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# Optimisation of the total population size for logistic diffusive equations: bang-bang property and fragmentation rate

Idriss Mazari\*      Grégoire Nadin†      Yannick Privat‡

## Abstract

In this article, we give an in-depth analysis of the problem of optimising the total population size for a standard logistic-diffusive model. This optimisation problem stems from the study of spatial ecology and amounts to the following question: assuming a species evolves in a domain, what is the best way to spread resources in order to ensure a maximal population size at equilibrium? In recent years, many authors contributed to this topic. We settle here the proof of two fundamental properties of optimisers: the bang-bang one which had so far only been proved under several strong assumptions, and the other one is the fragmentation of maximisers. Here, we prove the bang-bang property in all generality using a new spectral method. The technique introduced to demonstrate the bang-bang character of optimizers can be adapted and generalized to many optimization problems with other classes of bilinear optimal control problems where the state equation is semilinear and elliptic. We comment on it in a conclusion section. Regarding the geometry of maximisers, we exhibit a blow-up rate for the  $BV$ -norm of maximisers as the diffusivity gets smaller: if  $\Omega$  is an orthotope and if  $m_\mu$  is an optimal control, then  $\|m_\mu\|_{BV} \gtrsim \sqrt{\mu}$ . The proof of this results relies on a very fine energy argument.

**Keywords:** diffusive logistic equation, optimal control, bilinear optimal control, calculus of variations, shape optimization.

**AMS classification:** 35Q92,49J99,34B15.

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## 1 Introduction

This article is devoted to the study of a problem of calculus of variations motivated by questions of spatial ecology. This problem is related to the ubiquitous question of *optimal location of resources*. While we further specify what we mean by "optimal" in what follows, let us note that optimisation problems related to the location of resources are a possible way to tackle the question of *spatial*

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\*Technische Universität Wien, Institute of Analysis and Scientific Computing, 8-10 Wiedner Hauptstrasse, 1040 Wien ([idriss.mazari@tuwien.ac.at](mailto:idriss.mazari@tuwien.ac.at))

†CNRS, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France ([gregoire.nadin@sorbonne-universite.fr](mailto:gregoire.nadin@sorbonne-universite.fr))

‡Université de Strasbourg, CNRS UMR 7501, INRIA, Institut de Recherche Mathématique Avancée (IRMA), 7 rue René Descartes, 67084 Strasbourg, France ([yannick.privat@unistra.fr](mailto:yannick.privat@unistra.fr)).

*heterogeneity* in reaction-diffusion equations. In this context, spatial heterogeneity is interpreted as heterogeneity of the resources available to a given population.

In this paper, we thoroughly analyse the issue of optimising the total population size with respect to the resource distribution. The reaction-diffusion model we deal with is made precise in Section 1.1 and the precise statement of our main results in Section 1.2. In a nutshell, our results may be recast as follows:

- First, we give a characterisation of pointwise properties of optimal resource distributions (also called the *bang-bang property*) that has been partially tackled in [21, 25]; in these previous contributions, the contents of which we discuss in Sections 1.1 and 1.3, partial answers are provided under several technical assumptions. We present here a new method that we believe to be flexible and versatile enough to be applied to a wide class of *bilinear optimal control problem*, and that provides a positive answer to the question of knowing whether optimal resource distributions are bang-bang.
- Second, we prove a *fragmentation phenomenon*, with explicit blow-up rates: as has been noticed [24, 21, 22], for the optimisation of the total population size, the characteristic dispersal rate of the population has a drastic influence on the geometry of optimal resource distributions (in the sense that, the lower the characteristic dispersal rate, the more spread out the optimal resource distribution). Here, we provide an *explicit blow-up rates* for the  $BV$ -norm of optimal resource distribution, the  $BV$ -norm being a natural way to quantify the fragmentation or complexity of a resource distribution. We refer to Section 1.1 for further explanations.

## 1.1 Model and statement of the problems

**Statement of the problems** Let us first lay down the model and the optimisation problems under consideration. The following paragraph is dedicated to explaining which kind of properties we want to obtain for these optimisation problems.

We introduce the model we consider throughout the paper. We place ourselves in the framework of the Fisher-KPP equation which, since the seminal [8, 11], has been used at length: while its apparent simplicity makes it amenable to mathematical analysis, it is complex enough to capture several fundamental aspects of population dynamics [27]. This model reads:

$$\begin{cases} \mu\Delta\theta + \theta(m - \theta) = 0 & \text{in } \Omega, \\ \frac{\partial\theta}{\partial\nu} = 0 & \text{on } \partial\Omega, \\ \theta \geq 0, \theta \neq 0, \end{cases} \quad (\mathbf{E}_{m,\mu})$$

where  $\theta : \Omega \rightarrow \mathbb{R}_+$  is the population density,. The population access to resources is modelled by a function  $m \in L^\infty(\Omega)$  and  $\mu > 0$  is the dispersal rate.

Although we consider here Neumann boundary conditions, Theorems I and II below can be extended to Robin boundary conditions as well, the only difficulty being that one would need to ensure existence and uniqueness for the logistic-diffusive equations under these conditions. We comment on this in the conclusion (Section 4).

Provided that  $m \geq 0$  and  $m \not\equiv 0$ , there exists a unique solution to  $(\mathbf{E}_{m,\mu})$  [3, 4, 5]. We denote it  $\theta_{m,\mu}$ .

We can hence define the total-population size functional

$$\forall \mu > 0, \forall m \in \mathcal{M}(\Omega), \quad F_\mu(m) := \int_\Omega \theta_{m,\mu}. \quad (1.1)$$

We use the following class of constraints on the admissible resource distributions  $m$ , which was introduced in [17] and used, for instance, in [21, 25]:

$$\mathcal{M}(\Omega) := \left\{ m \in L^\infty(\Omega), 0 \leq m \leq 1, \int_{\Omega} m = m_0 \right\}. \quad (1.2)$$

The parameter  $m_0$  is a positive real number such that  $m_0 < |\Omega|$ , where  $|\Omega|$  denotes the volume of  $\Omega$ , in order to ensure that  $\mathcal{M}(\Omega) \neq \emptyset$ . The  $L^1$  constraint accounts for the fact that, in a given domain, only a limited amount of resources is available. The second constraint is a pointwise one, and accounts for natural limitations of the environment, i.e. the fact that, in a single spot, only a maximum amount of resources can be available.

The optimisation problem we consider reads

$$\boxed{\sup_{m \in \mathcal{M}(\Omega)} F_\mu(m)}, \quad (P_\mu)$$

where  $F_\mu(m)$  is given by (1.1).

**Remark 1** (Existence of maximisers). *For any  $\mu > 0$ , the existence of a solution  $m_\mu^*$  of  $(P_\mu)$  is an immediate consequence of the direct method in the calculus of variations.*

In the following paragraph, we present the fundamental properties we are interested in.

**Optimisation of spatial heterogeneity in mathematical biology: fundamental properties under consideration** Starting from spatially homogeneous models [8, 11], in which a population is assumed to live in a homogeneous environment, mathematical biology has over the past decades started considering the impact of *spatial heterogeneity* on population dynamics [5]. In most works, this spatial heterogeneity is modelled using resource distributions. Mathematically, this amounts to taking into account the heterogeneity in the reaction term of the equation. Given that it is hopeless, for a given resource distribution, to attain an explicit description of the ensuing population dynamics, the focus has, more recently, shifted to an optimisation point of view.

This approach has been initiated in [3, 10, 16] and has since received a considerable amount of attention [2, 6, 13, 24, 19, 21, 25]. The initial question that motivated most of these works was related to the optimal survival ability of a population [3, 26]. Namely:

*What is the best way to spread resources in a domain to ensure the optimal survival of a population?*

This problem is by now very well understood in several simple cases (we provide ampler references in Section 1.3). Among all the issues tackled by the authors of [3, 10, 13], let us single out the following ones, which have been deemed crucial in the study of spatial heterogeneity as they provide simple, qualitative information about the influence of heterogeneity: in a domain  $\Omega$ , if we consider resource distribution  $m$  belonging to  $\mathcal{M}(\Omega)$  defined by (1.2),

1. does the bang-bang property hold at the optimum? In other words, if one looks at maximising a criterion over resource terms in  $\mathcal{M}(\Omega)$ , does any optimal resource distribution  $m^*$  write  $m^* = \mathbf{1}_E$ , for some measurable subset  $E$  of  $\Omega$  of positive measure? Alternatively, this means that the underlying domain  $\Omega$  can be decomposed as

$$\Omega = \{m^* = 1\} \sqcup \{m = 0\}. \quad (1.3)$$

Despite several partial results [21, 25] which we detail in Remarks 4 and 5, this property is not known to hold in general for the optimisation of the total population size. In this article we prove that this property indeed holds for the optimal population size whatever the value of  $\mu > 0$  be (Theorem I).

2. do optimal resources tend to concentrate? In “simple” cases (i.e. in simple geometries and for specific boundary conditions), optimal resource distributions for the survival ability [3, 10] are known to be *concentrated*. For instance, considering an optimal resource distribution for the survival ability, which is known to write  $m^* = \mathbb{1}_E$ , then the set  $E$  is connected and enjoys moreover a symmetry property for Neumann boundary conditions in an orthotope [3, Proposition 2.9]. A similar conclusion holds whenever  $\Omega = \mathbb{B}(0; r)$  is a ball and if Dirichlet boundary conditions are imposed rather than Neumann. In that case,  $E = \mathbb{B}(0; r^*)$  is another centered ball, with a radius  $r^*$  chosen so as to satisfy the volume constraint. For general geometries and Robin boundary conditions, the situation is very involved and we refer to [13] for up to date qualitative properties. Such results are a mathematical formalisation of a paradigm first stated in [26]: fragmenting the set  $\{m = 1\}$  leaves less chance for survival: *concentrating resources is favorable to population dynamics*.

In the case of the total population size, it was first noticed in [21] that such results do not in general hold for small diffusivities, where the geometry of the optimal resource distribution tends to become more complicated. Recently, in [24], a complete treatment of a spatially discretised version of the problem was carried out, and precise fragmentation rules were established. However, these results cannot be extended to the present continuous version, since the optimiser they compute strongly depends on the discretization scale. In [22], it was shown that, the slower the dispersal rate of the population, the bigger the  $BV$ -norm<sup>1</sup> of the optimal resource distribution is.

**Remark 2.** When  $m \in W^{1,1}(\Omega)$ , the  $BV$ -norm and the  $W^{1,1}$  norm coincide. When  $m = \mathbb{1}_E$  and  $m$  is a Cacciopoli set (i.e. a set with finite Cacciopoli perimeter) then  $\|m\|_{BV(\Omega)} = |E| + \text{Per}(E)$ , where  $\text{Per}(E)$  is the Cacciopoli perimeter of the set. As a consequence, in our context, an information on the blow-up rate of the  $BV$ -norm yields an information on the blow-up rate of the  $TV$ -norm and, since Theorem I ensures that any maximiser  $m_\mu^*$  writes as  $\mathbb{1}_{E_\mu^*}$ , this implies a blow-up rate on  $\text{Per}(E_\mu^*)$  as  $\mu \rightarrow 0^+$ . We refer to [1] for more information regarding functions of bounded variations and perimeters of sets.

In [22], the main result reads:

**Theorem [22, Theorem 1].** Let  $\Omega = (0; 1)^d$ ,  $\mu > 0$ , and let  $m_\mu^*$  denote a solution of Problem  $(P_\mu)$ . Then,

$$\|m_\mu^*\|_{BV(\Omega)} \xrightarrow{\mu \rightarrow 0^+} +\infty.$$

In this article, we quantify this result by explicitly identifying blow-up rates in terms of the characteristic dispersal rate, and provide a scaling we expect to be optimal (Theorem III). The proof relies on fine energy estimates.

A more in-depth discussion of the bibliography is included in Section 1.3.

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<sup>1</sup>Recall that the total variation semi-norm of a function is

$$|m|_{TV(\Omega)} = \sup \left\{ \int_{\Omega} m \operatorname{div}(\varphi), \varphi \in \mathcal{C}_c^1(\Omega; \mathbb{R}^d), \|\varphi\|_{L^\infty} \leq 1 \right\} \quad (1.4)$$

and that the bounded variation norm of  $m$  is in turn defined as

$$\|m\|_{BV(\Omega)} = \|m\|_{L^1(\Omega)} + |m|_{TV(\Omega)}. \quad (1.5)$$

## 1.2 Main results

### 1.2.1 The bang-bang property

Let us first state that every solution of the optimal population size problem is *bang-bang*. This property, intrinsically interesting, has a practical interest: it allows us to reformulate the problem as a shape optimization one, the unknown being the set in which  $m$  takes its maximum value. One can then use adapted numerical approaches.

