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# REAL TORUS ACTIONS ON REAL AFFINE ALGEBRAIC VARIETIES

PIERRE-ALEXANDRE GILLARD

ABSTRACT. We extend the Altmann-Hausen presentation of normal affine algebraic  $\mathbb{C}$ -varieties endowed with effective torus actions to the real setting. In particular, we focus on actions of quasi-split real tori, in which case we obtain a simpler presentation.

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## INTRODUCTION

In the work of Altmann and Hausen in [2], normal affine algebraic varieties endowed with effective torus actions over an algebraically closed field of characteristic zero are determined by a geometrico-combinatorial datum on a certain rational quotient for the action. This geometrico-combinatorial presentation extends *mutatis mutandis* to actions of real split tori  $\mathbb{G}_{m,\mathbb{R}}^n$  on normal affine algebraic  $\mathbb{R}$ -varieties.

In contrast, for normal  $\mathbb{R}$ -varieties endowed with actions of a non-split torus  $T$ , much less is known regarding the existence of a presentation similar to the split case. However, this presentation was extended by Langlois in [12] for some complexity one<sup>1</sup> non-split torus actions on normal affine varieties  $X$  over an arbitrary field. This extension is based on a Galois descent construction specific to complexity one torus actions. On the other hand, the case where  $T$  is the real circle  $\mathbb{S}^1$  (of dimension 1) was studied by Dubouloz, Liendo and Petitjean in [6, 7]. They gave a complete description of  $\mathbb{S}^1$ -actions on normal affine  $\mathbb{R}$ -varieties based on the Altmann-Hausen presentation and on a Galois descent construction  $\mathbb{C}/\mathbb{R}$  specific to  $\mathbb{S}^1$ -actions, with no restriction on the complexity.

In view of these results, it is natural and reasonable to expect that a general presentation of geometrically normal affine varieties endowed with torus actions over arbitrary fields of characteristic zero can be obtained by combining Altmann-Hausen theory for split torus actions with appropriate Galois descent methods. In this context, we give a complete description of real torus actions on normal  $\mathbb{R}$ -varieties. The Weil restriction  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  of  $\mathbb{G}_{m,\mathbb{C}}$  is a real non-split torus (of dimension 2), and all real tori are isomorphic to a product of the three *elementary* real tori  $\mathbb{G}_{m,\mathbb{R}}$ ,  $\mathbb{S}^1$  and  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ . We treat the missing case of  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ -actions and more generally we extend the setting of Altmann-Hausen to real torus actions on normal affine  $\mathbb{R}$ -varieties. We will pay a special attention to actions of quasi-split tori, that is real tori with no  $\mathbb{S}^1$ -factors.

In view of extending the Altmann-Hausen presentation to the real setting, we use the language of  $\mathbb{R}$ -structures on algebraic  $\mathbb{C}$ -varieties. An  $\mathbb{R}$ -structure on an algebraic  $\mathbb{C}$ -variety  $X$  is an involution of  $\mathbb{R}$ -schemes  $\sigma$  on  $X$  such that the following diagram commutes:

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<sup>1</sup>That is, effective actions of a torus  $T$  such that  $\dim(X) = \dim(T) + 1$

$$\begin{array}{ccc}
X & \xrightarrow{\sigma} & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \bar{z})} & \mathrm{Spec}(\mathbb{C})
\end{array}$$

An  $\mathbb{R}$ -morphism between two  $\mathbb{C}$ -varieties  $X$  and  $X'$  endowed with  $\mathbb{R}$ -structures  $\sigma$  and  $\sigma'$  is a morphism of  $\mathbb{C}$ -varieties  $f : X \rightarrow X'$  such that  $\sigma' \circ f = f \circ \sigma$ . An  $\mathbb{R}$ -group structure  $\tau$  on a complex algebraic group  $G$  is an  $\mathbb{R}$ -structure on  $G$  such that the multiplication  $G \times G \rightarrow G$ , the inverse  $G \rightarrow G$  and the unity  $\mathrm{Spec}(\mathbb{C}) \rightarrow G$  are  $\mathbb{R}$ -morphisms (see §1 for details). Let us note that an  $\mathbb{R}$ -group structure  $\tau$  on a complex torus  $\mathbb{T}$  corresponds to a lattice involution  $\tilde{\tau}$  on its character lattice  $M := \mathrm{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ .

There is an equivalence of categories between the category of quasi-projective algebraic  $\mathbb{R}$ -varieties (resp. real algebraic groups) and the category of quasi-projective algebraic  $\mathbb{C}$ -varieties endowed with an  $\mathbb{R}$ -structure (resp. complex algebraic groups endowed with an  $\mathbb{R}$ -group structure); see Proposition 1.2 for the precise statement. Therefore we will often write  $(X, \sigma)$  to refer to an algebraic  $\mathbb{R}$ -variety and  $(G, \tau)$  to refer to a real algebraic group.

We now briefly explain Altman and Hausen's theory in order to state our main results. Let  $\mathbb{T}$  be an  $n$ -dimensional complex torus with character lattice  $M$ . Then any algebraic action of  $\mathbb{T}$  on an affine  $\mathbb{C}$ -variety  $X$  corresponds to an  $M$ -grading  $\mathbb{C}[X] = \bigoplus_{m \in M} \mathbb{C}[X]_m$  of its coordinate ring, the spaces  $\mathbb{C}[X]_m$  consisting of semi-invariant regular functions of weight  $m$  on  $X$ . Let  $\omega$  be a full dimensional cone in  $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ , let  $Y$  be a normal *semi-projective* variety (see Definition 2.7), and let  $\mathcal{D} := \sum \Delta_i \otimes D_i$  be a *proper polyhedral divisor*. This means that the  $D_i$  are prime divisors on  $Y$  and the coefficients  $\Delta_i$  are convex polyhedra in  $N_{\mathbb{Q}}$  having  $\omega$  as tail cone, where  $N$  is the cocharacter lattice (see Definition 2.4). Then, for every  $m \in \omega \cap M$ , we can evaluate  $\mathcal{D}$  in  $m$  to obtain a Weil  $\mathbb{Q}$ -divisor  $\mathcal{D}(m) := \sum \min\{m|\Delta_i\} \otimes D_i$ . From the datum  $(\omega, Y, \mathcal{D})$ , Altman and Hausen construct an  $M$ -graded  $\mathbb{C}$ -algebra:

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \omega \cap M} H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \subset \mathbb{C}(Y)[M].$$

The main results of [2] can be summarized as follows (see §3 for details):

**Theorem AH 1.** [2, Theorem 3.1]. *The affine scheme  $X[Y, \mathcal{D}] := \mathrm{Spec}(A[Y, \mathcal{D}])$  is a normal  $\mathbb{C}$ -variety endowed with an effective  $\mathbb{T}$ -action.*

**Theorem AH 2.** [2, Theorem 3.4]. *Let  $X$  be an affine normal variety endowed with an effective  $\mathbb{T}$ -action. There exists a datum  $(\omega, Y, \mathcal{D})$  such that the graded  $\mathbb{C}$ -algebras  $\mathbb{C}[X]$  and  $A[Y, \mathcal{D}]$  are isomorphic.*

As mentioned above, the present article focuses on real torus actions on normal affine  $\mathbb{R}$ -varieties. Our main results, Theorem A and Theorem C, give a presentation of real torus actions in the language of [2] extended to affine  $\mathbb{C}$ -varieties with  $\mathbb{R}$ -structures.

Let  $(\mathbb{T}, \tau)$  be a real torus, let  $M$  be the character lattice of  $\mathbb{T}$ , and let  $\omega$  be a full dimensional cone in  $M_{\mathbb{Q}}$ . Let  $(Y, \sigma_Y)$  be a semi-projective algebraic  $\mathbb{R}$ -variety and let  $\mathcal{D}$  be a proper polyhedral divisor on  $Y$ . The first main result gives a condition on  $\mathcal{D}$  for the existence of an  $\mathbb{R}$ -structure on the affine  $\mathbb{C}$ -variety  $X[Y, \mathcal{D}]$ . This result is the real analog of Theorem AH 1:

**Theorem A** (Theorem 3.3). *If there exists a monoid morphism  $h : \omega \cap M \rightarrow \mathbb{C}(Y)^*$  such that*

$$\forall m \in \omega \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \mathrm{div}_Y(h(\tilde{\tau}(m))),$$

*then there exists an  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on the normal affine variety  $X[Y, \mathcal{D}]$  such that the real torus  $(\mathbb{T}, \tau)$  acts on the  $\mathbb{R}$ -variety  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ .*

Conversely, given a  $\mathbb{T}$ -action on an affine algebraic  $\mathbb{C}$ -variety  $X$ , Altman and Hausen give in [2, §11] a method to construct a proper polyhedral divisor  $\mathcal{D}$  on a semi-projective variety  $Y$  based on the choice of an appropriate  $\mathbb{T}$ -equivariant closed immersion  $X \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$  and on the downgrading of the  $\mathbb{G}_{m, \mathbb{C}}^n$ -action on  $\mathbb{A}_{\mathbb{C}}^n$  to a  $\mathbb{T}$ -action.

Thus, for a  $(\mathbb{T}, \tau)$ -action on an affine algebraic  $\mathbb{R}$ -variety  $(X, \sigma)$ , a key ingredient to construct an  $\mathbb{R}$ -structure on the semi-projective variety  $Y$  mentioned in Theorem AH 2 is to find a certain  $\mathbb{T}$ -equivariant closed immersion  $X \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$  which is also  $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant:

**Proposition B** (Proposition 3.1). *There exists  $n \in \mathbb{N}$ ,  $n \geq \dim(\mathbb{T})$ , such that the following hold:*

- (i) *There exists an  $\mathbb{R}$ -group structure  $\tau'$  on  $\mathbb{G}_{m, \mathbb{C}}^n$  that extends to a  $\mathbb{R}$ -structure  $\sigma'$  on  $\mathbb{A}_{\mathbb{C}}^n$ ;*
- (ii)  *$(\mathbb{T}, \tau)$  is a closed subgroup of  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau')$ ; and*
- (iii)  *$(X, \sigma)$  is a closed subvariety of  $(\mathbb{A}_{\mathbb{C}}^n, \sigma)$  and  $(X, \sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$  is  $(\mathbb{T}, \tau)$ -equivariant.*

The immersion  $(\mathbb{T}, \tau) \hookrightarrow (\mathbb{G}_{m, \mathbb{C}}^n, \tau')$  induces an  $\mathbb{R}$ -group structure  $\tau_Y$  on the quotient torus  $\mathbb{T}_Y := \mathbb{G}_{m, \mathbb{C}}^n / \mathbb{T}$ . This  $\mathbb{R}$ -group structure  $\tau_Y$  induces in turn an  $\mathbb{R}$ -structure  $\sigma_Y$  on the semi-projective variety  $Y$  mentioned in Theorem AH 2. We downgrade  $\text{Gal}(\mathbb{C}/\mathbb{R})$ -equivariantly the  $\mathbb{G}_{m, \mathbb{C}}^n$ -action on  $\mathbb{A}_{\mathbb{C}}^n$  to a  $\mathbb{T}$ -action, which is a key ingredient in the proof of the following result (which is the real analogue of Theorem AH 2):

**Theorem C** (Theorem 3.6). *Let  $\omega \subset M_{\mathbb{Q}}$  be the weight cone of the  $\mathbb{T}$ -action on  $X$ . There exists a normal semi-projective  $\mathbb{R}$ -variety  $(Y, \sigma_Y)$ , a proper polyhedral divisor  $\mathcal{D}$  on  $Y$ , and a monoid morphism  $h : \omega \cap M \rightarrow \mathbb{C}(Y)^*$  such that*

$$\forall m \in \omega \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m))),$$

*and such that there is an isomorphism of  $\mathbb{R}$ -varieties between  $(X, \sigma)$  and  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ .*

In the case where the real torus  $T$  is quasi-split, our presentation simplifies. Indeed, if  $(X, \sigma)$  is endowed with a  $T$ -action and if  $(Y, \sigma_Y)$  is the variety mentioned in Theorem C, we see in Proposition 3.13 that there exists a proper polyhedral divisor  $\mathcal{D}$  on  $Y$  such that  $\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m))$  for all  $m \in \omega \cap M$ ; i.e we can take  $h = 1$ . From this result, we recover the Altmann-Hausen presentation for  $\mathbb{G}_{m, \mathbb{R}}^n$ -actions. On the other hand, this simplification is not always possible for  $\mathbb{S}^1$ -actions: see §4.3 for details and examples. In this case we recover the presentation for  $\mathbb{S}^1$ -actions given by Dubouloz and Liendo in [6].

After fixing our notation, the article is structured as follows.

In §1.1 we recall well-known facts about  $\mathbb{R}$ -structures on  $\mathbb{C}$ -varieties, and in §1.2 we see that tori inclusions corresponds to certain short exact sequences of lattices. Basic results on real torus actions and examples of real torus actions on affine toric  $\mathbb{R}$ -varieties are given in §1.3 and §1.4.

In §2, we briefly explain Altmann-Hausen's theory in view of extending it to the real case. We start by introducing polyhedral divisors in §2.1, and we recall the main results of [2] in §2.2.

In §3, after proving Proposition B in §3.1, we prove our main results in §3.2: Theorems A and C. Then, we give several cohomological results used to simplify the Altmann-Hausen presentation in the case where the acting torus is quasi-split.

In §4, we give examples of  $\mathbb{G}_{m, \mathbb{R}}$ -actions (see §4.1),  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m, \mathbb{C}})$ -actions (see §4.2) and  $\mathbb{S}^1$ -actions (see §4.3).

