Le Coz, D. Li, and T.-P. Tsai. Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(6):1251–1282, 2015.
- [6] S. Le Coz and T.-P. Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity*, 27(11):2689–2709, 2014.

- [7] S. Le Coz and Y. Wu. Stability of multisolitons for the derivative nonlinear Schrödinger equation. *Int. Math. Res. Not. IMRN*, (13):4120–4170, 2018.
- [8] X. Liu, G. Simpson, and C. Sulem. Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation. *J. Nonlinear Sci.*, 23(4):557–583, 2013.
- [9] Y. Martel, F. Merle, and T.-P. Tsai. Stability and asymptotic stability in the energy space of the sum of  $N$  solitons for subcritical gKdV equations. *Comm. Math. Phys.*, 231(2):347–373, 2002.
- [10] Y. Martel, F. Merle, and T.-P. Tsai. Stability in  $H^1$  of the sum of  $K$  solitary waves for some nonlinear Schrödinger equations. *Duke Math. J.*, 133(3):405–466, 2006.
- [11] T. Ozawa. On the nonlinear Schrödinger equations of derivative type. *Indiana Univ. Math. J.*, 45(1):137–163, 1996.
- [12] X. Tang and G. Xu. Stability of the sum of two solitary waves for (gDNLS) in the energy space. *J. Differential Equations*, 264(6):4094–4135, 2018.
- [13] P. van Tin. Construction of the multi-soliton trains, multi kink-soliton trains of the derivative nonlinear schrödinger equations by the fixed point method, 2021.

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