**Theorem I.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded connected domain with a  $\mathcal{C}^2$  boundary. Let  $m_\mu^*$  be a solution of  $(P_\mu)$ . Then there exists a measurable subset  $E \subset \Omega$  such that*

$$m_\mu^* = \mathbf{1}_E. \quad (1.6)$$

**Remark 3** (Sketch of the proof). *The idea of the proof rests upon the following fact: we can actually show that the second order Gâteaux derivative of the criterion  $F_\mu$  at a point  $m \in \mathcal{M}(\Omega)$  in a direction  $h$  (such that  $m + th \in \mathcal{M}(\Omega)$  for  $t$  small enough) writes*

$$\ddot{F}_\mu(m)[h, h] = \int_\Omega \Psi_m(x) |\nabla \dot{\theta}_{m,\mu}|^2 - \int_\Omega \Phi_m(x) \dot{\theta}_{m,\mu}^2, \quad (1.7)$$

where  $\Psi_m, \Phi_m \in L^\infty(\Omega)$ ,  $\inf_\Omega \Psi_m > 0$  and  $\dot{\theta}_{m,\mu}$  solves a PDE of the kind

$$\begin{cases} \mathcal{L}_m \dot{\theta}_{m,\mu} = h \theta_{m,\mu} & \text{in } \Omega \\ \partial_\nu \dot{\theta}_{m,\mu} = 0 & \text{in } \partial\Omega, \end{cases}$$

where  $\mathcal{L}_m$  denotes an elliptic operator of second order. We then argue by contradiction, assuming the existence of a maximiser  $m_\mu^*$  that is not a bang-bang function, meaning that the set  $\{0 < m_\mu^* < 1\}$  is of positive Lebesgue measure. Using the expression of  $\ddot{F}_\mu(m)[h, h]$  above, we exhibit a function  $h$  in  $L^\infty$  supported in  $\{0 < m_\mu^* < 1\}$ , with  $\int_\Omega h = 0$ , such that  $\int_\Omega |\nabla \dot{\theta}_{m_\mu^*,\mu}|^2$  is much larger than  $\int_\Omega \dot{\theta}_{m_\mu^*,\mu}^2$ . This is done by using the Fourier (spectral) expansion of  $\theta_{m,\mu}$ , associated with the operator  $\mathcal{L}_{m_\mu^*}$ , and by choosing  $h$  as above, and such that,  $h\theta_{m_\mu^*,\mu}$  only has high Fourier modes in this basis.

**Remark 4** (Comparison with the results of [25]). *In [25], the following result is proved: if  $m \in \mathcal{M}(\Omega)$  is such that  $\{0 < m < 1\}$  has a non-empty interior, then it is not a solution of  $(P_\mu)$ . This in particular implies that, if a maximiser  $m_\mu^*$  of the total population size functional is Riemann integrable, then  $m_\mu^*$  is continuous almost everywhere in  $\Omega$  and is thus necessarily of bang-bang type. However, such regularity is usually extremely hard to prove, and it is unclear to us whether it is attainable in this context. Furthermore, we believe we have located a slight mistake in their proof and we hence provide a correction in Section 2.2, where we also comment on the comparison between our two proofs.*

**Remark 5.** *In [21], the bang-bang property is proved to hold whenever the diffusivity  $\mu$  is large enough, using a proof that is also based on a second derivative calculation, but whose philosophy is completely different from that of Theorem I. Our present result does not require such an assumption.*

**Remark 6.** *A minor adaptation of our proof allows us to handle more general admissible sets and criteria:*

- *let us consider a function  $j$  satisfying*

$$j \in \mathcal{C}^2([0; 1]; \mathbb{R}), j \text{ is increasing in } [0; 1]: j' > 0. \quad (\mathbf{H}_j)$$

We define, for any  $\mu > 0$ ,

$$\mathcal{J}_{j,\mu} : \mathcal{M}(\Omega) \ni m \mapsto \int_{\Omega} j(\theta_{m,\mu}) \quad (1.8)$$

and the optimisation problem

$$\boxed{\sup_{m \in \mathcal{M}(\Omega)} \mathcal{J}_{j,\mu}(m)}. \quad (P_{j,\mu})$$

Then proving a bang-bang property for this problem is amenable to analysis using our technique.

- If one were to change the  $L^\infty$  bounds on  $m$  to  $0 \leq m \leq \kappa$  for some positive  $\kappa$ , the only modification would be to replace  $[0; 1]$  with the interval  $[0; \kappa]$  in assumption  $(\mathbf{H}_j)$  above.

We claim that our method of proof fits immediately to show the following result:

**Theorem II.** Let  $\Omega \subset \mathbb{R}^d$  be a  $\mathcal{C}^2$  bounded domain and let  $j$  satisfying  $(\mathbf{H}_j)$ . Let  $m_{\mu,j}^*$  be a solution of  $(P_{j,\mu})$ . Then  $m_{\mu,j}^*$  is a bang-bang function: there exists a measurable subset  $E \subset \Omega$  such that

$$m_{\mu,j}^* = \mathbb{1}_E. \quad (1.9)$$

A short paragraph explaining how to adapt the proof of Theorem I is provided in Section 2.3.

### 1.2.2 Quantifying the fragmentation for small diffusivities

Our second main result deals with the aforementioned fragmentation property for low diffusivities. Here, we will be led to make stronger assumptions on  $\Omega$ , namely, that  $\Omega$  is an orthotope:  $\Omega = (0; 1)^d$ . Hence, according to [22, Lemma 2], one has

$$\liminf_{\mu \rightarrow 0^+} \left( \sup_{m \in \mathcal{M}(\Omega)} F_\mu(m) \right) > m_0 = \inf_{\mu > 0, m \in \mathcal{M}(\Omega)} F_\mu(m). \quad (1.10)$$

The equality on the right-hand side is obtained in [16, Theorem 1.2].

**Remark 7.** If we consider another domain  $\tilde{\Omega}$  such that (1.10) holds, then the main result below, Theorem III, holds in  $\tilde{\Omega}$ .

We provide hereafter an explicit blow-up rate that we believe to be optimal. Once again, let us emphasize that it this rate does not depend on the space dimension  $d$ .

**Theorem III.** Let  $d \geq 1$  and let  $\Omega = (0; 1)^d$ . There exists  $C_0 > 0$  such that the following holds: there exists  $\mu_0 > 0$  such that, for any  $\mu \in (0; \mu_0)$ , if  $m_\mu^*$  is a solution of  $(P_\mu)$ , then

$$\|m_\mu^*\|_{BV(\Omega)} \geq \frac{C_0}{\sqrt{\mu}}. \quad (1.11)$$

**Remark 8** (Comment on the proof of Theorem III). The crux of the proof is the variational formulation of  $(\mathbf{E}_{m,\mu})$ , which ensures that  $\theta_{m,\mu}$  is the unique minimiser of

$$\mathcal{E}_{m,\mu} : \{u \in W^{1,2}(\Omega), u \geq 0\} \ni u \mapsto \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{2} \int_{\Omega} mu^2 + \frac{1}{3} \int_{\Omega} u^3, \quad (1.12)$$

and which needs to be carefully estimated as  $\mu \rightarrow 0^+$ . We prove that a "shifted" version of this energy controls the quantity  $\|\theta_{m,\mu} - m\|_{L^1(\Omega)}$  (Lemma 18). Therefore, using estimate (1.10), we

aim at controlling  $\mathcal{E}_{m,\mu}(\theta_{m,\mu})$  as  $\mu \rightarrow 0^+$ . Using Modica-type estimates, one can show that, for a fixed  $m \in \mathcal{M}(\Omega)$  that writes  $m = \mathbf{1}_E$ , there holds

$$\sqrt{\mu}\mathcal{E}_{m,\mu}(\theta_{m,\mu}) \xrightarrow{\mu \rightarrow 0^+} \text{Per}(E).$$

However, this convergence is non-uniform with respect to  $m$  (or, more precisely to  $E$ ) and, since we are working with a maximisation problem, it is not possible to conclude using the convergence result above. In the one dimensional case, we propose, in the appendix, an adaptation of [23] that makes this strategy work nonetheless. In higher dimension, we estimate the energy using a regularisation of  $m$  as a test function in the energy formulation of the equation.

### 1.3 Bibliographical comments on $(P_\mu)$

In this section, we gather a discussion on references connected to the optimisation of the total population size in logistic-diffusive models. For a presentation of the literature devoted to the optimal survival ability, we refer to [18, Introduction].

**Influence of the diffusivity  $\mu$  on  $F_\mu$ .** Problem  $(P_\mu)$  has been first introduced in [17] and several properties had been derived in [16], one of which is the following: for every  $\mu > 0$ , the unique minimiser of  $F_\mu$  in  $\mathcal{M}(\Omega)$  is  $m_0$ ; in other words

$$\forall \mu > 0, \forall m \in \mathcal{M}(\Omega), \quad m(\cdot) \neq m_0 \Rightarrow F_\mu(m) > m_0. \quad (1.13)$$

This result means that spatial homogeneity is detrimental to the population size. Furthermore, it is proved in [16] that, if  $m \in \mathcal{M}(\Omega)$  is given, then

$$F_\mu(m) \xrightarrow{\mu \rightarrow 0^+} m_0, \quad \text{and} \quad F_\mu(m) \xrightarrow{\mu \rightarrow \infty} m_0. \quad (1.14)$$

Hence, for a given  $m \in \mathcal{M}(\Omega)$ , the low and high diffusivity limits of the functional correspond to global minima. However, it was proved in [22, Lemma 2] that

$$\liminf_{\mu \rightarrow 0^+} \left( \sup_{m \in \mathcal{M}(\Omega)} F_\mu(m) \right) > m_0,$$

showing the intrinsic difficulty of passing to the low-diffusivity limit in problem  $(P_\mu)$ .

This point of view, where the resource distribution is considered fixed and the diffusivity is taken as a variable, was later deeply analysed in several articles. Notable among these are the following results:

1. In [2], for a fixed  $m \in L^\infty(\Omega)$  such that  $m(\cdot) \geq 0$  and  $m(\cdot) \neq 0$ , the authors consider the optimisation problem

$$\sup_{\mu > 0} \left( E_\mu(m) := \frac{F_\mu(m)}{\int_\Omega m} \right) \quad (1.15)$$

and observe that, in the one-dimensional case  $\Omega = (0; 1)$ , there holds

$$E_\mu(m) \leq 3. \quad (1.16)$$

This bound is sharp (a maximising sequence is explicitly constructed) and is not reached by any function  $m$ . This work has been later extended to the higher-dimensional case in [9] and the authors prove that, in that case (i.e. in dimension  $d \geq 2$ ), there holds

$$\sup_{\substack{m \in L^\infty(\Omega) \\ m \geq 0, m \neq 0}} \sup_{\mu > 0} E_\mu(m) = +\infty. \quad (1.17)$$



2. In [14], a function  $m$  such that the map  $\mu \mapsto F_\mu(m)$  has several local maxima is constructed. It emphasizes the intrinsic complexity of the interplay between the population size functional and the parameter  $\mu > 0$ .

Finally, let us also note that a related problem, where the underlying model is a system of ODEs with identical migration rates, was considered in [15].

We also point out to two surveys [12, 20] and to the references therein for up-to-date considerations about the influence of spatial heterogeneity for single or multiple species models or for optimisation problems in mathematical biology.

## 2 Proofs of Theorems I and II

### 2.1 Proof of Theorem I

The proof of this Theorem relies on a new formulation of the second order optimality conditions for the problem  $(P_\mu)$ . Let us first compute the necessary optimality conditions of the first and second orders.