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### Notation

Throughout the entire paper, we call a  $\mathbb{C}$ -variety a separated integral scheme of finite type over  $\mathbb{C}$ , and an  $\mathbb{R}$ -variety a separated geometrically integral scheme of finite type over  $\mathbb{R}$ . We denote by  $\Gamma := \{id, \gamma\}$  the Galois group of the field extension  $\mathbb{C}/\mathbb{R}$ , it is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . The group of regular automorphisms of a  $\mathbb{C}$ -variety  $X$  is denoted by  $\text{Aut}(X)$ , and the group of regular group automorphisms of a complex algebraic group  $G$  is denoted by  $\text{Aut}_{gr}(G)$ .

From here on,  $N$  denotes a lattice, i.e. a finitely generated free abelian group, and  $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  denotes its dual lattice. The associated  $\mathbb{Q}$ -vector spaces are denoted by  $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$

and  $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$  respectively, and the the corresponding pairing by:

$$M \times N \rightarrow \mathbb{Z}, \quad (u, v) \mapsto \langle u, v \rangle := u(v).$$

Let us recall some results of [9, §1.2]. Let  $N'$  be a lattice, and let  $f : N \rightarrow N'$  be a lattice homomorphism. It induces a unique  $\mathbb{Q}$ -linear map  $N_{\mathbb{Q}} \rightarrow N'_{\mathbb{Q}}$ , also denoted by  $f$ . A subset  $\omega_N \subset N_{\mathbb{Q}}$  is called a *convex polyhedral cone* if there exists a finite set  $S \subset N_{\mathbb{Q}}$  such that

$$\omega_N = \text{Cone}(S) := \left\{ \sum_{v \in S} \lambda_v v \mid \lambda_v \in \mathbb{Q}_{\geq 0} \right\} \subset N_{\mathbb{Q}}.$$

A cone  $\omega_N$  is *strongly convex* if  $\omega_N \cap (-\omega_N) = \{0\}$ . For us, a *cone* in  $N_{\mathbb{Q}}$  is always a convex polyhedral cone. The dual cone of  $\omega_N$  is defined by

$$\omega_N^{\vee} := \{u \in M_{\mathbb{Q}} \mid \forall v \in \omega_N, \langle u, v \rangle \geq 0\};$$

it is a cone in  $M_{\mathbb{Q}}$ . Let  $\omega_N$  be a cone in  $N_{\mathbb{Q}}$ . A *face*  $\tau_N$  of  $\omega_N$  is given by  $\tau_N = \omega_N \cap u^{\perp}$ , for some  $u \in \omega_N^{\vee}$ , where  $u^{\perp} := \{v \in \omega_N \mid \forall u \in \omega_N^{\vee}, \langle u, v \rangle = 0\}$ . Recall that a face of a cone is a cone. The relative interior  $\text{Relint}(\omega_N)$  of a cone  $\omega_N$  is obtained by removing all proper faces from  $\omega_N$ .

A *quasifan*  $\Lambda$  in  $N_{\mathbb{Q}}$  (or in  $M_{\mathbb{Q}}$ ) is a finite collection of cones in  $N_{\mathbb{Q}}$  (or in  $M_{\mathbb{Q}}$ ) such that, for any  $\lambda \in \Lambda$ , all the faces of  $\lambda$  belong to  $\Lambda$ , and for any  $\lambda_1, \lambda_2 \in \Lambda$ , the intersection  $\lambda_1 \cap \lambda_2$  is a face of both  $\lambda_i$ . The *support* of a quasifan is the union of all its cones. A quasifan is called a *fan* if all its cones are strongly convex.

A complex torus is an affine algebraic group isomorphic to  $\mathbb{G}_{m, \mathbb{C}}^n$ . There is a one-to-one correspondence between lattices and complex tori. To a lattice  $M \cong \mathbb{Z}^n$ , we associate the affine variety  $\text{Spec}(\mathbb{C}[M])$ , with  $\mathbb{C}[M] := \{\sum_{m \in M} c_m \chi^m \mid c_m \in \mathbb{C}\}$  and where  $\chi^m$  are indeterminate such that  $\chi^{m+m'} = \chi^m \chi^{m'}$ . It is a complex torus isomorphic to  $\mathbb{G}_{m, \mathbb{C}}^n$ . Conversely, to a complex torus  $\mathbb{T}$  isomorphic to  $\mathbb{G}_{m, \mathbb{C}}^n$ , we associate its character lattice  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ . It is isomorphic to  $\mathbb{Z}^n$ . Let us recall that  $\text{Aut}_{gr}(\mathbb{G}_{m, \mathbb{C}}^n) \cong \text{GL}_n(\mathbb{Z})$ .

The action of a complex torus  $\mathbb{T}$  on a  $\mathbb{C}$ -variety  $X$  is called *effective* if the neutral element of  $\mathbb{T}$  is the only element acting trivially on  $X$ . In this paper, we only consider effective torus actions. Let  $X$  be a  $\mathbb{C}$ -variety endowed with an action of the torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[M])$ . The *weight monoid* of this action is  $S := \{m \in M \mid \mathbb{C}[X]_m \neq \{0\}\}$  and the cone  $\omega_M$  of  $M_{\mathbb{Q}}$  spanned by the weight monoid  $S$  is called the *weight cone*. The algebra  $\mathbb{C}[X]$  is  $M$ -graded:  $\mathbb{C}[X] = \bigoplus_{m \in \omega_M \cap M} \mathbb{C}[X]_m$ . There is a bijective correspondence between the  $\mathbb{T}$ -actions on  $X$  and the  $M$ -gradings on  $\mathbb{C}[X]$  [11, §2.1].

We recall some definitions and results useful for the proof of Lemma 1.10. Let  $(G, \cdot)$  be a group. A  $G$ -module is an abelian group  $(M, +)$  endowed with an action  $(g, m) \mapsto g \cdot m$  of  $G$  such that the induced map  $\varphi_g : m \mapsto g \cdot m$  is an abelian group automorphism. Recall that this data is equivalent to a left module  $(M, +)$  over the ring  $\mathbb{Z}[G]$ . Indeed, if  $M$  is a module over the ring  $\mathbb{Z}[G]$ , we define a  $G$ -module structure on  $M$  via  $g \cdot m := \chi^g m$  for all  $(g, m) \in G \times M$ . Conversely, if  $M$  is a  $G$ -module, we construct a  $\mathbb{Z}[G]$ -module structure on  $M$  via  $(\sum_{g \in G} n_g \chi^g) m := \sum_{g \in G} n_g g \cdot m$ .

## 1. GALOIS DESCENT $\mathbb{C}/\mathbb{R}$ AND ALGEBRAIC TORI

We recall basic definitions and well-known facts about  $\mathbb{R}$ -structures on  $\mathbb{C}$ -varieties and  $\mathbb{R}$ -group structures on complex algebraic groups in view of studying torus actions on  $\mathbb{R}$ -varieties. See [3, §3.1.3] and [4].

**1.1. Galois descent  $\mathbb{C}/\mathbb{R}$ .** Let us briefly recall the classical correspondence between quasi-projective  $\mathbb{R}$ -varieties and quasi-projective  $\mathbb{C}$ -varieties endowed with an  $\mathbb{R}$ -structure. Every  $\mathbb{C}$ -variety  $X$  can be viewed as an  $\mathbb{R}$ -scheme via the composition of its structure morphism  $X \rightarrow \text{Spec}(\mathbb{C})$  with the morphism  $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$  induced by the inclusion  $\mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[i]/(i^2 + 1)$ . The Galois group  $\Gamma$  acts on  $\text{Spec}(\mathbb{C})$  by the usual complex conjugation  $z \mapsto \bar{z}$ .

**Definition 1.1.** (i) An  $\mathbb{R}$ -form of a  $\mathbb{C}$ -variety  $X$  is an  $\mathbb{R}$ -variety  $X_0$  together with an isomorphism  $X_0 \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C}) \cong X$  of  $\mathbb{C}$ -varieties. By abuse of notation we will often write:  $X_0$  is an  $\mathbb{R}$ -form of  $X$  instead of  $(X_0, \cong)$ .

- (ii) An  $\mathbb{R}$ -structure  $\sigma$  on a  $\mathbb{C}$ -variety  $X$  is an antiregular involution, i.e, an involution of  $\mathbb{R}$ -scheme  $\sigma : X \rightarrow X$  which makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow & & \downarrow \\ \text{Spec}(\mathbb{C}) & \xrightarrow{\text{Spec}(z \mapsto \bar{z})} & \text{Spec}(\mathbb{C}) \end{array}$$

- (iii) Two  $\mathbb{R}$ -structures  $\sigma$  and  $\sigma'$  on  $X$  are *equivalent* if there exists  $\varphi \in \text{Aut}(X)$  such that  $\sigma' = \varphi \circ \sigma \circ \varphi^{-1}$ .
- (iv) An  $\mathbb{R}$ -morphism between two  $\mathbb{C}$ -varieties  $X$  and  $X'$  with  $\mathbb{R}$ -structures  $\sigma$  and  $\sigma'$  is a morphism of  $\mathbb{C}$ -varieties  $f : X \rightarrow X'$  such that  $\sigma' \circ f = f \circ \sigma$  as morphisms of  $\mathbb{R}$ -schemes.

If a quasi-projective  $\mathbb{C}$ -variety  $X$  is endowed with an  $\mathbb{R}$ -structure  $\sigma$ , then the quotient  $X/\langle\sigma\rangle$  exists in the category of  $\mathbb{R}$ -varieties and the structure morphism  $X \rightarrow \text{Spec}(\mathbb{C})$  descends to a morphism  $X/\langle\sigma\rangle \rightarrow \text{Spec}(\mathbb{R})$  making  $X/\langle\sigma\rangle$  into an  $\mathbb{R}$ -variety such that  $X \cong (X/\langle\sigma\rangle)_{\mathbb{C}}$ . If  $f : (X, \sigma) \rightarrow (X', \sigma')$  is an  $\mathbb{R}$ -morphism between quasi-projective  $\mathbb{C}$ -varieties, and if  $\pi' : X' \rightarrow X'/\langle\sigma'\rangle$  denotes the quotient morphism, we obtain from the invariant morphism  $\pi' \circ f : X \rightarrow X'/\langle\sigma'\rangle$  a morphism  $f_0 : X/\langle\sigma\rangle \rightarrow X'/\langle\sigma'\rangle$  of  $\mathbb{R}$ -varieties.

**Proposition 1.2.** *The functor  $(X, \sigma) \mapsto X/\langle\sigma\rangle$  induces an equivalence of categories between the category of pairs  $(X, \sigma)$  consisting of a quasi-projective  $\mathbb{C}$ -variety  $X$  endowed with an  $\mathbb{R}$ -structure  $\sigma$  and the category of quasi-projective  $\mathbb{R}$ -varieties. Moreover,  $\sigma$  is equivalent to  $\sigma'$  if and only if  $X/\langle\sigma\rangle$  is  $\mathbb{R}$ -isomorphic to  $X/\langle\sigma'\rangle$ .*

Using this equivalence, we often write  $(X, \sigma)$  to refer to an algebraic  $\mathbb{R}$ -variety.

*Proof.* We give a sketch of the proof for the sake of completeness. If  $(X_0, \cong)$  is an  $\mathbb{R}$ -form of  $X$ , the  $\mathbb{C}$ -variety  $(X_0)_{\mathbb{C}} := X_0 \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C})$  is endowed with a canonical  $\mathbb{R}$ -structure given by the action of  $\Gamma$  by complex conjugation on the second factor, this gives an  $\mathbb{R}$ -structure on  $X \cong (X_0)_{\mathbb{C}}$ . If  $(X_0, \cong)$  and  $(X'_0, \cong)$  are  $\mathbb{R}$ -forms of  $X$  and  $X'$  respectively, and if  $f_0 : X_0 \rightarrow X'_0$  is a morphism of  $\mathbb{R}$ -varieties, then  $f_0 \times id : (X_0)_{\mathbb{C}} \rightarrow (X'_0)_{\mathbb{C}}$  is a morphism of  $\mathbb{C}$ -varieties, so we obtain a morphism  $f : X \rightarrow X'$  such that  $f \circ \sigma = \sigma' \circ f$ .  $\square$

We have similar definitions and properties for affine algebraic groups.

- Definition 1.3.** (i) Let  $G$  be a complex algebraic group. A real algebraic group  $G_0$  together with an isomorphism  $G_0 \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C}) \cong G$  is called a  $\mathbb{R}$ -form of  $G$ .
- (ii) An  $\mathbb{R}$ -group structure  $\tau$  on a complex algebraic group  $G$  is an  $\mathbb{R}$ -structure  $\tau : G \rightarrow G$  such that the multiplication  $G \times G \rightarrow G$ , the inverse  $G \rightarrow G$  and the unity  $\text{Spec}(\mathbb{C}) \rightarrow G$  are  $\mathbb{R}$ -morphisms.
- (iii) Two  $\mathbb{R}$ -group structures  $\tau$  and  $\tau'$  on  $G$  are *equivalent* if there exists  $\varphi \in \text{Aut}_{gr}(G)$  such that  $\tau' = \varphi \circ \tau \circ \varphi^{-1}$ .
- (iv) An  $\mathbb{R}$ -morphism between two complex algebraic groups  $G$  and  $G'$  with  $\mathbb{R}$ -structures  $\tau$  and  $\tau'$  is a morphism of complex algebraic groups  $f : G \rightarrow G'$  such that  $\tau' \circ f = f \circ \tau$  as morphisms of  $\mathbb{R}$ -schemes.

