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Mesoscale modeling and experimental analyses for pantographic cells: effect of hinge deformation

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Abstract

In order to synthesize exotic metamaterials, it was proposed to use as elementary components of metamaterial microstructures so-called pantographic cells. The use of a mesomodel based on Euler–Bernoulli nonlinear beam theory is utilized for describing their behavior when their characteristic sizes are of the order of millimeters. With an in-house code, in which the connecting hinges were modeled as elastic extensional and torsional elements, simulations were performed and compared with experimental data. The comparison was carried out for compression *and* extension tests. It is proven, both numerically and experimentally, that the hinges undergo different deformation mechanisms. As a consequence, for compression and extension tests, the distribution and the total value of the strain energy vary significantly. The analysis is made possible in parts via Digital Image Correlation, which allows kinematic fields to be measured and the hypotheses employed for developing the proposed model to be probed experimentally.

Keywords: Digital Image Correlation, Euler–Bernoulli beam model, Extensible metamaterials, Large microstructural torsion, Pantographic cells

1. Introduction

In the history of Science, the construction of models describing physical phenomena has been triggered by the actual feasibility of generating experimental proofs. In accordance with this approach, the Cauchy version of Continuum Mechanics has seen a successful amount of validations. Classic Continuum Mechanics assumes some *ad hoc* limitations in the selection of the terms to be used in the energy of deformable media [11]. These hypotheses were generalized to obtain new mechanical models. They are reasonable when looking for simple models that lead to a wide domain of validity. As every mathematical model, Cauchy Continuum Mechanics has a limited range of applicability. In the present paper, the mechanical behavior of pantographic cells is investigated as they constitute an important microstructural element of second gradient metamaterials [1].

As an example of aforementioned limitations, it was established for Cauchy continua that the macroscopic strain energy *a priori* includes only the first displacement gradients. This choice may be accepted without any doubt

on the basis of experimental evidences for many materials. However, it becomes too restrictive when modeling different, tailored-designed materials, e.g. for so-called metamaterials [3]). In the present work, it is proven that, when using pantographic cells as metamaterial microstructure, it is necessary, after homogenization, to use generalized micro-stretched continua. In particular, when the micro-deformability of connecting hinges is not negligible, it may be appropriate to specify the kinematics of introduced generalized continua with two placement fields [26]. It is noteworthy that such micro mechanisms play significant role in the deformation of natural as well as synthesized granular materials [17].

In the classic works [16, 21], the authors have demonstrated that the presence of a microstructure can be taken into account from the macroscopic viewpoint by also considering (at least) second gradients of displacements in the strain energy of the microstructured (architected) material. This observation has led to the development of various constitutive postulates [14, 15].

Pantographic structures are particular metamaterials that can be modeled by means of such second gradient continua [12, 24]. The underlying microstructure (sometimes simply called pantographic structure) consists of a grid made up of two layers of parallel fibers (or beams) in-

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47 terconnected by perfect hinges (free rotations) or deform-
 48 ing hinges (with torsional stiffness), which are sometimes
 49 referred to as pivots. The pantographic metamaterial is
 50 normally studied by means of two types of models, one
 51 discrete, based on a system of springs, and one continuum,
 52 obtained from the discrete one by a suitable homogeniza-
 53 tion procedure, as it is shown in [12]. The model we use in
 54 this work is a mesoscale model in the sense that it models
 55 each fibre as a continuum medium (a non-linear Euler-
 56 Bernoulli beam) and therefore this model is in a middle
 57 position between the two above mentioned. However, for
 58 the hinges we chose to consider a model that is in effect
 59 discrete because it models the torsion deformation with a
 60 rotational spring and the pivot shear deformation with an
 61 extensional spring. On the other hand, modeling pivots
 62 with a beam element does not seem appropriate. In this
 63 sense, the model presented in this paper is multiscale.

64 In literature many examples of similar architected
 65 metamaterials are present. In [10] the authors study the
 66 effects of the architecture of mechanical metamaterials on
 67 their mechanical properties, showing experimental exam-
 68 ples and numerical simulations. In [30], on the other hand,
 69 the authors study the deformations of stretchable meta-
 70 beams. The cited examples provide, from a model point of
 71 view, first gradient continua. The case of the metamaterial
 72 studied in this paper is slightly different. As we have al-
 73 ready remarked, the pantographic metamaterial has been
 74 specifically designed to give rise, after homogenization, to
 75 a second gradient continuum model. The model based on
 76 the non-linear Euler-Bernoulli beam model is limited to
 77 the study of cases where there are few fibers and these are
 78 arranged in such a way as to obtain a low-density lattice.
 79 Furthermore, in this paper a study is conducted to directly
 80 model the deformation of individual hinges. The informa-
 81 tion obtained by studying this case with only three cells
 82 will be important later for the study of denser architec-
 83 tures, where the mechanical response of the single hinge is
 84 difficult to separate from that of the rest of the elementary
 85 components.

86 The minimal pantographic structure that can be de-
 87 signed (and that preserves the basic mechanical features
 88 of this metamaterial) consists of a three-element panto-
 89 graph, namely a structure that is globally similar to that
 90 of three successive crosses (Figure 1).

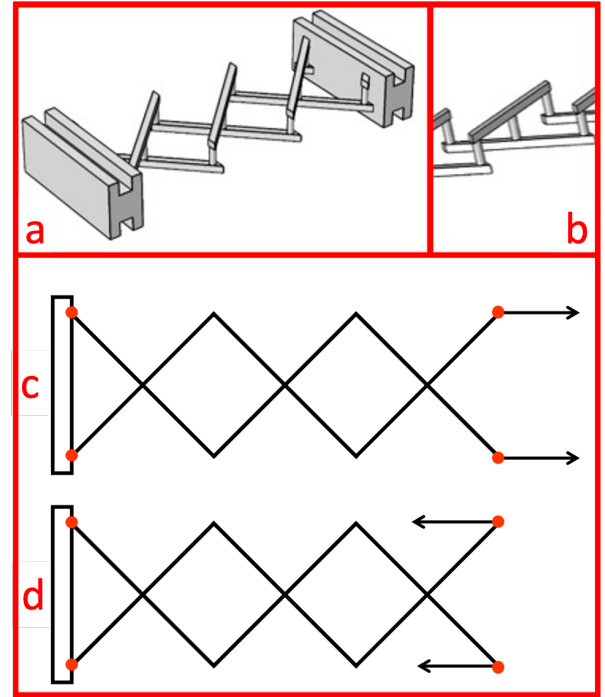


Figure 1: Minimal pantographic structure consisting of three suc-
 cessive cells. CAD model used for 3D printing the specimen (a) and
 detail of the pivots (b). The test studied in this work are a bias
 extension test (c) and a compression test (d).

In addition to macroscopic second gradient models
 of pantographic structures, mesoscopic descriptions are
 based on nonlinear Euler–Bernoulli beams for describing
 the “arms” of the pantograph. This mesomodel has been
 successfully used for interpreting experimental measure-
 ments (e.g. bias extension tests) performed on a fabric-
 like pantograph [2]. In the following, the mesomodel was
 selected in the analysis of pantographic cells. The very low
 number of unit cells does not seem sufficient for justifying
 the use of a macroscopic (i.e. continuum) model [19]. In
 order to validate second gradient models used to describe
 such metamaterials, it is necessary to conduct experimen-
 tal tests. It is therefore essential to produce samples to be
 used in experiments. This has been significantly simplified
 by the rapid development of Additive Manufacturing.