**Computation of optimality conditions** It is established in [6, Lemma 4.1] that, for any  $\mu > 0$  the map  $\mathcal{M}(\Omega) \ni m \mapsto \theta_{m,\mu}$  is differentiable at the first order in the sense of Gâteaux. Adapting their proof yields without difficulty its second order Gâteaux-differentiability. Let us fix  $m \in \mathcal{M}(\Omega)$  and an admissible perturbation<sup>2</sup>  $h \in L^\infty(\Omega)$ . Let us denote by  $\dot{\theta}_{m,\mu}$  (resp.  $\ddot{\theta}_{m,\mu}$ ) the first (resp. second) Gâteaux-derivative of  $\theta_{\cdot,\mu}$  at  $m$  in the direction  $h$ . It is standard (we refer to [6, Lemma 4.1]) to see that  $\dot{\theta}_{m,\mu}$  solves

$$\begin{cases} \mu \Delta \dot{\theta}_{m,\mu} + (m - 2\theta_{m,\mu}) \dot{\theta}_{m,\mu} = -h \theta_{m,\mu} & \text{in } \Omega, \\ \frac{\partial \dot{\theta}_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

**Remark 9.** *The fact that  $\dot{\theta}_{m,\mu}$  is uniquely determined by that equation (in other words, that (2.1) has a unique solution) can be proved as in [6, 21]. For the sensitivity analysis and computation of the Gâteaux-derivatives, we also refer to [25].*

To derive a tractable equation for the Gâteaux derivative  $\dot{F}_\mu(m)[h]$  of the functional  $F_\mu$  at  $m$  in the direction  $h$ , let us introduce the adjoint state  $p_{m,\mu}$  as the solution of

$$\begin{cases} \mu \Delta p_{m,\mu} + p_{m,\mu}(m - 2\theta_{m,\mu}) = -1 & \text{in } \Omega, \\ \frac{\partial p_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.2)$$

so that, multiplying (2.1) by  $p_{m,\mu}$  and integrating by parts readily gives

$$\int_{\Omega} p_{m,\mu} \theta_{m,\mu} h = \int_{\Omega} \dot{\theta}_{m,\mu} = \dot{F}_\mu(m)[h]. \quad (2.3)$$

It is standard in optimal control theory (see e.g. [28]) that, if  $m_\mu^*$  is a solution of  $(P_\mu)$  then there exists a constant  $c$  such that

$$\{0 < m_\mu^* < 1\} \subset \{\theta_{m_\mu^*} p_{m_\mu^*} = c\}. \quad (2.4)$$

---

<sup>2</sup>The wording ‘‘admissible perturbation’’ means that  $h$  belongs to the tangent cone to the set  $\mathcal{M}(\Omega)$  at  $m$ . It corresponds to the set of functions  $h \in L^\infty(\Omega)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^\infty(\Omega)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $m + \varepsilon_n h_n \in \mathcal{M}(\Omega)$  for every  $n \in \mathbb{N}$ .

**Remark 10.** As is done in [21], the sets  $\{m_\mu^* = 1\}$  and  $\{m_\mu^* = 0\}$  can be described in terms of level sets of the so-called switching function  $\theta_{m,\mu} p_{m,\mu}$  but we do not detail it since these are not informations we will use in the proof.

Let us turn to the computation of the second order Gâteaux derivative of the functional  $F_\mu$  in the direction  $h$ , which will be denoted  $\ddot{F}_\mu(m)[h, h]$ . To obtain it, we first recall (see [21, Equation (18)]) that  $\ddot{\theta}_{m,\mu}$  solves

$$\begin{cases} \mu \Delta \ddot{\theta}_{m,\mu} + (m - 2\theta_{m,\mu}) \ddot{\theta}_{m,\mu} = -2h \dot{\theta}_{m,\mu} + 2\dot{\theta}_{m,\mu}^2 & \text{in } \Omega, \\ \frac{\partial \ddot{\theta}_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.5)$$

Multiplying (2.5) by  $p_{m,\mu}$  and integrating by parts yields

$$\begin{aligned} \int_{\Omega} \ddot{\theta}_{m,\mu} &= 2 \int_{\Omega} (h \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu}^2) p_{m,\mu} = 2 \int_{\Omega} \left( \frac{-\mu \Delta \dot{\theta}_{m,\mu} - (m - 2\theta_{m,\mu}) \dot{\theta}_{m,\mu}}{\theta_{m,\mu}} \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu}^2 \right) p_{m,\mu} \\ &= 2 \int_{\Omega} \left( -\mu \Delta \dot{\theta}_{m,\mu} - (m - \theta_{m,\mu}) \dot{\theta}_{m,\mu} \right) \frac{p \dot{\theta}_{m,\mu}}{\theta_{m,\mu}}. \end{aligned}$$

Let us introduce  $u_{m,\mu} := \frac{p_{m,\mu}}{\theta_{m,\mu}}$ . We thus obtain

$$\begin{aligned} \int_{\Omega} \ddot{\theta}_{m,\mu} &= 2 \int_{\Omega} \left( \mu \nabla(u_{m,\mu} \dot{\theta}_{m,\mu}) \nabla \dot{\theta}_{m,\mu} - (m - \theta_{m,\mu}) \dot{\theta}_{m,\mu}^2 u_{m,\mu} \right) \\ &= 2 \int_{\Omega} u_{m,\mu} \left( \mu |\nabla \dot{\theta}_{m,\mu}|^2 - \left( m - \theta_{m,\mu} + \frac{\Delta u_{m,\mu}}{2u_{m,\mu}} \right) \dot{\theta}_{m,\mu}^2 \right). \end{aligned} \quad (2.6)$$

Furthermore, it is straightforward to see that

$$\forall m \in \mathcal{M}(\Omega), \quad \inf_{\Omega} \theta_{m,\mu} > 0. \quad (2.7)$$

Furthermore, we have the following result:

**Lemma 11.** For every  $m \in \mathcal{M}(\Omega)$ ,

$$\inf_{\Omega} p_{m,\mu} > 0. \quad (2.8)$$

*Proof of Lemma 11.* We start from the observation that  $\theta_{m,\mu}$  solves  $(\mathbf{E}_{m,\mu})$  implies that the principal eigenvalue  $\lambda(m - \theta_{m,\mu}, \mu)$  of the operator  $-\mu \Delta - (m - \theta) \text{Id}$  is zero [16]. Since  $\theta_{m,\mu} > 0$  in  $\Omega$ , the first eigenvalue  $\lambda(m - 2\theta_{m,\mu}, \mu)$  of the operator  $\mathcal{L}_m := -\mu \Delta - (m - 2\theta_{m,\mu}) \text{Id}$  satisfies

$$\lambda(m - 2\theta_{m,\mu}, \mu) > 0, \quad (2.9)$$

as a consequence of the monotonicity of eigenvalues [7]. Since  $p_{m,\mu}$  satisfies  $\mathcal{L}_m p_{m,\mu} = 1 > 0$  with Neumann boundary conditions, the conclusion follows from multiplying the equation on  $p_{m,\mu}$  by the negative part  $(p_{m,\mu})_-$  and integrating by parts: it yields

$$\mu \int_{\Omega} |\nabla (p_{m,\mu})_-|^2 - \int_{\Omega} (p_{m,\mu})_-^2 (m - 2\theta_{m,\mu}) = - \int_{\Omega} (p_{m,\mu})_- < 0 \text{ if } (p_{m,\mu})_- \neq 0. \quad (2.10)$$

However, according to the Courant-Fischer principle,

$$\lambda(m - 2\theta_{m,\mu}, \mu) = \inf_{\substack{u \in W^{1,2}(\Omega) \\ \int_{\Omega} u^2 = 1}} \mu \int_{\Omega} |\nabla u|^2 - \int_{\Omega} u^2 (m - 2\theta_{m,\mu}) > 0 \quad (2.11)$$

and therefore, it follows that  $p_{m,\mu}(\cdot) \geq 0$  and  $p_{m,\mu}(\cdot) \neq 0$  in  $\Omega$ . To conclude, it suffices to apply the strong maximum principle.  $\square$

According to Lemma 11 and (2.7), it follows that  $u_{m,\mu}$  satisfies

$$\inf_{\Omega} u_{m,\mu} > 0. \quad (2.12)$$

Furthermore, standard elliptic estimates entail

$$\forall p \in (1; +\infty), \quad \theta_{m,\mu}, p_{m,\mu} \in W^{2,p}(\Omega), \quad (2.13)$$

and from Sobolev embeddings, we get

$$\theta_{m,\mu}, p_{m,\mu} \in \mathcal{C}^{1,\alpha}(\bar{\Omega}) \quad (2.14)$$

for any  $\alpha \in (0; 1)$ . Using the equations on  $\theta_{m,\mu}$  and  $p_{m,\mu}$ , this gives, in turn that  $\Delta\theta_{m,\mu}$  and  $\Delta p_{m,\mu}$  belong to  $L^\infty(\Omega)$ . It follows, by computing explicitly  $\Delta u_{m,\mu}$ , that  $\Delta u_{m,\mu}$  belongs to  $L^\infty(\Omega)$ .

If we then define  $V_{m,\mu} := \left(m - \theta_{m,\mu} + \frac{\Delta u_{m,\mu}}{2u_{m,\mu}}\right)$  we have, as a consequence, that

$$V_{m,\mu} \in L^\infty(\Omega). \quad (2.15)$$

Starting from (2.6),  $\ddot{F}_\mu(m)[h, h]$  rewrites in the more tractable form

$$\ddot{F}_\mu(m)[h, h] = \int_{\Omega} \ddot{\theta}_{m,\mu} = \mu \int_{\Omega} u_{m,\mu} |\nabla \dot{\theta}_{m,\mu}|^2 - \int_{\Omega} V_{m,\mu} \dot{\theta}_{m,\mu}^2. \quad (2.16)$$

This expression is crucial to proving Theorem I.

*Proof of Theorem I.* Let us argue by contradiction, assuming the existence of a maximiser  $m$  (for the sake of readability, we drop the subscript  $m_\mu^*$ ) of  $F_\mu$  in  $\mathcal{M}(\Omega)$  such that the set  $\tilde{\Omega} := \{0 < m < 1\}$  is of positive Lebesgue measure.

Our goal is now to construct an admissible perturbation  $h \in L^\infty(\Omega)$  (see Footnote 2) such that

$$h \text{ is supported in } \tilde{\Omega}, \quad \ddot{F}_\mu(m)[h, h] > 0. \quad (2.17)$$

Let us first note that from the optimality conditions (2.4), if  $h$  is supported in  $\tilde{\Omega}$  and satisfies  $\int_{\tilde{\Omega}} h = 0$ , then, for the constant  $c$  given in (2.4) we have

$$\dot{F}_\mu(m)[h] = \int_{\Omega} h \theta_{m,\mu} p_{m,\mu} = c \int_{\tilde{\Omega}} h = 0.$$

Hence, if  $h$  satisfies (2.17), then a Taylor expansion yields  $F_\mu(m + \varepsilon h) - F_\mu(m) = \frac{\varepsilon^2}{2} \ddot{F}_\mu(m)[h, h] + o(\varepsilon^2)$ , which leads to a contradiction whenever  $\varepsilon > 0$  is chosen small enough. It is standard to show that any perturbation  $h$  supported in  $\Omega$  is admissible if, and only if  $\int_{\Omega} h = 0$ .

**Remark 12.** *To implement the previous construction, it suffices in fact to construct  $h \in L^2(\Omega)$  so that (2.17) is satisfied and  $\int_{\Omega} h = 0$ , forgetting that  $h$  has to belong to  $L^\infty(\Omega)$ . Indeed, let us assume that such a  $h \in L^2(\Omega)$  exists. Then, we introduce the sequence  $h_n := h \mathbb{1}_{|h| \leq n} - \int_{\Omega} h \mathbb{1}_{|h| \leq n} \in L^\infty(\Omega)$ , which converges weakly in  $L^2(\Omega)$  to  $h$  as  $n \rightarrow \infty$ . By elliptic regularity, it entails strong  $W^{1,2}$ -regularity of  $\dot{\theta}_{m,\mu}[h_n]$  to  $\dot{\theta}_{m,\mu}$  as  $n \rightarrow \infty$  and thus the convergence of second order derivatives. Choosing  $n$  large enough yields the required contradiction. In what follows, we will hence look for a function  $h \in L^2(\Omega)$  satisfying (2.17) and  $\int_{\Omega} h = 0$ .*

According to (2.16), by using (2.15) and (2.12), there exist two positive constants  $A_1$  and  $A_2$  such that

$$\ddot{F}_\mu(m)[h, h] \geq A_1 \int_\Omega |\nabla \dot{\theta}_{m,\mu}|^2 - A_2 \int_\Omega \dot{\theta}_{m,\mu}^2. \quad (2.18)$$

To obtain a contradiction, it hence suffices to construct a perturbation  $h \in L^2(\Omega)$  with support in  $\tilde{\Omega}$  satisfying  $\int_\Omega h = 0$  and such that

$$\int_\Omega |\nabla \dot{\theta}_{m,\mu}|^2 > \frac{A_2}{A_1} \int_\Omega \dot{\theta}_{m,\mu}^2. \quad (2.19)$$

Let us prove that such a perturbation  $h$  exists. To this aim, let us introduce the operator  $\mathcal{L}$  defined by

$$\mathcal{L}_m : H^2(\Omega) \ni \psi \mapsto -\mu \Delta \psi - (m - 2\theta_{m,\mu})\psi \in L^2(\Omega). \quad (2.20)$$

This operator is self-adjoint and of compact inverse in  $L^2(\Omega)$ , as a consequence of the spectral estimate (2.9). As a consequence, there exists a sequence of eigenvalues

$$\lambda_1(\mathcal{L}_m) < \lambda_2(\mathcal{L}_m) \leq \dots \leq \lambda_k(\mathcal{L}_m) \xrightarrow[k \rightarrow \infty]{} +\infty, \quad (2.21)$$

each of these eigenvalues being associated with a  $L^2$ -normalised eigenfunction  $\psi_k$  solving

$$\begin{cases} \mathcal{L}_m \psi_k = \lambda_k(\mathcal{L}) \psi_k \text{ in } \Omega, \\ \frac{\partial \psi_k}{\partial \nu} = 0 \text{ on } \partial\Omega, \\ \int_\Omega \psi_k^2 = 1. \end{cases} \quad (2.22)$$

Let us fix  $K \in \mathbb{N} \setminus \{0\}$  that will be fixed later and consider the family of linear functionals  $\{R_k\}_{k=0,\dots,K} \subset \left(L^2(\tilde{\Omega})\right)^K$  defined by

$$\forall f \in L^2(\tilde{\Omega}), \quad R_0(f) := \int_\Omega \mathbf{1}_{\tilde{\Omega}} f \quad \text{and} \quad R_k(f) := \int_\Omega \mathbf{1}_{\tilde{\Omega}} \theta_{m,\mu} \psi_k f \quad (2.23)$$

for every  $k \in \llbracket 1, K \rrbracket$ .

