



**HAL**  
open science

# On Domain Decomposition Methods with optimized transmission conditions and cross-points

Bruno Després, Anouk Nicolopoulos, Bertrand Thierry

► **To cite this version:**

Bruno Després, Anouk Nicolopoulos, Bertrand Thierry. On Domain Decomposition Methods with optimized transmission conditions and cross-points. *SIAM Journal on Numerical Analysis*, 2022, 60 (5), pp.2482-2507. hal-03230250

**HAL Id: hal-03230250**

**<https://hal.science/hal-03230250>**

Submitted on 19 May 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On Domain Decomposition Methods with optimized transmission conditions and cross-points

B. Després\*, A. Nicolopoulos† B. Thierry‡

May 19, 2021

## Abstract

We study a mathematical framework for the construction of domain decomposition methods (DDM) for the model Helmholtz equation with high order transmission conditions (TC) and cross-points. A compatibility condition is formulated for cross-points matrices so that the DDM is proved to be convergent under general conditions. The proof is based on a new global energy formulated on the skeleton of the DDM decomposition. It allows to use transmission operators which have a non trivial anti-symmetric part.

## 1 Context

We propose a mathematical framework for the construction of domain decomposition methods (DDM) for the model Helmholtz equation with high order transmission conditions (TC) and cross-points. It is an elaboration on a previous work [12] on the treatment of corners. We use the same definitions and notations in dimension two as in [12]: a cross-point is a point at the intersection of 3 subdomains or more; points at angular edges between two subdomains or on the exterior boundary are called corners.

The issue of cross-points is unavoidable for the development of optimized DDM with enhanced convergence or stability properties. Important mathematical and technical issues are obstacles to the development of these techniques, see [14, 16, 18, 19, 20]. Some partial solutions have been described in [2, 4, 13]. The purely discrete case is considered in [15, 16]. An optimization procedure based on quasi-local operators with convenient regularity is proposed in [17], but ultimately the problem posed by mesh singularities is not solved due to incompatibility between some function spaces. An important step towards a general solution is currently being investigated on the basis of the recent works of Claeys and Hiptmair [7] in which a distinct approach is used through the multi-trace formalism and a convenient definition of some function spaces defined on the skeleton of the DDM decomposition. This path has been extended in the PhD thesis of Parolin [21] and in [9, 8]. However, in this work, we are interested in DDM based on high order transmission conditions such as the ones that are commonly used in numerical DDM softwares, see [3, 20, 22]. To the best of our understanding, none of the works quoted above can provide a sound mathematical treatment of the cross-point issues for such DDM with high order transmission

---

\*Laboratoire Jacques Louis Lions, Sorbonne Université, 4 place Jussieu, 75005 Paris, France, despres@ann.jussieu.fr

†Laboratoire Jacques Louis Lions, Sorbonne Université, 4 place Jussieu, 75005 Paris, France, nicolopoulos@jll.math.upmc.fr

‡CNRS & Laboratoire Jacques Louis Lions, Sorbonne Université, 4 place Jussieu, 75005 Paris, France, bertrand.thierry@sorbonne-universite.fr

conditions. The most notable outputs of our construction are a new global energy defined on the skeleton of the DDM decomposition and a new notion of compatibility for the coupling of DDM at cross-points. Even if it is completely out of the scope of the present work, it is imaginable that the multi-trace formalism could also benefit of the tools developed herein.

This work is concerned only with the mathematical structure of the algorithms. The reader interested by numerical results can consult our previous work [12] where numerical illustrations are provided for simpler problems with external and internal corners. These can be seen as a particular case of our theory, see Section 4.1.

The model problem considered will be the Helmholtz equation written in an open bounded domain  $\Omega \subset \mathbb{R}^2$  with regular boundary  $\Gamma = \partial\Omega$ . The Helmholtz equation models time-harmonic acoustic waves and is representative of most time-harmonic linear equations. The complex valued unknown is  $u \in H^1(\Omega)$  weak solution of

$$\begin{cases} \Delta u + \omega^2 u &= f & \text{in } \Omega, \\ \partial_{\mathbf{n}} u - i\omega u &= 0 & \text{on } \Gamma, \end{cases} \quad (1)$$

where  $\mathbf{n}$  is the outgoing unit normal vector to  $\Omega$  and the right hand side  $f$  models a source with frequency  $\omega > 0$ . The domain  $\Omega$  is partitioned into a finite number  $N$  of non-overlapping polygonal open subdomains denoted as  $\Omega_i$ , that is

$$\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i, \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } j \neq i.$$

The boundary of a subdomain  $\Omega_i$  is split in two parts: the exterior part  $\Gamma_i := \partial\Omega_i \cap \Gamma$  and the interior part  $\Sigma_i := \partial\Omega_i \setminus \Gamma_i$ . The exterior normal vector is  $\mathbf{n}_i$ . We consider only cross-points, and not corners, to avoid notational complications since the broken line case has already been treated in our previous work [12]: for the sake of clarity, we make the hypothesis that  $\partial\Omega_i \cap \partial\Omega_j$  is a straight line for  $i \neq j$  when it is not empty. The interior part of the boundary can thus be decomposed into straight lines  $\Sigma_i = \cup_{i,j} \Sigma_{ij}$ , for  $\Sigma_{ij} := \partial\Omega_i \cap \partial\Omega_j$ . As a convention, we associate the normal vector  $\mathbf{n}_i$  to  $\Sigma_{ij}$ , making a difference between  $\Sigma_{ij}$  and  $\Sigma_{ji}$ . The skeleton of the decomposition is now defined by

$$\Sigma := \bigcup_{i,j} \Sigma_{ij}.$$

Each interior edge is contained twice, with opposite orientations. A function space that plays an important role in our analysis is the  $H^1$ -broken Sobolev space on the skeleton

$$H_{\text{br}}^1(\Sigma) := \{v \in L^2(\Sigma) \text{ such that } v|_{\Sigma_{ij}} \in H^1(\Sigma_{ij}) \text{ for all } i \neq j\}.$$

The natural exchange operator over the skeleton

$$\Pi: L^2(\Sigma) \rightarrow L^2(\Sigma)$$

is defined such that  $(\Pi v)|_{\Sigma_{ij}} = v|_{\Sigma_{ji}}$  for all  $v \in L^2(\Sigma)$ . The operator  $\Pi$  is isometric in  $L^2(\Sigma)$ , as well as its restriction to  $H_{\text{br}}^1(\Sigma)$ .

For a collection of  $u_i \in H^1(\Omega_i)$ ,  $1 \leq i \leq N$ , we denote the Dirichlet trace on an edge  $\Sigma_{ij}$  as  $u_{ij} := u_i|_{\Sigma_{ij}} \in L^2(\Sigma_{ij})$ . Assuming that some recurrence relations in the DDM guarantees extra-regularity on  $u_i$ , one can also define the Neumann trace on each edge  $\partial_{\mathbf{n}_i} u_{ij} := (\partial_{\mathbf{n}_i} u_i)|_{\Sigma_{ij}} \in L^2(\Sigma_{ij})$ . The Neumann traces will actually be more regular in our approach, see Remark 1. These local traces and normal derivative traces are then recast as vectors: we define global notations that correspond to the collection of Dirichlet and Neumann data on the skeleton

$$u_{\Sigma} := \bigoplus_{i,j} u_{ij} \in L^2(\Sigma) \quad \text{and} \quad \partial_{\mathbf{n}} u_{\Sigma} := \bigoplus_{i,j} \partial_{\mathbf{n}_i} u_{ij} \in L^2(\Sigma). \quad (2)$$

Similarly, we define vectors of Dirichlet and Neumann data on the exterior boundary  $\Gamma = \cup_i \Gamma_i$

$$u_\Gamma := \oplus_i u_i|_{\Gamma_i} \in L^2(\Gamma) \quad \text{and} \quad \partial_{\mathbf{n}} u_\Gamma := \oplus_i \partial_{\mathbf{n}_i} u_i|_{\Gamma_i} \in L^2(\Gamma). \quad (3)$$

Now, to approximate the solution  $u$  and its restrictions  $u_i = u|_{\Omega_i}$  in each subdomain, the DDM relies on the computation of iterates

$$u_i^p \in H^1(\Omega_i) \quad 1 \leq i \leq N, p \geq 0. \quad (4)$$

The question is how to define good transmission operators between subdomains. Extending the work on corners [12], we consider here transmission operators written as second order differential operators in the tangential direction  $\mathbf{t}_i$ , defined such that  $(\mathbf{n}_i, \mathbf{t}_i)$  be a local direct basis. On the skeleton, the transmission condition is written in differential form as

$$\left(1 - \frac{1}{2\omega^2} \partial_{\mathbf{t}_i \mathbf{t}_i}\right) \partial_{\mathbf{n}_i} u_{ij}^{p+1} - \mathbf{i}\omega u_{ij}^{p+1} = - \left(1 - \frac{1}{2\omega^2} \partial_{\mathbf{t}_j \mathbf{t}_j}\right) \partial_{\mathbf{n}_j} u_{ji}^p - \mathbf{i}\omega u_{ji}^p \quad \text{on } \Sigma_{ij}. \quad (5)$$

A first remark is that  $\partial_{\mathbf{t}_i \mathbf{t}_i} = \partial_{\mathbf{t}_j \mathbf{t}_j}$ , and the tangential differential operators are the same. More important for the development of our technique, a second remark is that the operator  $1 - \frac{1}{2\omega^2} \partial_{\mathbf{t}_i \mathbf{t}_i}$  is symmetric and non negative when restricted to functions with compact support contained in the edge  $\partial\Omega_i \cap \partial\Omega_j$ :

$$1 - \frac{1}{2\omega^2} \partial_{\mathbf{t}_i \mathbf{t}_i} = \left(1 - \frac{1}{2\omega^2} \partial_{\mathbf{t}_i \mathbf{t}_i}\right)^* \geq 0, \quad \text{in } \mathcal{D}(\partial\Omega_i \cap \partial\Omega_j). \quad (6)$$

This differential operator comes from [12], where it has been shown that it is a second order approximation of artificial radiation conditions on flat exterior boundaries.

The goal of this work is to add cross-point conditions to (5). It will define a global transmission operator  $T: L^2(\Sigma) \rightarrow L^2(\Sigma)$ , and we will rewrite and complete (5) in the abstract form

$$(\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_\Sigma^{p+1} = -(\Pi \partial_{\mathbf{n}} + \mathbf{i}\omega T \Pi) u_\Sigma^p. \quad (7)$$

Hence the abstract formulation of the family of domain decomposition iterative processes studied in this work:

Initialize  $u_i^0 \in H^1(\Omega_i)$  for all  $i$ .

Iterating on  $p \in \mathbb{N}$ , compute  $u_i^{p+1} \in H^1(\Omega_i)$  for all  $i$ , solutions of

$$\begin{cases} (\Delta + \omega^2) u_i^{p+1} = f & \text{in } \Omega_i, \forall i, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_\Sigma^{p+1} = -(\Pi \partial_{\mathbf{n}} + \mathbf{i}\omega T \Pi) u_\Sigma^p & \text{on } \Sigma, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega) u_\Gamma^{p+1} = 0 & \text{on } \Gamma. \end{cases} \quad (8)$$

**Remark 1** (Propagation of regularity). *Provided the normal derivatives are square integrable at the initial iteration  $(\partial_{\mathbf{n}_i} u_i^0)|_{\Sigma_{ij}} \in L^2(\Sigma_{ij})$  for all  $i \neq j$ , then the square integrability of the normal derivatives is propagated by the iterative process (8). Therefore, adding the regularity condition  $\partial_{\mathbf{n}} u^0 \in L^2(\Sigma)$  at the initial stage is enough to guarantee that  $\partial_{\mathbf{n}} u^p \in L^2(\Sigma)$  for all  $p \geq 0$ .*

*The operator  $T$  constructed in Section 2.2 will actually have more regularity*

$$T(L^2(\Sigma)) \subset H_{\text{br}}^1(\Sigma).$$

*In this case, adding the regularity condition  $\partial_{\mathbf{n}} u^0 \in H_{\text{br}}^1(\Sigma)$  at the initial stage is enough to guarantee that*

$$\partial_{\mathbf{n}} u^p \in H_{\text{br}}^1(\Sigma) \quad \text{for all } p \geq 0. \quad (9)$$

*This requirement is natural if one considers a variational formulation in the space  $H_{\text{br}}^1(\Sigma)$  for the differential identity (6).*

**Remark 2.** *By definition, one has that  $\Pi\partial_{\mathbf{n}} = \partial_{\mathbf{n}}\Pi$ . However, since  $T$  is a global operator on the skeleton  $\Sigma$ ,  $\Pi$  and  $T$  a priori do not commute and  $\Pi T \neq T\Pi$ . The study of the modified commutation property  $\Pi T = T^*\Pi$  will be key for this work.*

Provided the operator  $T$  is defined in a consistent way, most of the iterative DDM already quoted in the literature [3, 6, 9, 14, 17, 20, 21, 22, 8] can be recast under the form (8). The original DDM for the Helmholtz equation [1, 10, 11] corresponds to the simplest choice  $T = I_{\Sigma}$ .

We now sketch the organization and main results of this work.

- In Section 2 we derive a simple family of cross-point conditions to supplement the operator formulation (5) and we set up a mathematical framework for the analysis of general operators  $T$  which have cross-point contributions. The operator  $T$  appears as the solution of a variational problem written on the skeleton  $\Sigma$ , and  $T \neq T^*$  in the general case. A norm related to  $T$  is defined. This norm is global on the skeleton, but equivalent to the  $H_{\text{br}}^1(\Sigma)$  norm. The key estimate of Lemma 11 shows an isometry result for this special norm, provided the cross-point matrices have a special compatible structure.
- Next, Section 3 is devoted to proving the convergence of the DDM for the cross-point matrices under the compatibility property. For the sake of simplicity, the proof of Theorem 19 concentrates on the under-relaxed version of (8).
- Finally, in Section 4, we show that it is possible to use plane waves to calculate cross-point matrices which verify the compatibility property. For these cross-point matrices, convergence of the under-relaxed DDM is guaranteed by Theorem 19.