If  $G$  is a complex affine algebraic group endowed with an  $\mathbb{R}$ -group structure  $\tau$ , then the quotient scheme  $G_0 := G/\langle\tau\rangle$  is a real algebraic group which satisfies  $(G_0)_{\mathbb{C}} \cong G$  as complex algebraic groups.

**Remark 1.4.** There is an equivalence between the category of pairs  $(G, \tau)$  consisting of a complex affine algebraic group endowed with an  $\mathbb{R}$ -group structure, and the category of real affine algebraic groups. This induces a one-to-one correspondence between the  $\mathbb{R}$ -forms of  $G$ , up to isomorphism in the category of real algebraic groups, and the equivalence classes of  $\mathbb{R}$ -group structures on  $G$ .

**1.2. Real tori.** We define real tori and we recall that any real torus is isomorphic to a product of copies of three *elementary* real tori.

**Definition 1.5.** A *real torus*  $T$  is a real affine algebraic group such that  $T_{\mathbb{C}}$  is a complex torus. It is called a *split torus* if  $T \cong \mathbb{G}_{m, \mathbb{R}}^n$  for some integer  $n$ .

The torus  $\mathbb{G}_{m,\mathbb{C}}$  has two non isomorphic  $\mathbb{R}$ -forms: the real split torus  $\mathbb{G}_{m,\mathbb{R}}$  and the real circle  $\mathbb{S}^1 := \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 - 1))$ . Since  $\text{Aut}_{gr}(\mathbb{G}_{m,\mathbb{C}}) = \{id, -id\}$ , the equivalence class of an  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}$  has only one element. So, the  $\mathbb{R}$ -group structures on  $\mathbb{G}_{m,\mathbb{C}}$  associated to  $\mathbb{G}_{m,\mathbb{R}}$  and  $\mathbb{S}^1$  are respectively:

$$\tau_0 : z \mapsto \bar{z} \quad \text{and} \quad \tau_1 : z \mapsto \bar{z}^{-1}.$$

The group structure on  $\mathbb{S}^1$  is given by

$$(x, y) \cdot (x', y') = (xx' - yy', xy' + yx').$$

The *Weil restriction* of  $\mathbb{G}_{m,\mathbb{C}}$  is  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) := \text{Spec}(\mathbb{R}[x_1, y_1, x_2, y_2]/(x_1y_1 - x_2y_2 - 1, x_2y_1 + x_1y_2))$ . It is an  $\mathbb{R}$ -form of  $\mathbb{G}_{m,\mathbb{C}}^2$ . An  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^2$  associated to  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  is:

$$\tau_2 : (z, w) \mapsto (\bar{w}, \bar{z}).$$

The group structure on  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  is given by

$$(x_1, y_1, x_2, y_2) \cdot (x'_1, y'_1, x'_2, y'_2) = (x_1x'_1 - x_2x'_2, y_1y'_1 - y_2y'_2, x_1x'_2 + x_2x'_1, y_1y'_2 + y_2y'_1).$$

By abuse, we call Weil restriction any real torus isomorphic to  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ . Here,  $\text{Aut}_{gr}(\mathbb{G}_{m,\mathbb{C}}^2) \cong \text{GL}_2(\mathbb{Z})$ , so the equivalence class of an  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^2$  has infinitely many elements. For instance

$$\tau'_2 : (z, w) \mapsto (\bar{w}^{-1}, \bar{z}^{-1}) \quad \text{and} \quad \tau''_2 : (z, w) \mapsto (\bar{z}^{-1}\bar{w}, \bar{w})$$

are  $\mathbb{R}$ -group structures equivalent to  $\tau_2$ .

**Remark 1.6.** An  $\mathbb{R}$ -group structure  $\tau$  on a complex torus  $\mathbb{T}$  induces lattice involutions  $\tilde{\tau}$  and  $\hat{\tau}$  on  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$  and  $N := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}, \mathbb{T})$  respectively. For  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ , the involutions  $\tilde{\tau}_2$  and  $\hat{\tau}_2$  are both given by  $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2, (k, l) \mapsto (l, k)$ . For  $\mathbb{G}_{m,\mathbb{R}}$ ,  $\tilde{\tau}_0$  and  $\hat{\tau}_0$  are both given by  $id : \mathbb{Z} \rightarrow \mathbb{Z}$ , and for  $\mathbb{S}^1$ ,  $\tilde{\tau}_1$  and  $\hat{\tau}_1$  are both given by  $-id : \mathbb{Z} \rightarrow \mathbb{Z}$ .

These three *elementary* real tori form the building blocks of every real torus, that is:

**Proposition 1.7.** [14, Proposition 1.5]. *Every  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^n$  is equivalent to exactly one  $\mathbb{R}$ -group structure of the form  $\tau_0^{\times n_0} \times \tau_1^{\times n_1} \times \tau_2^{\times n_2}$ , with  $n_0 + n_1 + 2n_2 = n$ .*

**Remark 1.8.** Let  $(\mathbb{T}, \tau)$  be a subtorus of the real torus  $(\mathbb{G}_{m,\mathbb{C}}^n, \tau')$ . Let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$  and  $M' := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^n, \mathbb{G}_{m,\mathbb{C}})$ . The inclusion  $\mathbb{T} \hookrightarrow \mathbb{G}_{m,\mathbb{C}}^n$  induces a surjective lattice homomorphism  $M' \rightarrow M$ . Let  $M_Y$  be the kernel of this homomorphism, it is a sublattice of  $M'$ . Moreover, the lattice involution  $\tilde{\tau}'$  on  $M'$  induces a lattice involution  $\tilde{\tau}_Y$  on  $M_Y$ . Let  $\tau_Y$  be the induced  $\mathbb{R}$ -group structure on  $\mathbb{T}_Y$ . The following diagram of complex algebraic groups commutes:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{G}_{m,\mathbb{C}}^n & \longrightarrow & \mathbb{T}_Y := \text{Spec}(\mathbb{C}[M_Y]) \longrightarrow 1 \\ & & \downarrow \tau & & \downarrow \tau' & & \downarrow \tau_Y \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{G}_{m,\mathbb{C}}^n & \longrightarrow & \mathbb{T}_Y := \text{Spec}(\mathbb{C}[M_Y]) \longrightarrow 1 \end{array}$$

There exists an injective morphism  $F : N \rightarrow N'$  and a surjective homomorphism  $P : N' \rightarrow N_Y$ , and the following diagrams of free  $\mathbb{Z}$ -modules commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y \longrightarrow 0 \\ & & \downarrow \hat{\tau} & & \downarrow \hat{\tau}' & & \downarrow \hat{\tau}_Y \\ 0 & \longrightarrow & N & \xrightarrow{F} & N' & \xrightarrow{P} & N_Y \longrightarrow 0 \\ & & & & & & \\ 0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M \longrightarrow 0 \\ & & \downarrow \tilde{\tau}_Y & & \downarrow \tilde{\tau}' & & \downarrow \tilde{\tau} \\ 0 & \longrightarrow & M_Y & \xrightarrow{P^*} & M' & \xrightarrow{F^*} & M \longrightarrow 0 \end{array}$$

There always exists a section  $s^* : M \rightarrow M'$ , but not always a  $\Gamma$ -equivariant one. Therefore, we obtain a section  $\mathbb{T}_Y \rightarrow \mathbb{G}_{m,\mathbb{C}}^n$ , but not always a  $\Gamma$ -equivariant one. In other words,  $\mathbb{G}_{m,\mathbb{C}}^n \cong \mathbb{T} \times \mathbb{T}_Y$ , but this isomorphism is not always  $\Gamma$ -equivariant.

**Example 1.9.** The real tori  $\mathbb{G}_{m,\mathbb{R}}$  and  $\mathbb{S}^1$  are real subtori of  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ . The inclusion are given by:

$$(\mathbb{G}_{m,\mathbb{C}}, \tau_0) \rightarrow (\mathbb{G}_{m,\mathbb{C}}^2, \tau_2), t \mapsto (t, t) \quad \text{and} \quad (\mathbb{G}_{m,\mathbb{C}}, \tau_1) \rightarrow (\mathbb{G}_{m,\mathbb{C}}^2, \tau_2), t \mapsto (t, t^{-1}).$$

We obtain the diagrams of Remark 1.8 with  $M' = \mathbb{Z}^2$ ,  $M = \mathbb{Z}$ ,  $M_Y = \mathbb{Z}$ ,  $F^* = [1, 1]$  and  $P^* = [1, -1]$  for  $\mathbb{G}_{m,\mathbb{R}}$ , and  $F^* = [1, -1]$  and  $P^* = [1, 1]$  for  $\mathbb{S}^1$ . In these two cases, there does not exist a  $\Gamma$ -equivariant section since  $\tau_2$  is not equivalent to  $\tau_1 \times \tau_0$ .

In the case where  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  is a subtorus of a real torus  $(\mathbb{G}_{m,\mathbb{C}}^n, \tau')$ , we have the following result:

**Lemma 1.10.** *Let  $(\mathbb{G}_{m,\mathbb{C}}^{2q}, \tau_2^{\times q})$  be a subtorus of  $(\mathbb{G}_{m,\mathbb{C}}^n, \tau')$ . Let  $M := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^{2q}, \mathbb{G}_{m,\mathbb{C}})$  and  $M' := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^n, \mathbb{G}_{m,\mathbb{C}})$ . Then, there exists a  $\Gamma$ -equivariant section  $s^* : M \rightarrow M'$  (i.e.  $F^* \circ s^* = id$  and  $\tau' \circ s^* = s^* \circ \tau_2^{\times q}$ ).*

*Proof.* The Galois group  $\Gamma = \{id, \gamma\}$  acts on  $M$  via  $\tilde{\tau}_2^{\times q}$ , so  $M$  is a  $\Gamma$ -module. We have the following short exact sequences of  $\mathbb{Z}[\Gamma]$ -modules:

$$0 \longrightarrow M_Y \xrightarrow{P^*} M' \xrightarrow{F^*} M \longrightarrow 0.$$

Note that we have an isomorphism of  $\mathbb{Z}[\Gamma]$ -module

$$M \rightarrow \mathbb{Z}[\Gamma]^q, \quad (k_1, l_1, \dots, k_q, l_q) \mapsto (k_1\chi^{id} + l_1\chi^\gamma, \dots, k_q\chi^{id} + l_q\chi^\gamma).$$

Hence  $M$  is free  $\mathbb{Z}[\Gamma]$ -module of rank  $q$ , so it is a projective  $\mathbb{Z}[\Gamma]$ -module. By [8, Proposition A3.1], there exists a morphism  $s^* : M \rightarrow M'$  of  $\mathbb{Z}[\Gamma]$ -module such that  $F^* \circ s^* = id_M$ .  $\square$

**Remark 1.11.**

- (i) The interpretation of Lemma 1.10 is :  $(\mathbb{G}_{m,\mathbb{C}}^n, \tau') \cong R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})^q \times T'$ , where  $T'$  is a real torus of dimension  $n - 2q$ .
- (ii) For  $\mathbb{G}_{m,\mathbb{R}}$ -actions, the  $\mathbb{Z}[\Gamma]$ -module  $M = \mathbb{Z}$ , with  $\Gamma$ -action given by  $\tilde{\tau}_0 = id$ , is not a projective  $\mathbb{Z}[\Gamma]$ -module. Indeed, we have a  $\Gamma$ -equivariant isomorphism

$$M \rightarrow \mathbb{Z}[\Gamma]/(\chi^\gamma), \quad m \mapsto [m\chi^{id} + m\chi^\gamma].$$

- (iii) For  $\mathbb{S}^1$ -actions, the  $\mathbb{Z}[\Gamma]$ -module  $M = \mathbb{Z}$ , with  $\Gamma$ -action given by  $\tilde{\tau}_1 = -id$ , is not a projective  $\mathbb{Z}[\Gamma]$ -module. Indeed, we have a  $\Gamma$ -equivariant isomorphism

$$M \rightarrow \mathbb{Z}[\Gamma]/(\chi^\gamma), \quad m \mapsto [m\chi^{id} + (-m)\chi^\gamma].$$

**1.3. Real torus actions.** We now consider actions of real tori on  $\mathbb{R}$ -varieties.

**Lemma 1.12.** *Let  $T$  be a real torus. There is a one-to-one correspondence between quasi-projective  $\mathbb{R}$ -varieties endowed with a  $T$ -action and tuples  $(\mathbb{T}, \tau, X, \sigma, \mu)$  consisting of:*

- (i) a complex torus  $\mathbb{T}$  endowed with an  $\mathbb{R}$ -group structure  $\tau$  such that  $\mathbb{T}/\langle\tau\rangle \cong T$ ;
- (ii) a quasi-projective  $\mathbb{C}$ -variety  $X$  endowed with an  $\mathbb{R}$ -structure  $\sigma$ ;
- (iii) an action  $\mu : \mathbb{T} \times X \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} \times X & \xrightarrow{\mu} & X \\ \tau \times \sigma \downarrow & & \downarrow \sigma \\ \mathbb{T} \times X & \xrightarrow{\mu} & X \end{array}$$

*Proof.* Let  $(\mathbb{T}, \tau, X, \sigma, \mu)$  be such a tuple. By proposition 1.2, the morphism  $\mu : \mathbb{T} \times X \rightarrow X$  induces a morphism  $\mu_0 : (\mathbb{T} \times X)/\langle\tau \times \sigma\rangle \rightarrow X/\langle\sigma\rangle$ . Since  $(\mathbb{T} \times X)/\langle\tau \times \sigma\rangle \cong \mathbb{T}/\langle\tau\rangle \times X/\langle\sigma\rangle$ , we have a  $\mathbb{T}/\langle\tau\rangle$ -action on  $X/\langle\sigma\rangle$ .