Let us define  $E_k := \ker(R_k)$  for every  $k \in \llbracket 0, K \rrbracket$ . Observe that each space  $E_k$  is of codimension at most 1. In particular,

$$E := \bigcap_{k=0}^K E_k \subset L^2(\tilde{\Omega}) \quad (2.24)$$

is of codimension at most  $(K+1)$  in  $L^2(\tilde{\Omega})$  and is non-empty. Let us hence pick  $F_K \in E \setminus \{0\}$  and assume by homogeneity, that

$$\int_\Omega |F_K \mathbf{1}_{\tilde{\Omega}} \theta_{m,\mu}|^2 = 1. \quad (2.25)$$

Let us extend  $F_K$  to  $\Omega$  by setting  $F_K = F_K \mathbf{1}_{\tilde{\Omega}}$ . According to the definition of  $F_K$  it follows that

- (i)  $\int_{\tilde{\Omega}} F_K = \int_\Omega \mathbf{1}_{\tilde{\Omega}} F_K = 0$ ,
- (ii)  $\forall k \in \llbracket 0, K \rrbracket$ , one has  $\int_\Omega (F_K) \mathbf{1}_{\tilde{\Omega}} \theta_{m,\mu} = 0$  so that, defining  $h_K := F_K \mathbf{1}_{\tilde{\Omega}}$  we have that  $h_K$  is supported in  $\tilde{\Omega}$ , belongs to  $L^2(\Omega)$  and satisfies  $\int_\Omega h_K = 0$ . Furthermore, the function  $\eta_K = -F_K \mathbf{1}_{\tilde{\Omega}} \theta_{m,\mu}$  expands as

$$\eta_K = \sum_{\ell \geq K+1} \alpha_\ell \psi_\ell \quad \text{with} \quad \sum_{\ell \geq K+1} \alpha_\ell^2 = 1. \quad (2.26)$$

Finally, we observe that, for this perturbation  $h_K, \dot{\theta}_{m,\mu}$  solves

$$\begin{cases} \mathcal{L}_m \dot{\theta}_{m,\mu} = \eta_K, \\ \frac{\partial \dot{\theta}_{m,\mu}}{\partial \nu} = 0, \end{cases} \quad (2.27)$$

whence

$$\dot{\theta}_{m,\mu} = \sum_{\ell \geq K+1} \frac{\alpha_\ell}{\lambda_\ell(\mathcal{L}_m)} \psi_\ell. \quad (2.28)$$

Using the  $L^2(\Omega)$ -orthogonality property of the eigenfunctions, we get

$$\int_{\Omega} \dot{\theta}_{m,\mu}^2 = \sum_{\ell \geq K+1} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)^2}. \quad (2.29)$$

and, similarly,

$$\mu \int_{\Omega} |\nabla \dot{\theta}_{m,\mu}|^2 - \int_{\Omega} (m - 2\theta_{m,\mu}) \dot{\theta}_{m,\mu}^2 = \sum_{\ell=K+1}^{\infty} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)}.$$

We infer the existence of  $M > 0$  such that

$$\begin{aligned} \mu \int_{\Omega} |\nabla \dot{\theta}_{m,\mu}|^2 &\geq \sum_{\ell=K+1}^{\infty} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)} - M \int_{\Omega} \dot{\theta}_{m,\mu}^2 \geq \sum_{\ell=K+1}^{\infty} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)} - M \sum_{\ell=K+1}^{\infty} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)^2} \\ &= \sum_{\ell=K+1}^{\infty} \frac{\alpha_\ell^2}{\lambda_\ell(\mathcal{L}_m)^2} (\lambda_\ell(\mathcal{L}_m) - M) \\ &\geq (\lambda_{K+1}(\mathcal{L}_m) - M) \int_{\Omega} \dot{\theta}_{m,\mu}^2. \end{aligned}$$

The conclusion follows by taking  $K$  large enough so that  $\lambda_{K+1}(\mathcal{L}_m) > M$ , which concludes the proof.  $\square$

## 2.2 Comparison with the results of [25]

This section is dedicated to an explanation of the main difference with the proof of [25]. As recalled in Remark 4, the main result of [25] reads: *if  $\tilde{\Omega} = \{0 < m < 1\}$  has an interior point, then it cannot be a solution of Problem  $(P_\mu)$ .*

Although they do not use the expression (2.16) but an alternative expression of the second order Gâteaux-derivative  $\ddot{F}_\mu$ , their idea, to reach a contradiction, is to reason backwards, by finding a function  $\psi$ , that "should" act as  $\dot{\theta}_{m,\mu}$ , well chosen to yield a contradiction, and then constructing an admissible perturbation  $h$  supported in the interior of  $\{0 < m < 1\}$  such that  $\dot{\theta}_{m,\mu} = \psi$ .

However, we think that the authors, when carrying out their reasoning to reach a contradiction, made a slight mistake in the last line of the second set of equalities on [25, Page 11]. Indeed, it seems to us that it is implicitly assumed that if  $\dot{\theta}_{m,\mu}$  is compactly supported in  $\tilde{\Omega}$ , then so is  $\ddot{\theta}_{m,\mu}$ , which is in general wrong.

Nevertheless, we propose hereafter an alternative proof of their result that uses their idea of first fixing a desirable function  $\varphi$ , and then proving the existence of an admissible perturbation  $h$ , compactly supported in  $\tilde{\Omega} = \{0 < m < 1\}$  such that  $\varphi = \dot{\theta}_{m,\mu}$ , leading to a positive second order derivative.

Let us argue by contradiction, considering a solution  $m$  of  $(P_\mu)$  such that the set

$$\tilde{\Omega} := \{0 < m < 1\} \quad (2.30)$$

has a non-empty interior (in particular, it is of positive measure). As a consequence of (2.4) there exists  $c$  such that

$$\theta_{m,\mu} p_{m,\mu} = c \text{ in } \tilde{\Omega}. \quad (2.31)$$

Let us pick two interior points  $x_0, y_0$  of  $\tilde{\Omega}$  and let  $r > 0$  be such that

$$\mathbb{B}(x_0; r), \mathbb{B}(y_0; r) \subset \tilde{\Omega}, \quad \mathbb{B}(x_0; r) \cap \mathbb{B}(y_0; r) = \emptyset. \quad (2.32)$$

Let  $\chi \in \mathcal{D}(\mathbb{R}^d)$  be a  $\mathcal{C}^\infty$ , radially symmetric, non-negative function with compact support in  $\mathbb{B}(0; r)$  such that  $\chi(0) = 1$ . For every  $k \in \mathbb{N}$ , let us introduce  $\psi_k$  defined by

$$\psi_k(x) := \chi(x - x_0) \cos(k|x - x_0|) - \chi(x - y_0) \cos(k|x - y_0|). \quad (2.33)$$

**Lemma 13.** *For any  $k \in \mathbb{N}$ , there exists an admissible perturbation  $h_k$  supported in  $\tilde{\Omega}$ , such that*

$$\psi_k = \dot{\theta}_{m,\mu}[h_k], \quad (2.34)$$

where  $\dot{\theta}_{m,\mu}[h_k]$  denotes the unique solution of (2.1) associated to the perturbation choice  $h = h_k$ .

*Proof of Lemma 13.* Let us introduce  $h_k$ , defined by

$$h_k := \frac{1}{\theta_{m,\mu}} (-\mu \Delta \psi_k - (m - 2\theta_{m,\mu}) \psi_k). \quad (2.35)$$

Since, by construction  $\psi_k \in W^{2,\infty}(\Omega)$  and since  $\inf_{\Omega} \theta_{m,\mu} > 0$  we get that  $h_k \in L^\infty(\Omega)$ . Moreover, since  $\chi$  is compactly supported in  $\tilde{\Omega}$ , so is  $h_k$ . Since  $\tilde{\Omega} = \{0 < m < 1\}$ , the only condition we have to check to ensure that  $h_k$  is admissible at  $m$  is that

$$\int_{\Omega} h_k = 0. \quad (2.36)$$

By construction, one has

$$\mu \Delta \psi_k + \psi_k (m - 2\theta_{m,\mu}) = -h_k \theta_{m,\mu} \quad \text{in } \Omega, \quad (2.37)$$

so that, by multiplying this equation by  $p_{m,\mu}$  and integrating twice by parts we obtain

$$\int_{\Omega} \psi_k = - \int_{\Omega} \theta_{m,\mu} p_{m,\mu} h_k = -c \int_{\Omega} h_k \quad (2.38)$$

where the last equality comes from (2.31). Since by construction,  $\int_{\Omega} \psi_k = 0$  the conclusion follows and hence  $h_k$  is an admissible perturbation.  $\square$

Now, it remains to prove that

$$\exists k \in \mathbb{N}^*, \quad \ddot{F}_\mu(m)[h_k, h_k] = \mu \int_{\Omega} u_{m,\mu} |\nabla \psi_k|^2 - \int_{\Omega} V_m \psi_k^2 > 0. \quad (2.39)$$

Since  $\sup_{k \in \mathbb{N}} \|\psi_k\|_{L^\infty} \leq \|\chi\|_{L^\infty}$  and since  $V_{m,\mu} \in L^\infty(\Omega)$  according to (2.15), it is enough to show that

$$\int_{\Omega} u_{m,\mu} |\nabla \psi_k|^2 \xrightarrow[k \rightarrow \infty]{} +\infty. \quad (2.40)$$

Using the fact that  $\inf_{\Omega} u_{m,\mu} > 0$  from Estimate (2.12), (2.40) is implied by

$$\int_{\Omega} |\nabla \psi_k|^2 \xrightarrow[k \rightarrow \infty]{} +\infty. \quad (2.41)$$

Finally, since  $\mathbb{B}(x_0; r) \cap \mathbb{B}(y_0; r) = \emptyset$ , (2.41) is in turn implied by

$$\int_{\mathbb{B}(x_0; r)} |\nabla \psi_k|^2 \xrightarrow[k \rightarrow \infty]{} +\infty. \quad (2.42)$$

Let us now establish (2.42). By using polar coordinates, one has

$$\begin{aligned} \int_{\mathbb{B}(x_0; r)} |\nabla \psi_k|^2 &= (2\pi)^{d-1} \int_0^r s^{d-1} \left( k \sin(ks) \chi(s) + \cos(ks) \frac{\partial \chi}{\partial s}(s) \right)^2 ds \\ &= (2\pi)^{d-1} \int_0^r s^{d-1} k^2 \sin^2(ks) \chi(s) ds \end{aligned} \quad (I_{1,k})$$

$$+ 2(2\pi)^{d-1} k \int_0^r s^{d-1} \sin(ks) \cos(ks) \chi(s) \frac{\partial \chi}{\partial s}(s) ds \quad (I_{2,k})$$

$$+ (2\pi)^{d-1} \int_0^r s^{d-1} \cos^2(ks) \left( \frac{\partial \chi}{\partial s} \right)^2 ds. \quad (I_{3,k})$$

Since  $\sin^2(k\cdot)$  converges weakly to  $\frac{1}{2}$  in  $L^2(0, r)$ , since  $\chi(0) = 1$  and  $\|\chi\|_{\mathcal{C}^1} \leq M$  for some  $M > 0$ , it follows that

$$I_{1,k} \underset{k \rightarrow +\infty}{\sim} k^2 C_0, C_0 > 0 \quad \text{and} \quad I_{2,k} = \underset{k \rightarrow \infty}{o} (I_{1,k}).$$

Finally, (I<sub>3,k</sub>) remains bounded and we get

$$\int_{\mathbb{B}(x_0; r)} |\nabla \psi_k|^2 \underset{k \rightarrow \infty}{\sim} k^2 C_0$$

for some constant  $C_0 > 0$ , which concludes the proof.