## 2 Mathematical framework

In order to construct the operator  $T$  that will be used to formulate the DDM as in (8), we start from the variational formulation associated to the second order tangential operators from (6) coupled with the mixed boundary conditions (11). A sesquilinear form  $a$  and its associated operator  $T$  are obtained subsequently by integrating by parts.

### 2.1 Preliminary considerations on cross-points

As already stated, a cross-point is a node of the skeleton of the decomposition in polygonal subdomains that lies at the intersection of 3 or more subdomains. We will consider only interior cross-points. A cross-point thus lays at the intersection of at least two edges  $\Sigma_{ij}$  and  $\Sigma_{kl}$ ,  $1 \leq i, j, k, \ell \leq N$ , where  $i \neq j \neq k \neq \ell$ . Let  $N_X$  be the number of cross-points of the decomposition. They are indexed independently of the surrounding subdomains to form the set  $\mathbf{X} := \{\mathbf{x}_r \in \mathbb{R}^2, 1 \leq r \leq N_X\} \in (\mathbb{R}^2)^{N_X}$ . The number of subdomains around a node  $\mathbf{x}_r$  is  $d_r := \text{card}\{i, \mathbf{x}_r \in \partial\Omega_i\}$ . The edges around a node  $\mathbf{x}_r$  are referenced to by pairs  $(ij)$  where  $\mathbf{x}_r \in \bar{\Sigma}_{ij}$ . There are  $2d_r$  of them (with  $\Sigma_{ij}$  and  $\Sigma_{ji}$  counting separately) and their ensemble is denoted  $\mathcal{E}_r$ . Finally, at a cross-point  $\mathbf{x}_r$ , we define the unit vector  $\boldsymbol{\tau}_{ij}$  tangential and outwardly directed to  $\Sigma_{ij}$  for  $(ij) \in \mathcal{E}_r$ . These notations are illustrated on Figure 1 for 3 subdomains.

Weak formulations associated to differential operators need to be completed by boundary conditions to be well-posed. Let  $u_i \in H^1(\Omega_i)$  for  $1 \leq i \leq N$ . Let  $\varphi = \partial_{\mathbf{n}}u_{\Sigma}$  on the skeleton as defined in (2), and let us assume the regularity (9) that is  $\varphi \in H_{\text{br}}^1(\Sigma)$ . Define  $\varphi_{ij} = \partial_{\mathbf{n}_i}u_{ij}$  on each edge  $\Sigma_{ij}$ . The iteration (5) is based on second order tangential derivatives of  $\varphi_{ij}$ . Therefore we are interested in a linear combination of the traces  $\varphi_{ij}(\mathbf{x}_r)$  and of the outgoing tangential

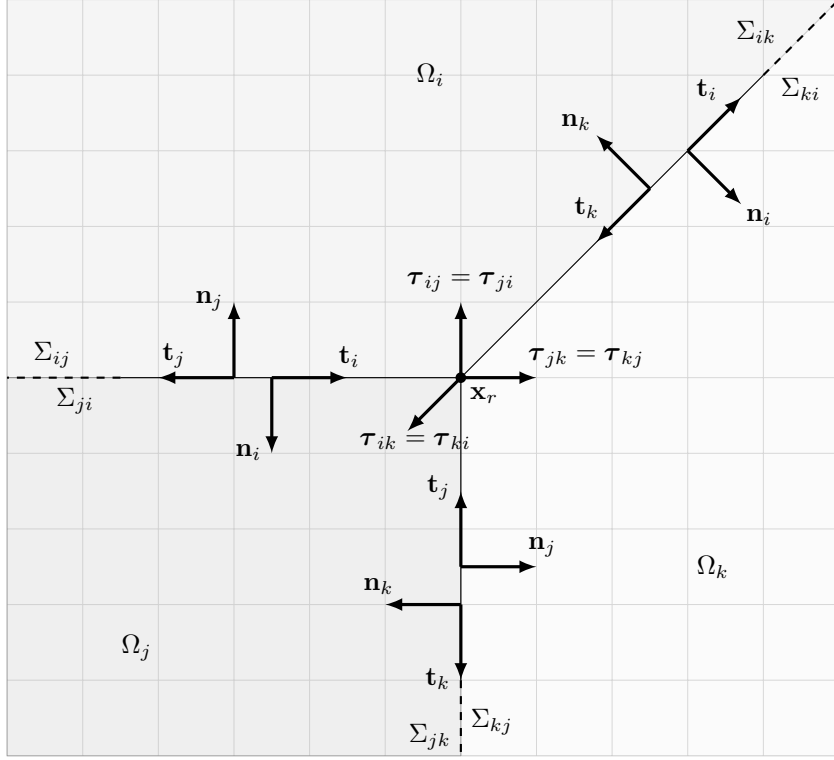


Figure 1: Some definitions around a cross point  $\mathbf{x}_r$

derivatives  $\partial_{\tau_{ij}} \varphi_{ij}(\mathbf{x}_r)$  on all edges intersecting at  $\mathbf{x}_r$ . Precisely, in this work, we assume that

$$\partial_{\tau_{ij}} \varphi_{ij}(\mathbf{x}_r) + \sum_{(kl) \in \mathcal{E}_r} \alpha_{ij,kl}^r \varphi_{kl}(\mathbf{x}_r) = 0, \quad \forall (ij) \in \mathcal{E}_r. \quad (10)$$

That is the outgoing tangential derivative at  $\mathbf{x}_r$  is expressed here as a linear combination of the traces, and the coefficients  $\alpha_{ij,kl}^r \in \mathbb{C}$  of the linear combination are unknowns at this stage of the construction. Define the vector  $\varphi_r \in \mathbb{C}^{2d_r}$  of Dirichlet traces of  $\varphi$  at  $\mathbf{x}_r$  over all  $\Sigma_{ij}$  intersecting at  $\mathbf{x}_r$ , assembled in a given order. Similarly, define the vector  $\partial_{\tau} \varphi_r \in \mathbb{C}^{2d_r}$  of the outgoing tangential derivative traces. They are vectors of size  $2d_r$  since every physical edge is counted twice in the skeleton. With these notations, the system of linear relations (10) rewrite

$$\partial_{\tau} \varphi_r + A^r \varphi_r = 0, \quad (11)$$

with the matrix  $A^r \in \mathcal{M}_{2d_r}(\mathbb{C})$  containing the unknowns  $(\alpha_{ij,kl}^r)_{ij,kl}$ .

Two natural requirements appear for the definition of the coefficients  $\alpha_{ij,kl}^r \in \mathbb{C}$ . A first requirement is that the associated DDM is well posed and convergent. Proving that this requirement holds is the main contribution of this work. The construction of the tools which justify the well-posedness of the DDM starts in this section. A second requirement is the optimality of coefficients  $\alpha_{ij,kl}^r \in \mathbb{C}$  in terms of accuracy for certain family of particular solutions, and this issue is examined in the last section.

## 2.2 Definition of $T$

Define the form  $a(\cdot, \cdot)$  such that for all  $\varphi, \psi \in H_{\text{br}}^1(\Sigma)$

$$a(\varphi, \psi) = \sum_{i,j=1}^N \int_{\Sigma_{ij}} \left( \varphi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \varphi_{ij} \overline{\partial_{\mathbf{t}_i} \psi_{ij}} \right) d\sigma + \frac{1}{2\omega^2} \sum_{r=1}^{N_{\mathbf{x}}} (A^r \varphi_r, \psi_r)_{\mathbb{C}^{2d_r}}. \quad (12)$$

We make the assumption that  $A^r$  has pure imaginary coefficients, which will be justified on examples in Section 4, and that it is skew-hermitian

$$A^r = \mathbf{i}H^r, \quad \text{where } H^r = (H^r)^\top \in \mathcal{M}_{2d_r}(\mathbb{R}). \quad (13)$$

Our results also hold for more general complex hermitian matrices  $H^r = (H^r)^* \in \mathcal{M}_{2d_r}(\mathbb{C})$ .

As a consequence of (13), the sesquilinear form  $a$  is coercive in  $L^2(\Sigma)$  and in  $H_{\text{br}}^1(\Sigma)$

$$\begin{cases} \|\varphi\|_{L^2(\Sigma)}^2 \leq \text{Re } a(\varphi, \varphi), \\ \|\varphi\|_{H_{\text{br}}^1(\Sigma)}^2 \leq \max\{1, 2\omega^2\} \text{Re } a(\varphi, \varphi), \end{cases} \quad (14)$$

where  $\text{Re } z$  denotes the real part of  $z \in \mathbb{C}$ .

The variational problem for a given  $v \in L^2(\Sigma)$  is

$$\begin{cases} \text{Find } \varphi \in H_{\text{br}}^1(\Sigma) \text{ such that} \\ a(\varphi, \psi) = (v, \psi)_{L^2(\Sigma)}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma). \end{cases} \quad (15)$$

This problem is well posed, there exists a unique solution  $\varphi \in H_{\text{br}}^1(\Sigma)$ . The strong form of this problem writes on the edges  $\Sigma_{ij}$

$$-\frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \varphi_{ij} + \varphi_{ij} = v_{ij}. \quad (16)$$

**Definition 3.** Let  $T: L^2(\Sigma) \rightarrow L^2(\Sigma)$  be the operator such that  $Tv = \varphi$ , where  $\varphi$  is the solution to the problem (15) for  $v \in L^2(\Sigma)$ .

By definition of  $\varphi$  in (15), one has the continuous embedding  $T(L^2(\Sigma)) \subset H_{\text{br}}^1(\Sigma)$ . One even has more regularity since (16) shows that  $\varphi$  is in the space  $H_{\text{br}}^2(\Sigma)$ , where

$$H_{\text{br}}^2(\Sigma) := \{v \in L^2(\Sigma) \text{ such that } v|_{\Sigma_{ij}} \in H^2(\Sigma_{ij}) \text{ for all } i \neq j\}.$$

Therefore one also has the continuous embedding  $T(L^2(\Sigma)) \subset H_{\text{br}}^2(\Sigma)$ . The operator  $T$  is thus compact.

The adjoint operator with respect to the sesquilinear product in  $L^2(\Sigma)$  is  $T^*: L^2(\Sigma) \rightarrow L^2(\Sigma)$ . If the cross-point matrices are non zero at least at one cross-point  $\mathbf{x}_r$  ( $H^r \neq 0$ ), then  $T^* \neq T$ . It is from this point that our analysis diverges from the entire literature, except from our previous paper on corners [12].

**Proposition 4.** It holds that

1.  $\|T\|_{L^2(\Sigma)} \leq 1$ .
2.  $T + T^* > 0$ .

*Proof.* Let  $\psi = \varphi = Tv$  in (15): it follows that  $a(Tv, Tv) = (v, Tv)_{L^2(\Sigma)}$ . Therefore,  $\|Tv\|_{L^2(\Sigma)}^2 \leq |a(Tv, Tv)| \leq \|v\|_{L^2(\Sigma)} \|Tv\|_{L^2(\Sigma)}$ , and the bound is obtained.

Using the definitions of  $T$  and  $T^*$  as well as (14), one has that

$$\frac{1}{2}(v, (T + T^*)v)_{L^2(\Sigma)} = \operatorname{Re}(v, Tv)_{L^2(\Sigma)} = \operatorname{Re} a(Tv, Tv) \geq \|Tv\|_{L^2(\Sigma)}^2. \quad (17)$$

Now assuming  $T + T^*$  is not positive means that there exists  $v \in L^2(\Sigma) \setminus \{0\}$  such that  $(T + T^*)v = 0$ . The inequality (17) then yields that  $Tv = 0$ , which in turn implies that  $T^*v = 0$  and that for all  $\psi \in H_{\text{br}}^1(\Sigma)$

$$(v, \psi)_{L^2(\Sigma)} = a(Tv, \psi) = 0.$$

Since  $H_{\text{br}}^1(\Sigma)$  is densely embedded in  $L^2(\Sigma)$ , the relation  $(v, \psi)_{L^2(\Sigma)} = 0$  actually holds for all  $\psi \in L^2(\Sigma)$ . Taking  $\psi = v$ , it yields the contradiction  $v = 0$ . The operator  $T + T^*$  is hence positive.  $\square$

We now define  $a^*$ , the sesquilinear form such that for all  $\varphi, \psi \in H_{\text{br}}^1(\Sigma)$

$$a^*(\varphi, \psi) = \sum_{i,j=1}^N \int_{\Sigma_{ij}} \left( \varphi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \varphi_{ij} \overline{\partial_{\mathbf{t}_i} \psi_{ij}} \right) d\sigma - \frac{\mathbf{i}}{2\omega^2} \sum_{r=1}^{N_X} (H^r \varphi_r, \psi_r)_{\mathbb{C}^{2d_r}}.$$

By construction, one has that  $a^*(\varphi, \psi) = \overline{a(\psi, \varphi)}$ .

Since  $a^*$  has the same structure as  $a$ , except that  $H^r$  is changed in  $-H^r$ , it defines an operator  $R: L^2(\Sigma) \rightarrow L^2(\Sigma)$  such that  $\varphi = Rv$  is the solution of the problem

$$\begin{cases} \text{Find } \varphi \in H_{\text{br}}^1(\Sigma) \text{ such that} \\ a^*(\varphi, \psi) = (v, \psi)_{L^2(\Sigma)}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma), \end{cases}$$

which inherits all properties of  $T$  stated above.

**Lemma 5.** *One has  $R = T^*$ .*

*Proof.* Take  $v, w \in L^2(\Sigma)$ . One has

$$a^*(Rv, Tw) = \overline{a(Tw, Rv)} \iff (v, Tw)_{L^2(\Sigma)} = \overline{(w, Rv)_{L^2(\Sigma)}} \iff (v, Tw)_{L^2(\Sigma)} = (Rv, w)_{L^2(\Sigma)}.$$

It shows that  $R = T^*$ .  $\square$

**Lemma 6.** *The ranges of  $T$  and of  $T^*$  are dense in  $H_{\text{br}}^1(\Sigma)$ .*

*Proof.* The proof follows a classical scheme in functional analysis. It is a combination of spectral decomposition and coercive and compact decomposition [5]. The proof relies on the structure of the sesquilinear form, and is therefore done only for  $T$ . The result for  $T^*$  follows.