From a modeling point of view, one can make appropri-
 ate simplifying choices depending on the type of mechan-
 ical behavior one expects for a given test. If, for example,
 one expects that the chosen test does not involve out-of-
 plane deformations, then one can model the pantographic
 structure with a 2D model, neglecting deformation mech-
 anisms characteristic of out-of-plane deformations (such
 as torsion of the fibers or their bending in the direction
 orthogonal to the plane in which they are initially con-
 tained). In a 2D model, the deformations of the fibers
 are taken into account by introducing in the deformation
 energy a term related to elongation and one related to
 bending. Pivots, on the other hand, are a more complex
 structural element: in order to observe experimen-

120 tally the energy contributions related to the second gradi-174
121 ent of the displacement (i.e. those related to the bending₁₇₅
122 of the fibers) it is necessary that the deformation energy
123 of the pivots is as small as possible. So the ideal would
124 be to have perfect hinges. This is of course possible in
125 numerical simulations, but it is not always feasible in the
126 production of test specimens (the possibility of producing
127 perfect working hinges depends very much on the scale at
128 which such mechanisms are printed and on the precision
129 of the 3D printer). So in some cases, as in the present
130 one, it is necessary to take into account the torsion of the
131 cylinders, as well as their shear deformation, which be-
132 comes non-negligible when they are slender. If, on the
133 other hand, the ratio between the height and diameter of
134 the pivot is very small, the torsion energy prevails over
135 the bending energy and makes the second gradient contri-
136 bution negligible. From a microstructure homogenization₁₇₆
137 point of view, the continuum obtained in this case is a first₁₇₇
138 gradient continuum as shown in [5]. 178

139 In this paper, the mechanical behavior of pantographic₁₇₉
140 cells is analyzed. The work is organized as follows: (i) the₁₈₀
141 adopted model, which was introduced in Ref. [2], is recalled₁₈₁
142 and adapted to pantographic cells; (ii) through the model,₁₈₂
143 numerical simulations are performed and the consequent₁₈₃
144 results (force-displacement curves and deformed shapes of₁₈₄
145 the structure) are compared to experimental data relative₁₈₅
146 to bias extension and compression tests; (iii) by means of₁₈₆
147 a newly conceived Digital Image Correlation (DIC) frame-₁₈₇
148 work [19], a theoretical “Ansatz” about the nature of de-₁₈₈
149 formations in the hinges is successfully corroborated. 189

150 Finally, we want to remark that in this work we have
151 expressly chosen to study a pantographic structure consist-
152 ing of only three cells. In fact, with larger structures,
153 it would be difficult to study the various energy terms
154 precisely and to separate the pivot terms. Clearly, the re-
155 sults obtained for a structure consisting of a few cells can
156 also be used when one wishes to study a structure consist-
157 ing of many cells. Note, however, that when the number
158 of cells becomes very high, the continuous model can be
159 used for pantographic structures. The “mesoscopic” model
160 presented in this paper is optimal in the case of structures
161 with a wide knit.

162 2. Euler–Bernoulli beam theory adapted to panto- 163 graphic cells

164 In this section, the model presented in Ref. [2] is dis-
165 cussed and adapted to the structure under study. First, a
166 summary is given of the physical system to be designed,
167 and then a description is provided of how to proceed in
168 modeling the observable quantities of the system. The
169 interested reader will find additional details of the beam
170 system presented herein in Refs. [22].

171 As stated in the introduction, the real system that is₁₉₀
172 the object of this work is a pantographic structure, which₁₉₁
173 consists of an assembly of two planar alignments of parallel₁₉₂

fibers (or beams), see Figure 2. It was produced by Addi-
tive Manufacturing and was made of polyamide PA2200.

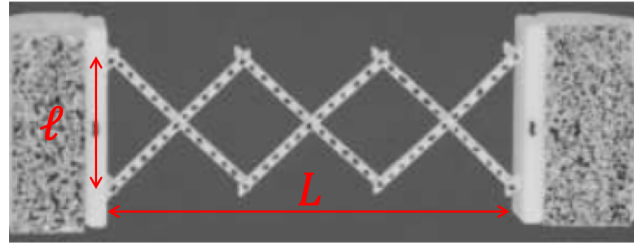


Figure 2: Picture of the reference configuration of a printed pantographic modular structure ($\ell = 10$ mm and $L = 30$ mm).

The geometrical and material characteristics of the sample are gathered in Tables 1-2. In this specific case, the pantographic structure is referred as ‘modular’ because its overall dimensions (i.e. ℓ and L , see Figure 2) are of the order of centimeters. In this particular case, the overall dimensions are 10 mm \times 30 mm. It consists of three basic pantograph cells with an X-shape. This structure represents the prototype of a minimal pantographic structure because, in order to observe its typical phenomenology, it is necessary to consider *at least* three X cells, two of which (i.e. the exterior ones) being clamped at their ends. The beams that constitute the pantograph are connected at their points of intersection by means of deformable cylinders that behave like hinges.

Table 1: Mechanical properties of the 3D printed polyamide PA2200

| Mechanical Properties |
|--|
| Young modulus $E = 0.5$ GPa |
| Poisson ratio $\nu = 0.33$ |
| Shear modulus $G = \frac{E}{2(1+\nu)}$ |

Table 2: Geometric dimensions of the beams composing the pantographic cells.

| Geometrical properties |
|--|
| Basis of the cross section $b = 1$ mm |
| Height of the cross section $h = 1$ mm |
| Cross sectional area $A = b \times h$ |
| Moment of inertia $J = \frac{1}{12}bh^3$ |
| Height of the pivots $H_P = 2$ mm |
| Radius of the pivots $R_P = 0.45$ mm |

The deformation observed experimentally can be schematized as follows, namely, the deformation of the beams, which are stretched and bent, and the deformation

of the hinges. The latter ones are subject to torsion due to the relative rotation of the beams of the two families and to shear deformation. The effect of torsion is observed by comparing the reaction force plot measured versus prescribed displacement when the hinges are elastic with certain torsional stiffness and when they are perfect (for example Ref. [27]). In this last case, the initial part of the plot is close to zero for the forces. When the hinges deform, the force departs from 0. The direct observation of the shear deformation mechanism of the hinges was deduced through Digital Image Correlation [19] as also shown below.

2.1. Mechanical model

The model adopted herein to describe the fiber system discussed above has been introduced in Ref. [2] and is summarized as shown in Figure 3. It characterizes the fibers as Euler-Bernoulli (nonlinear) beams, whilst the hinges are modeled as rotational springs (torsion model of the real cylinders) in addition to extensional springs (to take into account shear, which produces relative displacements between the two fibers connected by the hinge itself). It was observed in previous works [2, 26] that for ratios between the height and radius of the hinge in a certain range, the hinge may exhibit, in addition to the usual torsional deformation due to the change of angle between the fibers of the two families, deformations due to the relative motion of the two fibers, which was modeled by a linear spring. It is assumed that the constitutive law of the extensional spring, modeling the hinge in terms of energy, depends on the square of the relative displacement. Another cubic dependence on the relative displacement must also be added to obtain results in good agreement with experimental observations. This last deformation mechanism allows relative sliding to be introduced between the fibers of the two families in correspondence of the hinges. For this reason, the associated energy term is called *fiber connectivity*.

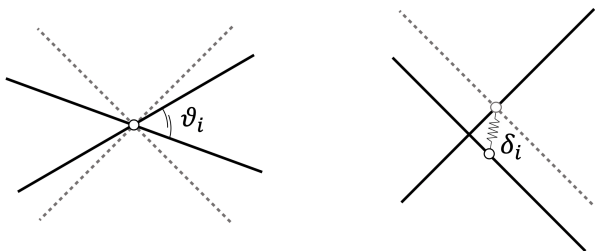


Figure 3: Kinematics of hinges depicting torsion (left) and fiber connectivity (right) mechanisms.

In Ref. [2], it was shown how, in order to obtain the deformed configuration and the reaction forces on the edges of the pantographic structure, the strain energy of the structure had to be written in terms of its individual constituents. Specifically, if each beam is considered as composed of a number of finite elements corresponding to the

number of beam elements between two consecutive hinges (in the simple case of pantographic cells, all beams are made up of two finite elements), then the strain energy of each element is given by extensional and bending contributions. In order to obtain the strain energy of the entire structure, all these energies of the finite elements are combined. It is also necessary to add the contribution of the hinges. For each hinge, there is a term due to torsion (which in the case of perfect hinges should vanish), and another one due to possible shear. These contributions to the strain energy are as numerous as the number of hinges in the pantographic structure. For a pantographic modular structure, the number of beam elements and the number of hinges is easily counted. For each fiber family, there are three fibers, each consisting of two elements, for a total number of 12 fiber elements for 7 hinges.

Let us call the ends of each beam element *nodes*. At these specific points, which correspond to the hinges, the nodal displacements and rotations are defined for the transverse sections [2], and are used to write the strain energy of the beams/fibers and the hinges. These displacements and rotations are therefore the unknowns to the problem, and are obtained by prescribing boundary conditions and minimizing the total strain energy. In the present case, which aims to analyze bias extension and compression tests, the boundary conditions consist in zero displacements for the nodes on one of the two short sides of the pantographic structure (corresponding to the clamped side), and in assigning displacements to the nodes on the other short side. The horizontal component of the displacements is designated as U_a and U_b (a and b for the two fiber families), and the vertical ones V_a and V_b , then the boundary conditions become

$$\begin{aligned}
 U_i(0, j) = 0, U_i(L, j) = \bar{U}, \quad (1) \\
 \forall i \in \{a, b\}, \forall j \in \{\text{set of nodes in the short side}\} \\
 V_i(0, j) = 0, V_i(L, j) = 0, \\
 \forall i \in \{a, b\}, \forall j \in \{\text{set of nodes in the short side}\}
 \end{aligned}$$

From boundary conditions (1), the displacements of the nodes on the two short side edges are assigned, while the other nodal displacements and rotations are calculated by means of the minimization problem. By minimizing the strain energy, it is possible to obtain the values of this energy for each deformation step and from these values, using Castigliano's theorem [25], the force-displacement curve is obtained.