**Remark 14.** *In this approach which, as we underline, works under the strong hypothesis that  $\tilde{\Omega}$  has a non-empty interior, the core point is to build a sequence of admissible perturbations  $\{h_k\}_{k \in \mathbb{N}}$  such that the family  $\mathcal{H} = \{h_k\}_{k \in \mathbb{N}}$  is uniformly bounded in  $W^{-2,2}$  but not in  $W^{-1,2}$ ; this guarantees the blow-up of the  $W^{1,2}$ -norm and the boundedness of the  $L^2$ -norm of the associated Gâteaux-derivatives  $\theta_{m,\mu}[h_k]$ . In the proof of Theorem I, the perturbation  $h$  that we construct has a fixed  $L^2$  norm, and hence the sequence of Gâteaux-derivatives is uniformly bounded in  $W^{2,2}(\Omega)$ .*

### 2.3 Proof of Theorem II

The proof of Theorem II follows the same lines as the one of Theorem I. For this reason, we only indicate hereafter the main steps, and point to the principal differences.

Following the same methodology for stating the first order optimality conditions for problem  $(P_{\mu})$ , let us introduce the adjoint state  $p_{j,m,\mu}$  solution of

$$\begin{cases} \mu \Delta p_{j,m,\mu} + p_{j,m,\mu} (m - 2\theta_{m,\mu}) = -j'(\theta_{m,\mu}) & \text{in } \Omega, \\ \frac{\partial p_{j,m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.43)$$

Since  $j' > 0$ , a direct adaptation of Lemma 11 yields

$$\forall m \in \mathcal{M}(\Omega), \quad \inf_{\Omega} p_{j,m,\mu} > 0. \quad (2.44)$$

It is straightforward to see that the Gâteaux derivative of the functional  $\mathcal{J}_j$  writes

$$\dot{\mathcal{J}}_j(m)[h] = \int_{\Omega} h \theta_{m,\mu} p_{j,m,\mu}, \quad (2.45)$$

for every  $m \in \mathcal{M}(\Omega)$  and any admissible perturbation  $h$  at  $m$ .

Let us compute the second order Gâteaux derivative of  $\mathcal{J}_j$ . Keeping track of the fact that  $\ddot{\theta}_{m,\mu}$  solves (2.5) and that by direct computation, we obtain

$$\ddot{\mathcal{J}}_j(m)[h, h] = \int_{\Omega} \left( \dot{\theta}_{m,\mu}^2 j''(\theta_{m,\mu}) + \ddot{\theta}_{m,\mu} j'(\theta_{m,\mu}) \right), \quad (2.46)$$

we get an expression analogous to (2.16). Indeed, multiplying (2.43) by  $\ddot{\theta}_{m,\mu}$  and integrating by parts yields

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \ddot{\theta}_{m,\mu} j'(\theta_{m,\mu}) &= \int_{\Omega} \left( h \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu}^2 \right) p_{j,m,\mu} \\ &= - \int_{\Omega} \dot{\theta}_{m,\mu}^2 p_{j,m,\mu} + \int_{\Omega} \left( \frac{-\mu \Delta \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu} (m - 2\theta_{m,\mu})}{\theta_{m,\mu}} \right) p_{j,m,\mu}. \end{aligned}$$

Let us introduce

$$u_{j,m,\mu} := \frac{p_{j,m,\mu}}{\theta_{m,\mu}}. \quad (2.47)$$

Notice that, using the same arguments as in the proof of Theorem I, we obtain

$$\inf_{\Omega} u_{j,m,\mu} > 0, \quad \Delta u_{j,m,\mu} \in L^\infty(\Omega). \quad (2.48)$$

Since  $j$  belongs to  $\mathcal{C}^2$ , there exists  $M_j > 0$  such that

$$\|j''(\theta_{m,\mu})\|_{L^\infty} \leq M_j. \quad (2.49)$$

We thus obtain the existence of a potential  $V_{j,m,\mu} \in L^\infty(\Omega)$  such that

$$\ddot{\mathcal{J}}_j(m)[h, h] = \mu \int_{\Omega} u_{j,m,\mu} |\nabla \dot{\theta}_{m,\mu}|^2 - \int_{\Omega} V_{j,m,\mu} \dot{\theta}_{m,\mu}^2. \quad (2.50)$$

As a consequence, by (2.48) and by the fact that  $V_{j,m,\mu} \in L^\infty(\Omega)$ , it suffices to find a perturbation  $h$  such that, for a large enough parameter  $M_0 > 0$ ,

$$\int_{\Omega} |\nabla \dot{\theta}_{m,\mu}|^2 \geq M_0 \int_{\Omega} \dot{\theta}_{m,\mu}^2. \quad (2.51)$$

We are now back to proving (2.19), and the proof reads the same way.

### 3 Proof of Theorem III

The core of this proof relies on fine energy estimates. In what follows, it will be convenient to introduce the set of bang-bang functions

$$\mathbb{M}(\Omega) := \{m \in \mathcal{M}(\Omega), \exists E \subset \Omega \mid m = \mathbf{1}_E\}.$$

To alleviate the reading, let us start with the presentation of the proof structure.



**Main idea** The proof rests upon the use of two ingredients:

(i) the first one reads

**Lemma 15** ([22, Lemma 2]). *There exists  $\delta > 0$  such that*

$$\liminf_{\mu \rightarrow 0^+} \left( \sup_{m \in \mathcal{M}(\Omega)} F_\mu(m) \right) \geq m_0 + \delta > 0. \quad (3.1)$$

(ii) the second one, on which the emphasis will be put throughout the proof, is an estimate of the following form: there exist a constant  $M > 0$  and two exponents  $\alpha, \beta > 0$  such that

$$\forall m \in \mathcal{M}(\Omega) \cap BV(\Omega), \quad \|\theta_{m,\mu} - m\|_{L^1(\Omega)}^\beta \leq M \|m\|_{BV(\Omega)} \mu^\alpha. \quad (3.2)$$

If Estimate (3.2) holds, then, assuming that the optimiser  $m_\mu^*$  is a  $BV(\Omega)$ -function (if it is not, then  $\|m\|_{BV(\Omega)} = +\infty$  and the statement of the theorem is trivial) then we have

$$\mu^\alpha \|m_\mu^*\|_{BV(\Omega)} \geq \frac{1}{M} \|\theta_{m,\mu} - m\|_{L^1(\Omega)}^\beta \geq \frac{1}{M} \left| \int_\Omega \theta_{m,\mu} - m_0 \right|^\beta \geq \frac{\delta^\beta}{M}, \quad (3.3)$$

according to Lemma 15, yielding that

$$\|m_\mu^*\|_{BV(\Omega)} \geq \frac{M'}{\mu^\alpha}, \quad (3.4)$$

with  $M' = \frac{\delta^\beta}{M}$ . To obtain convergence rates such as (3.2), we will proceed using energy arguments and prove that a rescaled, shifted version of the natural energy associated with the PDE ( $\mathbf{E}_{m,\mu}$ ) yields this kind of control.

**The rescaled energy functional** Let us first recall that the equation ( $\mathbf{E}_{m,\mu}$ ) admits a variational formulation: let us introduce

$$\mathcal{E}_{m,\mu} : W^{1,2}(\Omega) \ni \theta \mapsto \frac{\mu}{2} \int_\Omega |\nabla \theta|^2 + \frac{1}{3} \int_\Omega \theta^3 - \frac{1}{2} \int_\Omega m \theta^2,$$

then  $\theta_{m,\mu}$  is characterized as the unique minimiser of  $\mathcal{E}_\mu$  over  $W^{1,2}(\Omega)$ ; in other words

$$\mathcal{E}_{m,\mu}(\theta_{m,\mu}) = \inf_{\substack{u \in W^{1,2}(\Omega) \\ u \geq 0}} \mathcal{E}_{m,\mu}(u). \quad (3.5)$$

Since we could not locate this formulation in the literature, we give a proof:

**Lemma 16.**  *$\theta_{m,\mu}$  is the unique minimiser of*

$$\mathcal{E}_{m,\mu} : u \mapsto \frac{\mu}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\Omega m u^2 + \frac{1}{3} \int_\Omega u^3 \quad (3.6)$$

over the set  $\mathcal{K} := \{u \in W^{1,2}(\Omega), u \geq 0 \text{ in } \Omega\}$ .

For the sake of completeness, this lemma is proved in Appendix A

Let us introduce

$$\tilde{\mathcal{E}}_{m,\mu} : \{u \in W^{1,2}(\Omega), u \geq 0\} \ni \theta \mapsto \mathcal{E}_{m,\mu}(\theta) + \frac{1}{6} \int_\Omega m^3. \quad (3.7)$$

**Remark 17.** The definition of  $\tilde{\mathcal{E}}_{m,\mu}$  is justified by the following, formal computation: let us assume that  $m$  is a  $\mathcal{C}^1$  function. It is known [16] that  $\theta_{m,\mu} \xrightarrow{\mu \rightarrow 0^+} m$  in  $L^p(\Omega)$ , for  $p \in [1; +\infty)$ . Since we aim at obtaining a convergence rate for  $\|\theta_{m,\mu} - m\|_{L^1(\Omega)}$  as  $\mu \rightarrow 0^+$ , it is natural to consider the energy  $\mathcal{E}_{m,\mu}(m)$ . Explicit computations show that

$$\mathcal{E}_{m,\mu}(m) = \frac{\mu}{2} \int_{\Omega} |\nabla m|^2 - \frac{1}{6} \int_{\Omega} m^3 \xrightarrow{\mu \rightarrow 0} -\frac{1}{6} \int_{\Omega} m^3,$$

which justifies to consider the energy  $\tilde{\mathcal{E}}_{m,\mu}$ .

**Estimating  $\|\theta_{m,\mu} - m\|_{L^1(\Omega)}$  using  $\tilde{\mathcal{E}}_{m,\mu}$ .** The key point is then to prove that  $\|\theta_{m,\mu} - m\|_{L^1(\Omega)}$  can be estimated in terms of the rescaled energy, *via* the following two Lemmas.

**Lemma 18.** *There exists a constant  $M_1 > 0$  such that*

$$\forall m \in \mathcal{M}(\Omega), \quad \|\theta_{m,\mu} - m\|_{L^1(\Omega)} \leq M_1 \tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu})^{\frac{1}{3}} = M_1 \left( \inf_{u \in W^{1,2}(\Omega), u \geq 0} \tilde{\mathcal{E}}_{m,\mu}(u) \right)^{\frac{1}{3}}. \quad (3.8)$$

*Proof of Lemma 18.* We split the proof into two steps.

**Step 1.** *There exists  $M > 0$  such that*

$$\forall \mu > 0, \forall m \in \mathcal{M}(\Omega), \quad \int_{\Omega} \left( \frac{\theta_{m,\mu}}{3} + \frac{m}{6} \right) (\theta_{m,\mu} - m)^2 \leq \tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu}). \quad (3.9)$$

This follows from explicit computations. Setting  $A = \int_{\Omega} \left( \frac{\theta_{m,\mu}}{3} + \frac{m}{6} \right) (\theta_{m,\mu} - m)^2$ , one has

$$\begin{aligned} A &= \frac{1}{3} \int_{\Omega} \theta_{m,\mu} (\theta_{m,\mu}^2 - 2m\theta_{m,\mu} + m^2) + \frac{1}{6} \int_{\Omega} m (\theta_{m,\mu}^2 - 2m\theta_{m,\mu} + m^2) \\ &= \frac{1}{3} \int_{\Omega} \theta_{m,\mu}^3 - \frac{2}{3} \int_{\Omega} m\theta_{m,\mu}^2 + \frac{1}{3} \int_{\Omega} \theta_{m,\mu} m^2 + \frac{1}{6} \int_{\Omega} m^3 + \frac{1}{6} \int_{\Omega} m\theta_{m,\mu}^2 - \frac{1}{3} \int_{\Omega} \theta_{m,\mu} m^2 \\ &= \frac{1}{3} \int_{\Omega} \theta_{m,\mu}^3 + \frac{1}{6} \int_{\Omega} m^3 - \frac{1}{2} \int_{\Omega} m\theta_{m,\mu}^2 = \mathcal{E}_{\mu}(\theta_{m,\mu}) - \frac{\mu}{2} \int_{\Omega} |\nabla \theta_{m,\mu}|^2 + \frac{1}{6} \int_{\Omega} m^3 \\ &\leq \mathcal{E}_{\mu}(\theta_{m,\mu}) + \frac{1}{6} \int_{\Omega} m^3 = \tilde{\mathcal{E}}_{\mu}(\theta_{m,\mu}). \end{aligned}$$