Define the  $H_{\text{br}}^1(\Sigma)$  hermitian scalar product derived from the hermitian part of  $a$

$$a_1(\varphi, \psi) := \sum_{i,j=1}^N \int_{\Sigma_{ij}} \left( \varphi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \varphi_{ij} \overline{\partial_{\mathbf{t}_i} \psi_{ij}} \right) d\sigma, \quad \forall \varphi, \psi \in H_{\text{br}}^1(\Sigma).$$

Define the sesquilinear form  $a_2 := (a - a_1)/\mathbf{i}$ .



- One defines the associated solution operator  $S: L^2(\Sigma) \rightarrow H_{\text{br}}^1(\Sigma)$  such that

$$a_1(Sv, \psi) = (v, \psi)_{L^2(\Sigma)}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma). \quad (18)$$

This operator is compact in  $L^2(\Sigma)$  by compact embedding of  $H_{\text{br}}^1(\Sigma) \subset L^2(\Sigma)$ , and is also self-adjoint. There exists a Hilbertian family of eigenfunctions  $(v_n)_{n \in \mathbb{N}} \in H_{\text{br}}^1(\Sigma)$  that span  $L^2(\Sigma)$  such that  $Sv_n = \mu_n v_n$  for a decreasing family of positive eigenvalues  $\mu_n \searrow 0$ . The eigenfunctions are chosen orthonormal in  $L^2(\Sigma)$ . Since  $a_1(v_n, \psi) = \frac{1}{\mu_n} (v_n, \psi)_{L^2(\Sigma)}$  for all  $\psi \in H_{\text{br}}^1(\Sigma)$ , one has

$$\sqrt{a_1(\psi, \psi)} = \left( \sum_{n \geq 0} \frac{1}{\mu_n} |(\psi, v_n)_{L^2(\Sigma)}|^2 \right)^{1/2}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma). \quad (19)$$

One obtains  $\overline{S(L^2(\Sigma))}^{H_{\text{br}}^1(\Sigma)} = H_{\text{br}}^1(\Sigma)$ , which is the relation we seek but for  $S$  instead of  $T$ . The rest of the proof is dedicated to show that  $T - S$  is a compact operator.

- Define the operator  $Q: H_{\text{br}}^1(\Sigma) \rightarrow H_{\text{br}}^1(\Sigma)$  such that

$$a_1(Q\varphi, \psi) = a_2(\varphi, \psi), \quad \text{for all } \psi \in H_{\text{br}}^1(\Sigma).$$

Due to the continuity of the trace operator,  $Q$  is bounded from  $H_{\text{br}}^1(\Sigma)$  into  $H_{\text{br}}^1(\Sigma)$ . For  $\varphi \in H_{\text{br}}^1(\Sigma)$ , denote  $\phi = Q\varphi$ . It holds by definition that

$$\sum_{i,j=1}^N \int_{\Sigma_{ij}} \left( \phi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \phi_{ij} \overline{\partial_{\mathbf{t}_i} \psi_{ij}} \right) d\sigma = \frac{1}{2\omega^2} \sum_{r=1}^{N_X} (H^r \varphi_r, \psi_r)_{C^{2d_r}}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma).$$

The functions  $\phi_{ij}$  are solutions of  $-\frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \phi_{ij} + \phi_{ij} = 0$  in each  $\Sigma_{ij}$  with boundary conditions at the end points of  $\Sigma_{ij}$  provided by  $\partial_r \phi_r = H^r \varphi_r$ . Therefore  $\phi_{ij}$  is bounded in  $H_{\text{br}}^2(\Sigma)$ , and the operator  $Q$  is compact in  $H_{\text{br}}^1(\Sigma)$ .

- Let  $v \in L^2(\Sigma)$ . We denote by  $I$  the identity operator in  $H_{\text{br}}^1(\Sigma)$ . For all  $\psi \in H_{\text{br}}^1(\Sigma)$ , one has that

$$\begin{aligned} a_1((I + \mathbf{i}Q)Tv - Sv, \psi) &= a_1(Tv, \psi) + \mathbf{i}a_2(Tv, \psi) - a_1(Sv, \psi) \\ &= a(Tv, \psi) - a_1(Sv, \psi) \\ &= (v, \psi)_{L^2(\Sigma)} - (v, \psi)_{L^2(\Sigma)} \\ &= 0. \end{aligned}$$

Therefore  $(I + \mathbf{i}Q)Tv - Sv$  for all  $v \in L^2(\Sigma)$ . So  $(I + \mathbf{i}Q)T = S$ .

- We now prove that  $I + \mathbf{i}Q$  has a bounded inverse in  $H_{\text{br}}^1(\Sigma)$ . Since  $Q$  is compact, it is sufficient to check that if  $(I + \mathbf{i}Q)\varphi = 0$  for  $\varphi \in H_{\text{br}}^1(\Sigma)$ , then  $\varphi = 0$ . And indeed, we have in this case

$$0 = a_1((I + \mathbf{i}Q)\varphi, \psi) = a_1(\varphi, \psi) + \mathbf{i}a_2(\varphi, \psi) = a(\varphi, \psi), \quad \forall \psi \in H_{\text{br}}^1(\Sigma),$$

which implies that  $\varphi = 0$ . So one has  $T = (I + \mathbf{i}Q)^{-1}S$ .

- Finally,  $\overline{T(L^2(\Sigma))}^{H_{\text{br}}^1(\Sigma)} = (I + \mathbf{i}Q)^{-1} \overline{S(L^2(\Sigma))}^{H_{\text{br}}^1(\Sigma)} = (I + \mathbf{i}Q)^{-1} H_{\text{br}}^1(\Sigma) = H_{\text{br}}^1(\Sigma)$ , where the closure is taken with respect to the norm in  $H_{\text{br}}^1(\Sigma)$ .  $\square$

### 2.3 A norm based on the spectral decomposition of $T + T^*$

The ideas behind the following definitions take their origin from [12]. They are based on the fact that the operator  $T + T^*$  induces a norm in  $L^2(\Sigma)$  which is particularly well adapted to the study of stable DDMs.

From Lemma 6, one has the inclusion  $(T+T^*)(L^2(\Sigma)) \subset H_{\text{br}}^1(\Sigma)$ , which is compactly embedded in  $L^2(\Sigma)$ . So  $T + T^* > 0$  is a self-adjoint compact operator of  $L^2(\Sigma)$ . Therefore, referring to [5], it admits a spectral decomposition. Let  $(u_n)_{n \in \mathbb{N}} \subset H_{\text{br}}^1(\Sigma)$  be the Hilbertian basis such that for all  $n, m \in \mathbb{N}$ ,

$$\left(\frac{T + T^*}{2}\right) u_n = \lambda_n u_n, \quad (u_n, u_m)_{L^2(\Sigma)} = \delta_{nm} \quad \text{and} \quad \overline{\text{span}\{u_n\}_{n \in \mathbb{N}}}^{L^2(\Sigma)} = L^2(\Sigma). \quad (20)$$

The eigenvalues satisfy  $1 \geq \lambda_n \geq \lambda_{n+1} > 0$  according to Proposition 4, and  $\lambda_n$  converges towards zero as  $n$  goes to infinity. This leads to the definition of the operator  $\left(\frac{T+T^*}{2}\right)^{-1} : L^2(\Sigma) \rightarrow L^2(\Sigma)$  such that  $\left(\frac{T+T^*}{2}\right)^{-1} u_n = \frac{1}{\lambda_n} u_n$  and to the definition of the space

$$H_T^1(\Sigma) := \{u \in L^2(\Sigma), \|u\| < \infty\},$$

endowed with the norm  $\|\cdot\|$  defined as

$$\|u\| := \left( \sum_{n \geq 0} \frac{|(u, u_n)_{L^2(\Sigma)}|^2}{\lambda_n} \right)^{1/2}, \quad \forall u \in L^2(\Sigma).$$

The hermitian scalar product in  $H_T^1(\Sigma)$  is denoted

$$\langle u, v \rangle := \sum_{n \geq 0} \frac{1}{\lambda_n} (u, u_n)_{L^2(\Sigma)} \overline{(v, u_n)_{L^2(\Sigma)}} = \left( \left( \frac{T + T^*}{2} \right)^{-1} u, v \right)_{L^2(\Sigma)}, \quad \forall u, v \in H_T^1(\Sigma).$$

As a consequence,  $H_T^1(\Sigma)$  is a Hilbert space.

**Theorem 7.** *One has  $H_T^1(\Sigma) = H_{\text{br}}^1(\Sigma)$  with equivalence of norms.*

This result is not completely surprising. Indeed, on the one hand, if all cross-point matrices are zero, *i.e.*  $H^r = 0$  for all  $r$ , then the sesquilinear form (12) comes down to  $a_1$ , which is equivalent to the  $H_{\text{br}}^1$  hermitian scalar product, and  $T = S = S^* = T^*$ . On the other hand, if  $H^r \neq 0$  for some  $r$ , then  $T^* \neq T$  but the coupling at cross-points is a compact perturbation of the integral parts. The main difference between the two norms is that the  $H_{\text{br}}^1(\Sigma)$  norm is local because it can be localized on every interface, while the  $H_T^1(\Sigma)$  norm is global since the eigenfunctions  $u_n$  are global on the skeleton  $\Sigma$ .

The new norm is particularly adapted to the study of DDMs, as the isometry Lemma 11 will show.

The proof of the Theorem is divided in two steps which are Lemma 8 and Lemma 10. The first step is devoted to the continuous embedding  $H_T^1(\Sigma) \subset H_{\text{br}}^1(\Sigma)$ , the easiest one. The reverse continuous embedding  $H_{\text{br}}^1(\Sigma) \subset H_T^1(\Sigma)$  is proved in the second step.

**Lemma 8** (First part of Theorem 7). *One has  $H_T^1(\Sigma) \subset H_{\text{br}}^1(\Sigma)$  with  $\|u\|_{H_{\text{br}}^1(\Sigma)} \leq C \|u\|$  for all  $u \in H_T^1(\Sigma)$ , for  $C > 0$  a numerical constant.*

*Proof.* Take  $u \in H_T^1(\Sigma)$ , that is  $u \in L^2(\Sigma)$  with  $\sum_{n \geq 0} \frac{1}{\lambda_n} |(u, u_n)_{L^2(\Sigma)}|^2 < \infty$ . Define the approximation  $u^N \in H_T^1(\Sigma)$  by truncation of the series

$$u^N := \sum_{n=0}^N (u, u_n)_{L^2(\Sigma)} u_n, \quad \forall N \in \mathbb{N}.$$

The sequence  $(u^N)_{N \in \mathbb{N}}$  is a Cauchy sequence in  $H_T^1(\Sigma)$ : for  $M > N \geq 0$ , one has

$$\|u^M - u^N\|^2 = \sum_{n \geq 0} \frac{1}{\lambda_n} \left| \left( \sum_{m=N+1}^M (u, u_m)_{L^2(\Sigma)} u_m, u_n \right)_{L^2(\Sigma)} \right|^2 = \sum_{n=N+1}^M \frac{1}{\lambda_n} |(u, u_n)_{L^2(\Sigma)}|^2.$$

We now prove that it is also a Cauchy sequence in  $H_{\text{br}}^1(\Sigma)$ . For  $M > N \geq 0$ , define  $v := \left(\frac{T+T^*}{2}\right)^{-1} (u^M - u^N)$ . One has that

$$v = \sum_{n=N+1}^M (u, u_n)_{L^2(\Sigma)} \left(\frac{T+T^*}{2}\right)^{-1} u_n = \sum_{n=N+1}^M \frac{1}{\lambda_n} (u, u_n)_{L^2(\Sigma)} u_n,$$

so that it follows

$$\left( \left(\frac{T+T^*}{2}\right) v, v \right)_{L^2(\Sigma)} = \left( u^M - u^N, \left(\frac{T+T^*}{2}\right)^{-1} (u^M - u^N) \right)_{L^2(\Sigma)} = \|u^M - u^N\|^2. \quad (21)$$

Due to definition of the sesquilinear form (12), it also holds

$$\left( \left(\frac{T+T^*}{2}\right) v, v \right)_{L^2(\Sigma)} = \text{Re} \left( (Tv, v)_{L^2(\Sigma)} \right) = \text{Re} \left( a(Tv, Tv) \right) \geq \frac{1}{\max\{1, 2\omega^2\}} \|Tv\|_{H_{\text{br}}^1(\Sigma)}^2,$$

and similarly

$$\left( \left(\frac{T+T^*}{2}\right) v, v \right)_{L^2(\Sigma)} = \text{Re} \left( (T^*v, v)_{L^2(\Sigma)} \right) = \text{Re} \left( a^*(T^*v, T^*v) \right) \geq \frac{1}{\max\{1, 2\omega^2\}} \|T^*v\|_{H_{\text{br}}^1(\Sigma)}^2.$$

So one can write

$$\begin{aligned} \|u^M - u^N\|_{H_{\text{br}}^1(\Sigma)}^2 &= \left\| \left(\frac{T+T^*}{2}\right) v \right\|_{H_{\text{br}}^1(\Sigma)}^2 \leq \frac{1}{2} \left( \|Tv\|_{H_{\text{br}}^1(\Sigma)}^2 + \|T^*v\|_{H_{\text{br}}^1(\Sigma)}^2 \right) \\ &\leq C^2 \left( \left(\frac{T+T^*}{2}\right) v, v \right)_{L^2(\Sigma)}, \end{aligned} \quad (22)$$

for the numerical constant  $C^2 := \max\{1, 2\omega^2\}$ . Combining (21) with (22) yields

$$\|u^M - u^N\|_{H_{\text{br}}^1(\Sigma)} \leq C \|u^M - u^N\|.$$

The sequence  $(u^N)_{N \in \mathbb{N}}$  is therefore also a Cauchy sequence in  $H_{\text{br}}^1(\Sigma)$ . It immediately follows from the completeness of these spaces that  $H_T^1(\Sigma) \subset H_{\text{br}}^1(\Sigma)$  with the explicit bound  $\|u\|_{H_{\text{br}}^1(\Sigma)} \leq C \|u\|$ .  $\square$

The proof of the reverse embedding requires more technical material: in Lemma 9, we prove that in addition to being compact and self-adjoint, the operator  $\frac{T+T^*}{2}$  has a particular structure that can be interpreted as a compact perturbation of the Helmholtz solution operator  $S$  defined in (18).