Conversely, let  $X$  be an  $\mathbb{R}$ -variety endowed with a  $T$ -action  $T \times X \rightarrow X$ . Since  $(T \times X)_{\mathbb{C}} \cong T_{\mathbb{C}} \times X_{\mathbb{C}}$ , we obtain an action  $\mu := T_{\mathbb{C}} \times X_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$  satisfying the commutative diagram:

$$\begin{array}{ccc} T_{\mathbb{C}} \times X_{\mathbb{C}} & \xrightarrow{\mu} & X_{\mathbb{C}} \\ id \times (z \mapsto \bar{z}) \downarrow & id \times (z \mapsto \bar{z}) & \downarrow id \times (z \mapsto \bar{z}) \\ T_{\mathbb{C}} \times X_{\mathbb{C}} & \xrightarrow{\mu} & X_{\mathbb{C}} \end{array}$$



Which ends the proof.  $\square$

**Example 1.13.** Consider the action of  $\mathbb{G}_{m,\mathbb{C}}^2$  on  $\mathbb{A}_{\mathbb{C}}^3$  given by  $(s, t) \cdot (x, y, z) = (sx, ty, stz)$ . The Weil restriction  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$  acts on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$ , where  $\sigma'(x, y, z) = (\bar{y}, \bar{x}, \bar{z})$ .

**Example 1.14.** Consider the hypersurface  $X$  of  $\mathbb{A}_{\mathbb{C}}^4 := \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4])$  defined by  $x_1x_3 = x_2x_4$ . The torus  $\mathbb{G}_{m,\mathbb{C}}^2$  acts on  $\mathbb{A}_{\mathbb{C}}^4$  by  $(s, t) \cdot (x_1, x_2, x_3, x_4) := (sx_1, tx_2, st^2x_3, s^2tx_4)$ . Since the polynomial  $x_1x_3 - x_2x_4$  is homogeneous,  $\mathbb{G}_{m,\mathbb{C}}^2$  acts on  $X$ . Let  $\sigma'$  be the  $\mathbb{R}$ -structure on  $\mathbb{A}_{\mathbb{C}}^4$  defined by  $\sigma'(x_1, x_2, x_3, x_4) = (\bar{x}_2, \bar{x}_1, \bar{x}_4, \bar{x}_3)$  and let  $\sigma$  be the induced  $\mathbb{R}$ -structure on  $X$ . Then, the real torus  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$  acts on  $(\mathbb{A}_{\mathbb{C}}^4, \sigma')$  and on  $(X, \sigma)$ .

Let us note that if a real torus  $(\mathbb{T}, \tau)$  acts on an affine variety  $(X, \sigma)$ , then the comorphism  $\sigma^\#$  of  $\sigma$  preserves the  $M$ -grading of the algebra  $\mathbb{C}[X]$ , where  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$ . This observation will be useful in the proof of Proposition 3.1.

**Lemma 1.15.** *Let  $(\mathbb{T}, \tau)$  be a real torus acting on the affine  $\mathbb{R}$ -variety  $(X, \sigma)$ . Let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$  and let  $\omega_M$  be the weight cone of the  $\mathbb{T}$ -action on  $X$ . Then  $\tilde{\tau}(\omega_M) = \omega_M$  and for all  $m \in M$ :*

$$\sigma^\#(\mathbb{C}[X]_m) = \mathbb{C}[X]_{\tilde{\tau}(m)}.$$

*Proof.* Let  $m \in M$  and let  $f \in \mathbb{C}[X]_m$ . We obtain from the diagram of Lemma 1.12:

$$(\mu^\# \circ \sigma^\#)(f) = ((\tau^\# \times \sigma^\#) \circ \mu^\#)(f) = \tau^\#(\chi^m) \otimes \sigma^\#(f) = \chi^{\tilde{\tau}(m)} \otimes \sigma^\#(f).$$

Hence  $\sigma^\#(\mathbb{C}[X]_m) \subset \mathbb{C}[X]_{\tilde{\tau}(m)}$ . Moreover, if  $g \in \mathbb{C}[X]_{\tilde{\tau}(m)}$ , then  $g = \sigma^\#(\sigma^\#(g))$ . Hence,  $\sigma^\#(\mathbb{C}[X]_m) = \mathbb{C}[X]_{\tilde{\tau}(m)}$ .  $\square$

**1.4. The case of affine toric  $\mathbb{R}$ -varieties.** In this subsection, we consider the particular case of affine toric  $\mathbb{R}$ -varieties, i.e. affine  $\mathbb{R}$ -varieties  $X$  such that  $X_{\mathbb{C}}$  is an affine toric  $\mathbb{C}$ -variety.

**Proposition 1.16.** *Let  $(\mathbb{T}, \tau)$  be a real torus, let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$  and  $N$  be its dual lattice. Let  $\delta$  be a pointed cone in  $N_{\mathbb{Q}}$  and let  $X_\delta$  be the associated affine toric  $\mathbb{C}$ -variety. The torus  $(\mathbb{T}, \tau)$  acts on the affine toric  $\mathbb{R}$ -variety  $(X_\delta, \sigma)$ , where  $\sigma$  is an  $\mathbb{R}$ -structure on  $X_\delta$ , if and only if there exists an  $\mathbb{R}$ -group structure  $\tau'$  on  $\mathbb{T}$  equivalent to  $\tau$  such that  $\hat{\tau}'(\delta) = \delta$ .*

*Proof.* (Compare with [10, Proposition 1.19]). Assume that  $\tau$  is equivalent to  $\tau'$  and  $\hat{\tau}'(\delta) = \delta$ . Recall that we denote  $\mathbb{C}[M] := \{\sum_{m \in M} a_m \chi^m, a_m \in \mathbb{C}\}$  and  $\mathbb{C}[S_\delta] := \{\sum_{m \in \delta^\vee \cap M} a_m \chi^m, a_m \in \mathbb{C}\}$  the coordinate rings of  $\mathbb{G}_{m,\mathbb{C}}^n$  and  $X_\delta$  respectively. Since  $\tilde{\tau}'(\delta^\vee) = \delta^\vee$ , the algebra automorphism

$$\tau^\# : \mathbb{C}[M] \rightarrow \mathbb{C}[M], \quad \sum_{m \in M} a_m \chi^m \mapsto \sum_{m \in M} \bar{a}_m \chi^{\tilde{\tau}'(m)}$$

can be restricted to  $\mathbb{C}[S_\delta] \subset \mathbb{C}[M]$ . It is the comorphism of an  $\mathbb{R}$ -structure  $\sigma$  on  $X_\delta$ .

Conversely, let  $\sigma$  be an  $\mathbb{R}$ -structure on  $X_\delta$  such that  $(\mathbb{T}, \tau)$  acts on  $(X_\delta, \sigma)$ . Let  $m \in \delta^\vee \cap M$ , then there exists  $\tilde{\varphi} \in \text{GL}(M)$  such that  $\chi^m \in \mathbb{C}[S_\delta]_{\tilde{\varphi}(m)}$ . Let  $\varphi \in \text{Aut}_{gr}(\mathbb{T})$  be the corresponding automorphism. By lemma 1.15,  $\sigma^\#(\chi^m) \in \mathbb{C}[S_\delta]_{\tilde{\tau}(\tilde{\varphi}(m))}$ , so  $\sigma^\#(\chi^m) = \chi^{(\tilde{\varphi}^{-1} \circ \tilde{\tau} \circ \tilde{\varphi})(m)}$ . Let  $\tau' := \varphi^{-1} \circ \tau \circ \varphi$  be an  $\mathbb{R}$ -group structure on  $\mathbb{T}$  equivalent to  $\tau$ , then  $\sigma^\#(\chi^m) = \chi^{\tilde{\tau}'(m)}$ . Hence,  $\tilde{\tau}'(m) \in \delta^\vee \cap M$  and  $\hat{\tau}'(\delta) = \delta$ .  $\square$

**Remark 1.17.** The weight cone of the  $\mathbb{T}$ -action on  $X_\delta$  does not always coincide with  $\delta^\vee$  (compare with Lemma 1.15).

**Remark 1.18.** The Weil restriction  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  acts on a 2-dimensional affine toric  $\mathbb{R}$ -variety  $(X_\delta, \sigma)$  if and only if there exists a basis  $\{e_1, e_2\}$  of the lattice  $N$  such that the cone  $\delta$  is symmetric with respect to the line  $\mathbb{Q}(e_1 + e_2)$ . Let  $\sigma$  be the  $\mathbb{R}$ -structure on  $\mathbb{A}_{\mathbb{C}}^2$  defined by  $\sigma(x, y) = (\bar{y}, \bar{x})$ . The toric  $\mathbb{R}$ -variety  $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$  is endowed with an  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ -action since  $\hat{\tau}_2(\mathbb{Q}_{\geq 0}^2) = \mathbb{Q}_{\geq 0}^2$ .

**Example 1.19.** Consider the cone  $\delta = \mathbb{Q}_{\geq 0}^3$  in  $N_{\mathbb{Q}} = \mathbb{Q}^3$  spanned by the canonical basis of  $N$ . Let  $X_\delta := \mathbb{A}_{\mathbb{C}}^3$  be the associated toric  $\mathbb{C}$ -variety. The cone  $\delta$  is stable under the lattice involution  $\hat{\tau}' := \hat{\tau}_2 \times \hat{\tau}_0$  induced by the  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^3$  defined by  $\tau' := \tau_2 \times \tau_0$ . Consider the  $\mathbb{R}$ -structure  $\sigma'$  on  $\mathbb{A}_{\mathbb{C}}^3$  defined by  $\sigma'(x, y, z) = (\bar{y}, \bar{x}, \bar{z})$ . The natural action of  $\mathbb{G}_{m,\mathbb{C}}^3$  on  $\mathbb{A}_{\mathbb{C}}^3$  is compatible

with the  $\mathbb{R}$ -structures  $\tau'$  and  $\sigma'$ , i.e.  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \times \mathbb{G}_{m,\mathbb{R}}$  acts on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$ . Note that the action of  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$  on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$  given in Example 1.13 comes from the action of  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \times \mathbb{G}_{m,\mathbb{R}}$  on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$  (details in Example 3.2).

**Counter-example 1.20.** Consider the cone  $\delta = \text{Cone} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$  in  $N_{\mathbb{Q}} = \mathbb{Q}^2$ . There are no  $\mathbb{R}$ -group structure  $\tau$  equivalent to  $\tau_2$  such that  $\tau(\delta) = \delta$ , so we cannot endow  $X_{\delta}$  with an  $\mathbb{R}$ -structure compatible with a  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ -action.

Let  $(\mathbb{T}, \tau)$  be a real torus and let  $\sigma$  be an  $\mathbb{R}$ -structure on an  $n$ -dimensional toric  $\mathbb{C}$ -variety  $X_{\delta}$  induced by an  $\mathbb{R}$ -group structure  $\tau'$  on  $\mathbb{G}_{m,\mathbb{C}}^n$ . By a  $(\mathbb{T}, \tau)$ -action on  $(X_{\delta}, \sigma)$ , we mean a  $(\mathbb{T}, \tau)$ -action such that  $(\mathbb{T}, \tau)$  is a real subtorus of  $(\mathbb{G}_{m,\mathbb{C}}^n, \tau')$ . Let's now have a look at  $(\mathbb{T}, \tau)$ -actions on  $(X_{\delta}, \sigma)$ .

**Corollary 1.21.** *Let  $M := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^n, \mathbb{G}_{m,\mathbb{C}})$  and  $N$  be its dual lattice. Let  $\delta$  be a pointed cone in  $N_{\mathbb{Q}}$  and let  $X_{\delta}$  be the associated affine toric  $\mathbb{C}$ -variety. The torus  $(\mathbb{G}_{m,\mathbb{C}}^{2q}, \tau_2^{\times q})$  acts on the affine toric  $\mathbb{R}$ -variety  $(X_{\delta}, \sigma)$ , where  $\sigma$  is an  $\mathbb{R}$ -structure on  $X_{\delta}$ , if and only if there exists an  $\mathbb{R}$ -group structure  $\tau'$  on  $\mathbb{G}_{m,\mathbb{C}}^n$  equivalent to an  $\mathbb{R}$ -group structure of the form  $\tau_2^{\times q} \times \tau''$  and such that  $\hat{\tau}'(\delta) = \delta$ , where  $\tau''$  is an  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^{n-2q}$ .*

*Proof.* It is a consequence of Lemma 1.10 and Proposition 1.16.  $\square$

The Corollary 1.21 is specific to Weil restriction actions:

**Example 1.22.** Let  $N = \mathbb{Z}^2$ , let  $\delta = \mathbb{Q}_{\geq 0}^2$  be a pointed cone in  $N_{\mathbb{Q}}$ , and let  $\mathbb{A}_{\mathbb{C}}^2$  be the associated affine toric  $\mathbb{C}$ -variety. Since  $\hat{\tau}_2(\delta) = \delta$ , the  $\mathbb{R}$ -group structure  $\tau_2$  on  $\mathbb{G}_{m,\mathbb{C}}^2$  extends to an  $\mathbb{R}$ -structure  $\sigma$  on  $\mathbb{A}_{\mathbb{C}}^2$  defined by  $\sigma(x, y) = (\bar{y}, \bar{x})$ . Note that the real torus  $(\mathbb{G}_{m,\mathbb{C}}, \tau_0)$  acts on  $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$  by  $t \cdot (x, y) = (tx, ty)$ , but  $\tau_2$  is not equivalent to  $\tau_0 \times \tau_1$  (see Example 1.9).