We want to remark the fundamental role of the non-linearity of the adopted model for the pantographic architecture, which will be central for obtaining numerical results capable to properly describe the experimental data. The Euler-Bernoulli beam model used to model the fibers of the pantographic architecture is non-linear, as is specified in the following. Furthermore, the geometric arrangement of the fibers itself provides for a global energy that is non-linear. The terms that contribute to the strain energy of the fibers are: (i) elongation energy, written in terms

of the generalized non-linear axial strain; (ii) bending energy, written in terms of the non-linear curvature. These two terms alone would be sufficient to produce a non-linear global deformation energy, but to these must be added two terms relating to hinge deformations, the torsion energy (iii) and what we have called the “fiber connectivity” energy (iv). These last two contributions are themselves non-linear, as it will become manifest in equations (4) and (5) and considering that the assumed value for the parameter β is 1.55 (see Table 3).

2.2. Ritz approach for minimizing the energy

The strain energy is written as the sum of the contributions of beam elements for extensional and flexural terms, as well as hinges for torsional and fiber connectivity terms. Specifically, let N be the number of beam elements, and P the number of hinges (for the studied pantographic structure, $N = 6$ for two families of beams, and $P = 7$), the extensional energy reads

$$\mathcal{E}^{(e)} = \sum_{i=1}^{2N} \frac{1}{2} \int_0^{\lambda_i} EA\eta^2(x)dx \quad (2)$$

and the flexural energy

$$\mathcal{E}^{(b)} = \sum_{i=1}^{2N} \frac{1}{2} \int_0^{\lambda_i} EJ\chi^2(x)dx \quad (3)$$

where $\eta(x)$ and $\chi(x)$ denote generalized strains relative to the axis line of the beam. More precisely, $\eta(x)$ is the stretching strain, while $\chi(x)$ is the curvature. For the hinges, the torsional term depends on the angular variation between the two fibers that the hinge interconnects

$$\mathcal{E}^{(p)} = \sum_{i=1}^P \frac{1}{2} \mathbb{K}_p \left(\frac{\pi}{2} - \Delta\vartheta_i \right)^\beta \quad (4)$$

where \mathbb{K}_p accounts for the torsional stiffness and is related to the geometric characteristics of the hinge, $\Delta\vartheta_i$ the variation of the angle between the fibers crossing in the i -th hinge, β an exponent that in general is different from 2 and is obtained from fit of experimental data. The energy term related to fiber connectivity is written as

$$\mathcal{E}^{(c)} = \sum_{i=1}^P \frac{1}{2} \mathbb{K}_c (\delta_i^2 + \mathfrak{K}_3 \delta_i^3 + \dots) \quad (5)$$

where \mathbb{K}_c is related to the shear deformation of the hinge, δ_i to sliding of the fibers at the i -th hinge, and \mathfrak{K}_3 a coefficient that has to satisfy the positiveness of the energy. The nonlinear correction due to \mathfrak{K}_3 , and any other higher order correction, are to be considered to fit the experimental response of the system.

To understand in detail this last term, let us consider a hinge connecting two fibers of the A and B families. The ends of the hinge, (X_A, Y_A) and (X_B, Y_B) ,

in the undeformed configuration are superimposed (i.e. $(X_A, Y_A) = (X_B, Y_B)$). A deformation of the structure may result in a relative displacement of the hinge ends. By assigning the horizontal and vertical nodal shifts for the two families (U_A, V_A) and (U_B, V_B) , the relative shift of the fibers δ_i is expressed in terms of these positions and shifts as

$$\delta_i^2 = \left[\left(X_A^{(i)} + U_A^{(i)} \right) - \left(X_B^{(i)} + U_B^{(i)} \right) \right]^2 + \left[\left(Y_A^{(i)} + V_A^{(i)} \right) - \left(Y_B^{(i)} + V_B^{(i)} \right) \right]^2 \quad (6)$$

Like $\mathcal{E}^{(c)}$, the other energy terms are also written according to the components of displacements and rotations of the nodes. It suffices to minimize the total strain energy to find, for each assigned deformation step, the nodal displacements and rotations. With these variables and the shape functions used to define finite elements, the deformed configuration is obtained, and then the reaction forces. The selected shape functions are cubic Hermite polynomials. The detailed procedure of finite element decomposition and minimization is reported in Ref. [2]. Let us merely indicate the way to obtain the values of the displacements at equilibrium. The total strain energy reads \mathcal{E}

$$\mathcal{E} = \mathcal{E}^{(e)} + \mathcal{E}^{(b)} + \mathcal{E}^{(p)} + \mathcal{E}^{(c)} \quad (7)$$

and it is postulated that equilibrium is given by the minimum of the potential energy

$$\delta\mathcal{E} = 0 \quad (8)$$

In the present case, there is no need for adding an external work to the minimization because essential boundary conditions were prescribed.

In the next section, the results of numerical simulations based on the presented model are reported, and comparisons with experimental measurements are carried out. All details for the numerical implementation of the presented model is found in Ref. [2].

3. Numerical simulations and comparison with experimental measurements

The experimental tests have been performed with quasi-static loading conditions: the loading speed is 0.5mm/s . Moreover, we want to underline that the fiber element and hinge numbers, in general, can affect the accuracy and convergency of the numerical results. The number of hinges considered in the code used corresponds to the number of physical pivots that are present in the tested specimen. On the other hand, the numerical code we wrote to perform the simulations subdivides the individual fibers into non-linear Euler-Bernoulli beam finite elements. As we have already remarked, the nodes of the subdivision into finite elements correspond to the physical pivots. One could certainly choose to increase the number of nodes and finite elements. This increases the computational effort

337 considerably. The calculation time in the current case is
 338 only a few seconds. By increasing the number of nodes,
 339 the calculation time can also increase considerably. The
 340 result, however, does not change substantially, and, there-
 341 fore, that the chosen decomposition is sufficient for the
 342 present case.

343 Figures 4 and 5 show the comparison between the mea-
 344 sured reaction forces and those calculated for the extension
 345 and compression tests at the end of the calibration of the
 346 four parameters of the above model.

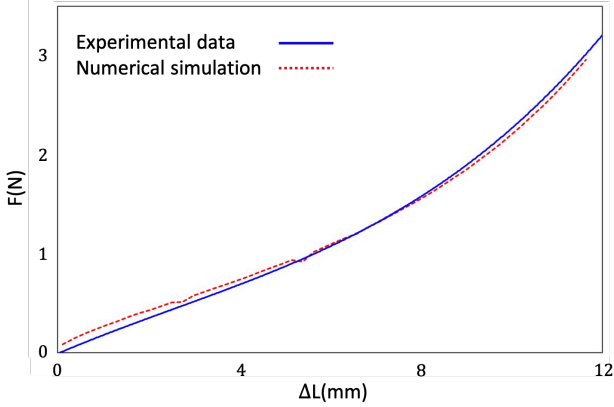


Figure 4: Bias extension test. Comparison between experiment (blue solid line) and model prediction (red dotted line).

347 These macroscopic predictions are in good agreement
 348 with the experimental results. To obtain them, some of
 349 the constitutive parameters have been calibrated. More
 350 specifically, the elongation and bending stiffness were as-
 351 sumed to be known from the Euler-Bernoulli beam theory.
 352 They are expressed in terms of the Young's modulus of
 353 the chosen material and of geometrical dimensions of the
 354 beams.

355 As we have specified, the parameters of the model were
 356 calibrated on the extension test, whose force-displacement
 357 graph is shown in Figure 4. The same parameters, thus
 358 calibrated expressly for the extension test, were used to
 359 carry out the numerical simulation of the compression test.
 360 Now, of course, in the case of Figure 5 the agreement be-
 361 tween numerical simulation and experimental data is not
 362 as strong as in Figure 4, but the results are reassuring.
 363 Clearly better results could be obtained by an iterative
 364 calibration that takes into account the experimental data
 365 of both tests: as far as the present study is concerned,
 366 we simply observe that the model adopted also predicts a
 367 lesser involvement of the fiber-connectivity mechanism in
 368 the compression test, a fact that is also revealed by the
 369 analysis of the experimental images carried out by means
 370 of Digital Image Correlation techniques. This last cir-
 371 cumstance will be clearer in the following.