As in the proof of Lemma 6, we split the sesquilinear form  $a = a_1 + \mathbf{i}a_2$ , where

$$\begin{aligned} a_1(\varphi, \psi) &= \sum_{i,j=1}^N \int_{\Sigma_{ij}} \left( \varphi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{t_i} \varphi_{ij} \overline{\partial_{t_i} \psi_{ij}} \right) d\sigma, & \forall \varphi, \psi \in H_{\text{br}}^1(\Sigma), \\ a_2(\varphi, \psi) &= \frac{1}{2\omega^2} \sum_{r=1}^{N_X} (H^r \varphi_r, \psi_r)_{\mathbb{C}^{2d_r}}, & \forall \varphi, \psi \in H_{\text{br}}^1(\Sigma). \end{aligned}$$

Both  $a_1$  and  $a_2$  are hermitian. The first sesquilinear form  $a_1$  is coercive in  $H_{\text{br}}^1(\Sigma)$ .

It is convenient to introduce the square root  $S^{\frac{1}{2}}$  of the solution operator  $S$  to  $a_1$  (see (18)). Referring to the quantities defined in Lemma 6, it is the operator  $S^{\frac{1}{2}} : L^2(\Sigma) \rightarrow L^2(\Sigma)$  such that

$$S^{\frac{1}{2}} v_n = \mu_n^{\frac{1}{2}} v_n, \quad \forall n \in \mathbb{N}. \quad (23)$$

Parseval's identity, relations (23) and (19) and the coercivity of  $a_1$  yield

$$\|v\|_{L^2(\Sigma)}^2 = \sum_{n \geq 0} \left| (v, v_n)_{L^2(\Sigma)} \right|^2 = \sum_{n \geq 0} \frac{1}{\mu_n} \left| \left( S^{\frac{1}{2}} v, v_n \right)_{L^2(\Sigma)} \right|^2 = a_1(S^{\frac{1}{2}} v, S^{\frac{1}{2}} v).$$

Since the sesquilinear form  $a_1$  defines an equivalent norm in  $H_{\text{br}}^1(\Sigma)$  the operator  $S^{\frac{1}{2}}$  is an isomorphism between  $L^2(\Sigma)$  and  $H_{\text{br}}^1(\Sigma)$ .

The inverse operator  $S^{-\frac{1}{2}} : H_{\text{br}}^1(\Sigma) \rightarrow L^2(\Sigma)$  is defined by

$$S^{-\frac{1}{2}} v_n = \mu_n^{-\frac{1}{2}} v_n, \quad \forall n \in \mathbb{N}.$$

By construction, one has the identity  $a_1(\varphi, \psi) = (S^{-\frac{1}{2}} \varphi, S^{-\frac{1}{2}} \psi)_{L^2(\Sigma)}$  for all  $\varphi, \psi \in H_{\text{br}}^1(\Sigma)$ .

Finally, we define the operator  $K : L^2(\Sigma) \rightarrow L^2(\Sigma)$  such that

$$(Ku, v)_{L^2(\Sigma)} = a_2(S^{\frac{1}{2}} u, S^{\frac{1}{2}} v), \quad \forall u, v \in L^2(\Sigma).$$

For all  $u, v \in L^2(\Sigma)$  and for a given continuity constant  $C > 0$  of the sesquilinear form  $a_2$ , one has

$$(Ku, v)_{L^2(\Sigma)} = a_2(S^{\frac{1}{2}} u, S^{\frac{1}{2}} v) \leq C \|S^{\frac{1}{2}} u\|_{H_{\text{br}}^1(\Sigma)} \|S^{\frac{1}{2}} v\|_{H_{\text{br}}^1(\Sigma)} \leq C \|u\|_{L^2(\Sigma)} \|v\|_{L^2(\Sigma)}.$$

The operator  $K$  is therefore bounded from  $L^2(\Sigma)$  to  $L^2(\Sigma)$ . Lastly, we note that  $K^* = K$  in  $L^2(\Sigma)$ .

**Lemma 9.** *One has the identity  $\frac{T+T^*}{2} = S^{\frac{1}{2}} (I + K^2)^{-1} S^{\frac{1}{2}}$ .*

*Proof.* Let  $u, v \in L^2(\Sigma)$ . By definition of  $S^{\frac{1}{2}}$  and of  $a, a_1$  and  $a_2$ , it follows

$$(S^{\frac{1}{2}} u, v)_{L^2(\Sigma)} = (u, S^{\frac{1}{2}} v)_{L^2(\Sigma)} = a(Tu, S^{\frac{1}{2}} v) = a_1(Tu, S^{\frac{1}{2}} v) + \mathbf{i}a_2(Tu, S^{\frac{1}{2}} v).$$

One introduces the truncation of  $Tu \in H_{\text{br}}^1(\Sigma)$

$$Tu|_N = \sum_{n=0}^N (Tu, v_n)_{H_{\text{br}}^1(\Sigma)} \mu_n v_n = \sum_{n=0}^N (Tu, v_n)_{H_{\text{br}}^1(\Sigma)} \frac{v_n}{\|v_n\|_{H_{\text{br}}^1(\Sigma)}^2},$$

such that  $\lim_{N \rightarrow \infty} \|Tu - Tu|_N\|_{H_{\text{br}}^1(\Sigma)} = 0$  by definition. Therefore, one has the decomposition

$$(S^{\frac{1}{2}}u, v)_{L^2(\Sigma)} = a_1(Tu - Tu|_N, S^{\frac{1}{2}}v) + a_1(Tu|_N, S^{\frac{1}{2}}v) + \mathbf{i}a_2(Tu, S^{\frac{1}{2}}v),$$

where  $a_1(Tu - Tu|_N, S^{\frac{1}{2}}v) \xrightarrow{N \rightarrow \infty} 0$ . The operator  $S$  is a bijection in the finite dimensional space  $\text{span}\{v_n\}_{n=0}^N$ . Therefore, the operator  $S^{-1}$  is correctly defined in this space. So one can write

$$\begin{aligned} a_1(Tu|_N, S^{\frac{1}{2}}v) + \mathbf{i}a_2(Tu, S^{\frac{1}{2}}v) &= a_1(SS^{-1}Tu|_N, S^{\frac{1}{2}}v) + \mathbf{i}a_2(S^{\frac{1}{2}}S^{-\frac{1}{2}}Tu, S^{\frac{1}{2}}v) \\ &= (S^{-1}Tu|_N, S^{\frac{1}{2}}v)_{L^2(\Sigma)} + \mathbf{i}(KS^{-\frac{1}{2}}Tu, v)_{L^2(\Sigma)} \\ &= (S^{-\frac{1}{2}}Tu|_N, v)_{L^2(\Sigma)} + \mathbf{i}(KS^{-\frac{1}{2}}Tu, v)_{L^2(\Sigma)}. \end{aligned}$$

Hence  $(S^{\frac{1}{2}}u, v)_{L^2(\Sigma)} = ((I + \mathbf{i}K)S^{-\frac{1}{2}}Tu, v)_{L^2(\Sigma)}$  for all  $u, v \in L^2(\Sigma)$ , and  $S^{\frac{1}{2}} = (I + \mathbf{i}K)S^{-\frac{1}{2}}T$ , so that

$$(I + \mathbf{i}K)^{-1}S^{\frac{1}{2}} = S^{-\frac{1}{2}}T. \quad (24)$$

The inverse operator  $(I + \mathbf{i}K)^{-1}$  is correctly defined, since  $(I + \mathbf{i}K)^{-1} = (I + K^2)^{-1}(I - \mathbf{i}K)$  and

$$I + K^2 = I + K^*K \geq I,$$

so that the operator  $I + K^2$  is coercive and continuous in  $L^2(\Sigma)$  and has a continuous inverse in  $L^2(\Sigma)$  according to the Lax-Milgram theorem.

Referring to Lemma 5, a similar formula holds for the operator  $T^*$ . One obtains

$$(I - \mathbf{i}K)^{-1}S^{\frac{1}{2}} = S^{-\frac{1}{2}}T^*. \quad (25)$$

Adding (24) to (25), one gets  $S^{-\frac{1}{2}}(T + T^*) = 2(I + K^2)^{-1}S^{\frac{1}{2}}$ . Applying  $\frac{1}{2}S^{\frac{1}{2}}$  to this relation yields the claim.  $\square$

**Lemma 10** (Second part of Theorem 7). *One has  $H_{\text{br}}^1(\Sigma) \subset H_T^1(\Sigma)$  with  $\|u\| \leq C\|u\|_{H_{\text{br}}^1(\Sigma)}$  for all  $u \in H_{\text{br}}^1(\Sigma)$ , for  $C > 0$  a numerical constant which depends on the operator  $K$ .*

*Proof.* Let  $\varepsilon > 0$ . The operator  $\frac{T+T^*}{2} + \varepsilon I$  is coercive, continuous and with continuous inverse in  $L^2(\Sigma)$ . One also has that  $(\frac{T+T^*}{2} + \varepsilon I)u_n = (\lambda_n + \varepsilon)u_n$  for all  $n \in \mathbb{N}$ .

Let  $u \in H_{\text{br}}^1(\Sigma)$ . Then, for  $v_\varepsilon$  the solution of

$$\left(\frac{T+T^*}{2} + \varepsilon I\right)v_\varepsilon = u,$$

it follows that

$$(v_\varepsilon, u)_{L^2(\Sigma)} = \sum_{n \geq 0} \frac{1}{\lambda_n + \varepsilon} |(u, u_n)_{L^2(\Sigma)}|^2. \quad (26)$$

According to Lemma 9, one has that

$$\begin{aligned} (v_\varepsilon, u)_{L^2(\Sigma)} &= (v_\varepsilon, (S^{\frac{1}{2}}(I + K^2)^{-1}S^{\frac{1}{2}} + \varepsilon I)v_\varepsilon)_{L^2(\Sigma)} \\ &= (S^{\frac{1}{2}}v_\varepsilon, (I + K^2)^{-1}S^{\frac{1}{2}}v_\varepsilon)_{L^2(\Sigma)} + \varepsilon\|v_\varepsilon\|_{L^2(\Sigma)}^2. \end{aligned}$$

One gets the upper and lower bounds

$$\|S^{\frac{1}{2}}v_\varepsilon\|_{L^2(\Sigma)}\|S^{-\frac{1}{2}}u\|_{L^2(\Sigma)} \geq (v_\varepsilon, u)_{L^2(\Sigma)} \geq \alpha\|S^{\frac{1}{2}}v_\varepsilon\|_{L^2(\Sigma)}^2,$$

where  $\alpha$  is the coercivity constant associated to  $(I + K^2)^{-1}$ . Since  $u \in H_{\text{br}}^1(\Sigma)$ , according to the definition of  $S^{-\frac{1}{2}}$ , one has that  $\|S^{-\frac{1}{2}}u\|_{L^2(\Sigma)} \leq \tilde{C}\|u\|_{H_{\text{br}}^1(\Sigma)}$ . Hence for all  $\varepsilon > 0$

$$\|S^{\frac{1}{2}}v_\varepsilon\|_{L^2(\Sigma)} \leq \frac{\tilde{C}}{\alpha}\|u\|_{H_{\text{br}}^1(\Sigma)}. \quad (27)$$

Combining (26) to (27) yields the uniform upper bound

$$\sum_{n \geq 0} \frac{1}{\lambda_n + \varepsilon} |(u, u_n)_{L^2(\Sigma)}|^2 \leq C^2 \|u\|_{H_{\text{br}}^1(\Sigma)}^2, \quad \forall \varepsilon > 0,$$

where  $C^2 := \tilde{C}/\alpha$ . Taking  $\varepsilon \rightarrow 0^+$ , one obtains that  $u \in H_T^1(\Sigma)$  with the explicit bound  $\|u\| \leq C\|u\|_{H_{\text{br}}^1(\Sigma)}$ .  $\square$

## 2.4 Application to the stability of DDM

The interest of the norm  $\|\cdot\|$  is highlighted in the next result for functions which solve the homogeneous  $f = 0$  Helmholtz equation in all subdomains.

**Lemma 11.** *Let  $u_i \in H^1(\Omega_i)$  for  $1 \leq i \leq N$  be a collection of solutions to the homogeneous Helmholtz equation in each subdomain. Assume  $\partial_{\mathbf{n}}u_\Sigma \in H_{\text{br}}^1(\Sigma)$ . The following formal equality holds*

$$\|\partial_{\mathbf{n}}u_\Sigma - \mathbf{i}\omega T u_\Sigma\|^2 + \|\partial_{\mathbf{n}}u_\Gamma - \mathbf{i}\omega u_\Gamma\|_{L^2(\Gamma)}^2 = \|\partial_{\mathbf{n}}u_\Sigma + \mathbf{i}\omega T^* u_\Sigma\|^2 + \|\partial_{\mathbf{n}}u_\Gamma + \mathbf{i}\omega u_\Gamma\|_{L^2(\Gamma)}^2.$$

*Proof.* Define  $\Delta$  as the subtraction of the right hand side to the left hand side. It holds

$$\begin{aligned} \Delta &= \|\omega T u_\Sigma\|^2 - \|\omega T^* u_\Sigma\|^2 - 2 \operatorname{Re} \langle \partial_{\mathbf{n}}u_\Sigma, \mathbf{i}\omega(T + T^*)u_\Sigma \rangle - 4 \operatorname{Re} (\partial_{\mathbf{n}}u_\Gamma, \mathbf{i}\omega u_\Gamma)_{L^2(\Gamma)} \\ &= \omega^2 (\langle T u_\Sigma, T u_\Sigma \rangle - \langle T^* u_\Sigma, T^* u_\Sigma \rangle) - 2 \operatorname{Re} \langle \partial_{\mathbf{n}}u_\Sigma, \mathbf{i}\omega(T + T^*)u_\Sigma \rangle - 4 \operatorname{Re} (\partial_{\mathbf{n}}u_\Gamma, \mathbf{i}\omega u_\Gamma)_{L^2(\Gamma)} \\ &= \omega^2 \left( \left( \left( \frac{T + T^*}{2} \right)^{-1} T u_\Sigma, T u_\Sigma \right)_{L^2(\Sigma)} - \left( \left( \frac{T + T^*}{2} \right)^{-1} T^* u_\Sigma, T^* u_\Sigma \right)_{L^2(\Sigma)} \right) \\ &\quad - 2 \operatorname{Re} \left( \left( \frac{T + T^*}{2} \right)^{-1} \partial_{\mathbf{n}}u_\Sigma, \mathbf{i}\omega(T + T^*)u_\Sigma \right)_{L^2(\Sigma)} - 4 \operatorname{Re} (\partial_{\mathbf{n}}u_\Gamma, \mathbf{i}\omega u_\Gamma)_{L^2(\Gamma)} \\ &= 2\omega^2 (\langle (T^*(T + T^*)^{-1}T - T(T + T^*)^{-1}T^*)u_\Sigma, u_\Sigma \rangle)_{L^2(\Sigma)} \\ &\quad - 4 \operatorname{Re} (\partial_{\mathbf{n}}u_\Sigma, \mathbf{i}\omega u_\Sigma)_{L^2(\Sigma)} - 4 \operatorname{Re} (\partial_{\mathbf{n}}u_\Gamma, \mathbf{i}\omega u_\Gamma)_{L^2(\Gamma)}. \end{aligned}$$

One also has the identity

$$\begin{aligned} T^*(T + T^*)^{-1}T - T(T + T^*)^{-1}T^* &= (T^* + T)(T + T^*)^{-1}T - T(T + T^*)^{-1}(T + T^*) \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} \Delta &= -4 \operatorname{Re} \left( (\partial_{\mathbf{n}}u_\Sigma, \mathbf{i}\omega u_\Sigma)_{L^2(\Sigma)} + (\partial_{\mathbf{n}}u_\Gamma, \mathbf{i}\omega u_\Gamma)_{L^2(\Gamma)} \right) \\ &= -4 \operatorname{Re} \left( \mathbf{i}\omega \sum_i \int_{\partial\Omega_i} \partial_{\mathbf{n}_i} u_i \bar{u}_i \right) \\ &= -4 \operatorname{Re} \left( \mathbf{i}\omega \sum_i \int_{\Omega_i} (|\nabla u_i|^2 - \omega^2 |u_i|^2) \right) = 0. \end{aligned}$$

$\square$

Next we consider the iterative solution of the abstract DDM (8), under a fundamental compatibility assumption on  $T$ .