## 2. ALTMANN-HAUSEN PRESENTATION FOR NORMAL AFFINE $\mathbb{C}$ -VARIETIES

In this section, we introduce the group of tailed polyhedra, which will serve as the group of coefficients for the polyhedral divisors, and we recall the main results obtained by Altmann-Hausen in [2]. We also recall some basic facts about convex geometry. Our main references for this are Altmann-Hausen article's [2] and Fulton book's [9].

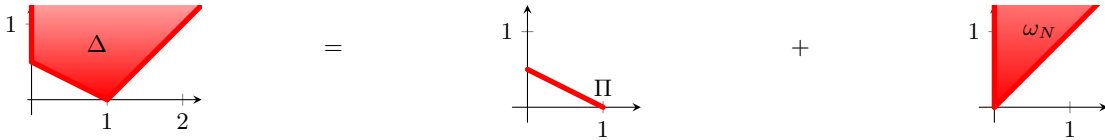
**2.1. Tailed polyhedra and polyhedral divisors.** A subset  $\Pi \subset N_{\mathbb{Q}}$  is called a *polytope* if there exists a finite set  $S \subset N_{\mathbb{Q}}$  such that  $\Pi$  is the convex hull of  $S$ , and it is called a *rational polytope* if  $S$  can be taken inside the lattice  $N$ . A proper face  $\Pi'$  of  $\Pi$  is the intersection of  $\Pi$  with a supporting affine hyperplane.

A *convex polyhedron* is the intersection of finitely many closed affine half spaces in  $N_{\mathbb{Q}}$ . For us, a *polyhedron* in  $N_{\mathbb{Q}}$  is always a convex polyhedron. The *relative interior* of a polyhedron  $\Delta$ , denoted by  $\text{Relint}(\Delta)$ , is obtained by removing all proper faces from  $\Delta$ . Moreover, any polyhedron  $\Delta$  in  $N_{\mathbb{Q}}$  admits a Minkowski sum decomposition:

$$\Delta = \Pi + \omega_N,$$

where  $\Pi \subset N_{\mathbb{Q}}$  is a polytope and  $\omega_N \subset N_{\mathbb{Q}}$  is a cone. In this decomposition, the cone  $\omega_N$  is unique and called the *tail cone* of  $\Delta$  (see [2, §1]).

**Example 2.1.**  $\Delta = \Pi + \omega_N$



**Definition 2.2.** Let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$ . By a  $\omega_N$ -polyhedron in  $N_{\mathbb{Q}}$ , we mean a polyhedron in  $N_{\mathbb{Q}}$  having the cone  $\omega_N$  as its tail cone. We denote the set of all  $\omega_N$ -polyhedra in  $N_{\mathbb{Q}}$  by  $\text{Pol}_{\omega_N}^+(N_{\mathbb{Q}})$ .

The Minkowski sum of two  $\omega_N$ -polyhedra in  $N_{\mathbb{Q}}$  is again a  $\omega_N$ -polyhedron in  $N_{\mathbb{Q}}$ . Thus, endowed with Minkowski sum,  $\text{Pol}_{\omega_N}^+(N_{\mathbb{Q}})$  is an abelian monoid, whose neutral element is  $\omega_N$  (see [2, §1]).

We now introduce the language of polyhedral divisors and proper polyhedral divisors. The idea is to replace rational coefficient by tailed polyhedra (see [2, §2]).

Let  $Y$  be a normal  $\mathbb{C}$ -variety. The group of Weil divisors on  $Y$  is denoted  $\text{WDiv}(Y)$  and the group of Cartier divisors on  $Y$  is denoted by  $\text{CDiv}(Y)$ . Since  $Y$  is normal, we have an inclusion  $\text{CDiv}(Y) \subset \text{WDiv}(Y)$ . A Cartier (resp. Weil)  $\mathbb{Q}$ -divisor is an element of  $\mathbb{Q} \otimes_{\mathbb{Z}} \text{CDiv}(Y)$  (resp  $\mathbb{Q} \otimes_{\mathbb{Z}} \text{WDiv}(Y)$ ). The sheaf of sections  $\mathcal{O}(D)$  of a Weil  $\mathbb{Q}$ -divisor  $D$  on  $Y$  is defined by:

$$H^0(V, \mathcal{O}(D)) := \{f \in \mathbb{C}(Y) \mid \text{div}_V(f|_V) + D|_V \geq 0\} \cup \{0\},$$

where  $V \subset Y$  is an open subset. Now we turn to divisors with tailed polyhedra coefficients. Let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$ . An  $\omega_N$ -polyhedral divisor on  $Y$  is a formal sum:

$$\mathcal{D} = \sum_Z \Delta_Z \otimes Z \in \text{Pol}_{\omega_N}^+(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \text{WDiv}(Y)$$

over all prime divisors  $Z \subset Y$ , and  $\Delta_Z = \omega_N$  for all but finitely prime divisors  $Z$ .

Let  $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$  be a  $\omega_N$ -polyhedral divisor on  $Y$ . For a prime divisor  $Z$  on  $Y$  we denote the support function of  $\Delta_Z$  by

$$h_Z : \omega_N^{\vee} \rightarrow \mathbb{Q}, \quad m \mapsto \min\{\langle m, v \rangle \mid v \in \Delta_Z\}.$$

For every  $m \in \omega_N^{\vee}$  we can evaluate  $\mathcal{D}$  in  $m$  by letting  $\mathcal{D}(m)$  be the Weil  $\mathbb{Q}$ -divisor on  $Y$  defined by:

$$\mathcal{D}(m) := \sum_Z h_Z(m) \otimes Z \in \mathbb{Q} \otimes_{\mathbb{Z}} \text{WDiv}(Y).$$

Before introducing proper polyhedral divisors, we recall the following definitions:

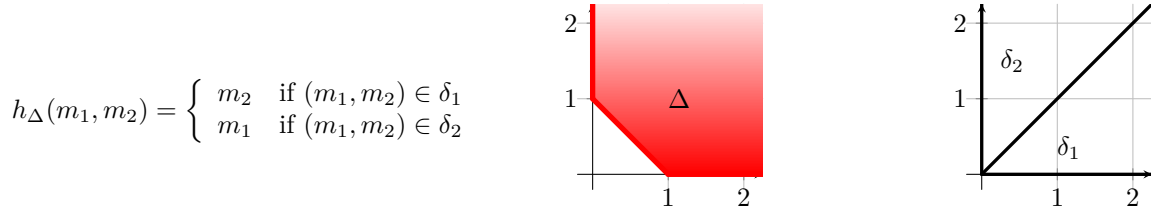
**Definition 2.3.** A Cartier  $\mathbb{Q}$ -divisor  $D$  on  $Y$  is called *semi-ample* if, for some  $n \in \mathbb{N}^*$ , the set of open subsets  $Y_f := Y \setminus \text{Supp}(\text{div}(f) + D)$ , with  $f \in H^0(Y, \mathcal{O}_Y(nD))$ , cover  $Y$ . A Cartier  $\mathbb{Q}$ -divisor  $D$  on  $Y$  is called *big* if, for some  $n \in \mathbb{N}^*$ , there exists a section  $f \in H^0(Y, \mathcal{O}_Y(nD))$  with an affine non-vanishing locus  $Y_f$ .

**Definition 2.4.** A *proper  $\omega_N$ -polyhedral divisor* on  $Y$ , abbreviated an  $\omega_N$ -pp-divisor, is an  $\omega_N$ -polyhedral divisor  $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$  on  $Y$  satisfying the following properties:

- (i) for all  $m \in \omega_N^{\vee} \cap M$ ,  $\mathcal{D}(m)$  is a semi-ample Cartier  $\mathbb{Q}$ -divisor on  $Y$ ; and
- (ii) for all  $m \in \text{Relint}(\omega_N^{\vee}) \cap M$ ,  $\mathcal{D}(m)$  is big.

The sum of two  $\omega_N$ -pp-divisors with respect to a given cone  $\omega_N$  is again an  $\omega_N$ -pp-divisor. Thus,  $\omega_N$ -pp-divisors form a monoid denoted by  $\text{PPDiv}_{\mathbb{Q}}(Y, \omega_N)$ .

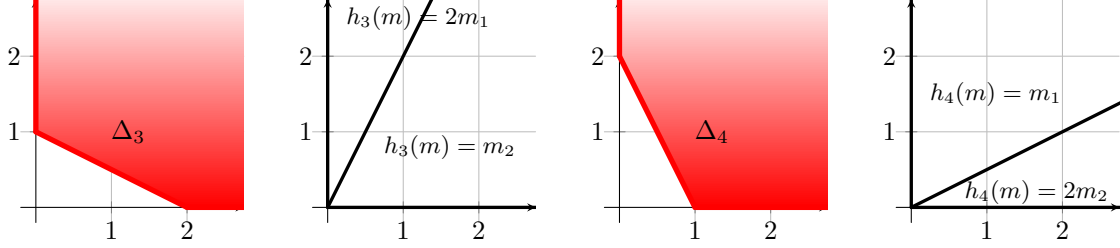
**Example 2.5.** Let  $N = \mathbb{Z}^2$ , let  $\omega_N = \mathbb{Q}_{\geq 0}^2$ , and let  $\Delta$  be the  $\omega_N$ -polyhedron defined below. The normal quasifan associated to  $\Delta$  consists of the two cones  $\delta_1$  and  $\delta_2$  refining the cone  $\omega_N^{\vee} = \mathbb{Q}_{\geq 0}^2$  of the dual lattice  $M = \mathbb{Z}^2$  (see [13, §1.1.2] for details). The support function  $h_{\Delta} : \omega_N^{\vee} \rightarrow \mathbb{Q}$ ,  $m \mapsto \min\{\langle m, v \rangle \mid v \in \Delta\}$  is linear on each  $\delta_i$ , and we obtain:



Consider a divisor  $\mathcal{D} := \Delta \otimes D$  on a normal variety  $Y$ , where  $D$  is a prime divisor. Then,

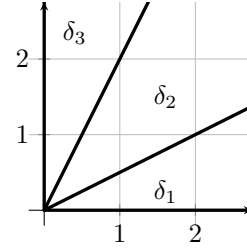
$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes D & \text{if } (m_1, m_2) \in \delta_1 \\ m_1 \otimes D & \text{if } (m_1, m_2) \in \delta_2 \end{cases}$$

**Example 2.6.** Let  $N = \mathbb{Z}^2$ , let  $\omega_N = \mathbb{Q}_{\geq 0}^2$ , let  $\Delta_1 = \Delta_2 = \omega_N$  and let  $\Delta_3$  and  $\Delta_4$  be the  $\omega_N$ -polyhedra defined in the following illustrations. The normal quasifan associated to  $\Delta_3$  (resp.  $\Delta_4$ ) consists of two cones refining the cone  $\omega_N^\vee = \mathbb{Q}_{\geq 0}^2$  of the dual lattice  $M = \mathbb{Z}^2$ . The support function of the polyhedron  $\Delta_i$  is denoted  $h_i : \omega_N^\vee \rightarrow \mathbb{Q}$ ,  $m \mapsto \min\{\langle m|v \rangle \mid v \in \Delta_i\}$ . Note that  $h_1 = h_2 = 0$ .



Consider the divisor  $\mathcal{D} := \Delta_1 \otimes D_1 + \Delta_2 \otimes D_2 + \Delta_3 \otimes D_3 + \Delta_4 \otimes D_4$  on a normal variety  $Y$ , where the  $D_i$  are prime divisors. We have  $\mathcal{D} = \Delta_3 \otimes D_3 + \Delta_4 \otimes D_4$ . Considering the fan refining these two normal fan, we obtain:

$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes D_3 + 2m_2 \otimes D_4 & \text{if } (m_1, m_2) \in \delta_1 \\ m_2 \otimes D_3 + m_1 \otimes D_4 & \text{if } (m_1, m_2) \in \delta_2 \\ 2m_1 \otimes D_3 + m_1 \otimes D_4 & \text{if } (m_1, m_2) \in \delta_3 \end{cases}$$



**2.2. Altmann-Hausen presentation.** Let us present the main results of [2] about the geometrico-combinatorial presentation of normal affine  $\mathbb{C}$ -varieties endowed with a torus action.

**Definition 2.7.** A  $\mathbb{C}$ -variety  $Y$  is said to be *semi-projective* if its  $\mathbb{C}$ -algebra of global functions  $H^0(Y, \mathcal{O})$  is finitely generated and  $Y$  is projective over  $Y_0 = \text{Spec}(H^0(Y, \mathcal{O}))$ .

**Remark 2.8.** Note that affine varieties and projective varieties are semi-projective. A semi-projective variety is quasi-projective. Indeed,  $Y \rightarrow Y_0$  is a projective morphism, moreover  $Y_0$  is an affine variety (so quasi-projective). Then the morphism  $Y \rightarrow \text{Spec}(\mathbb{C})$  is quasi-projective.