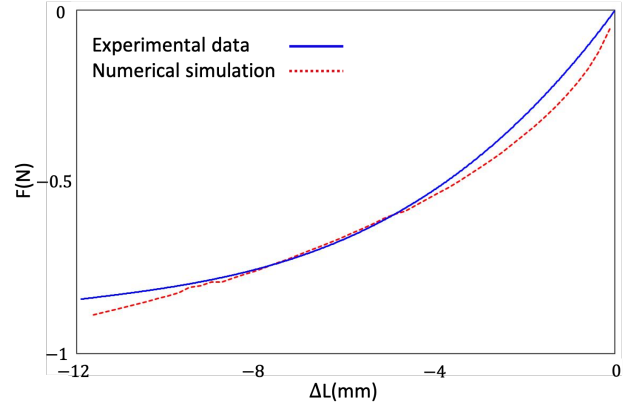


Figure 5: Bias compression test. Comparison between experiment (blue solid line) and model prediction (red dotted line).

372 As it can be seen in the following Fig. 6, the elongation
 373 and bending energies of the fibers are much smaller than
 374 the energies associated with the pivots. The largest energy
 375 contribution is due to the torsion of the pivots and, in a
 376 second step, to the fibre connectivity energy term. The
 377 latter two terms are clearly non-linear, although, as we
 378 mentioned earlier, all four energy contributions are non-
 379 linear and therefore partly contribute to the non-linearity
 380 found in the curves in Figures 4 and 5. Furthermore, we
 381 would like to point out that the non-linear character of
 382 the force-displacement graphs is not due to the material
 of which the specimen is made, but to the architecture
 according to which the fibers are organized. In the con-
 sidered tests, no plasticization phenomena are detected:
 the fibers definitely did not undergo plasticization because
 they are not sufficiently elongated (or compressed) and
 not sufficiently flexed; the pivots could plasticize, but in
 that case plasticization would be easily observable in the
 force-displacement graph (for example this is well visible
 in metal specimens, as shown in [27]). Furthermore, the spec-
 imen was unloaded and no permanent deformation was
 visible.

The parameters related to the hinges are those that
 need to be calibrated since the two energy components
 related to the hinges were deduced phenomenologically.
 The calibration was performed by comparing the com-
 puted force-displacement curve with the experimental one
 for the bias extension test (see Figure 4). As the study
 we present in this paper consists of a qualitative analy-
 sis of the mechanical behavior of a minimal pantographic
 structure in order to investigate the individual mechanisms
 occurring at the level of the microstructure, the model pa-
 rameters are optimized “by hand” in order to obtain both a
 force-displacement curve compatible with the experimen-
 tal one and deformed shapes as close as possible to those
 observed experimentally. As we have already pointed out,
 in the second part of the paper, some analyses carried out
 by means of Digital Image Correlation are presented. The
 perspective of the joint work of the various co-authors is to
 obtain a numerical tool able to quantitatively calibrate the

412 parameters of the model on the basis of the experimental
413 data. In this perspective, the use of Digital Image Corre-
414 lation is central. Above all, the authors' intent is to obtain
415 an Integrated Digital Image Correlation [20] that can inter-
416 act in an automated way with the numerical program
417 and obtain the most suitable values for the constituent
418 parameters of the model. At present, this is not yet im-
419 plemented and requires a very articulated work, which the
420 authors are carrying out. The calibrated parameters are
421 reported in Table 3. An important point to note is that
422 the *same* constitutive parameters were used in the model
423 to describe tension and compression tests.

Table 3: Constitutive parameters of the studied pantographic cells.

| Parameter | Value |
|--|----------------------------------|
| Shear stiffness \mathbb{K}_p | $4.60 \cdot 10^{-3} \text{ N/m}$ |
| Fiber connectivity stiffness \mathbb{K}_c | $1.20 \cdot 10^3 \text{ N/m}$ |
| Fiber connectivity correction \mathfrak{R}_3 | $1.00 \cdot 10^2 \text{ N/m}^2$ |
| Exponent in shear energy β | 1.55 |

424 With the calibrated parameters, the contributions of
425 the various parts of the strain energy were computed for
426 the two considered experiments (Figure 6). In these plots,
427 in addition to the total energy (black solid line), the dif-
428 ferent contributions are shown, namely, stretching energy⁴⁵¹
429 (red dashed line), bending energy (green dotted line),⁴⁵²
430 shear energy (blue dashed line) and fiber connectivity en-⁴⁵³
431 ergy (yellow dot-dashed line). It is observed that the total⁴⁵⁴
432 energy in extension is about twice as high as that in com-⁴⁵⁵
433 pression for the same elongation magnitude. In both tests,⁴⁵⁶
434 the prevalent component is shear (i.e., most of the strain⁴⁵⁷
435 energy is due to hinge torsion).⁴⁵⁸

436 The difference between the two total energies lies in the⁴⁵⁹
437 fact that the deformation mechanisms mainly involved in⁴⁶⁰
438 extension and compression are not identical. In an exten-⁴⁶¹
439 sion test, there is a non-negligible component of the fiber⁴⁶²
440 connectivity energy, while in a compression test this com-⁴⁶³
441 ponent is of the same order of magnitude as the bending⁴⁶⁴
442 energy. In both tests, as expected from a theoretical point⁴⁶⁵
443 of view, the stretching energy is negligible. Further, the⁴⁶⁶
444 levels of shear energy are very similar for both tests when⁴⁶⁷
445 compared to other contributions. This observation reveals
446 that in both tests the relative rotation of the fibers in cor-
447 respondence of the hinges is roughly the same, thereby re-
448 vealing a certain “symmetry” between the two tests. The
449 lack of overall symmetry at the global scale is then inter-
450 preted as the result of fiber connectivity.

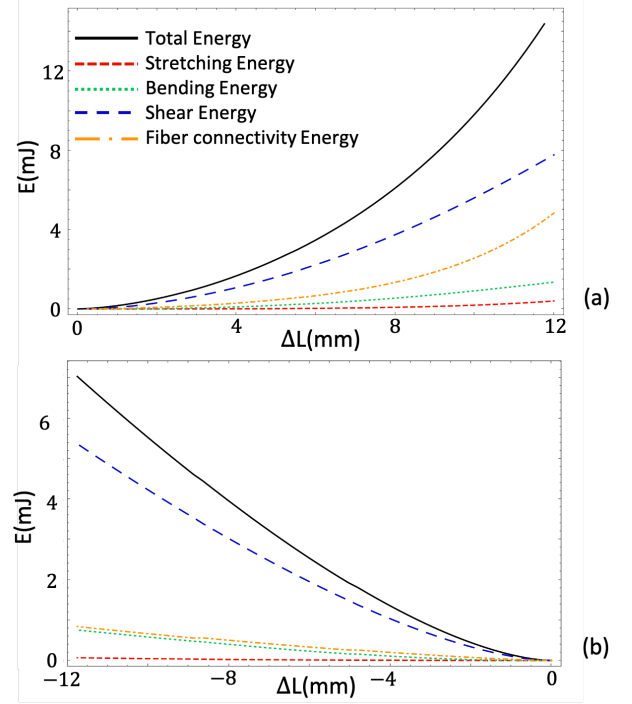


Figure 6: Energy contributions for bias extension (a) and compression (b) tests.

In Figures 7-9, the deformed configurations of the pantographic cells are compared to the simulated shapes for different deformation steps. More precisely, in Figure 7, a qualitative comparison between experiment and simulation was carried out for the bias extension test. The deformed shapes were numerically determined using Hermite cubic polynomials, which were mentioned in the previous section. The order of such shape functions was sufficient to ensure that the global deformed shape and that the calculation of the reaction force were well predicted (Figure 4). For the computation of the internal actions in beams (i.e. local analysis of deformation), the chosen order of the shape functions was not sufficient to obtain reliable results. In order to numerically calculate the internal actions, it would be desirable to choose shape functions of higher order or, alternatively, to increase the number of finite elements to discretize each beam.