**Definition 12.** We say that operator  $T$  is compatible if  $\Pi T \Pi = T^*$ , or equivalently if  $T \Pi = T^* \Pi$ .

Under the compatibility assumption, DDM (8) can be rewritten as

$$\begin{cases} (\Delta + \omega^2) u_i^{p+1} = f & \text{in } \Omega_i \text{ for all } i, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^{p+1} = -\Pi (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p & \text{on } \Sigma, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega) u_{\Gamma}^{p+1} = 0 & \text{on } \Gamma. \end{cases} \quad (28)$$

The importance of the compatibility condition is evidenced with the energy

$$E^p := \|\|(\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p\|\|^2. \quad (29)$$

**Proposition 13.** Assume that  $T$  verifies the compatibility condition from Definition 12, and that at each stage of the algorithm (28)  $u^p \in \oplus_i H^1(\Omega_i)$  and  $\partial_{\mathbf{n}} u_{\Sigma}^p \in H_{\text{br}}^1(\Sigma)$ . Then, algorithm (28) is stable in the sense that its energy is decreasing:  $E^{p+1} = E^p - 4\omega^2 \|u^p\|_{L^2(\Gamma)}^2$ .

*Proof.* By definition of the DDM (28) and of the energy (29) one has

$$\begin{aligned} E^{p+1} &= \left( \left( \frac{T + T^*}{2} \right)^{-1} (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^{p+1}, (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^{p+1} \right)_{L^2(\Sigma)} \\ &= \left( \left( \frac{T + T^*}{2} \right)^{-1} \Pi (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p, \Pi (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p \right)_{L^2(\Sigma)}. \end{aligned}$$

One also has  $\Pi(T + T^*)\Pi = T + T^*$ , so that  $\Pi(T + T^*)^{-1}\Pi = (T + T^*)^{-1}$ , hence

$$E^{p+1} = \left( \left( \frac{T + T^*}{2} \right)^{-1} (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p, (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p \right)_{L^2(\Sigma)} = \|\|(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p\|\|^2.$$

Applying Lemma 11 gives

$$\begin{aligned} E^{p+1} &= E^p + \|\partial_{\mathbf{n}} u_{\Gamma}^p - \mathbf{i}\omega u_{\Gamma}^p\|_{L^2(\Gamma)}^2 - \|\partial_{\mathbf{n}} u_{\Gamma}^p + \mathbf{i}\omega u_{\Gamma}^p\|_{L^2(\Gamma)}^2 \\ &= E^p - 4\omega^2 \|u^p\|_{L^2(\Gamma)}^2, \end{aligned}$$

and the result is proven.  $\square$

It now remains to identify sesquilinear forms  $a$  and cross-point matrices  $H_r$  such that the compatibility condition on the associated operator  $T$  from Definition 12 is satisfied.

**Lemma 14.** Assume  $a^*(\varphi, \psi) = a(\Pi\varphi, \Pi\psi)$  for all  $\varphi, \psi \in H_{\text{br}}^1(\Sigma)$ . Then  $\Pi T \Pi = T^*$ .

*Proof.* Let  $v \in L^2(\Sigma)$  and  $\varphi = Tv$ , and let  $\psi \in H_{\text{br}}^1(\Sigma)$ . One has  $a(\varphi, \psi) = (v, \psi)$ , therefore the hypothesis implies

$$a(\Pi(\Pi\varphi), \Pi(\Pi\psi)) = (v, \psi) \implies a^*(\Pi\varphi, \Pi\psi) = (v, \psi).$$

Equivalently, one has  $a^*(\Pi\varphi, \psi) = (v, \Pi\psi)$ , which in turn implies that  $a^*(\Pi\varphi, \psi) = (\Pi v, \psi)$ . It follows that  $\Pi T v = \Pi\varphi = T^* \Pi v$  for any  $v \in L^2(\Sigma)$ , so that  $\Pi T \Pi = T^*$ .  $\square$

Let  $\mathbf{x}_r$  be a cross-point. We define the local indices  $i_1, \dots, i_{d_r}$  corresponding to the counter-clockwise count of the surrounding subdomains, where  $i_1 = \min_{1 \leq n \leq d_r} \{i_n\}$ . For practical reasons, we choose to order the elements of  $\varphi_r \simeq \partial_{\mathbf{n}} u_{\Sigma}(\mathbf{x}_r)$  as follows:

$$\begin{aligned} \varphi_r &= \left( (\varphi_{i_n, i_{n-1}}(\mathbf{x}_r))_{n=1}^{d_r}, (\varphi_{i_{n-1}, i_n}(\mathbf{x}_r))_{n=1}^{d_r} \right) \\ &\simeq \left( (\partial_{\mathbf{n}^{i_n}} u_{i_n, i_{n-1}}(\mathbf{x}_r))_{n=1}^{d_r}, (\partial_{\mathbf{n}^{i_{n-1}}} u_{i_{n-1}, i_n}(\mathbf{x}_r))_{n=1}^{d_r} \right), \end{aligned} \quad (30)$$

where  $i_0 := i_{d_r}$ . An illustration is provided in Figure 3 for three subdomains. Let  $\Pi^r$  be the local exchange operator around the node  $\mathbf{x}_r$ .

**Lemma 15.** *With the convention (30), one has*

$$\Pi^r = \begin{pmatrix} 0 & I_{d_r} \\ I_{d_r} & 0 \end{pmatrix}. \quad (31)$$

*Proof.* Indeed  $\Pi_r$  exchanges  $\varphi_{i_n, i_{n-1}}(\mathbf{x}_r)$  and  $\varphi_{i_{n-1}, i_n}(\mathbf{x}_r)$  for all  $1 \leq n \leq d_r$ .  $\square$

**Lemma 16.** *Assume  $\Pi^r H^r \Pi^r = -H^r$  for all corners  $\mathbf{x}_r$ . Then  $\Pi T \Pi = T^*$ .*

*Proof.* Let  $\varphi, \psi \in H_{\text{br}}^1(\Sigma)$ . One has

$$\begin{aligned} a(\Pi\varphi, \Pi\psi) &= \sum_{i,j} \int_{\Sigma_{ij}} \left( \varphi_{ij} \overline{\psi_{ij}} + \frac{1}{2\omega^2} \partial_{\mathbf{t}_i} \varphi_{ij} \overline{\partial_{\mathbf{t}_i} \psi_{ij}} \right) d\gamma + \mathbf{i} \sum_{r=1}^{N_X} (H^r \Pi^r \varphi_r, \Pi^r \psi_r)_{\mathbb{C}^{2d_r}} \\ &= a^*(\varphi, \psi) + \mathbf{i} \sum_{r=1}^{N_X} ((H^r + \Pi^r H^r \Pi^r) \varphi_r, \psi_r)_{\mathbb{C}^{2d_r}}. \end{aligned}$$

The last term vanishes due to the assumption, and Lemma 14 yields the claim.  $\square$

### 3 Convergence of the under-relaxed DDM

In this Section we assume that the compatibility condition from Definition 12 is verified by  $T$ . We introduce a parameter of under relaxation  $0 < \alpha \leq 1$ . The DDM writes

$$\begin{cases} (\Delta + \omega^2) u^{p+1} &= f & \text{in } \Omega_i \text{ for all } i, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^{p+1} &= -\alpha \Pi (\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p + (1 - \alpha) (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p & \text{on } \Sigma, \\ (\partial_{\mathbf{n}} - \mathbf{i}\omega) u_{\Gamma}^{p+1} &= 0 & \text{on } \Gamma. \end{cases} \quad (32)$$

For  $\alpha = 1$  we recover DDM (28), but in this Section we will prove the convergence of DDM (32) for  $\alpha < 1$ .

**Lemma 17.** *Let  $u_i^0 \in H^1(\Omega_i)$  for all  $1 \leq i \leq N$ , and  $\partial_{\mathbf{n}} u_{\Sigma}^0 \in H_{\text{br}}^1(\Sigma)$ . Then, the homogeneous relaxed DDM (32) with  $f = 0$  is well defined at all stages  $p \in \mathbb{N}$ , with  $u_i^p \in H^1(\Omega_i)$  for all  $1 \leq i \leq N$  and  $\partial_{\mathbf{n}} u_{\Sigma}^p \in H_{\text{br}}^1(\Sigma)$ . Moreover, there exists  $C > 0$  such that the following uniform in  $p$  estimate holds*

$$\sum_{i=1}^N \|u_i^p\|_{H^1(\Omega_i)} + \|\partial_{\mathbf{n}} u_{\Sigma}^p\|_{H_{\text{br}}^1(\Sigma)} \leq C \|\partial_{\mathbf{n}} u_{\Sigma}^p - \mathbf{i}\omega T u_{\Sigma}^p\|. \quad (33)$$



*Proof.* We start by proving that  $u^1$  can indeed be calculated from  $u^0$ . It is the solution to the weak formulation in the space  $\oplus_{i=1}^N H^1(\Omega_i)$

$$\left\{ \begin{array}{l} \text{Find } u^1 = (u_i^1)_{i=1}^N \in \oplus_i H^1(\Omega_i) \text{ such that for all } v = (v_i)_{i=1}^N \in \oplus_i H^1(\Omega_i), \\ \sum_{i=1}^N \left( \int_{\Omega_i} (\nabla u_i^1 \cdot \overline{\nabla v_i} - \omega^2 u_i^1 \overline{v_i}) - \mathbf{i}\omega \int_{\partial\Omega_i \cap \Sigma} (Tu^1)_i \overline{v_i} - \mathbf{i}\omega \int_{\partial\Omega_i \cap \Gamma} u_i^1 \overline{v_i} \right) = \sum_{i=1}^N \int_{\partial\Omega_i \cap \Sigma} g_i \overline{v_i}, \end{array} \right. \quad (34)$$

where the right hand side  $g \in H_{\text{br}}^1(\Sigma)$  is defined in each  $\partial\Omega_i \cap \Sigma$  by

$$g_i := (-\alpha \Pi(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^0 + (1 - \alpha)(\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^0), \quad 1 \leq i \leq N.$$

The sesquilinear form (34) is bi-continuous in the space  $\oplus_{i=1}^N H^1(\Omega_i)$  and can be easily decomposed into a coercive part plus a compact part. So according to Fredholm's alternative [5], uniqueness of the solution is equivalent to the well-posedness of this variational problem.

Uniqueness of the solution is a consequence of the positivity of  $T + T^*$ . If  $g_i = 0$  for all  $i$ , then by taking  $v = u^1$ , one gets  $\text{Re}(Tu^1, u^1)_{L^2(\Sigma)} = -\|u^1\|_{L^2(\Gamma)}^2$ . Since  $\text{Re}(Tu^1, u^1)_{L^2(\Sigma)} = \text{Re}(T^*u^1, u^1)_{L^2(\Sigma)}$ , it follows

$$\text{Re} \left( \frac{(T + T^*)}{2} u^1, u^1 \right)_{L^2(\Sigma)} = -\|u^1\|_{L^2(\Gamma)}^2,$$

and thus  $u_{\Sigma}^1 = 0$  and  $u_{\Gamma}^1 = 0$ . Therefore the normal derivative along  $\Sigma$  (resp.  $\Gamma$ ) also vanishes as  $\partial_{\mathbf{n}} u_{\Sigma}^1 = \mathbf{i}\omega T u_{\Sigma}^1 + g = 0$  (resp.  $\partial_{\mathbf{n}} u_{\Gamma}^1 = \mathbf{i}\omega u_{\Gamma}^1 = 0$ ). Finally, a unique continuation principle yields that  $u^1$  vanishes as a function in  $\oplus_i H^1(\Omega_i)$ .