Let  $X$  be a  $\mathbb{C}$ -variety endowed with an action of the torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[M])$  of *weight cone*  $\omega_M \subset M_{\mathbb{Q}}$ . We write  $\mathbb{C}[X] = \bigoplus_{m \in \omega_M \cap M} \mathbb{C}[X]_m$ . For all  $m \in \omega_M \cap M$ , we denote:

$$\mathbb{C}(X)_m := \left\{ \frac{f}{g} \mid \exists k \in M, f \in \mathbb{C}[X]_{m+k}, g \in \mathbb{C}[X]_k \right\} \subset \mathbb{C}(X).$$

Let  $Y$  be a normal semi-projective variety, let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$  and let  $\mathcal{D} = \sum_{Z \in \omega_N} \Delta_Z \otimes Z$  be an  $\omega_N$ -pp-divisor on  $Y$ . By [2, proposition 2.11], for all  $m, m' \in M \cap \omega_N^\vee$ , we have  $\mathcal{D}(m + m') \geq \mathcal{D}(m) + \mathcal{D}(m')$ . So, for all  $m, m' \in \omega_N^\vee \cap M$ , we have a map:

$$H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \otimes H^0(Y, \mathcal{O}_Y(\mathcal{D}(m'))) \rightarrow H^0(Y, \mathcal{O}_Y(\mathcal{D}(m + m'))).$$

This ensures that the  $H^0(Y, \mathcal{O}_Y)$ -sub-modules  $H^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))$  of  $\mathbb{C}(Y)$  can be put together into an  $M$ -graded  $\mathbb{C}$ -algebra:

$$A[Y, \mathcal{D}] := \bigoplus_{m \in \omega_N^\vee \cap M} H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \mathfrak{X}_m \subset \mathbb{C}(Y)[M],$$

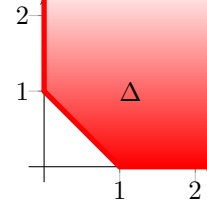
where  $\mathfrak{X}_m$  is an indeterminate of weight  $m$ . We denote by  $X[Y, \mathcal{D}] := \text{Spec}(A[Y, \mathcal{D}])$  the associated  $\mathbb{T}$ -scheme. The general idea of the construction of Altmann-Hausen is to identify  $\mathbb{C}(X)_0$  with  $\mathbb{C}(Y)$  for some semi-projective variety  $Y$ , and use an appropriate pp-divisor on  $Y$  to construct the grading of  $\mathbb{C}[X]$  via an identification between  $\mathbb{C}[X]_m$  and  $H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) \mathfrak{X}_m$ .

**Theorem 2.9.** [2, Theorems 3.1 and 3.4]. *Fix a torus  $\mathbb{T}$ . Let  $M$  its character lattice.*

- (i) *Let  $Y$  be a normal semi-projective variety, let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$ , and let  $\mathcal{D}$  be a  $\omega_N$ -pp-divisor on  $Y$ . The affine scheme  $X[Y, \mathcal{D}]$  is a normal variety, of dimension  $\dim(Y) + \dim(\mathbb{T})$ , endowed with a  $\mathbb{T}$ -action of weight cone  $\omega_N^\vee$ .*
- (ii) *Conversely, let  $X$  be an affine normal variety endowed with a  $\mathbb{T}$ -action, and let  $\omega_N$  be the cone in  $N_{\mathbb{Q}}$  dual to the weight cone. There exists a normal semi-projective variety  $Y$  and a  $\omega_N$ -pp-divisor  $\mathcal{D}$  on  $Y$  such that the graded  $\mathbb{C}$ -algebras  $\mathbb{C}[X]$  and  $A[Y, \mathcal{D}]$  are isomorphic.*

**Example 2.10.** Consider the example 1.13. The affine variety  $\mathbb{A}_{\mathbb{C}}^3$  endowed with the action of  $\mathbb{G}_{m, \mathbb{C}}^2$  given by  $(s, t) \cdot (x, y, z) = (sx, ty, stz)$  is described by a semi-projective variety  $Y := \mathbb{P}_{\mathbb{C}}^1 = \mathbb{A}_{\mathbb{C}}^1 \cup \{\infty\}$ , and a pp-divisor on  $Y$  defined by  $\mathcal{D} := \Delta \otimes \{\infty\}$ , where  $\Delta$  is the polyhedral defined below. Using example 2.5, we have:

$$\mathcal{D}(m_1, m_2) = \begin{cases} m_2 \otimes \{\infty\} & \text{if } (m_1, m_2) \in \delta_1 \\ m_1 \otimes \{\infty\} & \text{if } (m_1, m_2) \in \delta_2 \end{cases}$$



### 3. ALTMANN-HAUSEN PRESENTATION FOR NORMAL AFFINE $\mathbb{R}$ -VARIETIES

**3.1. Equivariant toric downgrading.** Given an action of a complex torus  $\mathbb{T}$  on a normal affine  $\mathbb{C}$ -variety  $X$ , Altmann and Hausen indicate in [2, §11] a recipe on how to determine a semi-projective variety  $Y_X$  and a pp-divisor  $\mathcal{D}_X$  mentioned in Theorem 2.9. The idea is to embed  $\mathbb{T}$ -equivariantly  $X$  into a toric variety  $\mathbb{A}_{\mathbb{C}}^n$  such that  $X$  intersects the dense open orbit of  $\mathbb{A}_{\mathbb{C}}^n$  for the natural  $\mathbb{G}_{m, \mathbb{C}}^n$ -action. They construct a normal semi-projective variety  $Y$  and a pp-divisor  $\mathcal{D}$  describing the  $\mathbb{T}$ -action on  $\mathbb{A}_{\mathbb{C}}^n$ . From these data, they obtain  $Y_X$  and  $\mathcal{D}_X$  describing the  $\mathbb{T}$ -action on  $X$ .

In this section, we describe a  $\mathbb{T}$ -equivariant embedding  $X \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$  which is also  $\Gamma$ -equivariant (Proposition 3.1), and we use this embedding to extend the Altmann-Hausen presentation to the case of real torus actions on affine  $\mathbb{R}$ -varieties (Theorems 3.3 and 3.6).

**Proposition 3.1.** *Let  $M$  be a lattice of rank  $d$ . Let  $X$  be an affine  $\mathbb{C}$ -variety endowed with an action of  $\mathbb{T}$  and let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ . Let  $\sigma$  be an  $\mathbb{R}$ -structure on  $X$ , and let  $\tau$  be an  $\mathbb{R}$ -group structure on  $\mathbb{T}$ . If the real torus  $(\mathbb{T}, \tau)$  acts on  $(X, \sigma)$ , then there exist  $n \in \mathbb{N}$ ,  $n \geq d$  such that:*

- (i) *There is an  $\mathbb{R}$ -group structure  $\tau'$  on  $\mathbb{G}_{m, \mathbb{C}}^n$  that extends to an  $\mathbb{R}$ -structure  $\sigma'$  on  $\mathbb{A}_{\mathbb{C}}^n$ ;*
- (ii)  *$(\mathbb{T}, \tau)$  is a closed subgroup of  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau')$ ; and*
- (iii)  *$(X, \sigma)$  is a closed subvariety of  $(\mathbb{A}_{\mathbb{C}}^n, \sigma')$  and  $(X, \sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$  is  $(\mathbb{T}, \tau)$ -equivariant. Moreover,  $X$  intersects the dense open orbit of  $\mathbb{A}_{\mathbb{C}}^n$  for the natural  $\mathbb{G}_{m, \mathbb{C}}^n$ -action, and the weight cone of  $\mathbb{A}_{\mathbb{C}}^n$  is the weight cone of  $X$ .*

*Proof.* (i) The algebra  $\mathbb{C}[X]$  is finitely generated, so we can write  $\mathbb{C}[X] = \mathbb{C}[\tilde{g}_1, \dots, \tilde{g}_k]$  with  $\tilde{g}_i \in \mathbb{C}[X] \setminus \{0\}$ . Since  $\mathbb{C}[X] = \bigoplus_{m \in M} \mathbb{C}[X]_m$ , there exists homogeneous elements  $\tilde{g}_{i,j}$  such that  $\tilde{g}_i = \tilde{g}_{i,1} + \dots + \tilde{g}_{i,k_i}$ , hence  $\mathbb{C}[X] = \mathbb{C}[\tilde{g}_{i,j}]$ . Note that  $\mathbb{C}[X] = \mathbb{C}[\tilde{g}_{i,j}, \sigma^\#(\tilde{g}_{i,j})]$ . Moreover, by lemma 1.15, an homogeneous element is sent to an homogeneous element by  $\sigma^\#$ . Hence we can assume that there exists  $n \in \mathbb{N}$  such that  $\mathbb{C}[X] = \mathbb{C}[g_1, \dots, g_n]$ , where the  $g_i$  are homogeneous of degree  $m_i \in M$  and such that the set  $\{g_i \mid 1 \leq i \leq n\}$  is stable under the involution  $\sigma^\#$ . Let  $\tau'$  and  $\sigma'$  be the maps induced by the antilinear maps  $\tau'^\#(x_i) = x_j$ , and  $\sigma'^\#(x_i) = x_j$ , where  $\sigma^\#(g_i) = g_j$ . This induces an  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m, \mathbb{C}}^n = \text{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$  and a  $\mathbb{R}$ -structure on  $\mathbb{A}_{\mathbb{C}}^n = \text{Spec}(\mathbb{C}[x_1, \dots, x_n])$ .

(ii) The  $\mathbb{C}$ -algebra morphism  $\psi : \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \mathbb{C}[M]$ ,  $x_i \mapsto \chi^{m_i}$  is surjective since the  $\mathbb{T}$ -action on  $X$  is effective. Since  $(\mathbb{T}, \tau)$  acts on  $(X, \sigma)$ ,  $\psi$  is  $\Gamma$ -equivariant. So, the  $\mathbb{R}$ -algebra morphism  $\psi^\Gamma : \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\Gamma \rightarrow \mathbb{C}[M]^\Gamma$  is well defined and surjective. Hence,  $(\mathbb{T}, \tau)$  is a closed subgroup of  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau')$ .

(iii) The  $\mathbb{C}$ -algebra morphism  $\varphi : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[X]$ ,  $x_i \mapsto g_i$  is surjective and induces a  $\mathbb{C}$ -algebra isomorphism  $\mathbb{C}[g_1, \dots, g_n] \cong \mathbb{C}[x_1, \dots, x_n]/\mathfrak{a}$ , with  $\mathfrak{a} = \text{Ker}(\varphi)$ . Moreover, the morphism

$\varphi$  is  $\Gamma$ -equivariant. So, the  $\mathbb{R}$ -algebra morphism  $\varphi^\Gamma : \mathbb{C}[x_1, \dots, x_n]^\Gamma \rightarrow \mathbb{C}[X]^\Gamma$  is well defined and surjective. Hence,  $(X, \sigma)$  is a closed subvariety of  $(\mathbb{A}_{\mathbb{C}}^n, \sigma')$ .

Note that  $\varphi$  is  $\mathbb{T}$ -equivariant, so the closed immersion  $X \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$  is  $\mathbb{T}$ -equivariant. Moreover, the comorphism of the  $\mathbb{T}$ -action on  $\mathbb{A}_{\mathbb{C}}^n$  is given by:

$$\tilde{\mu}^\sharp : \mathbb{C}[x_1, \dots, x_n] \rightarrow \mathbb{C}[M] \otimes \mathbb{C}[x_1, \dots, x_n], \quad x_i \mapsto \chi^{m_i} \otimes x_i$$

Then, the following diagram commutes:

$$\begin{array}{ccccc}
 & & \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] & \xrightarrow{\tilde{\mu}^\sharp} & \mathbb{C}[M] \otimes \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] \\
 & \swarrow \sigma'^\sharp & \vdots & \nearrow \tau^\sharp \times \sigma'^\sharp & \downarrow id \times \varphi \\
 \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] & \xrightarrow{\tilde{\mu}^\sharp} & \mathbb{C}[M] \otimes \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] & & \\
 \downarrow \varphi & & \downarrow id \times \varphi & & \downarrow id \times \varphi \\
 & \swarrow \sigma^\sharp & \mathbb{C}[X] & \xrightarrow{\mu^\sharp} & \mathbb{C}[M] \otimes \mathbb{C}[X] \\
 \mathbb{C}[X] & \xrightarrow{\mu^\sharp} & \mathbb{C}[M] \otimes \mathbb{C}[X] & \nearrow \tau^\sharp \times \sigma^\sharp & \\
 & & \downarrow & & \\
 & & \mathbb{C}[M] \otimes \mathbb{C}[X] & & 
 \end{array}$$

Hence, the morphism  $\varphi$  is  $(\mathbb{T}, \tau)$ -equivariant, so  $(X, \sigma)$  is a closed subvariety of  $(\mathbb{A}_{\mathbb{C}}^n, \sigma')$ , and  $(X, \sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$  is  $(\mathbb{T}, \tau)$ -equivariant.