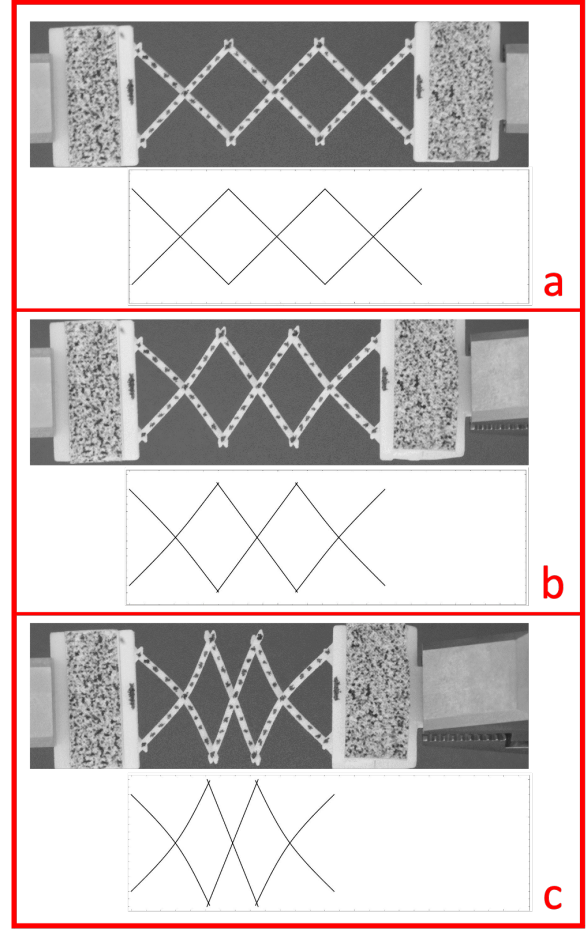
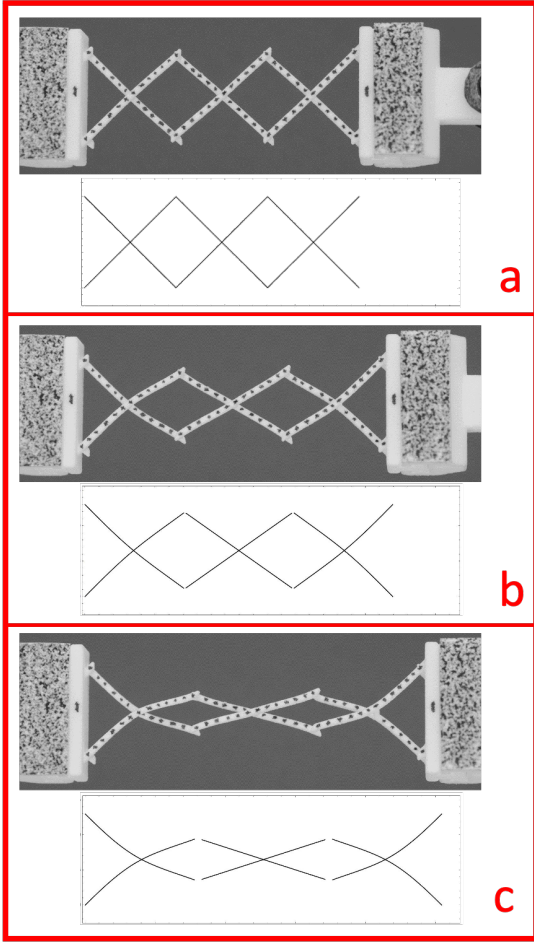


Figure 7: Bias extension test: comparison between experimental deformed shapes and the numerically simulated ones for three selected values of prescribed displacement. Imposed displacement on the horizontal axis and measured reaction force on the vertical axis.

Figure 8: Bias compression test: comparison between experimental deformed shapes and the deformed shapes obtained by numerical simulation for three selected values of prescribed displacement.

468 In Figure 8, the same type of comparison is performed⁴⁸⁹
 469 for the compression test. As it was clear by observing Fig-⁴⁹⁰
 470 ure 6, extension and compression tests were not completely⁴⁹¹
 471 symmetric, due to the fiber-connectivity energy term. If,⁴⁹²
 472 on the one hand, this is simply accountable from the model,⁴⁹³
 473 point of view (in fact the fiber-connectivity energy term⁴⁹⁴
 474 depends also on the cube of the relative displacement), on⁴⁹⁵
 475 the other hand one has to carefully interpret this lack of⁴⁹⁶
 476 symmetry in the context of the experiments. Figure 7-⁴⁹⁷
 477 8 only show the face view of the two experiments. It is
 478 likely that, in the bias extension test, the deformation of
 479 the specimen is always in the plane. On the contrary, the
 480 specimen deformations are unlikely to remain in-plane in
 481 the compression test. For example, in Figure 8(c), the
 482 relative displacements of the four hinges at the ends of
 483 the central cell seem very pronounced, but they could be
 484 affected also from parallax induced by out-of-place de-
 485 formations (specifically, by torsion of the fibers themselves).
 486 This point will be further discussed when dealing with the
 487 comparisons between computed and measured relative dis-
 488 placements.

In Figure 9, the predicted deformed shape (in red) is laid over the image of the last deformation step for both tests. This result probes in a more quantitative way the trustworthiness of the computed deformed shapes, which are in very good agreement with experimental observations. An even more quantitative comparison would be possible using Digital Image Correlation (see Section 4). This kind of comparison has been recently performed in Ref. [4].

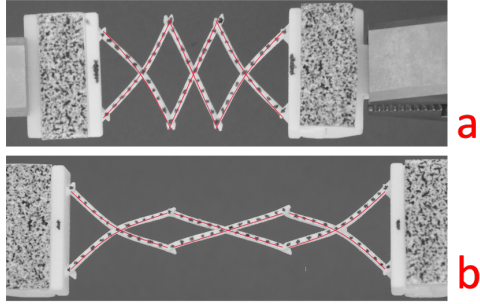


Figure 9: Superposition of computed (red) and experimental deformed shapes at the end of bias compression (a) and extension (b) tests.

498 In Figures 10-12, additional details about the numerical
 499 results related to the fiber connectivity are shown. This
 500 contribution was introduced in the strain energy in order
 501 to take into account the slender ratio between height and
 502 radius of the hinges. The relative displacements of the
 503 beams in correspondence of the hinges, which are referred
 504 to as displacement jumps, are indicated by the distance of
 505 the two red and blue points that correspond to the upper
 506 and lower ends of the hinges. A symmetry is observed in
 507 these displacement jumps. In Figure 10, the fiber connect-
 508 ivity is studied for four different deformation steps for the
 509 bias extension test, analogous results are reported for the
 510 compression test in Figure 11.

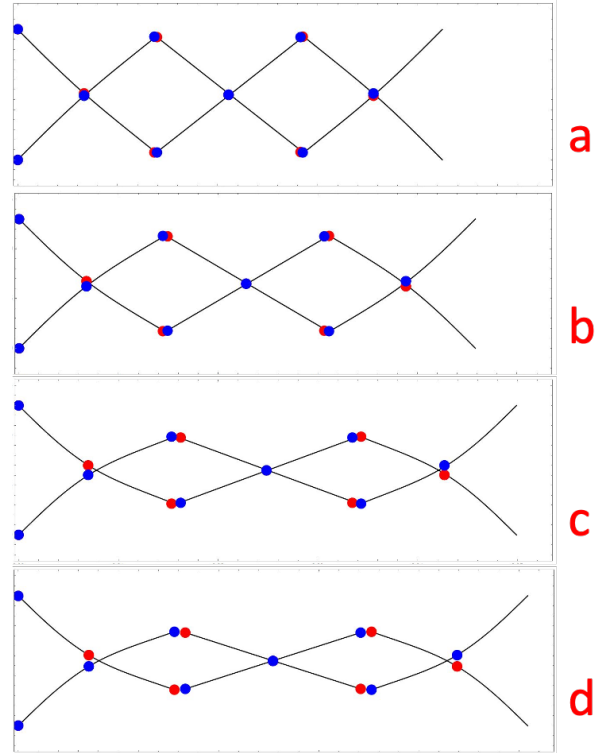


Figure 10: Bias extension test: numerically computed deformed shapes of the structure. The red and blue points, which are superimposed in the undeformed configuration, represent the two ends of the hinges and their subsequent misalignment is the result of the fiber connectivity term.

511 Two aspects emerge from the observation of Figures 10
 512 and 11. In both tests, for reasons of symmetry, the central
 513 hinge is not sheared at all (and for the same reasons,
 514 it is also the hinge that experiences the maximum relative
 515 rotation between the beams). The displacement jumps calcu-
 516 lated in the bias extension test are in amplitude greater
 517 than those of the compression test, even if the absolute
 518 range of deformation is the same for both tests. The sec-
 519 ond observation is a direct consequence of the fact that the
 520 energy term associated with the fiber connectivity is not
 521 quadratic but consists of quadratic and cubic terms. In the
 522 next section, this asymmetry is experimentally probed.