**Step 2.** *There exists  $M_0 > 0$  such that for every  $\mu > 0$  and  $m \in \mathbb{M}(\Omega)$ , one has*

$$\|\theta_{m,\mu} - m\|_{L^1(\Omega)} \leq M_0 \left( \int_{\Omega} \left( \frac{\theta_{m,\mu}}{3} + \frac{m}{6} \right) (\theta_{m,\mu} - m)^2 \right)^{\frac{1}{3}}.$$

According to the Jensen inequality, one has  $\|\theta_{m,\mu} - m\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} \left( \int_{\Omega} (\theta_{m,\mu} - m)^2 \right)^{\frac{1}{2}}$ . Since  $m \in \mathbb{M}(\Omega)$ , one has

$$\int_{\Omega} \left( \frac{\theta_{m,\mu}}{3} + \frac{m}{6} \right) (\theta_{m,\mu} - m)^2 = \int_{\{m=0\}} \frac{\theta_{m,\mu}^3}{3} + \int_{\{m=1\}} \frac{1}{6} (\theta_{m,\mu} - 1)^2.$$

According to the Jensen inequality, there exists a constant  $C > 0$  such that

$$\int_{\{m=0\}} \frac{\theta_{m,\mu}^3}{3} \geq C \left( \int_{\{m=0\}} \theta_{m,\mu}^2 \right)^{\frac{3}{2}}. \quad (3.10)$$

Furthermore, for any  $N_0 \in \mathbb{R}_+^*$ , there exists  $C_{N_0} > 0$  such that

$$\forall x \in [0; N_0), \quad C_{N_0} x^{\frac{3}{2}} \leq x.$$

Since  $0 \leq |\theta_{m,\mu} - m| \leq 2$  a.e. in  $\Omega$ , this last inequality yields the existence of  $C' > 0$  such that

$$\frac{1}{6} \int_{\{m=1\}} (\theta_{m,\mu} - m)^2 \geq C' \left( \int_{\{m=1\}} (\theta_{m,\mu} - m)^2 \right)^{\frac{3}{2}} \quad (3.11)$$

As a consequence, there exists  $C'' > 0$  such that

$$\begin{aligned} \int_{\Omega} \left( \frac{\theta_{m,\mu}}{2} + \frac{m}{6} \right) (\theta_{m,\mu} - m)^2 &\geq \int_{\{m=0\}} \frac{\theta_{m,\mu}^3}{2} + \int_{\{m=1\}} \frac{1}{6} (\theta_{m,\mu} - m)^2 \\ &\geq C'' \left( \int_{\{m=0\}} \theta_{m,\mu}^2 \right)^{\frac{3}{2}} + C'' \left( \int_{\{m=1\}} (\theta_{m,\mu} - m)^2 \right)^{\frac{3}{2}} \\ &\geq C'' \left( \int_{\Omega} (\theta_{m,\mu} - m)^2 \right)^{\frac{3}{2}}, \end{aligned}$$

whence the conclusion.

The lemma follows from a combination of the two steps.  $\square$

As a consequence, we will aim at proving an estimate of the form

$$\tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu}) \leq \|m\|_{BV(\Omega)} \mu^\beta \quad (3.12)$$

which, with Lemma 18, will lead to an estimate of the type (3.2). As explained in the introduction, we provide in Section B an alternative proof of (3.12) in the one-dimensional case following the method of Modica [23], that cannot unfortunately be straightforwardly extended to higher dimensions.

Let us now concentrate on the multidimensional case, assuming that  $\Omega = (0;1)^d$  with  $d \geq 1$ . We aim at obtaining an estimate of the form (3.2) which, by Lemma 18 amounts to determine a bound on  $\tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu})$ . Let us first consider  $m \in \mathcal{M}(\Omega) \cap W^{1,2}(\Omega)$ , that we will use as a test function in the energy. We get

$$\tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu}) \leq \tilde{\mathcal{E}}_{m,\mu}(m) = \frac{\mu}{2} \int_{\Omega} |\nabla m|^2. \quad (3.13)$$

We now consider a convolution kernel defined with the help of an approximation of unity. Namely, we consider a  $\mathcal{C}^\infty$  function  $\chi$  with compact support in  $\mathbb{B}(0;1)$  satisfying

$$0 \leq \chi \leq 1 \text{ a.e. in } \mathbb{B}(0;1), \quad \int_{\mathbb{B}(0;1)} \chi = 1.$$

For every  $\varepsilon > 0$ , we define

$$\chi_\varepsilon(x) := \frac{1}{\varepsilon^d} \chi\left(\frac{x}{\varepsilon}\right). \quad (3.14)$$

Every  $m \in \mathcal{M}(\Omega)$  is extended outside of  $\Omega$  by a compactly supported function of bounded variation, according to [1, Proposition 3.21]. We define

$$m_\varepsilon := m \star \chi_\varepsilon \quad (3.15)$$

for every  $\varepsilon > 0$ , where  $\star$  stands for the convolution product in  $L^1(\mathbb{R}^d)$ . It is standard that

$$\forall p \in (1; +\infty), \quad \|m - m_\varepsilon\|_{L^p(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.16)$$

According to [16, Equation (2.4)], there exists a constant  $M$  such that for every  $m, m' \in \{f \in L^\infty(\Omega), f \geq 0\}$ , there holds

$$\|\theta_{m,\mu} - \theta_{m',\mu}\|_{L^1(\Omega)} \leq C \|m - m'\|_{L^1(\Omega)}^{\frac{1}{3}}. \quad (3.17)$$

By the triangle inequality, for any  $m \in \mathcal{M}(\Omega) \cap W^{1,2}(\Omega)$  and any  $\mu, \varepsilon > 0$ ,

$$\left| \int_{\Omega} \theta_{m,\mu} - m_0 \right| \leq \|\theta_{m,\mu} - \theta_{m_\varepsilon,\mu}\|_{L^1(\Omega)} + \|\theta_{m_\varepsilon,\mu} - m_\varepsilon\|_{L^1(\Omega)} + \left| \int_{\Omega} m_\varepsilon - m_0 \right|. \quad (3.18)$$

Note that one has, for any  $i \in \llbracket 1, d \rrbracket$ ,

$$\begin{aligned} \|\partial_i m_\varepsilon\|_{L^2(\Omega)}^2 &= \int_{\Omega} \left( \frac{1}{\varepsilon^{d+1}} \int_{\mathbb{R}^d} \partial_i \chi \left( \frac{x-y}{\varepsilon} \right) m(y) dy \right)^2 dx \\ &\leq \frac{M}{\varepsilon^{2(d+1)}} \int_{\mathbb{R}^d} \int_{\Omega} |\nabla \chi|^2 \left( \frac{x-y}{\varepsilon} \right) |m(y)| dy dx \quad \text{by Jensen's inequality} \\ &\leq \frac{M}{\varepsilon^2} \|\nabla \chi\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Hence, there exists  $M > 0$  such that

$$\|\nabla m_\varepsilon\|_{L^2(\Omega)} \leq \frac{M}{\varepsilon}.$$

As a consequence, by (3.13) and Lemma 18, we have

$$\|\theta_{m,\mu} - m\|_{L^1(\Omega)} \leq M_1 \mu^{\frac{1}{3}} \|\nabla m\|_{L^2(\Omega)}^{\frac{2}{3}} \leq M_1' \frac{\mu^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}. \quad (3.19)$$

It follows that there exists  $M > 0$  such that

$$\begin{aligned} \left| \int_{\Omega} \theta_{m,\mu} - m_0 \right| &\leq \|\theta_{m,\mu} - \theta_{m_\varepsilon,\mu}\|_{L^1(\Omega)} + \|\theta_{m_\varepsilon,\mu} - m_\varepsilon\|_{L^1(\Omega)} + \left| \int_{\Omega} m_\varepsilon - m_0 \right| \\ &\leq M \left( \|m - m_\varepsilon\|_{L^1(\Omega)}^{\frac{1}{3}} + \frac{\mu^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}} + \left| \int_{\Omega} m_\varepsilon - m_0 \right| \right) \end{aligned}$$

To end the proof, we need the following lemma, whose proof is postponed to the end of this section for the sake of clarity.

**Lemma 19.** *There exists a constant  $d_0 > 0$  such that*

$$\forall m \in \mathcal{M}(\Omega), \|m - m_\varepsilon\|_{L^1(\Omega)} \leq d_0 \varepsilon \|\nabla m\|_{L^1(\Omega)}. \quad (3.20)$$

As a consequence, we have  $\|m - m_\varepsilon\|_{L^1(\Omega)} \lesssim \varepsilon \|m\|_{BV(\mathbb{R}^d)}$ . Since the extension operator is bounded, we thus get  $\|m - m_\varepsilon\|_{L^1(\Omega)} \leq M \varepsilon \|m\|_{BV(\Omega)}$ . Plugging all these estimates together and using (3.18), we finally obtain

$$\left| \int_{\Omega} \theta_{m,\mu} - m_0 \right| \leq M \left( \varepsilon \|m\|_{BV} \right)^{\frac{1}{3}} + \frac{\mu^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}} + \varepsilon \|\nabla m\|_{L^1(\Omega)}$$

$$\leq M (\varepsilon \|m\|_{BV})^{\frac{1}{3}} + \varepsilon \|\nabla m\|_{L^1(\Omega)} + M \frac{\mu^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}.$$

Taking  $\varepsilon = N_0 \mu^{\frac{1}{2}}$  with  $N_0 > 0$ , one gets

$$\left| \int_{\Omega} \theta_{m,\mu} - m_0 \right| - \frac{M}{N_0^{2/3}} \leq \varepsilon \|m_{\mu}^*\|_{BV}^{\frac{1}{3}} + \varepsilon \|m_{\mu}^*\|_{BV},$$

and choosing  $N_0$  large enough yields

$$\liminf_{\varepsilon \rightarrow 0} (\varepsilon \|m_{\mu}^*\|_{BV})^{\frac{1}{3}} + (\varepsilon \|m_{\mu}^*\|_{BV}) > 0. \quad (3.21)$$

We infer the existence of  $C_0 > 0$  such that for all solution  $m_{\mu}^*$  of  $(P_{\mu})$ , one has

$$C_0 \leq (\varepsilon \|m_{\mu}^*\|_{BV})^{\frac{1}{3}} + (\varepsilon \|m_{\mu}^*\|_{BV}) \quad (3.22)$$

and therefore, there exists  $c_0 > 0$  such that, for every  $\mu > 0$  small enough

$$c_0 \leq \varepsilon \|m_{\mu}^*\|_{BV} = N_0 \mu^{\frac{1}{2}} \|m_{\mu}^*\|_{BV(\Omega)}. \quad (3.23)$$

The desired conclusion follows

*Proof of Lemma 19.* As is customary in convolution, one has

$$m(x) - \chi_{\varepsilon} \star m(x) \lesssim \sup_{|h|=\varepsilon} \|\tau_h m - m\|_{L^1(\mathbb{R}^d)}$$

for a.e.  $x \in \Omega$ , where  $\tau_h$  stands for the translation operator. However, we claim that

$$\sup_{|h|=\varepsilon} \|\tau_h m - m\|_{L^1(\mathbb{R}^d)} \leq \|h\|_{L^{\infty}} \|\nabla m\|_{L^1(\mathbb{R}^d)}. \quad (3.24)$$

It suffices to prove (3.24) for  $m \in \mathcal{C}^1$ , and the general result follows from the density of  $\mathcal{C}^1$  functions in  $BV(\mathbb{R}^d)$ . For any  $h \in \mathbb{R}^d$  we have

$$\begin{aligned} \int_{\mathbb{R}^d} |m(x+h) - m(x)| dx &= \int_{\mathbb{R}^d} \left| \int_0^1 \frac{d}{d\xi} [m(x+\xi h)] d\xi \right| dx = \int_{\mathbb{R}^d} \left| \int_0^1 \langle \nabla m(x+\xi h), h \rangle \right| dx \\ &\leq \|h\|_{\infty} \int_0^1 \int_{\mathbb{R}^d} |\nabla m(x+\xi h)| d\xi dx \\ &= \|h\|_{\infty} \int_{\mathbb{R}^d} |\nabla m| = \|h\|_{\infty} \|\nabla m\|_{L^1}. \end{aligned}$$

The desired result follows.  $\square$

## 4 Conclusion: possible extensions of the bang-bang property to other state equations

We conclude this article with a discussion on possible generalisations of our method. Indeed, an interesting question is to know whether or not the methods put forth in the proof of Theorem I could be applied to other types of boundary conditions, for instance Dirichlet or Robin, or for

other kinds of non-linearities. We justify below that it is the case, and that the main difficulty lies in the well-posedness of the equation acting as a constraint on the optimisation problem ( $P_\mu$ ).