We now prove the inequality (33). The problem being well-posed for any  $g \in H_{\text{br}}^1(\Sigma)$ , the open mapping theorem [5] states that there exists a constant  $C > 0$  such that

$$\sum_{i=1}^N \|u_i^1\|_{H^1(\Omega_i)} \leq C \|g\|_{H_{\text{br}}^1(\Sigma)}.$$

Combined to the continuity of the operator  $T$  and of the Dirichlet trace, it yields  $\|Tu_{\Sigma}^1\|_{H_{\text{br}}^1(\Sigma)} \leq C' \|g\|_{H_{\text{br}}^1(\Sigma)}$ , for a given  $C' > 0$ , and finally, since  $\partial_{\mathbf{n}} u_{\Sigma}^1 = \mathbf{i}\omega T u_{\Sigma}^1 + g$ ,

$$\|\partial_{\mathbf{n}} u_{\Sigma}^1\|_{H_{\text{br}}^1(\Sigma)} \leq C'' \|g\|.$$

According to the equivalence of  $\|g\|_{H_{\text{br}}^1(\Sigma)}$  and  $\|g\|$  stated in Theorem 7, the inequality (33) is obtained for  $p = 1$ .

By iteration on  $p \geq 1$ , the DDM is well posed at all stages, and the iterates verify (33).  $\square$

**Lemma 18.** *Consider the homogeneous relaxed DDM (32). Then the energy  $E^p$  (29) decreases*

$$E^{p+1} \leq E^p - 4\alpha\omega^2 \|u\|_{L^2(\Gamma)}^2 - \alpha(1 - \alpha) \left\| \Pi(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p + (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p \right\|^2, \quad \forall p \geq 1. \quad (35)$$

*Proof.* For  $p \geq 1$  and  $u^p$  defined by DDM (32), let  $v^p \in \oplus_i H^1(\Omega_i)$  be the solution of the non-relaxed DDM (28). One has that  $u^{p+1} = \alpha v^p + (1 - \alpha)u^p$ . Therefore,

$$\begin{aligned} E^{p+1} &= \left\| (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^{p+1} \right\|^2 \\ &= \alpha \left\| (\partial_{\mathbf{n}} - \mathbf{i}\omega T) v_{\Sigma}^p \right\|^2 + (1 - \alpha) \left\| (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p \right\|^2 \\ &\quad - \alpha(1 - \alpha) \left\| (\partial_{\mathbf{n}} - \mathbf{i}\omega T) v_{\Sigma}^p - (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p \right\|^2. \end{aligned}$$

Proposition 13 applies to  $v^p$  and yields  $\|(\partial_{\mathbf{n}} - \mathbf{i}\omega T) v_{\Sigma}^p\|^2 = E^p - 4\omega^2 \|u^p\|_{L^2(\Gamma)}$ . Thus

$$E^{p+1} = \alpha (E^p - 4\omega^2 \|u^p\|_{L^2(\Gamma)}) + (1 - \alpha)E_p - \alpha(1 - \alpha) \|\Pi(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p - (\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p\|$$

which yields the claim.  $\square$

Iterating the identity (35) yields the a priori estimates

$$E^p \leq E^1 \text{ for all } p \geq 1, \quad (36)$$

$$4\alpha\omega^2 \sum_{p \geq 1} \|u^p\|_{L^2(\Gamma)}^2 \leq E^1, \quad (37)$$

$$\alpha(1 - \alpha) \sum_{p \geq 1} \|(\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p + \Pi(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p\|^2 \leq E^1. \quad (38)$$

The inequality (37) implies that  $\|u^p\|_{L^2(\Gamma)} \xrightarrow{p \rightarrow \infty} 0$ . Assuming that the under relaxation parameter is active  $0 < \alpha < 1$ , (38) implies that  $\|(\partial_{\mathbf{n}} - \mathbf{i}\omega T) u_{\Sigma}^p + \Pi(\partial_{\mathbf{n}} + \mathbf{i}\omega T^*) u_{\Sigma}^p\| \xrightarrow{p \rightarrow \infty} 0$ .

**Theorem 19.** *Under the assumptions of Lemma 17 and for  $0 < \alpha < 1$ , the solution of the homogeneous relaxed DDM (32) converges weakly to zero:  $u^p \rightharpoonup 0$  in  $\oplus_i H^1(\Omega_i)$ .*

*Proof.* Due to (33), one controls  $\sum_i \|u_i^p\|_{H^1(\Omega_i)} + \|\partial_{\mathbf{n}} u_{\Sigma}^p\|_{H_{\text{br}}^1(\Sigma)}$  by  $E^p$ . Moreover  $E^p \leq E^1$  (36). Therefore, up to the extraction of subsequences,  $u_i^p$  and  $\partial_{\mathbf{n}} u_{\Sigma}^p$  converge weakly towards some given functions  $u_i^* \in H^1(\Omega_i)$  and  $\partial_{\mathbf{n}} u_{\Sigma}^* \in H_{\text{br}}^1(\Sigma)$ . The notation  $\partial_{\mathbf{n}} u_{\Sigma}^*$  for the limit of  $\partial_{\mathbf{n}} u_{\Sigma}^p$  is a priori a slight abuse of notation, that will be dissipated as we will verify that it is equal to the trace of the normal derivative of  $u^*$ .

Since  $u_i^p$  verifies the Helmholtz equation in  $\Omega_i$ , and since  $\partial_{\mathbf{n}} u^p - \mathbf{i}\omega u^p = 0$  on  $\Gamma$ , one has

$$\int_{\Omega_i} \nabla u_i^p \cdot \overline{\nabla v_i} - \omega^2 u_i^p \overline{v_i} = \int_{\partial\Omega_i \cap \Sigma} \partial_{\mathbf{n}_j} u_i^p \overline{v_i} + \mathbf{i}\omega \int_{\partial\Omega_i \cap \Gamma} u_i^p \overline{v_i}, \quad \forall v_i \in H^1(\Omega_i).$$

One has that  $u_{\Gamma}^* = 0$  according to (37), therefore passing to the weak limit as  $p \rightarrow \infty$  yields

$$\int_{\Omega_i} \nabla u_i^* \cdot \overline{\nabla v_i} - \omega^2 u_i^* \overline{v_i} = \int_{\partial\Omega_i \cap \Sigma} \partial_{\mathbf{n}_j} u_i^* \overline{v_i}, \quad \forall v_i \in H^1(\Omega_i).$$

It follows that the weak limit satisfies the homogeneous Helmholtz equation separately in all  $\Omega_i$ , with  $\partial_{\mathbf{n}_i} u_i^* = 0$  on  $\Gamma$ , and  $(\partial_{\mathbf{n}_i} u_{ij}^*)_{ij} = \partial_{\mathbf{n}} u_{\Sigma}^*$ , dissipating the slight abuse of notation for the limit of  $\partial_{\mathbf{n}} u_{\Sigma}^p$ . Inequality (38) yields

$$\partial_{\mathbf{n}} u_{\Sigma}^* - \mathbf{i}\omega T u_{\Sigma}^* + \Pi(\partial_{\mathbf{n}} u_{\Sigma}^* + \mathbf{i}\omega T^* u_{\Sigma}^*) = 0. \quad (39)$$

We now exploit this identity to show the continuity of traces and normal derivatives over  $\Sigma$ , as (39) is equivalent to

$$(I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^* = \mathbf{i}\omega T(I - \Pi)u_{\Sigma}^*. \quad (40)$$

Going back to the definition of the operator  $T$ , one gets

$$a((I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^*, \psi) = \mathbf{i}\omega ((I - \Pi)u_{\Sigma}^*, \psi)_{L^2(\Sigma)}, \quad \forall \psi \in H_{\text{br}}^1(\Sigma).$$

Taking  $\psi = (I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^* \in H_{\text{br}}^1(\Sigma)$  one has

$$\begin{aligned} a((I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^*, (I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^*) &= \mathbf{i}\omega ((I - \Pi)u_{\Sigma}^*, (I + \Pi)\partial_{\mathbf{n}} u_{\Sigma}^*)_{L^2(\Sigma)} \\ &= \mathbf{i}\omega ((I + \Pi)(I - \Pi)u_{\Sigma}^*, \partial_{\mathbf{n}} u_{\Sigma}^*)_{L^2(\Sigma)} \\ &= 0. \end{aligned}$$

The coercivity of the sesquilinear form  $a$  yields the continuity of the Neumann traces  $(I + \Pi)\partial_{\mathbf{n}}u_{\Sigma}^* = 0$ . Coming back to (40), it follows  $\mathbf{i}\omega T(I - \Pi)u_{\Sigma}^* = 0$ . The positivity of  $T + T^*$  yields the continuity of the Dirichlet traces  $(I - \Pi)u_{\Sigma}^* = 0$ .

In summary,  $u^*$  satisfies the homogeneous Helmholtz equation in all subdomains  $\Omega_i$ , with a homogeneous boundary condition on  $\Gamma$ , and with the continuity of its trace and normal derivative trace over  $\Sigma$ . Therefore  $u^* = 0$ . By unicity of the weak limit, the whole sequence  $u^p$  converges weakly towards  $u^* = 0$ .  $\square$

By compactness one has the strong convergence of the sequence  $u^p \rightarrow 0$  in  $L^2(\Omega_i)$ . Refining the proof, one can get additional properties such as the strong convergence  $(I + \Pi)\partial_{\mathbf{n}}u_{\Sigma}^p \rightarrow 0$  in  $L^2(\Sigma)$ .

## 4 A class of admissible matrices

In this Section, an optimality condition (46) is introduced which expresses that some outgoing plane waves must satisfy natural linear relations at the cross-point. This condition is composed of  $2d_r < d_r^2$  linear equations. After a short digression about corners where  $d_r = 2$  so that  $2d_r = d_r^2$ , we will show that it is always possible to construct admissible cross-point matrices which also verify this condition.

### 4.1 The particular case of corners

In a previous work [12], we defined and studied similar matrices for DDM but for corners and not cross-points. The difference is that a corner shows up at a broken interface between only two subdomains, *i.e.*  $d_r = 2$ , which is a priori excluded in this study. However the same formalism applies, and it is possible to check that the corner matrices from [12] verify the general symmetry and compatibility conditions (41).

At a corner point  $\mathbf{Q}_r$  on the boundary between two subdomains  $\Omega_i$  and  $\Omega_j$ , the interior angle inside  $\Omega_i$  is denoted by  $\theta_r \in (-2\pi, 0)$ , see Figure 2. We derived in (46) the formula

$$(H^r \varphi_r, \varphi_r)_{\mathbb{C}^{2d_r}} = \frac{-1}{4\omega} \left( \cos \frac{\theta_r}{2} |\varphi_{ij}^1(Q_r) + \varphi_{ji}^2(Q_r)|^2 + \frac{\cos \theta_r}{\cos(\theta_r/2)} |\varphi_{ij}^1(Q_r) - \varphi_{ji}^2(Q_r)|^2 \right. \\ \left. - \cos \frac{\theta_r}{2} |\varphi_{ji}^1(Q_r) + \varphi_{ij}^2(Q_r)|^2 - \frac{\cos \theta_r}{\cos(\theta_r/2)} |\varphi_{ji}^1(Q_r) - \varphi_{ij}^2(Q_r)|^2 \right),$$

where the vector  $\varphi_r$  is defined by the values of  $\varphi_{ij}$  and of  $\varphi_{ji}$  along each segment of the broken line (hence the superscript 1 or 2) :

$$\varphi_r = (\varphi_{ij}^1(Q_r), \varphi_{ji}^1(Q_r), \varphi_{ij}^2(Q_r), \varphi_{ji}^2(Q_r)) \in \mathbb{C}^4.$$

The bilinear form has been obtained in [12] by algebraic manipulations in a way that the optimality condition (46) is verified up to an error of order 2 for plane waves which direction are a small variation around the angle bisector. Denoting  $\gamma := \frac{-1}{4\omega} \left( \cos \frac{\theta_r}{2} + \frac{\cos \theta_r}{\cos(\theta_r/2)} \right)$  and  $\delta := \frac{-1}{4\omega} \left( \cos \frac{\theta_r}{2} - \frac{\cos \theta_r}{\cos(\theta_r/2)} \right)$ , the matrix  $H^r$  is

$$H^r = \begin{pmatrix} \gamma & 0 & 0 & \delta \\ 0 & -\gamma & -\delta & 0 \\ 0 & -\delta & -\gamma & 0 \\ \delta & 0 & 0 & \gamma \end{pmatrix}.$$

This matrix satisfies the four conditions of (41).

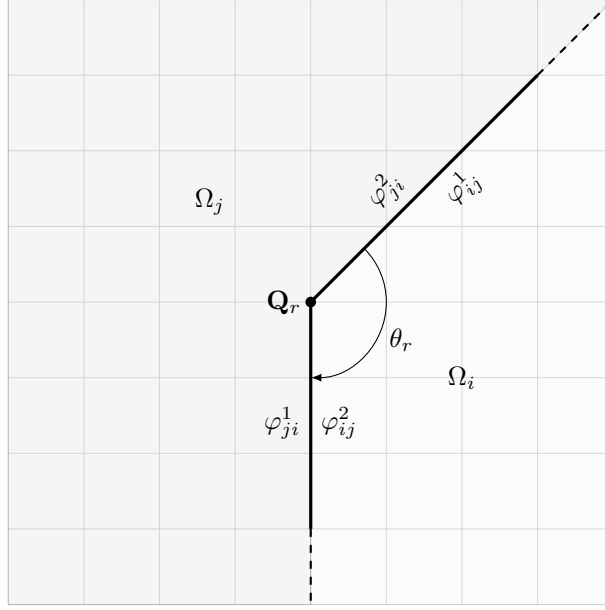


Figure 2: Description of a corner  $\mathbf{Q}_r$  lying on a broken line separating  $\Omega_i$  from  $\Omega_j$ , and corresponding quantities.

## 4.2 The general case

We come back to cross-points where  $d_r \geq 3$  and  $2d_r < d_r^2$ .

### 4.2.1 Local numbering

To describe the situation, we use a *local numbering* around a given node  $\mathbf{x}_r$ . This numbering is displayed in Figure 3 in the case of  $d_r = 3$  subdomains, and has already been defined in (30). It corresponds to the operator  $\Pi^r$  given by (31). The subdomains around the node are reindexed as  $i_1, i_2, \dots, i_{d_r}$  in a counter-clockwise order. For each neighboring subdomain having two incidental edges at  $\mathbf{x}_r$ , the normal and tangential vectors are reindexed using superscript  $\mathbf{n}^n$  and  $\mathbf{t}^n$  for  $1 \leq n \leq 2d_r$  such that, for  $n \leq d_r$ ,  $\mathbf{n}^n = \mathbf{n}_{i_n}|_{\Sigma_{i_n, i_{n-1}}}$  and  $\mathbf{n}^{n+d_r} = -\mathbf{n}_{i_n}|_{\Sigma_{i_n, i_{n-1}}}$ , with the convention  $i_0 = i_{d_r}$ . The same numbering is applied to  $\mathbf{t}^n$  for  $1 \leq n \leq 2d_r$ . The outgoing tangential unit vectors are also reindexed, such that for  $1 \leq n \leq d_r$ ,  $\boldsymbol{\tau}^n = \boldsymbol{\tau}_{i_n, i_{n-1}} = -\mathbf{t}^n(\mathbf{x}_r)$  and  $\boldsymbol{\tau}^{n+d_r} = \boldsymbol{\tau}_{i_n, i_{n+1}} = \mathbf{t}^{n+d_r}(\mathbf{x}_r)$ .