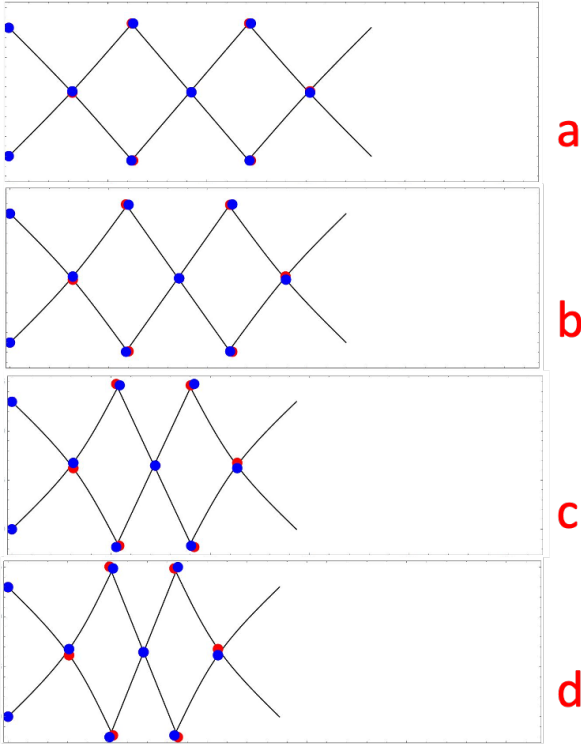


Figure 11: Bias compression test: computed deformed shapes of the cells. The red and blue points, which are superimposed in the undeformed configuration, represent the two ends of the hinges and their subsequent misalignment is the result of the fiber connectivity term.

In Figure 12, for the last deformation step of the bias extension test, the displacement jumps are highlighted by adding yellow arrows. These arrows have the advantage of indicating not only the amplitude of the displacement jump, but also its direction. A careful observation of these arrows shows that some displacements, which would otherwise appear completely horizontal, have a vertical component as well. An analysis of these displacement jumps in a more quantitative way is reported in Figures 14-15.

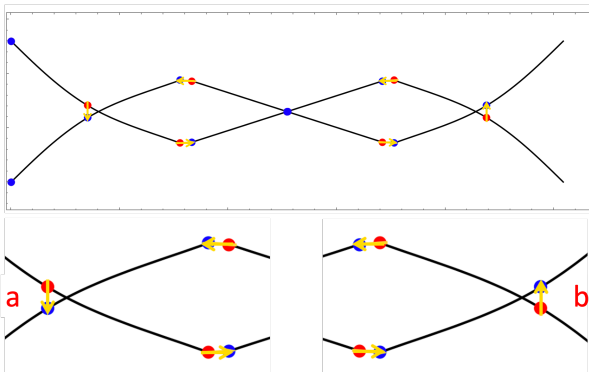


Figure 12: Numerically computed deformed shape of the pantographic cells for the last extension step. The direction of the relative displacement between the beams of the two families is depicted by yellow arrow. In (a) and (b) the extremes of the figure are magnified for showing the antisymmetry of the arrows.

For the hinges labelled in Figure 13, displacement jumps are plotted in components for the bias extension test in Figure 14 and for the compression test in Figure 15.

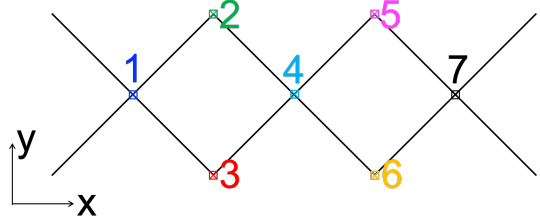


Figure 13: Location of the 7 hinges where the relative displacement has been computed for a comparison with the measured values obtained via DIC.

In Figures 14-15 the displacement jumps at the hinges as labelled in Figure 13 are plotted for the two numerical simulations (respectively, bias extension and compression). These plots show that there is a certain symmetry in the microshear deformation of the hinges. Observing Figures 14-15, in both tests the hinges behave similarly. Specifically, the hinges labelled 2, 3, 5 and 6 mainly experience horizontal displacement jumps, while hinges 1 and 7 are sheared along the y (vertical) direction. Last, for symmetry reasons, the central hinge, labelled 4 does not undergo any displacement jump.

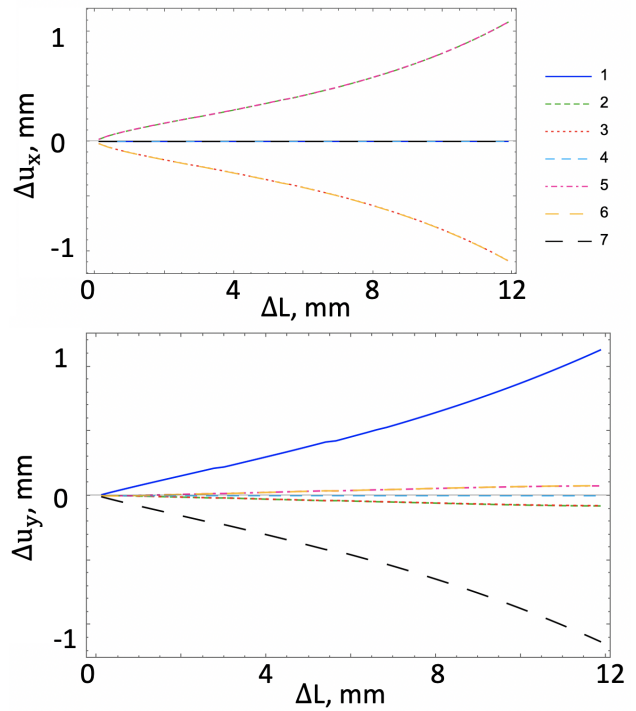


Figure 14: Predicted history of displacement jump for the 7 labelled pivots in the horizontal (up) and vertical (down) directions for the bias extension.

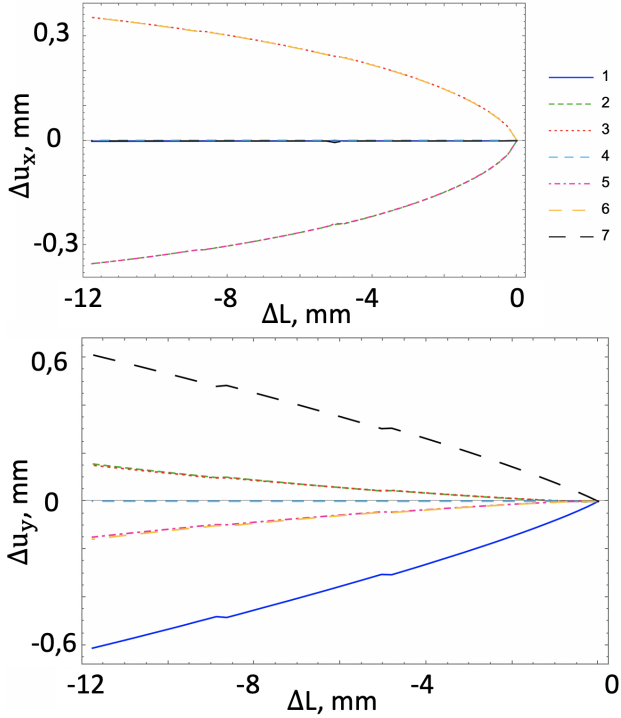


Figure 15: History of displacement jump for the 7 labelled pivots in the horizontal (up) and vertical (down) directions for the compression test numerical simulation.

A further study could investigate the distribution of the displacement jumps for pantographic structures with more cells. A deeper analysis should be dedicated to a study of these jumps in multi-layered pantographic structures (which until now have been considered only for their performing properties in three-point bending tests [29]).

Before presenting the results obtained by DIC, we want to remark that the discrepancies that will be observed between the Figs. 14-15 and the Figs. 16-17 can be attributed to many simplifications assumed in the present study, as, for example, the fact that both in the numerical model and in the DIC analysis are not taken into account out-of-plane deformations, which can clearly influence the shear of the hinges. Out-of-plane deformations are non prevalent in the presented experimental tests. This, of course, does not avoid that such deformations can occur and, indeed, can be observed in some tests, expressly designed for the study of such a phenomenon. However, we have to consider some aspects: out-of-plane deformation and buckling are phenomena that can be studied by means of stability criteria of a structure. In the case of the tests presented here, an out-of-plane deformation may occur during the compression test. The main deformation mechanisms involved in the compression test are those related to fibre bending and pivot torsion and shear. In particular, it has been observed by some co-authors of this work (in a paper that is currently being written) that the torsional stiffness of the pivots greatly influences the presence or absence of out-of-plane deformation.

4. Digital Image Correlation analyses

In the previous section, the acquired pictures were used for qualitative comparisons between the deformed shapes observed experimentally and predicted numerically with the calibrated model (Figures 7-9). These images can also be used to measure displacement fields via Digital Image Correlation [28]. Of the various approaches applied to the bias extension test [19], microscale analyses were run in which each beam was meshed and backtracked to fit the reference configuration of each test. Since displacement jumps were sought, no constraints (e.g. via Lagrange multipliers) were applied to the displacement of the connecting ends of each hinge. These kinematic hypotheses led to the lowest registration residuals and thus were deemed the closer to the experiment [19].