Let us consider a boundary operator  $\mathcal{B}$ , that may be of Neumann ( $\mathcal{B}u = \frac{\partial u}{\partial \nu}$ ) or of Robin type ( $\mathcal{B}u = \frac{\partial u}{\partial \nu} + \beta u$  for some  $\beta > 0$ ). Let us consider a non-linearity  $F = F(x, u)$  of class  $\mathcal{C}^2$ , and consider, for a given  $m \in \mathcal{M}(\Omega)$ , the solution  $u_m$  of

$$\begin{cases} -\Delta u_m = mu_m + F(x, u_m) & \text{in } \Omega, \\ \mathcal{B}u_m = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

The first assumption on  $F$  one has to make is:

$$\begin{aligned} &\text{For any } m \in \mathcal{M}(\Omega), (4.1) \text{ has a unique positive solution } u_m. \\ &\text{Furthermore, } \inf_{m \in \mathcal{M}(\Omega)} \inf_{\Omega} u_m > 0, \quad \sup_{m \in \mathcal{M}(\Omega)} \|u_m\|_{\mathcal{C}^1} < \infty. \end{aligned} \quad (\mathbf{H})$$

It is notable that  $(\mathbf{H})$  is satisfied whenever  $F$  satisfies:

1.  $F(x, 0) = 0$  and the steady state  $z(\cdot) = 0$  is unstable.
2. Uniformly w.r.t.  $x \in \Omega$ , one has  $\lim_{y \rightarrow \infty} F(x, y)/y = -\infty$ .

This is for instance problematic when considering Dirichlet boundary conditions for the logistic-diffusive equation: depending on the range of  $\mu$  the equation may only have trivial solutions.

We also assume

$$\text{The map } m \mapsto u_m \text{ is twice G\^ateaux-differentiable.} \quad (\mathbf{H}')$$

which is for instance ensured whenever  $F \in W^{2,\infty}$  and if the solution  $u = u_m$  is linearly stable.

Consider then the following optimisation problem, where the function  $j$  satisfies  $(\mathbf{H}_j)$

$$\sup_{m \in \mathcal{M}(\Omega)} J(m), \quad \text{with } J(m) = \int_{\Omega} j(u_m). \quad (\mathbf{P})$$

Our methods enable us to prove that any solution of  $(\mathbf{P})$  is a bang-bang function. To do so, we need to write down the first and second order G\^ateaux-derivative of  $u_m$  with respect to  $m$ : using the same notations as in the rest of this article, if  $(\mathbf{H})$ - $(\mathbf{H}')$  hold, then it can be shown that

$$\begin{cases} -\Delta \dot{u}_m - m\dot{u}_m - \dot{u}_m \frac{\partial F}{\partial u}(x, u_m) = hu_m & \text{in } \Omega, \\ \mathcal{B}\dot{u}_m = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.2)$$

and

$$\begin{cases} -\Delta \ddot{u}_m - m\ddot{u}_m - \ddot{u}_m \frac{\partial F}{\partial u}(x, u_m) = 2h\dot{u}_m + (\dot{u}_m)^2 \frac{\partial^2 F}{\partial u^2}(x, u_m) & \text{in } \Omega, \\ \mathcal{B}\ddot{u}_m = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.3)$$

This allows us to compute the derivative of  $J$ . Under  $(\mathbf{H})$ - $(\mathbf{H}')$ , we have

$$\dot{J}(m)[h] = \int_{\Omega} \dot{u}_m j'(u_m) \quad \text{and} \quad \ddot{J}(m)[h, h] = \int_{\Omega} (\dot{u}_m)^2 j''(u_m) + \int_{\Omega} \ddot{u}_m j'(u_m). \quad (4.4)$$

Let us introduce the adjoint state  $p_m$ , solving

$$\begin{cases} -\Delta p_m - mp_m - p_m \frac{\partial F}{\partial u}(x, u_m) = j'(u_m) & \text{in } \Omega, \\ \mathcal{B}p_m = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5)$$

Here, we need to make another assumption on  $F$ :

$$\text{For any } m \in \mathcal{M}(\Omega), \quad \inf_{\Omega} p_m > 0. \quad (\mathbf{H}'')$$

Given the assumption on  $j$ ,  $(\mathbf{H}'')$  is for instance implied if the first eigenvalue of  $-\Delta - m - \frac{\partial F}{\partial u}(\cdot, u_m)$  is positive (linear stability condition), that also ensures the Gâteaux differentiability of  $m \mapsto u_m$ . This in turn holds if  $F(x, u) = -ug(x, u)$  with  $g \in W^{2,\infty}$  non-decreasing. Using the adjoint state to compute  $\ddot{J}(m)[h, h]$  in a more tractable form, one has

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\Omega} (\dot{u}_m)^2 j''(u_m) + \int_{\Omega} \ddot{u}_m j'(u_m) \\ &= \int_{\Omega} (\dot{u}_m)^2 j''(u_m) + \int_{\Omega} p_m \left( 2h\dot{u}_m + (\dot{u}_m)^2 \frac{\partial^2 F}{\partial u^2}(x, u_m) \right) \\ &= \int_{\Omega} (\dot{u}_m)^2 \left( j''(u_m) + \frac{\partial^2 F}{\partial u^2}(x, u_m) \right) + 2 \int_{\Omega} \left( \frac{p_m}{u_m} \right) \dot{u}_m \left( -\Delta \dot{u}_m - m\dot{u}_m - \dot{u}_m \frac{\partial F}{\partial u}(x, u_m) \right) \\ &= \int_{\Omega} (\dot{u}_m)^2 \left( j''(u_m) + \frac{\partial^2 F}{\partial u^2}(x, u_m) - m \frac{p_m}{u_m} - \frac{p_m}{u_m} \frac{\partial F}{\partial u}(x, u_m) \right) \\ &\quad + 2 \int_{\Omega} \frac{p_m}{u_m} (-\dot{u}_m \Delta \dot{u}_m). \end{aligned}$$

Let us set  $\Psi_m := \frac{p_m}{u_m}$  and  $V_m := j''(u_m) + \frac{\partial^2 F}{\partial u^2}(x, u_m) - m\Psi_m - \Psi_m \frac{\partial F}{\partial u}(x, u_m)$ . Then we obtain

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\Omega} (\dot{u}_m)^2 V_m + 2 \int_{\Omega} \dot{u}_m \langle \nabla \Psi_m, \nabla \dot{u}_m \rangle + 2 \int_{\Omega} \Psi_m |\nabla \dot{u}_m|^2 \\ &= \int_{\Omega} (\dot{u}_m)^2 (V_m - \Delta \Psi_m) + 2 \int_{\Omega} \Psi_m |\nabla \dot{u}_m|^2. \end{aligned}$$

Defining  $V'_m = V_m - \Delta \Psi_m$ , we finally get

$$\ddot{J}(m)[h, h] = \int_{\Omega} 2\Psi_m |\nabla \dot{u}_m|^2 + \int_{\Omega} (\dot{u}_m)^2 V'_m \quad (4.6)$$

and the assumptions we made on  $F$  allow us to conclude that  $V'_m$  belongs to  $L^\infty$  and that  $\inf_{\Omega} \Psi_m > 0$ . To obtain the bang-bang property, we argue by contradiction and assume that the set  $\tilde{\Omega} := \{0 < m^* < 1\}$  is of positive measure. To reach a contradiction, it suffices to exhibit a perturbation  $h$  that is supported in  $\tilde{\Omega}$  such that

$$\int_{\Omega} h = 0 \quad \text{and} \quad \int_{\Omega} |\nabla \dot{u}_m|^2 > \frac{\|V'_m\|_{L^\infty(\Omega)}}{\inf_{\Omega} \Psi_m} \int_{\Omega} (\dot{u}_m)^2. \quad (4.7)$$

Following the proof of Theorem I, we introduce the sequence of eigenfunctions and eigenvalues  $\{\varphi_k, \lambda_k\}_{k \in \mathbb{N}}$  associated to the operator

$$\mathcal{L}_m := -\Delta - \left( m + \frac{\partial F}{\partial u}(x, u_m) \right) \quad (4.8)$$

with  $\mathcal{B}\varphi_k = 0$ . Adapting hence the proof of Theorem I, we show that for any  $K \in \mathbb{N}$ , there exists an admissible perturbation  $h$  such that

$$hu_m = \sum_{k \geq K} \alpha_k \varphi_k, \quad \sum_{k \geq K} \alpha_k^2 = 1. \quad (4.9)$$

It follows that for such a perturbation,

$$\int_{\Omega} |\nabla \dot{u}_m|^2 \geq \lambda_K \int_{\Omega} (\dot{u}_m)^2. \quad (4.10)$$

Choosing  $K \in \mathbb{N}$  large enough so that  $\lambda_K \geq \frac{\|V'_m\|_{L^\infty(\Omega)}}{\inf_{\Omega} \Psi_m}$  yields the expected conclusion.

# Appendix

## A Proof of Lemma 16

Let us first recall that since  $\theta_{m,\mu}$  is non-negative and does not vanish in  $\Omega$ , we have

$$\mathcal{E}_{m,\mu}(\theta_{m,\mu}) = -\frac{1}{6} \int_{\Omega} \theta_{m,\mu}^3 < 0, \quad (\text{A.1})$$

so that  $u(\cdot) = 0$  is not a minimiser of  $\mathcal{E}_{m,\mu}$ .

In order to prove this Lemma, let us introduce the energy functional

$$\mathcal{F}_{m,\mu} : W^{1,2}(\Omega) \ni u \mapsto \frac{\mu}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{2} \int_{\Omega} m u^2 + \frac{1}{3} \int_{\Omega} |u|^3. \quad (\text{A.2})$$

Observe that

$$\forall u \in W^{1,2}(\Omega), \quad \mathcal{F}_{m,\mu}(u) = \mathcal{F}_{m,\mu}(|u|) = \mathcal{E}_{m,\mu}(|u|). \quad (\text{A.3})$$

In particular, if  $\mathcal{F}_{m,\mu}$  has a minimiser  $u^*$ , then  $|u^*|$  also minimises  $\mathcal{F}_{m,\mu}$ , and  $|u^*|$  solves

$$\inf_{u \in \mathcal{X}} \mathcal{E}_{m,\mu}(u). \quad (\text{A.4})$$

Conversely, if  $u_* \geq 0$  is a minimiser of  $\mathcal{E}_{m,\mu}$  then for any  $z \in W^{1,2}(\Omega)$ ,

$$\mathcal{F}_{m,\mu}(z) = \mathcal{F}_{m,\mu}(|z|) = \mathcal{E}_{m,\mu}(|z|) \geq \mathcal{E}_{m,\mu}(u_*) = \mathcal{F}_{m,\mu}(u_*) \quad (\text{A.5})$$

and so  $u_*$  is a minimiser of  $\mathcal{F}_{m,\mu}$ .

Let us then prove that  $\theta_{m,\mu}$  is a minimiser of  $\mathcal{F}_{m,\mu}$ . Consider a minimising sequence  $\{y_k\}_{k \in \mathbb{N}}$  of  $\mathcal{F}_{m,\mu}$ . Up to replacing  $y_k$  with  $|y_k|$  which, thanks to (A.3), would still yield a minimising sequence, we can assume that for every  $k \in \mathbb{N}$ ,  $y_k$  is non-negative. Let us introduce  $\lambda(m)$  as the first eigenvalue of the operator  $-\Delta - m$  with Neumann boundary conditions. According to the Courant-Fischer principle, one has

$$\lambda(m) = \inf_{\substack{u \in W^{1,2}(\Omega) \\ \int_{\Omega} u^2 = 1}} \left( \mu \int_{\Omega} |\nabla u|^2 - \int_{\Omega} m u^2 \right) \quad (\text{A.6})$$

and therefore

$$\mathcal{F}_{m,\mu}(y_k) \geq \lambda(m) \|y_k\|_{L^2(\Omega)}^2 + \frac{1}{3} \|y_k\|_{L^3(\Omega)}^3. \quad (\text{A.7})$$

Since the embedding  $L^3(\Omega) \hookrightarrow L^2(\Omega)$  is continuous, there exists  $C > 0$  such that

$$\sup_{k \in \mathbb{N}} \left( C \lambda(m) \|y_k\|_{L^3(\Omega)}^2 + \frac{1}{3} \|y_k\|_{L^3(\Omega)}^3 \right) < \infty. \quad (\text{A.8})$$

As a consequence,  $\{y_k\}_{k \in \mathbb{N}}$  is bounded in  $L^3(\Omega)$  and then also in  $L^2$  by using the same argument. Finally, by definition of  $\mathcal{F}_{m,\mu}$ , it is also uniformly bounded in  $W^{1,2}(\Omega)$ .