The class of matrices  $H^r$  admissible for our construction is given by the symmetry condition (13) along with the compatibility condition of Lemma 16

$$\mathcal{H}^r := \left\{ H^r \in \mathcal{M}_{2d_r}(\mathbb{R}) \mid H^r = H^{r\top} \text{ and } \Pi^r H^r \Pi^r H^r = -H^r \right\}, \quad 1 \leq r \leq N_X.$$

Because of the chosen structure for  $\varphi_r$ , it is useful to decompose the matrices  $H^r$  into block matrices

$$H^r = \begin{pmatrix} H_{1,1}^r & H_{1,2}^r \\ H_{2,1}^r & H_{2,2}^r \end{pmatrix}.$$

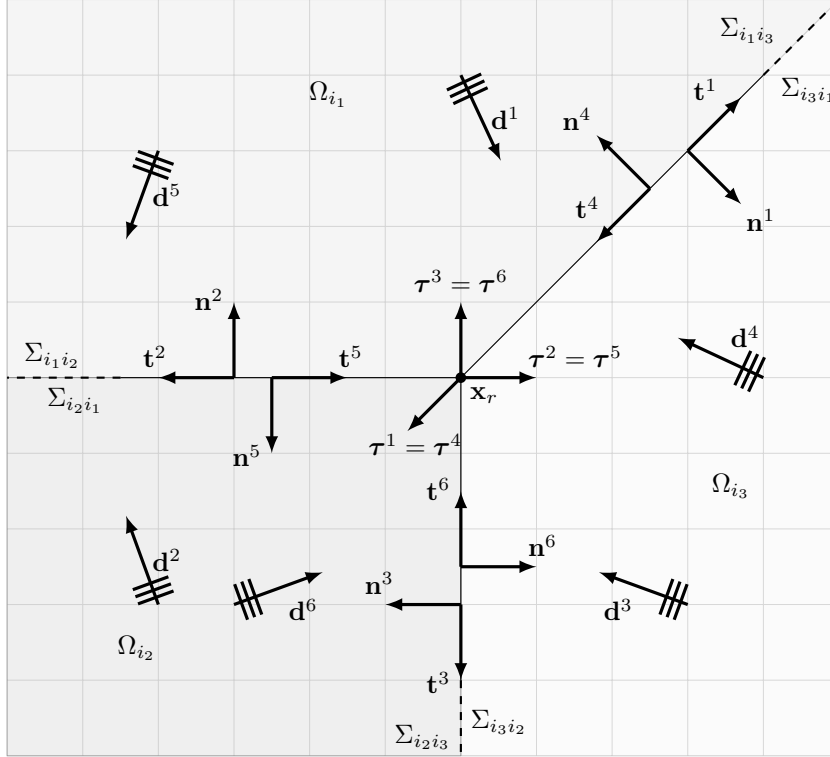


Figure 3: Local indexing around a cross-point (here with 3 subdomains) based on Figure 1. The 6 directions  $\mathbf{d}^n$  have here been drawn only for illustrative purpose and could be different as far as they satisfy (42).

The condition  $H^r \in \mathcal{H}^r$  is equivalent to

$$\begin{aligned} H_{1,1}^r &= H_{1,1}^{r\top}, & H_{1,2}^r &= H_{2,1}^{r\top}, & H_{2,2}^r &= H_{2,2}^{r\top}, & & \text{(symmetry),} \\ H_{2,2}^r &= -H_{1,1}^r & \text{and} & & H_{2,1}^r &= -H_{1,2}^r & & \text{(compatibility).} \end{aligned}$$

These five conditions can be recast into the system of four equations

$$H_{1,1}^r = H_{1,1}^{r\top}, \quad H_{2,2}^r = -H_{1,1}^r, \quad H_{1,2}^r = -H_{1,2}^{r\top}, \quad H_{2,1}^r = -H_{1,2}^r. \quad (41)$$

**Remark 20.** For a given  $r$ , the number of linear constraints to characterize admissible matrices in  $\mathcal{H}^r$  is equal to  $d_r^2$ . Conditions (41) correspond to  $d_r(d_r + 1)/2$  degrees of freedom to define both  $H_{1,1}^r$  and  $H_{2,2}^r = -H_{1,1}^r$ , and to  $(d_r - 1)d_r/2$  degrees of freedom to define both  $H_{1,2}^r$  and  $H_{2,1}^r = -H_{1,2}^{r\top}$ : the sum is equal to the number of linear constraints.

#### 4.2.2 Compatibility condition

Following a standard approach for wave problems, we choose plane waves as family of particular solutions. We consider a collection of unitary directions  $\mathbf{d}^n \in \mathbb{R}^2$  and the associated collection of plane waves defined by

$$u_{\mathbf{d}^n}(\mathbf{x}) = e^{i\omega(\mathbf{d}^n \cdot \mathbf{x} - \mathbf{x}_r)}, \quad 1 \leq n \leq 2d_r.$$

A collection of directions is endowed to a specific node  $\mathbf{x}_r$ , but we chose not to precise it with an  $r$  index for the sake of notational clarity. For the construction proposed below, we will choose independent directions

$$\mathbf{d}^n \text{ and } \mathbf{d}^{n+d_r} \text{ linearly independent for all } 1 \leq n \leq d_r. \quad (42)$$

At the node  $\mathbf{x}_r$ , one has for  $1 \leq n \leq d_r$  and  $1 \leq m \leq 2d_r$

$$\partial_{\mathbf{n}^m} u_{\mathbf{d}^n}(\mathbf{x}_r) = \mathbf{i}\omega(\mathbf{d}^n, \mathbf{n}^m) u_{\mathbf{d}^n}(\mathbf{x}_r) \text{ and } \partial_{\mathbf{n}^m} u_{\mathbf{d}^{n+d_r}}(\mathbf{x}_r) = \mathbf{i}\omega(\mathbf{d}^{n+d_r}, \mathbf{n}^m) u_{\mathbf{d}^n}(\mathbf{x}_r). \quad (43)$$

For  $1 \leq n \leq d_r$ , one has the two relations

$$\begin{aligned} \partial_{\boldsymbol{\tau}^n} \partial_{\mathbf{n}^n} u_{\mathbf{d}^n}(\mathbf{x}_r) &= -\partial_{\mathbf{t}^n} \partial_{\mathbf{n}^n} u_{\mathbf{d}^n}(\mathbf{x}_r) = \omega^2(\mathbf{d}^n, \mathbf{t}^n)(\mathbf{d}^n, \mathbf{n}^n) u_{\mathbf{d}^n}(\mathbf{x}_r), \\ \partial_{\boldsymbol{\tau}^{n+d_r}} \partial_{\mathbf{n}^{n+d_r}} u_{\mathbf{d}^{n+d_r}}(\mathbf{x}_r) &= \partial_{\mathbf{t}^{n+d_r}} \partial_{\mathbf{n}^{n+d_r}} u_{\mathbf{d}^{n+d_r}}(\mathbf{x}_r) = -\omega^2(\mathbf{d}^{n+d_r}, \mathbf{t}^{n+d_r})(\mathbf{d}^{n+d_r}, \mathbf{n}^{n+d_r}) u_{\mathbf{d}^n}(\mathbf{x}_r). \end{aligned} \quad (44)$$

Imposing the Robin coupling conditions (10) for each plane wave  $u_{\mathbf{d}^n}$  on the boundaries of the corresponding domain  $\Omega_{i_n}$  comes down to imposing

$$\begin{cases} \partial_{\boldsymbol{\tau}^n} \partial_{\mathbf{n}^n} u_{\mathbf{d}^n} & + \mathbf{i} \sum_{m=1}^{2d_r} h_{nm} \partial_{\mathbf{n}^m} u_{\mathbf{d}^n} & = 0, & 1 \leq n \leq d_r, \\ \partial_{\boldsymbol{\tau}^{n+d_r}} \partial_{\mathbf{n}^{n+d_r}} u_{\mathbf{d}^{n+d_r}} & + \mathbf{i} \sum_{m=1}^{2d_r} h_{n+d_r, m} \partial_{\mathbf{n}^m} u_{\mathbf{d}^{n+d_r}} & = 0, & 1 \leq n \leq d_r. \end{cases}$$

for  $h_{nm} := -\mathbf{i}\alpha_{nm}$ . Using the formulas (43)-(44) one gets

$$\begin{cases} \omega(\mathbf{d}^n, \mathbf{t}^n)(\mathbf{d}^n, \mathbf{n}^n) & - \sum_{m=1}^{2d_r} h_{nm}(\mathbf{d}^n, \mathbf{n}^m) & = 0, & 1 \leq n \leq d_r, \\ -\omega(\mathbf{d}^{n+d_r}, \mathbf{t}^{n+d_r})(\mathbf{d}^{n+d_r}, \mathbf{n}^{n+d_r}) & - \sum_{m=1}^{2d_r} h_{n+d_r, m}(\mathbf{d}^{n+d_r}, \mathbf{n}^m) & = 0, & 1 \leq n \leq d_r. \end{cases} \quad (45)$$

For mathematical convenience, this system (45) is rewritten in matrix form in real algebra

$$\mathbf{diag}(H^r E^{\text{pw}}) = F^{\text{pw}}, \quad (46)$$

where  $H^r = (h_{nm})_{1 \leq n, m \leq 2d_r}$  is the unknown,  $E^{\text{pw}} := (E_{nm}^{\text{pw}})_{1 \leq n, m \leq 2d_r}$  where

$$E_{nm}^{\text{pw}} := \frac{1}{\mathbf{i}\omega} \partial_{\mathbf{n}^n} u_{\mathbf{d}^m}(\mathbf{x}_r) = (\mathbf{d}^m, \mathbf{n}^n), \quad (47)$$

and  $\mathbf{diag}(H^r E^{\text{pw}}) = ((H^r E^{\text{pw}})_{nn})_{1 \leq n \leq 2d_r}$  is the vector of the diagonal entries of  $H^r E^{\text{pw}}$ . The right hand side  $F^{\text{pw}}$  contains the first elements of each line in (45). The system (46) can be considered as an optimality condition on the unknown coefficients of  $H^r$ . We will prove in the next paragraph that this linear system of  $2d_r$  equations for  $4d_r^2$  unknowns is solvable in real algebra under the conditions required for stability.

### 4.2.3 Existence of a class of admissible matrices

The objective is to show that equation (46) can be solved with  $H^r \in \mathcal{H}^r$ . We rewrite this in a different form by introducing the linear operator  $\mathcal{L}_r : \mathcal{M}_{2d_r}(\mathbb{R}) \rightarrow \mathbb{R}^{2d_r} \times \mathcal{M}_{2d_r}(\mathbb{R}) \times \mathcal{M}_{2d_r}(\mathbb{R})$  defined by

$$\mathcal{L}_r H := \begin{pmatrix} \mathbf{diag}(H E^{\text{pw}}) \\ H - H^\top \\ \Pi^r H + H^\top \Pi^r \end{pmatrix}, \quad \forall H \in \mathcal{M}_{2d_r}(\mathbb{R}).$$

Proving that (46) is solvable for  $H^r \in \mathcal{H}^r$  is equivalent to showing that there is a solution  $H^r \in \mathcal{M}_{2d_r}(\mathbb{R})$  to

$$\mathcal{L}_r H = \mathbf{g}_r^{\text{pw}}, \quad (48)$$

where  $\mathbf{g}_r^{\text{pw}} := (F^{\text{pw}}, 0, 0)$ . We denote the matrix scalar product  $A:B$  of  $A, B \in \mathcal{M}_{2d_r}(\mathbb{R})$  as the contraction  $A:B := \text{trace}(AB^\top)$ . We also denote the natural scalar product in the space  $\mathbb{R}^{2d_r} \times \mathcal{M}_{2d_r}(\mathbb{R}) \times \mathcal{M}_{2d_r}(\mathbb{R})$ , as  $\langle \cdot, \cdot \rangle$ . With these notations one has

$$\langle \mathcal{L}_r H, (\mathbf{a}, B, C) \rangle = (\mathbf{diag}(HE^{\text{pw}}), \mathbf{a}) + (H - H^\top):B + (\Pi^r H + H^\top \Pi^r):C.$$

The linear system has a solution if and only if  $\mathbf{g}_r^{\text{pw}} \in \text{range}(\mathcal{L}_r)$ . We use the closed range theorem [5] which states that  $\text{range}(\mathcal{L}_r) = \ker(\mathcal{L}_r^\top)^\perp$ , and show in what follows that  $\mathbf{g}_r^{\text{pw}} \in \ker(\mathcal{L}_r^\top)^\perp$ .

**Lemma 21.** *The adjoint linear operator  $\mathcal{L}_r^\top: \mathbb{R}^{2d_r} \times \mathcal{M}_{2d_r}(\mathbb{R}) \times \mathcal{M}_{2d_r}(\mathbb{R}) \rightarrow \mathcal{M}_{2d_r}(\mathbb{R})$  is defined for all  $(\mathbf{a}, B, C) \in \mathbb{R}^{2d_r} \times \mathcal{M}_{2d_r}(\mathbb{R}) \times \mathcal{M}_{2d_r}(\mathbb{R})$  by*

$$\mathcal{L}_r^\top(\mathbf{a}, B, C) := \mathbf{diag}(\mathbf{a})E^{\text{pw}\top} + B - B^\top + \Pi^r C + \Pi^r C^\top,$$

where the diagonal matrix  $\mathbf{diag}(\mathbf{a}) \in \mathcal{M}_{2d_r}(\mathbb{R})$  has for diagonal the vector  $\mathbf{a}$ .