Figure 16 shows the displacement jumps for the bias extension test. For the longitudinal component (Figure 16(a)), the overall trends are identical to those obtained by numerical simulation (Figure 14), namely, the two lower hinges (3 and 6) experience positive jumps, the two upper hinges (2 and 5) undergo negative displacement jumps, the two extreme hinges (1 and 7) have virtually no displacement jumps. Conversely, the middle hinge experiences some negative displacement jump in both directions, which was not expected from the numerical simulation. This observation shows that the response of the pantographic cells did not possess all the symmetries found in the simulation. This is due to imperfections induced by the printing process of such small structures, and alignment of the sample in the testing machine. This trend is further confirmed when comparing the vertical component of the displacement jumps (Figures 14 and 16(b)).

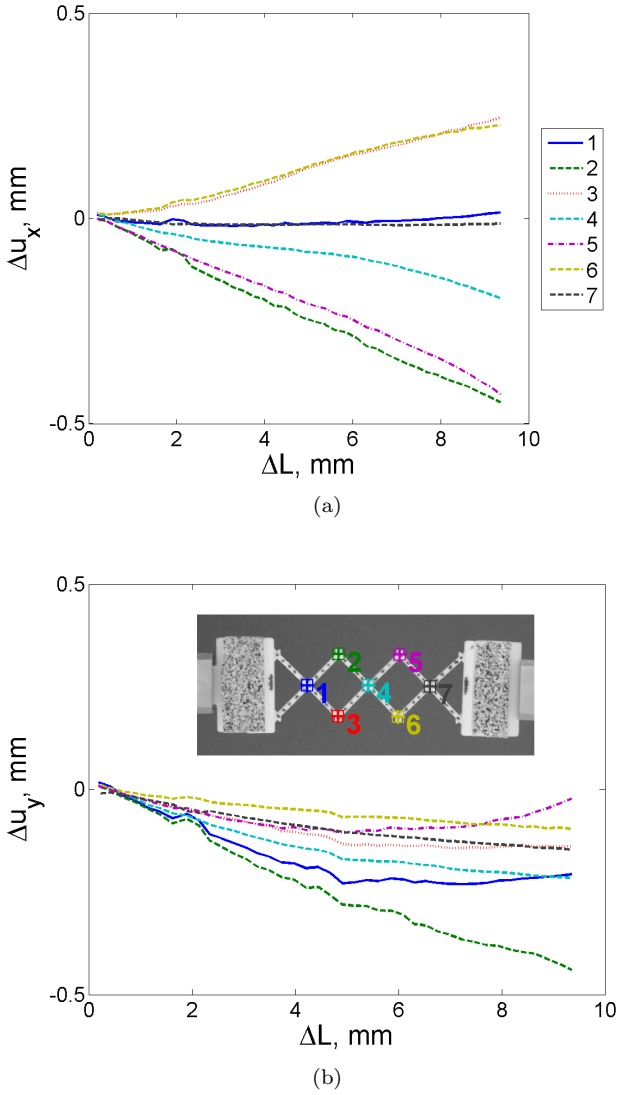


Figure 16: DIC analyses at the microscale of the bias extension test. History of displacement jumps in the horizontal (a) and vertical (b) directions. The inset shows the location and labels of the 7 hinges.

Let us also note that on a more quantitative way the experimental amplitudes are lower than the predicted levels. This fact can be explained by considering that (i) at the current state of research the fiber connectivity stiffness \mathbb{K}_c and the constant \mathfrak{k}_3 were calibrated together with \mathbb{K}_p and the parameter β for obtaining the best fit of the experimental force-displacement curve. The predicted deformed shapes were consistent with experimental pictures. (ii) DIC analyses were performed on 2D pictures of experimental tests that were not totally in-plane. (iii) Last, the experimental conditions (e.g. 3D printed specimen with imperfections, clamping) are different from the ideal ones, which were assumed in the numerical simulations.

Figure 17 shows the displacement jumps for the bias compression test. For the vertical component (Figure 17(b)), the general trends are close to those obtained by numerical simulation (Figure 15). However, the lev-

els are higher and nonlinearities are observed for the two extreme hinges (1 and 7). For the horizontal component of the displacement jumps (Figures 14 and 16(a)), the order of magnitude is closer even though more complex responses are observed in comparison with the simulations. It is believed that such differences point toward imperfections that induce 3D effects that were not accounted for in the 2D numerical simulations.

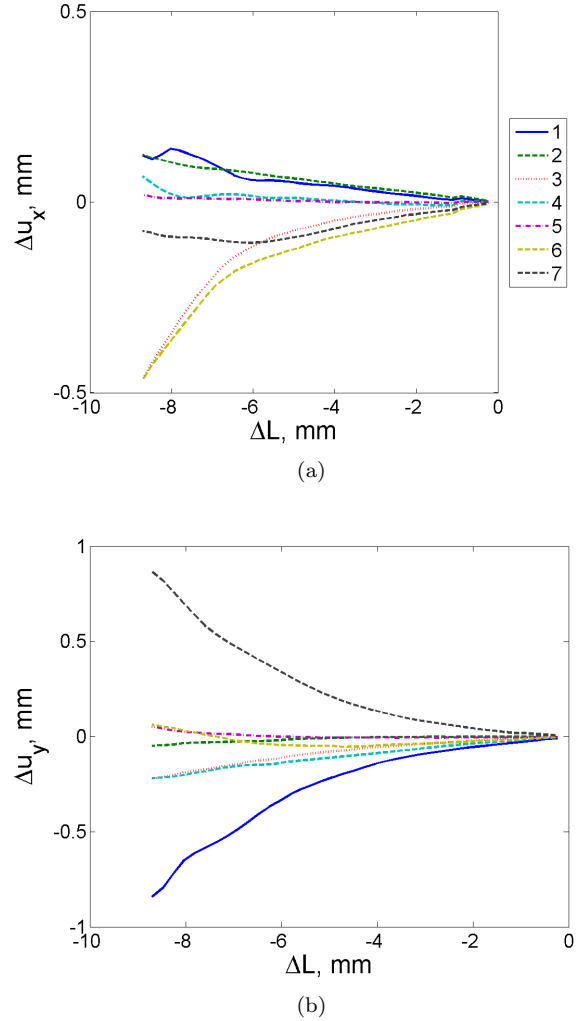


Figure 17: DIC analyses at the microscale of the bias compression test. The location and labels of the 7 hinges are identical to those of Figure 16. History of displacement jumps in the horizontal (a) and vertical (b) directions.

5. Conclusion

A model of a pantographic structure composed of nonlinear Euler-Bernoulli beams has been considered in this study. In this study the focus is on the mechanical behavior of hinges, which play a dramatically important role in the mechanics of pantographic structures. For this reason, we have chosen to study a structure consisting of only three cells, which makes it easier to focus on the hinges. As pointed out in the text, there is an asymmetry between

641 the extension and compression tests. This asymmetry is 698
 642 related to the shear deformation of the pivots, which de- 699
 643 form differently and not symmetrically in the two tests: 700
 644 this result would be difficult to detect in the case of struc- 701
 645 tures with many beams and many pivots. In the present 702
 646 paper, a more in-depth study of the deformation mecha- 703
 647 nisms of pivots is presented. The energy term denoted as 704
 648 fibre-connectivity was in fact only mentioned in previous 705
 649 works. Analyses conducted with the Digital Image Corre- 706
 650 lation technique allowed us to detect, in a qualitative way 707
 651 at the current state of research, many details of the pivot 708
 652 deformations, thus providing an indication of the way for-
 653 ward in the modeling process.