Hence, there exists a strong  $L^2(\Omega)$ , weak  $L^3(\Omega)$  and weak  $W^{1,2}$  closure point  $y_{\infty} \in W^{1,2}(\Omega)$  of  $\{y_k\}_{k \in \mathbb{N}}$ . Since the map  $\mathbb{R} \ni x \mapsto |x|^3$  is convex, the map  $L^3(\Omega) \ni y \mapsto \int_{\Omega} |y|^3$  is lower semi-continuous. Hence it follows that

$$\liminf_{k \rightarrow \infty} \mathcal{F}_{m,\mu}(y_k) \geq \mathcal{F}_{m,\mu}(y_{\infty}), \quad (\text{A.9})$$



and  $y_\infty$  minimises  $\mathcal{F}_{m,\mu}$  over  $\mathcal{K}$ . Since  $0_{L^2(\Omega)}$  is not a minimiser, we have  $y_\infty \geq 0$  and  $y_\infty(\cdot) \neq 0$ . The map  $x \mapsto |x|^3$  is  $\mathcal{C}^1$  and the Euler-Lagrange equation on  $y_\infty$  writes

$$\begin{cases} \Delta y_\infty + y_\infty(m - y_\infty) = 0 & \text{in } \Omega, \\ \frac{\partial y_\infty}{\partial \nu} = 0 & \text{in } \partial\Omega, \\ y_\infty \geq 0. \end{cases} \quad (\text{A.10})$$

From uniqueness for non-zero, non-negative solutions of the logistic-diffusive PDE, it follows that  $y_\infty = \theta_{m,\mu}$ . As a consequence:

$$\mathcal{E}_{m,\mu}(\theta_{m,\mu}) = \mathcal{F}_{m,\mu}(\theta_{m,\mu}) = \min_{W^{1,2}(\Omega)} \mathcal{F}_{m,\mu} = \min_{\mathcal{K}} \mathcal{E}_{m,\mu}, \quad (\text{A.11})$$

which concludes the proof.

## B Proof of (3.12) in the one-dimensional case

We assume in this section that  $\Omega = (0, 1)$ . Let us prove (3.12). The proof relies on ideas by Modica [23]. Given Lemma 15 and (3.3), it is enough to establish a uniform convergence rate of  $\theta_{m,\mu}$  to  $m$  in  $L^1(\Omega)$  with respect to the  $BV(\Omega)$  norm of  $m$ , as  $\mu \rightarrow 0$ .

We proceed in several steps, first considering the case where  $m$  is the characteristic function of a set of finite perimeter before encompassing the general case. Let us introduce

$$\mathcal{M}_M(\Omega) := \{m \in \mathcal{M}(\Omega), \|m\|_{BV(\Omega)} \leq M\} \quad \text{and} \quad \mathbb{M}_M(\Omega) := \mathbb{M}(\Omega) \cap \mathcal{M}_M(\Omega), \quad (\text{B.1})$$

for every  $M > 0$ . The following Proposition is the key point of the proof.

**Proposition 20.** *There exists  $C_1 > 0$  such that*

$$\forall M > 0, \forall m \in \mathbb{M}(\Omega), \quad \tilde{\mathcal{E}}_{m,\mu}(\theta_{m,\mu}) \leq C_1 \sqrt{\mu} \|m\|_{BV(\Omega)}. \quad (\text{B.2})$$

We can now prove Theorem 1. First of all, the maximiser  $m_\mu^*$  of  $(P_\mu)$  is a bang-bang function by Theorem I and belongs therefore to  $\mathbb{M}(\Omega)$ . We thus obtain

$$\delta^3 \leq C_1^3 \sqrt{\mu} \|m_\mu^*\|_{BV(\Omega)}, \quad (\text{B.3})$$

where  $\delta > 0$  is given by Lemma 15. The conclusion follows.

*Proof of Proposition 20.* In what follows, we will bypass the distinction between the interior perimeter of a subset  $A \subset (0, 1)$ , denoted  $\text{Per}_{int}(A)$ , and its perimeter denoted  $\text{Per}(A)$  when seen as a subset of  $\mathbb{R}$ . Since we have obviously

$$\text{Per}_{int}(A) \leq \text{Per}(A) \leq \text{Per}_{int}(A) + 2, \quad (\text{B.4})$$

it follows that there exists  $c_0 > 0$  such that, for any set  $A$  of finite perimeter

$$\text{Per}_{int}(A) \geq 2 \quad \Rightarrow \quad c_0 \text{Per}(A) \leq \text{Per}_{int}(A) \leq \text{Per}(A). \quad (\text{B.5})$$

Furthermore, since we know from [22] that the  $BV(\Omega)$  norm of maximisers blows-up as  $\mu \rightarrow 0$ , we can always assume that the set of finite perimeter  $A$  we are working with satisfies  $\text{Per}_{int}(A) \geq 2$ .

Since  $m \in \mathbb{M}_M(\Omega)$ , we know that  $m$  writes  $m = \mathbb{1}_A$  where  $A$  is a set of bounded perimeter.

Let us then consider such a subset  $A$ . Since  $A$  is of finite perimeter, it writes

$$A = \bigsqcup_{i=1}^n (a_i; b_i) \quad (\text{B.6})$$

with  $0 \leq a_i < b_i < a_{i+1} \leq 1$  for every  $i \in \llbracket 1, \dots, n \rrbracket$ .

To obtain the conclusion of the Proposition, it suffices to exhibit a constant  $C_1$  that does not depend on  $\mu, m$ , and a function  $u_\mu \in W^{1,2}(\Omega)$  such that

$$\tilde{\mathcal{E}}_{m,\mu}(u_\mu) \leq C_1 \sqrt{\mu} \text{Per}(A). \quad (\text{B.7})$$

Let us introduce  $h_A$ , the so-called signed-distance function to the set  $A$ , defined by

$$h_A : x \mapsto \begin{cases} \text{dist}(x, \partial A) & \text{if } x \notin A, \\ 0 & \text{if } x \in \partial A, \\ -\text{dist}(x, \partial A) & \text{if } x \in A, \end{cases} \quad (\text{B.8})$$

as well as the auxiliary function

$$\phi_\varepsilon : \mathbb{R} \ni t \mapsto \begin{cases} 1 & \text{if } t < 0, \\ 0 & \text{if } t \geq \eta_\varepsilon, \\ 1 - \frac{t}{\eta_\varepsilon} & \text{otherwise,} \end{cases} \quad (\text{B.9})$$

for some regularization parameter  $\varepsilon \geq 0$ , where  $\eta_\varepsilon = \varepsilon^{\frac{1}{4}}$ . We combine these two functions and introduce  $u_\varepsilon = \phi_\varepsilon \circ h_A$ . Let us use  $u_\varepsilon$  as a test function in the variational formulation (3.5). We will estimate separately the gradient term and the remainder term of the energy functional.

**Estimate of the gradient term.** Since  $h_A$  is differentiable a.e. and  $|h'_A| = 1$ , we have  $(u'_\varepsilon)^2 = \phi'_\varepsilon(h_A(\cdot))^2$  a.e. in  $\Omega$ . Using the decomposition of  $A$ , we get

$$\int_0^1 (u'_\varepsilon)^2 = \int_0^{a_1} \phi'_\varepsilon(h_A(t))^2 dt + \sum_{i=1}^n \left\{ \int_{a_i}^{b_i} \phi'_\varepsilon(h_A(t))^2 dt + \int_{b_i}^{a_{i+1}} \phi'_\varepsilon(h_A(t))^2 dt \right\} + \int_{a_{n+1}}^1 \phi'_\varepsilon(h_A(t))^2 dt. \quad (\text{B.10})$$

Let us focus on the term

$$\sum_{i=1}^n \left\{ \int_{a_i}^{b_i} \phi'_\varepsilon(h_A(t))^2 dt + \int_{b_i}^{a_{i+1}} \phi'_\varepsilon(h_A(t))^2 dt \right\}.$$

The main interest of this decomposition is that on each interval  $(a_i; b_i)$  or  $(b_i; a_{i+1})$ , the function  $h$  is symmetric with respect to the midpoint of the interval. As a consequence, two cases may occur when considering the interval  $(a_i; b_i)$  (the case  $(b_i; a_{i+1})$  being exactly identical):

- (i) either  $|b_i - a_i| \leq 2\eta_\varepsilon$ , in which case, since  $\|\phi'_\varepsilon\|_{L^\infty} = \frac{1}{\eta_\varepsilon}$  it follows that

$$\int_{a_i}^{b_i} \phi'_\varepsilon(h_A(t))^2 dt \leq 2\eta_\varepsilon \|\phi'_\varepsilon\|_{L^\infty}^2 \leq \frac{2}{\eta_\varepsilon}. \quad (\text{B.11})$$

- (ii) or  $|b_i - a_i| > 2\eta_\varepsilon$ , in which case  $|\{u'_\varepsilon \neq 0\} \cap (a_i; b_i)| \leq 2\eta_\varepsilon$  and so

$$\int_{a_i}^{b_i} \phi'_\varepsilon(h_A(t))^2 dt \leq 2\eta_\varepsilon \|\phi'_\varepsilon\|_{L^\infty}^2 \leq \frac{2}{\eta_\varepsilon}. \quad (\text{B.12})$$

As such, we have

$$\sum_{i=1}^n \left\{ \int_{a_i}^{b_i} \phi'_\varepsilon(h_A(t))^2 dt + \int_{b_i}^{a_{i+1}} \phi'_\varepsilon(h_A(t))^2 dt \right\} \leq \frac{4n}{\eta_\varepsilon} \leq 2 \frac{\text{Per}(A)}{\eta_\varepsilon}. \quad (\text{B.13})$$

The end terms

$$\int_0^{a_1} \phi'_\varepsilon(h_A(t))^2 dt + \int_{b_n}^1 \phi'_\varepsilon(h_A(t))^2 dt$$

are handled in the same way, and we finally obtain

$$\int_0^1 (u'_\varepsilon)^2(t) dt \leq C \frac{\text{Per}(A)}{\eta_\varepsilon} \quad (\text{B.14})$$

for some constant  $C > 0$ .

**Estimate of the potential term.** It remains to deal with the quantity

$$\frac{1}{3} \int_0^1 u_\varepsilon^3(t) dt - \frac{1}{2} \int_0^1 m u_\varepsilon^2(t) dt + \frac{1}{6} \int_0^1 m^3. \quad (\text{B.15})$$

If we define  $\psi_\varepsilon = \frac{1}{3}u_\varepsilon^3 - \frac{1}{2}m u_\varepsilon^2 + \frac{1}{6}m^3$  we have the following decomposition: in the set  $\{m = 1\}$ , we have  $h_A \leq 0$ , hence  $u_\varepsilon = 1$  and we infer that  $\psi_\varepsilon = 0$  in  $\{m = 1\}$ .

The integral to estimate boils down to

$$\int_0^1 \psi_\varepsilon(t) \mathbf{1}_{\{m=0\}} dt = \int_0^1 \frac{1}{3} u_\varepsilon^3 \mathbf{1}_{\{m=0\}}. \quad (\text{B.16})$$

However, we can do exactly the same distinction as for the analysis of the gradient part of the energy: for any  $i \in \llbracket 1, n \rrbracket$  (the end intervals are handled in the same way) we either have  $|a_{i+1} - b_i| \leq 2\eta_\varepsilon$ , in which case

$$\int_{b_i}^{a_{i+1}} \psi_\varepsilon(t) dt \leq 2\eta_\varepsilon \quad (\text{B.17})$$

or  $|a_{i+1} - b_i| > 2\eta_\varepsilon$ , in which case the same conclusion holds since  $0 \leq \psi_\varepsilon \leq 1$  a.e. in  $\Omega$ . As a consequence, we obtain

$$\int_0^1 \psi_\varepsilon(t) dt \leq 2\eta_\varepsilon \text{Per}(A). \quad (\text{B.18})$$

Combining (B.14) and (B.18) yields the existence of  $C_1 > 0$  such that

$$\tilde{\mathcal{E}}_{m,\mu}(u_\varepsilon) \leq C_1 \left( \frac{\mu}{\eta_\varepsilon} + \eta_\varepsilon \right) \text{Per}(A). \quad (\text{B.19})$$

Picking  $\eta_\varepsilon = \sqrt{\mu}$ , we obtain

$$\tilde{\mathcal{E}}_{m,\mu}(u_\varepsilon) \leq 2C_1 \sqrt{\mu} \text{Per}(A), \quad (\text{B.20})$$

leading to the desired conclusion.  $\square$

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