*Proof.* Let  $H, B, C \in \mathcal{M}_{2d_r}(\mathbb{R})$  and  $\mathbf{a} = (\mathbf{a}_n)_{n=1}^{2d_r} \in \mathbb{R}^{2d_r}$ . We want to show that

$$\begin{aligned} \langle \mathcal{L}_r H, (\mathbf{a}, B, C) \rangle &= (\mathbf{diag}(HE^{\text{pw}}), \mathbf{a}) + (H - H^\top):B + (\Pi^r H + H^\top \Pi^r):C \\ &= H: \left( \mathbf{diag}(\mathbf{a})E^{\text{pw}\top} + B - B^\top + \Pi^r C + \Pi^r C^\top \right), \end{aligned}$$

and will prove it one term after the other. For the first term, one has

$$\begin{aligned} (\mathbf{diag}(HE^{\text{pw}}), \mathbf{a}) &= \text{trace}(HE^{\text{pw}} \mathbf{diag}(\mathbf{a})) \\ &= \text{trace} \left( H \left( \mathbf{diag}(\mathbf{a})E^{\text{pw}\top} \right)^\top \right) = H: \left( \mathbf{diag}(\mathbf{a})E^{\text{pw}\top} \right). \end{aligned}$$

Second, it holds that

$$(H - H^\top):B = H:B - H^\top:B = H:B - H:B^\top = H:(B - B^\top).$$

Finally, since  $\Pi^{r\top} = \Pi^r$ , we have

$$\begin{aligned} (\Pi^r H + H^\top \Pi^r):C &= (\Pi^r H):C + (H^\top \Pi^r):C = (H^\top \Pi^r)^\top:C + H:(\Pi^r C^\top) \\ &= H:(\Pi^r C) + H:(\Pi^r C^\top) = H:(\Pi^r C + \Pi^r C^\top). \end{aligned}$$

Regrouping the three relations leads to the claim.  $\square$

**Lemma 22.** *If  $(\mathbf{a}, B, C) \in \ker(\mathcal{L}_r^\top)$  then  $\mathbf{a} = 0$ .*

*Proof.* One has the equivalence

$$\begin{aligned} (\mathbf{a}, B, C) \in \text{Ker}(\mathcal{L}_r^\top) &\iff \mathbf{diag}(\mathbf{a})E^{\text{pw}\top} + B - B^\top + \Pi^r(C + C^\top) = 0 \\ &\iff (\mathbf{diag}(\mathbf{a})E^{\text{pw}\top})^\top + B^\top - B + (C + C^\top)\Pi^r = 0, \end{aligned}$$

Summing these two expressions implies that

$$\mathbf{diag}(\mathbf{a})E^{\text{pw}\top} + (\mathbf{diag}(\mathbf{a})E^{\text{pw}\top})^\top = -\Pi^r(C + C^\top) - (C + C^\top)\Pi^r. \quad (49)$$

Decomposing the symmetric matrix  $C + C^\top$  into blocks, it follows

$$(C + C^\top) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

where  $G_{11} = G_{11}^\top$ ,  $G_{21} = G_{12}^\top$  and  $G_{22} = G_{22}^\top$ . Using the structure of  $\Pi^r$  (31), it follows from (49) that  $\text{diag}(\mathbf{a})E^{\text{pw}\top} + (\text{diag}(\mathbf{a})E^{\text{pw}\top})^\top$  has the following structure

$$\text{diag}(\mathbf{a})E^{\text{pw}\top} + (\text{diag}(\mathbf{a})E^{\text{pw}\top})^\top = \begin{pmatrix} K_1 & K_2 \\ K_2 & K_1 \end{pmatrix} \quad (50)$$

where  $K_1$  and  $K_2$  are symmetric since

$$K_1 = K_1^\top = -G_{12} - G_{21} \text{ and } K_2 = K_2^\top = -G_{11} - G_{22}. \quad (51)$$

On another hand, one has the block decompositions

$$\text{diag}(\mathbf{a})E^{\text{pw}\top} = \begin{pmatrix} (a_n(\mathbf{d}^n, \mathbf{n}^m))_{n,m=1}^{d_r} & (a_n(\mathbf{d}^n, \mathbf{n}^{m+d_r}))_{n,m=1}^{d_r} \\ (a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^m))_{n,m=1}^{d_r} & (a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^{m+d_r}))_{n,m=1}^{d_r} \end{pmatrix}, \quad (52)$$

and

$$(\text{diag}(\mathbf{a})E^{\text{pw}\top})^\top = \begin{pmatrix} (a_m(\mathbf{d}^m, \mathbf{n}^n))_{n,m=1}^{d_r} & (a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^n))_{n,m=1}^{d_r} \\ (a_m(\mathbf{d}^m, \mathbf{n}^{n+d_r}))_{n,m=1}^{d_r} & (a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^{n+d_r}))_{n,m=1}^{d_r} \end{pmatrix}. \quad (53)$$

Combining (50) to (51)-(53), the constraints can be written for all  $1 \leq n, m \leq d_r$  as

$$\begin{aligned} a_n(\mathbf{d}^n, \mathbf{n}^m) + a_m(\mathbf{d}^m, \mathbf{n}^n) &= a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^{m+d_r}) + a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^{n+d_r}), \\ a_n(\mathbf{d}^n, \mathbf{n}^{m+d_r}) + a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^n) &= a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^m) + a_m(\mathbf{d}^m, \mathbf{n}^{n+d_r}). \end{aligned} \quad (54)$$

Due to the numbering (30) illustrated in Figure 3, one has the relations

$$\mathbf{n}^n = -\mathbf{n}^{d_r+n} \quad 1 \leq n \leq d_r.$$

The two lines of (54) thus rewrite

$$\begin{aligned} a_n(\mathbf{d}^n, \mathbf{n}^m) + a_m(\mathbf{d}^m, \mathbf{n}^n) &= -a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^m) - a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^n), \\ -a_n(\mathbf{d}^n, \mathbf{n}^m) + a_{m+d_r}(\mathbf{d}^{m+d_r}, \mathbf{n}^n) &= a_{n+d_r}(\mathbf{d}^{n+d_r}, \mathbf{n}^m) - a_m(\mathbf{d}^m, \mathbf{n}^n), \end{aligned}$$

and adding them gives for all  $1 \leq n, m \leq d_r$

$$(a_m \mathbf{d}^m + a_{m+d_r} \mathbf{d}^{m+d_r}, \mathbf{n}^n) = -(a_{m+d_r} \mathbf{d}^{m+d_r} + a_m \mathbf{d}^m, \mathbf{n}^n),$$

which in turn implies that  $a_m \mathbf{d}^m + a_{m+d_r} \mathbf{d}^{m+d_r} = 0$  for all  $1 \leq m \leq d_r$  since the  $\mathbf{n}^n$  span  $\mathbb{R}^2$ . Finally, using the hypothesis (42), this linear combination can only be zero for  $a_m = a_{m+d_r} = 0$ , which ends the proof.  $\square$

**Proposition 23.** *At a given node  $\mathbf{x}_r$ , under the conditions of Lemma 22, there exists an admissible matrix  $H^r$  solution to the linear system  $\mathcal{L}_r H^r = \mathbf{g}_r^{\text{pw}}$ . The solution is non unique for  $d_r \geq 3$ .*

*Proof.* The previous Lemma yields that  $\ker(\mathcal{L}_r^\top) \subset \{0\} \times \mathcal{M}_{2d_r}(\mathbb{R}) \times \mathcal{M}_{2d_r}(\mathbb{R})$ . So the right hand side  $\mathbf{g}_r^{\text{pw}} = (\mathbf{b}_r^{\text{pw}}, 0, 0)$  is orthogonal to  $\ker(\mathcal{L}_r^\top)$ , which implies that there always exists an admissible matrix  $H^r$  solution to the linear system  $\mathcal{L}_r H^r = \mathbf{g}_r^{\text{pw}}$ .

For  $d_r \geq 3$ , the space  $\mathcal{H}^r$ , which corresponds to the space of matrices such that  $\mathcal{L}_r H = (\mathbf{a}, 0, 0)$  for any  $\mathbf{a} \in \mathbb{R}^{2d_r}$  has dimension  $d_r^2$ , as stated in Remark 20. On the other hand, the number of linear equations imposed by taking  $\mathbf{a} = 0$  is  $2d_r$ , see (46). It implies there exists a linear vector subspace of dimension  $d_r^2 - 2d_r > 0$  which yields the right hand term  $\mathbf{g}_r^{\text{pw}}$ .  $\square$



## 5 Conclusion

We have developed a framework for the construction of convergent DDMs with transmission operators  $T$  which satisfy  $T \neq T^*$ . Nevertheless, many problems remain open at the end of this study. A non exhaustive list follows which can be the topic of future researches.

- With this approach, it is not guaranteed that the algorithms will converge faster. A sound study and numerical tests would be necessary.
- The transmission operators  $T$  do not satisfy the matching requirements in terms of functional analysis of the multi-trace theory [7, 9]. The question remains to know is it would be possible to construct operators  $T$  such that  $T \neq T^*$  that also realize the multi-trace formalisms bijections.
- A solution verifying the compatibility relations was obtained in the last Section, under the requirement that  $\mathbf{d}^n$  and  $\mathbf{d}^{n+d_r}$  are independent directions. It seems natural to ask whether taking one direction per edge  $\mathbf{d}^n = \pm \mathbf{d}^{n+d_r}$  would also be suitable.
- The multidimensional extension is of course of interest and challenging.
- The Maxwell extension is also challenging, especially working with the functional spaces  $H_{\text{div}}$  and  $H_{\text{rot}}$ .
- Heterogeneous media also raises new questions: how should the transmission operator be modified? How will the convergence be affected?

## References

- [1] J.-D. Benamou and B. Desprès. A domain decomposition method for the helmholtz equation and related optimal control problems. *Journal of Computational Physics*, 136(1):68 – 82, 1997.
- [2] A. Bendali and Y. Boubendir. Non-overlapping domain decomposition method for a nodal finite element method. *Numer. Math.*, 103(4):515–537, 2006.
- [3] Y. Boubendir, X. Antoine, and C. Geuzaine. A quasi-optimal non-overlapping domain decomposition algorithm for the Helmholtz equation. *J. Comput. Phys.*, 231(2):262–280, 2012.
- [4] Y. Boubendir, A. Bendali, and M. B. Fares. Coupling of a non-overlapping domain decomposition method for a nodal finite element method with a boundary element method. *Internat. J. Numer. Methods Engrg.*, 73(11):1624–1650, 2008.
- [5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [6] C. Chniti, F. Nataf, and F. Nier. Improved interface conditions for 2D domain decomposition with corners: numerical applications. *J. Sci. Comput.*, 38(2):207–228, 2009.
- [7] X. Claeys. A new variant of the optimised schwarz method for arbitrary non-overlapping subdomain partitions, 2019.

- [8] X. Claeys, F. Collino, P. Joly, and E. Parolin. A discrete domain decomposition method for acoustics with uniform exponential rate of convergence using non-local impedance operators. In Springer-Verlag, editor, *Domain Decomposition Methods in Science and Engineering XXV*, LNSCE, 2020.
- [9] X. Claeys and E. Parolin. Robust treatment of cross points in optimized schwarz methods. <https://arxiv.org/abs/2003.06657>, 2020.
- [10] B. Després. Décomposition de domaine et problème de Helmholtz. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(6):313–316, 1990.
- [11] B. Després. Domain decomposition method and the Helmholtz problem. In *Mathematical and numerical aspects of wave propagation phenomena (Strasbourg, 1991)*, pages 44–52. SIAM, Philadelphia, PA, 1991.
- [12] B. Després, A. Nicolopoulos, and B. Thierry. Corners and stable optimized domain decomposition methods for the helmholtz problem. <https://hal.archives-ouvertes.fr/hal-02612368/document>, 2020.
- [13] C. Farhat, P. Avery, R. Tezaur, and J. Li. FETI-DPH: a dual-primal domain decomposition method for acoustic scattering. *J. Comput. Acoust.*, 13(3):499–524, 2005.
- [14] M. J. Gander and F. Kwok. Best Robin parameters for optimized Schwarz methods at cross points. *SIAM J. Sci. Comput.*, 34(4):A1849–A1879, 2012.
- [15] M. J. Gander and F. Kwok. On the applicability of Lions’ energy estimates in the analysis of discrete optimized Schwarz methods with cross points. In *Domain decomposition methods in science and engineering XX*, volume 91 of *Lect. Notes Comput. Sci. Eng.*, pages 475–483. Springer, Heidelberg, 2013.
- [16] M. J. Gander and K. Santugini. Cross-points in domain decomposition methods with a finite element discretization. *Electron. Trans. Numer. Anal.*, 45:219–240, 2016.
- [17] M. Lecouvez, B. Stupfel, P. Joly, and F. Collino. Quasi-local transmission conditions for non-overlapping domain decomposition methods for the helmholtz equation. *Comptes Rendus Physique*, 15(5):403 – 414, 2014. Electromagnetism / Électromagnétisme.
- [18] S. Loisel. Condition number estimates for the nonoverlapping optimized Schwarz method and the 2-Lagrange multiplier method for general domains and cross points. *SIAM J. Numer. Anal.*, 51(6):3062–3083, 2013.
- [19] A. Modave, X. Antoine, and C. Geuzaine. An efficient domain decomposition method with cross-point treatment for Helmholtz problems. In *CSMA 2019 - 14e Colloque National en Calcul des Structures*, Giens (Var), France, May 2019.
- [20] A. Modave, C. Geuzaine, and X. Antoine. Corner treatments for high-order local absorbing boundary conditions in high-frequency acoustic scattering. *Journal of Computational Physics*, 401:109029, 2020.
- [21] E. Parolin. *Méthodes de décomposition de domaine sans recouvrement avec opérateurs de transmission non-locaux pour des problèmes de propagation d’ondes harmoniques*. PhD thesis, 2020.

- [22] B. Thierry, A. Vion, S. Tournier, M. El Bouajaji, D. Colignon, N. Marsic, X. Antoine, and C. Geuzaine. Getddm: An open framework for testing optimized schwarz methods for time-harmonic wave problems. *Computer Physics Communications*, 203:309 – 330, 2016.