654 The effect of hinge deformation on the mechanical re- 709
 655 sponse of pantographic cells was studied herein. As said, 710
 656 the model adopted for the analysis of such system was 711
 657 based on the use of nonlinear Euler–Bernoulli beams. A 712
 658 very important role is assumed by the shear deformation 713
 659 of individual hinges. From a macroscopic point of view, 714
 660 that deformation corresponds to relative motions between 715
 661 the beams of the two families that constitute the metama- 716
 662 terial. From a numerical point of view, the simulation of 717
 663 bias extension and compression tests would be symmetric 718
 664 in the absence of fiber connectivity deformation. However, 719
 665 this symmetry is lost thanks to fiber connectivity. From
 666 a purely experimental point of view, it was observed that
 667 the reaction forces measured in both the tests are not anti- 720
 668 symmetric but almost 5 times higher in magnitude in the
 669 bias extension test than in compression. 721

670 The model developed herein was calibrated with a 722
 671 unique set of parameters that could accurately describe 723
 672 the macroscopic load/displacement response of the stud- 724
 673 ied cells. Further, the deformed shapes were in qualitative 725
 674 agreement with the experimental observations. Since im- 726
 675 ages were acquired during the experiments studied herein, 727
 676 displacements could be measured via DIC. In the present 728
 677 case, relative motions at hinge ends could be measured 729
 678 thanks to a specially designed kinematic basis [19], which
 679 led to very low registration residuals. Displacement jumps 730
 680 could be measured and quantified for both tests. Some of
 681 the general trends predicted by the calibrated model were 731
 682 confirmed by the DIC measurements. Conversely, loss of 732
 683 symmetries points toward printing and experimental im- 733
 684 perfections of such centimetric objects. 735

685 The results obtained in this work deserve to be further 736
 686 investigated by means of other numerical tools. In par- 737
 687 ticular, the calibration of the model parameters may be 738
 688 performed at the hinge scale when based upon the kine- 740
 689 matic measurements reported herein. This new route may 741
 690 lead to a better description of the hinge response that was 742
 691 not used for the calibration performed herein. One ques- 744
 692 tion that will then arise is whether all the hinges have the 745
 693 same overall response given the printing imperfections. We 746
 694 recall that a model similar to the one developed in [2] and 748
 695 developed here was proposed in [6]. Suitable numerical 749
 696 algorithms, already available in the literature [7–9, 18],
 697 can be implemented for the detailed analysis of problems

similar to the one we discussed.

As shown in Ref. [26], fiber connectivity induces failure modes that were not previously observed in pantographic structures. This observation and, in general, the study of damage in pantographic structures have motivated a recent study on the optimization of such structures [13]. Various studies on damage in materials described by generalized models may be useful to refine the approach to damage in the specific case of pantographic structures. Some results available in the literature can be found in [23].

Authors’ contributions

M. Spagnuolo, U. Andreaus: Conceptualization, Methodology, Software.
 M. Spagnuolo: Data curation, Writing, Original draft preparation.
 A. Misra: Visualization, Investigation.
 I. Giorgio, F. Hild: Supervision.
 M. Spagnuolo, F. Hild: Software, Validation.
 M. Spagnuolo, F. Hild, I. Giorgio: Writing, Reviewing and Editing.

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References

- [1] Jean-Jacques Alibert, Pierre Seppecher, and Francesco dell’Isola. Truss modular beams with deformation energy depending on higher displacement gradients. *Mathematics and Mechanics of Solids*, 8(1):51–73, 2003.
- [2] Ugo Andreaus, Mario Spagnuolo, Tomasz Lekszycki, and Simon R Eugster. A Ritz approach for the static analysis of planar pantographic structures modeled with nonlinear Euler–Bernoulli beams. *Continuum Mechanics and Thermodynamics*, pages 1–21, 2018.
- [3] Emilio Barchiesi, Mario Spagnuolo, and Luca Placidi. Mechanical metamaterials: a state of the art. *Mathematics and Mechanics of Solids*, page 1081286517735695, 2018.
- [4] Emilio Barchiesi, François Hild, and Francesco dell’Isola. On the Validation of Homogenized Modeling for Bi-Pantographic Metamaterials via Digital Image Correlation. *International Journal of Solids and Structures*, In press.(hal-02954418), 2020.
- [5] Claude Boutin. Microstructural effects in elastic composites. *International Journal of Solids and Structures*, 33(7):1023–1051, 1996.

- [6] Giuseppe Capobianco, Simon R Eugster, and Tom Winandy. Modeling planar pantographic sheets using a nonlinear Euler-Bernoulli beam element based on B-spline functions. *PAMM*, 18(1):e201800220, 2018.
- [7] Antonio Cazzani, Marcello Malagù, and Emilio Turco. Isogeometric analysis of plane-curved beams. *Mathematics and Mechanics of Solids*, 21(5):562–577, 2016.
- [8] Antonio Cazzani, Flavio Stochino, and Emilio Turco. An analytical assessment of finite element and isogeometric analyses of the whole spectrum of timoshenko beams. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 96(10):1220–1244, 2016.
- [9] Antonio Cazzani, Mauro Serra, Flavio Stochino, and Emilio Turco. A refined assumed strain finite element model for statics and dynamics of laminated plates. *Continuum Mechanics and Thermodynamics*, 32:665–692, 2020.
- [10] Corentin Coulais, Chris Kettenis, and Martin van Hecke. A characteristic length scale causes anomalous size effects and boundary programmability in mechanical metamaterials. *Nature Physics*, 14(1):40–44, 2018.
- [11] F dell’Isola, G Maier, U Perego, U Andreaus, R Esposito, and S Forest, editors. *The complete works of Gabrio Piola: Volume I (Advanced Structured Materials, vol. 38)*. Springer Verlag, Berlin (Germany), 2014.
- [12] F dell’Isola, I Giorgio, Marek Pawlikowski, and NL Rizzi. Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proc. R. Soc. A*, 472(2185):20150790, 2016.
- [13] Boris Desmorat, Mario Spagnuolo, and Emilio Turco. Stiffness optimization in nonlinear pantographic structures. *Mathematics and Mechanics of Solids*, DOI: 10.1177/1081286520935503, 2020.
- [14] Victor A Eremeyev and Francesco dell’Isola. A note on reduced strain gradient elasticity. In *Generalized Models and Non-classical Approaches in Complex Materials 1*, pages 301–310. Springer, 2018.
- [15] Victor A Eremeyev, Giuseppe Rosi, and Salah Naili. Comparison of anti-plane surface waves in strain-gradient materials and materials with surface stresses. *Mathematics and mechanics of solids*, 24(8):2526–2535, 2019.
- [16] Paul Germain. The method of virtual power in continuum mechanics. part 2: Microstructure. *SIAM Journal on Applied Mathematics*, 25(3):556–575, 1973.
- [17] Ivan Giorgio, Francesco dell’Isola, and Anil Misra. Chirality in 2D Cosserat Media Related to Stretch-Micro-rotation coupling with links to Granular Micromechanics. *International Journal of Solids and Structures*, 2020.
- [18] L Greco, M Cuomo, and L Contrafatto. A quadrilateral g1-conforming finite element for the kirchhoff plate model. *Computer Methods in Applied Mechanics and Engineering*, 346:913–951, 2019.
- [19] François Hild, A Misra, and F dell’Isola. Multiscale DIC applied to Pantographic Structures. *Experimental Mechanics*, pages 1–13, 2020.
- [20] Florent Mathieu, Hugo Leclerc, François Hild, and Stéphane Roux. Estimation of elastoplastic parameters via weighted femu and integrated-dic. *Experimental Mechanics*, 55(1):105–119, 2015.
- [21] Raymond David Mindlin. Microstructure in linear elasticity. Technical report, Columbia Univ New York Dept of Civil Engineering and Engineering Mechanics, 1963.
- [22] N Nejadasadeghi, M De Angelo, R Drobnicki, T Lekszycki, F dell’Isola, and A Misra. Parametric experimentation on pantographic unit cells reveals local extremum configuration. *Experimental Mechanics*, 59(6):927–939, 2019.
- [23] Luca Placidi and Emilio Barchiesi. Energy approach to brittle fracture in strain-gradient modelling. *Proc. R. Soc. A*, 474(2210):20170878, 2018.
- [24] Daria Scerrato, Ivan Giorgio, and Nicola Luigi Rizzi. Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. *Zeitschrift für angewandte Mathematik und Physik*, 67(3):53, 2016.
- [25] RV Southwell. XVI. On Castigliano’s theorem of least work, and the principle of St. Venant. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 45(265):193–212, 1923.
- [26] M Spagnuolo, K Barcz, A Pfaff, F dell’Isola, and P Franciosi. Qualitative pivot damage analysis in aluminum printed pantographic sheets: Numerics and experiments. *Mechanics Research Communications*, 83:47–52, 2017.
- [27] Mario Spagnuolo, Patrice Peyre, and Corinne Dupuy. Phenomenological aspects of quasi-perfect pivots in metallic pantographic structures. *Mechanics Research Communications*, 101:103415, 2019.
- [28] Michael A Sutton. Computer vision-based, noncontacting deformation measurements in mechanics: a generational transformation. *Applied Mechanics Reviews*, 65(5), 2013.
- [29] M Erden Yildizdag, Emilio Barchiesi, and Francesco dell’Isola. Three-point bending test of pantographic blocks: numerical and experimental investigation. *Mathematics and Mechanics of Solids*, page 1081286520916911, 2020.
- [30] Hong Zhang, Xiaoyun Xie, Yiwei Xie, CM Wang, and Pengcheng Jiao. Extensible beam-like metastructures at the microscale: Theoretical and modified hencky bar-chain modeling. *International Journal of Mechanical Sciences*, 180:105636, 2020.