Finally, note that for all  $i \in \{1, \dots, n\}$ ,  $x_i \notin \mathfrak{a}$ , hence  $X$  intersects the dense open orbit of  $\mathbb{G}_{m, \mathbb{C}}^n$ . It follows that the weight cone of  $\mathbb{A}_{\mathbb{C}}^n$  is the weight cone of  $X$ .  $\square$

**Example 3.2.** We pursue example 2.10. The action of  $(\mathbb{G}_{m, \mathbb{C}}^2, \tau_2)$  on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$  comes from the  $\Gamma$ -equivariant inclusion of  $(\mathbb{G}_{m, \mathbb{C}}^2, \tau_2)$  in  $(\mathbb{G}_{m, \mathbb{C}}^3, \tau')$  given by  $(s, t) \mapsto (s, t, st)$ , where  $\tau'$  is the  $\mathbb{R}$ -group structure defined by  $\tau' = \tau_2 \times \tau_0$  (see Example 1.19). We denote by  $M$  and  $M'$  the character lattices of  $\mathbb{G}_{m, \mathbb{C}}^2$  and  $\mathbb{G}_{m, \mathbb{C}}^3$  respectively. Then, we obtain the diagrams of Remark 1.8 with:

$$F := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad P := \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \quad \hat{\tau}_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{\tau}' := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \hat{\tau}_Y := [1]$$

**3.2. Real torus actions on normal affine  $\mathbb{R}$ -varieties.** We present the main theoretical results of this article concerning the presentation of affine  $\mathbb{R}$ -varieties endowed with real torus actions:

**Theorem 3.3.** *Let  $(\mathbb{T}, \tau)$  be a real torus, let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ , and let  $(Y, \sigma_Y)$  be a normal semi-projective  $\mathbb{R}$ -variety. Let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$ , and let  $\mathcal{D}$  be an  $\omega_N$ -pp-divisor on  $Y$ . Assume that there exists a monoid morphism  $h : \omega_N^\vee \cap M \rightarrow \mathbb{C}(Y)^*$  such that*

$$\forall m \in \omega_N^\vee \cap M, \quad \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m))) \quad \text{and} \quad h(m)\sigma_Y^\sharp(h(\tilde{\tau}(m))) = 1, \quad (1)$$

then there exists an  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on the normal affine variety  $X[Y, \mathcal{D}]$  such that  $(\mathbb{T}, \tau)$  acts on  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ .

**Remark 3.4.** The datum  $(Y, \mathcal{D})$  is used to construct the affine  $\mathbb{T}$ -variety  $X[Y, \mathcal{D}]$ . The monoid morphism  $h$  is the additional datum that encodes the real structure  $\sigma_{X[Y, \mathcal{D}]}$  on  $X[Y, \mathcal{D}]$  such that  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  is a  $(\mathbb{T}, \tau)$ -variety.

**Remark 3.5.** Since  $Y$  is an integral scheme, for any affine open subset  $U \subset Y$ , the ring  $\mathcal{O}_Y(U)$  is integral and  $\mathbb{C}(Y) = \text{Frac}(\mathcal{O}_Y(U))$ . Hence,  $\sigma_Y$  induces an  $\mathbb{R}$ -field automorphism denoted  $\sigma_Y^\sharp : \mathbb{C}(Y) \rightarrow \mathbb{C}(Y)$ . The field of invariant rational functions is denoted by  $\mathbb{C}(Y)^\Gamma := \{f \in \mathbb{C}(Y) \mid \sigma_Y^\sharp(f) = f\}$ . A classical result, due to Artin, states that if  $G$  is a finite group of automorphisms of a field  $\mathbb{k}$ , then  $G = \text{Gal}(\mathbb{k}/\mathbb{k}^G)$ . So, the extension  $\mathbb{C}(Y)/\mathbb{C}(Y)^\Gamma$  is Galois, with Galois group  $\Gamma$ .

*Proof.* By Theorem 2.9 (1),  $X[Y, \mathcal{D}] := \text{Spec}(A[Y, \mathcal{D}])$  is a normal affine  $\mathbb{C}$ -variety endowed with a  $\mathbb{T}$ -action, of weight cone  $\omega_N^\vee$ . This action is obtained from the following comorphism:

$$\mu^\sharp : A[Y, \mathcal{D}] \rightarrow \mathbb{C}[M] \otimes A[Y, \mathcal{D}], \quad f\mathfrak{X}_m \mapsto \chi^m \otimes f\mathfrak{X}_m.$$

We now construct an  $\mathbb{R}$ -structure on  $X[Y, \mathcal{D}]$  such that  $(\mathbb{T}, \tau)$  acts on  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ . Condition (1) implies that, for all  $m \in \omega_N^\vee \cap M$ ,

$$\alpha_m : H^0(Y, \mathcal{O}_Y(\mathcal{D}(m)))\mathfrak{X}_m \rightarrow H^0(Y, \mathcal{O}_Y(\mathcal{D}(\tilde{\tau}(m))))\mathfrak{X}_{\tilde{\tau}(m)}, \quad f\mathfrak{X}_m \mapsto \sigma_Y^\sharp(f)h(\tilde{\tau}(m))\mathfrak{X}_{\tilde{\tau}(m)}$$

are isomorphisms of  $A[Y, \mathcal{D}]_0$ -modules and these isomorphisms collect into an involution  $\bigoplus_{m \in \omega_N^\vee \cap M} \alpha_m$  on the direct sum  $A[Y, \mathcal{D}]$ . The latter corresponds to a  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on  $X[Y, \mathcal{D}]$ .

Finally,  $(\mathbb{T}, \tau)$  acts on  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  since the following diagram commutes:

$$\begin{array}{ccc} A[Y, \mathcal{D}] & \xrightarrow{\mu^\sharp} & \mathbb{C}[M] \otimes A[Y, \mathcal{D}] \\ \sigma_{X[Y, \mathcal{D}]}^\sharp \downarrow & & \downarrow \tau^\sharp \otimes \sigma_{X[Y, \mathcal{D}]}^\sharp \\ A[Y, \mathcal{D}] & \xrightarrow{\mu^\sharp} & \mathbb{C}[M] \otimes A[Y, \mathcal{D}] \end{array}$$

□

**Theorem 3.6.** *Let  $(\mathbb{T}, \tau)$  be a real torus and let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ . Let  $(X, \sigma_X)$  be a normal affine  $\mathbb{R}$ -variety endowed with a  $(\mathbb{T}, \tau)$ -action. Let  $\omega_N$  be the cone in  $N_{\mathbb{Q}}$  dual to the weight cone  $\omega_M$ . There exists a normal semi-projective  $\mathbb{R}$ -variety  $(Y, \sigma_Y)$ , an  $\omega_N$ -pp-divisor  $\mathcal{D}$  on  $Y$ , and a monoid morphism  $h : \omega_M \cap M \rightarrow \mathbb{C}(Y)^*$  such that*

$$\forall m \in \omega_M \cap M, \quad \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m))) \quad \text{and} \quad h(m)\sigma_Y^\sharp(h(\tilde{\tau}(m))) = 1,$$

and such that the affine varieties  $(X, \sigma_X)$  and  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic.

*Proof.* • **Step 0: Preliminaries.**

Using proposition 3.1, there exists  $n \in \mathbb{N}$  such that  $(\mathbb{T}, \tau)$  is a closed subgroup of  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau')$  and  $(X, \sigma_X)$  is a closed  $(\mathbb{T}, \tau)$ -equivariant subvariety of  $(\mathbb{A}_{\mathbb{C}}^n, \sigma)$ . So, let  $\mathfrak{a}$  be the ideal of  $\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] = \mathbb{C}[x_1, \dots, x_n]$  such that  $\mathbb{C}[X]$  is  $(\Gamma \times \mathbb{T})$ -equivariantly isomorphic to  $\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]/\mathfrak{a}$ . We write  $\mathbb{C}[X] = \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]/\mathfrak{a}$ . Let  $M' := \text{Hom}_{gr}(\mathbb{G}_{m, \mathbb{C}}^n, \mathbb{G}_{m, \mathbb{C}})$  and let  $M_Y$  be the sublattice of  $M'$  constructed in Remark 1.8. We have the commutative diagrams of Remark 1.8. Recall that there always exists a section  $s^* : M \rightarrow M'$ , but not necessarily  $\Gamma$ -equivariant, and a cosection  $t^* : M' \rightarrow M_Y$ . These homomorphisms satisfy  $F^* \circ s^* = \text{Id}_M$ ,  $t^* \circ P^* = \text{Id}_{M_Y}$  and  $P^* \circ t^* = \text{Id}_{M'} - s^* \circ F^*$ .

Since  $\text{Frac}(\mathbb{C}[M']) = \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)^*$ , the section  $s^*$  induces a morphism

$$u : \omega_M \cap M \rightarrow \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)^*, \quad m \mapsto \chi^{s^*(m)}$$

such that for all  $m \in \omega_M \cap M$ ,  $u(m) \in \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_m$ .

Note that, for all  $m \in \omega_M \cap M$ ,  $\mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_m = \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0 u(m)$ . Indeed, since  $\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]_m \subset \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_m$ , we can write  $\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]_m = \widetilde{\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]_m} u(m)$ , with  $\widetilde{\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]_m} \subset \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0$ . Then, we can write:

$$\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] = \bigoplus_{m \in \omega_M \cap M} \widetilde{\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]_m} u(m) \subset \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0[M].$$

Since for all  $i$ ,  $x_i \notin \mathfrak{a}$ , for all  $m \in \omega_M \cap M$  we have  $u_X(m) \in \mathbb{C}(X)_m$ , where  $u_X(m)$  is obtained from the surjective morphism  $\mathbb{C}[\mathbb{A}_{\mathbb{C}}^n] \twoheadrightarrow \mathbb{C}[X]$ . It follows a morphism  $u_X : \omega_M \cap M \rightarrow \mathbb{C}(X)^*$ . Hence we can write

$$\mathbb{C}[X] = \bigoplus_{m \in \omega_M \cap M} \mathbb{C}[X]_m = \bigoplus_{m \in \omega_M \cap M} \widetilde{\mathbb{C}[X]_m} u_X(m) \subset \mathbb{C}(X)_0[M],$$

where, for all  $m \in \omega_M \cap M$ ,  $\mathbb{C}[X]_m = \widetilde{\mathbb{C}[X]_m} u_X(m)$  and  $\widetilde{\mathbb{C}[X]_m} \subset \mathbb{C}(X)_0$ .

• **Step 1: Altmann-Hausen quotient and divisors.**

Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $N'$ . The cone in  $N'_{\mathbb{Q}}$  of the toric variety  $\mathbb{A}_{\mathbb{C}}^n$  is  $\mathbb{Q}_{\geq 0}^n$ , and  $F^*(\mathbb{Q}_{\geq 0}^n) = \omega_M$ . Since the fan in  $N'_{\mathbb{Q}}$  generated by  $\{e_1, \dots, e_n\}$  is  $\Gamma$ -stable (for  $\hat{\tau}'$ ) and  $P$  is  $\Gamma$ -equivariant, the fan  $\Lambda_Y$  in  $(N_Y)_{\mathbb{Q}}$  generated by  $\{P(e_1), \dots, P(e_n)\}$  is  $\Gamma$ -stable (for  $\hat{\tau}_Y$ ).

Let  $Y$  be the toric variety obtained from the fan  $\Lambda_Y$ ; it is a semi-projective variety (see [5, Proposition 7.2.9]). Since  $\Lambda_Y$  is  $\Gamma$ -stable, the  $\mathbb{R}$ -group structure  $\tau_Y$  on  $\mathbb{T}_Y := \text{Spec}(\mathbb{C}[M_Y])$  extends to an  $\mathbb{R}$ -structure  $\sigma_Y$  on  $Y$  by [10, Proposition 1.19].

Let  $Y_X$  be the closure of the image of  $X \cap \mathbb{G}_{m,\mathbb{C}}^n$  in  $Y$  by the surjective group homomorphism  $\pi : \mathbb{G}_{m,\mathbb{C}}^n \rightarrow \mathbb{T}_Y$  composed with the inclusion  $\mathbb{T}_Y \hookrightarrow Y$ . Since these morphisms are  $\Gamma$ -equivariant, the  $\mathbb{R}$ -structure on  $Y$  restricts to an  $\mathbb{R}$ -structure  $\sigma_{Y_X}$  on  $Y_X$ . The normalization  $\tilde{Y}_X$  of  $Y_X$ , with morphism  $\nu : \tilde{Y}_X \rightarrow Y_X$ , is a semi-projective variety. Using universal property of normalization and the fact that  $\nu$  is an isomorphism on a dense open subset of  $Y_X$ , there exists an  $\mathbb{R}$ -structure  $\sigma_{\tilde{Y}_X}$  on  $\tilde{Y}_X$  which makes the following diagram commute:

$$\begin{array}{ccc} \tilde{Y}_X & \xrightarrow{\sigma_{\tilde{Y}_X}} & \tilde{Y}_X \\ \nu \downarrow & & \downarrow \nu \\ Y_X & \xrightarrow{\sigma_{Y_X}} & Y_X \end{array}$$

For each ray of the fan  $\Lambda_Y$ , we denote by  $v_i$  its first lattice vector. To a ray spanned by  $v_i$  corresponds a toric divisor  $D_{v_i}$  on  $Y$ .

The divisor  $\mathcal{D} = \sum_{v_i} \Delta_{v_i} \otimes D_{v_i}$ , where  $\Delta_{v_i} := s(\mathbb{Q}_{\geq 0}^n \cap P^{-1}(v_i))$ , is an  $\omega_N$ -pp-divisor on  $Y$ . Let  $\mathcal{D}_X$  be the divisor obtained by pulling back  $\mathcal{D}$  to  $\tilde{Y}_X$ . It is an  $\omega_N$ -pp-divisor on  $\tilde{Y}_X$  (see [1, Proposition 8.1] which is not in the published version).

• **Step 2:** *Isomorphisms*  $\mathbb{C}(Y) \cong \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0$  and  $\mathbb{C}(Y_X) \cong \mathbb{C}(X)_0$ .

Observe that  $\pi^\sharp : \mathbb{C}[M_Y] \rightarrow \mathbb{C}[M']_0$  is an isomorphism. Hence,  $\pi^\sharp$  induces an isomorphism:

$$\text{Frac}(\mathbb{C}[M_Y]) \rightarrow \text{Frac}(\mathbb{C}[M']_0), \quad \frac{f}{g} = \frac{\sum a_i \chi^{m_i}}{\sum b_j \chi^{m_j}} \mapsto \frac{\pi^\sharp(f)}{\pi^\sharp(g)} = \frac{\sum a_i \chi^{P^*(m_i)}}{\sum b_j \chi^{P^*(m_j)}}.$$

For all  $m \in M$ , note that  $(F^*)^{-1}(m) = s^*(m) + \text{Ker}(F^*)$ . Hence,  $\text{Frac}(\mathbb{C}[M']_0) = \text{Frac}(\mathbb{C}[M']_0)$ . Since  $\mathbb{T}_Y$  is a dense open subset of  $Y$ , we have  $\mathbb{C}(Y) = \mathbb{C}(\mathbb{T}_Y)$ . Since  $\mathbb{G}_{m,\mathbb{C}}^n$  is a dense open subset of  $\mathbb{A}_{\mathbb{C}}^n$ , we have  $\text{Frac}(\mathbb{C}[M']) = \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)$ . Therefore,  $\text{Frac}(\mathbb{C}[M']_0) = \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0$ . Finally, we obtain a  $\Gamma$ -equivariant isomorphism:

$$\varphi : \mathbb{C}(Y) \rightarrow \mathbb{C}(\mathbb{A}_{\mathbb{C}}^n)_0.$$

Moreover, the inclusion  $X \hookrightarrow \mathbb{A}_{\mathbb{C}}^n$  is  $\mathbb{T}$ -equivariant, the variety  $\mathbb{T}_Y \cap Y_X$  is affine and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{C}}^n & \xrightarrow{\pi} & \mathbb{T}_Y \\ \uparrow & & \uparrow \\ \mathbb{G}_{m,\mathbb{C}}^n \cap X & \xrightarrow{\quad} & \mathbb{T}_Y \cap Y_X = \pi(\mathbb{G}_{m,\mathbb{C}}^n \cap X) \end{array}$$

Therefore the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[\mathbb{T}_Y \cap Y_X] & \xleftarrow{\quad} & \mathbb{C}[\mathbb{G}_{m,\mathbb{C}}^n \cap X]_0 \\ \uparrow & & \uparrow \\ \mathbb{C}[\mathbb{T}_Y] & \xleftarrow{\pi^\sharp} & \mathbb{C}[\mathbb{G}_{m,\mathbb{C}}^n]_0 \end{array}$$

It follows an isomorphism  $\mathbb{C}(\mathbb{T}_Y \cap Y_X) \rightarrow \text{Frac}(\mathbb{C}[\mathbb{G}_{m,\mathbb{C}}^n \cap X]_0)$ . Since,

$$\text{Frac}(\mathbb{C}[\mathbb{G}_{m,\mathbb{C}}^n \cap X]_0) = \mathbb{C}(\mathbb{G}_{m,\mathbb{C}}^n \cap X)_0, \quad \mathbb{C}(Y_X) = \mathbb{C}(\mathbb{T}_Y \cap Y_X), \quad \text{and} \quad \mathbb{C}(X) = \mathbb{C}(\mathbb{G}_{m,\mathbb{C}}^n \cap X),$$

we obtain a  $\Gamma$ -equivariant isomorphism  $\mathbb{C}(Y_X) \cong \mathbb{C}(X)_0$ . Note that we have a  $\Gamma$ -equivariant isomorphism  $\mathbb{C}(Y_X) \cong \mathbb{C}(\tilde{Y}_X)$ , therefore, the isomorphism:

$$\varphi_X : \mathbb{C}(\tilde{Y}_X) \rightarrow \mathbb{C}(X)_0$$

is  $\Gamma$ -equivariant.

• **Step 3:** *Isomorphisms*  $A[Y, \mathcal{D}] \cong \mathbb{C}[\mathbb{A}_{\mathbb{C}}^n]$  and  $A[\tilde{Y}_X, \mathcal{D}_X] \cong \mathbb{C}[X]$ .



Let  $m \in \omega_M \cap M$ . Consider the polyhedron  $\Delta(m) := (F^*)^{-1}(m) \cap \mathbb{Q}_{\geq 0}^n \subset M'_\mathbb{Q}$  and the polyhedron  $\Delta_Y(m) := t^*(\Delta(m)) \subset (M_Y)_\mathbb{Q}$ . Note that

$$\widetilde{\mathbb{C}[\mathbb{A}_\mathbb{C}^n]}_m := \bigoplus_{m' \in \mathbb{Q}_{\geq 0}^n \cap (F^*)^{-1}(m) \cap M'} \mathbb{C}\chi^{m'} = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \mathbb{C}\chi^{P^*(m_Y)} = \varphi \left( \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \mathbb{C}\chi^{m_Y} \right).$$

It follows from Lemma [9, §3.4] and the proof of [1, Proposition 8.5] that:

$$H^0(Y, \mathcal{O}_Y(\mathcal{D}(m))) = \bigoplus_{m_Y \in \Delta_Y(m) \cap M_Y} \mathbb{C}\chi^{m_Y}.$$

Therefore, we obtain a graded algebra isomorphism:

$$\Phi : A[Y, \mathcal{D}] \rightarrow \mathbb{C}[\mathbb{A}_\mathbb{C}^n], \quad f\mathfrak{X}_m \mapsto \varphi(f)u(m).$$

Moreover, there is a natural surjective graded algebra morphism:

$$\Psi : A[Y, \mathcal{D}] \twoheadrightarrow A[\tilde{Y}_X, \mathcal{D}_X].$$

Let  $\Phi_X : A[\tilde{Y}_X, \mathcal{D}_X] \rightarrow \mathbb{C}(X)_0[M]$  be the morphism defined by  $f\mathfrak{X}_m \mapsto \varphi_X(f)u_X(m)$ . Since the following diagram commutes:

$$\begin{array}{ccc} A[Y, \mathcal{D}] & \xleftarrow{\Phi} & \mathbb{C}[\mathbb{A}_\mathbb{C}^n] \\ \Psi \downarrow & & \downarrow \\ A[\tilde{Y}_X, \mathcal{D}_X] & \xrightarrow{\Phi_X} & \mathbb{C}[X] \end{array}$$

we obtain a graded algebra isomorphism:

$$\Phi_X : A[\tilde{Y}_X, \mathcal{D}_X] \rightarrow \mathbb{C}[X], \quad f\mathfrak{X}_m \mapsto \varphi_X(f)u_X(m).$$

- **Step 4:** Equality  $\sigma_{\tilde{Y}_X}^*(\mathcal{D}_X(m)) = \mathcal{D}_X(\tilde{\tau}(m)) + \text{div}(h(\tilde{\tau}(m)))$ , for all  $m \in \omega_M \cap M$ .

Let  $m \in \omega_M \cap M$ . Let  $h' : \omega_M \cap M \rightarrow \mathbb{C}(\mathbb{A}_\mathbb{C}^n)_0$  be the monoid morphism defined by  $h'(m) := \frac{\sigma^\#(u(\tilde{\tau}(m)))}{u(m)}$ . By Lemma 1.15,  $\sigma^\#(\mathbb{C}[\mathbb{A}_\mathbb{C}^n]_{\tilde{\tau}(m)}) = \mathbb{C}[\mathbb{A}_\mathbb{C}^n]_m$ . It follows  $h'(m) \in \widetilde{\mathbb{C}[\mathbb{A}_\mathbb{C}^n]}_m$ . Moreover, note that  $h'(m)\sigma^\#(h'(\tilde{\tau}(m))) = 1$ . Consider the monoid morphism  $h := \tilde{\varphi}^{-1} \circ h' : \omega_M \cap M \rightarrow \mathbb{C}(Y)^*$ . We construct a  $\mathbb{R}$ -structure on  $X[Y, \mathcal{D}]$  using the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}[\mathbb{A}_\mathbb{C}^n] & \xrightarrow{\sigma^\#} & \mathbb{C}[\mathbb{A}_\mathbb{C}^n] \\ \Phi \uparrow & & \downarrow \Phi^{-1} \\ A[Y, \mathcal{D}] & \longrightarrow & A[Y, \mathcal{D}] \end{array} \quad \begin{array}{ccc} \varphi(f)u(m) & \longmapsto & \sigma^\#(\varphi(f))h'(\tilde{\tau}(m))u(\tilde{\tau}(m)) \\ \uparrow & & \downarrow \\ f\mathfrak{X}_m & \longmapsto & \varphi^{-1}(\sigma^\#(\varphi(f))h'(\tilde{\tau}(m)))\mathfrak{X}_{\tilde{\tau}(m)} \end{array}$$

Since  $\varphi$  is  $\Gamma$ -equivariant, we have  $\varphi^{-1}(\sigma^\#(\varphi(f))h'(\tilde{\tau}(m))) = \sigma_Y^\#(f)h(\tilde{\tau}(m))$ . Hence, the morphism

$$A[Y, \mathcal{D}] \rightarrow A[Y, \mathcal{D}], \quad f\mathfrak{X}_m \mapsto \sigma_Y^\#(f)h(\tilde{\tau}(m))\mathfrak{X}_{\tilde{\tau}(m)}$$

induces an  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on  $X[Y, \mathcal{D}]$ . From this we deduce that:

$$\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m))). \quad (2)$$

Moreover,  $h(m)\sigma_Y^\#(h(\tilde{\tau}(m))) = 1$ . By the same reasoning, we construct an  $\mathbb{R}$ -structure  $\sigma_{X[\tilde{Y}_X, \mathcal{D}_X]}$  on  $X[\tilde{Y}_X, \mathcal{D}_X]$  from the following morphism:

$$A[\tilde{Y}_X, \mathcal{D}_X] \rightarrow A[\tilde{Y}_X, \mathcal{D}_X], \quad f\mathfrak{X}_m \mapsto \sigma_{\tilde{Y}_X}^\#(f)h_X(\tilde{\tau}(m))\mathfrak{X}_{\tilde{\tau}(m)},$$

where  $h_X$  is obtained from  $h$  by the projection  $\mathbb{C}[\mathbb{A}_\mathbb{C}^n] \rightarrow \mathbb{C}[X]$ . Thus we obtain:

$$\sigma_{\tilde{Y}_X}^*(\mathcal{D}_X(m)) = \mathcal{D}_X(\tilde{\tau}(m)) + \text{div}_{\tilde{Y}_X}(h_X(\tilde{\tau}(m)))$$

and  $h_X(m)\sigma_{\tilde{Y}_X}^\#(h_X(\tilde{\tau}(m))) = 1$ . □

**Remark 3.7.** The construction of the real variety  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  does not depend on the choice of the cosection. Indeed, for  $j \in \{1, 2\}$ , let  $s_j : N' \rightarrow N$  be two cosections, let  $\mathcal{D}_j := \sum_i \Delta_{v_i}^j \otimes D_{v_i}$  be the two associated pp-divisors, and let  $h_j : \omega_M \cap M \rightarrow \mathbb{C}(Y)^*$  be the monoid morphisms constructed in Step 4). Note that for all  $m \in M$ ,  $(s_1^* - s_2^*)(m) \in \text{Ker}(F^*) = \text{Im}(P^*)$ , thus there exists a lattice homomorphism  $s_0 : N_Y \rightarrow N$  such that  $s_1 - s_2 = s_0 \circ P$ . Let  $g : M \rightarrow \mathbb{C}(Y)^*$  be the morphism defined by  $g(m) := \chi^{s_0^*(m)}$ . Let  $m \in \omega_M \cap M$ . Since  $\Delta_{v_i}^1 = s_0(v_i) + \Delta_{v_i}^2$ , we have :

$$\mathcal{D}_1(m) = \sum_i \langle s_0^*(m) | v_i \rangle \otimes D_{v_i} + \mathcal{D}_2(m) = \text{div}_Y(g(m)) + \mathcal{D}_2(m).$$

Therefore the  $M$ -graded algebras  $A[Y, \mathcal{D}_1]$  and  $A[Y, \mathcal{D}_2]$  are isomorphic via:

$$A[Y, \mathcal{D}_1] \rightarrow A[Y, \mathcal{D}_2], f\mathfrak{X}_m \mapsto fg(m)\mathfrak{X}_m.$$

Moreover, since

$$\frac{\sigma_Y^\sharp(g(\tilde{\tau}(m)))}{g(m)} = \frac{\sigma_Y^\sharp(\chi^{s_0^*(\tilde{\tau}(m))})}{\chi^{s_0^*(m)}} = \tilde{\varphi}_X^{-1} \left( \frac{\sigma_Y^\sharp(\chi^{P^* \circ s_0^*(\tilde{\tau}(m))})}{\chi^{P^* \circ s_0^*(m)}} \right) = \tilde{\varphi}_X^{-1} \left( \frac{\sigma_Y^\sharp(u_1(\tilde{\tau}(m))) u_2(m)}{u_1(m) \sigma_Y^\sharp(u_2(\tilde{\tau}(m)))} \right) = \frac{h_1(m)}{h_2(m)},$$

the following diagram commutes:

$$\begin{array}{ccc} A[Y, \mathcal{D}_1] & \xrightarrow{\sigma_{X[Y, \mathcal{D}_1]}^\sharp} & A[Y, \mathcal{D}_1] & & f\mathfrak{X}_m & \longmapsto & \sigma_Y^\sharp(f)h_1(\tilde{\tau}(m))\mathfrak{X}_{\tilde{\tau}(m)} \\ \cong \downarrow & & \downarrow \cong & & \downarrow & & \downarrow \\ A[Y, \mathcal{D}_2] & \xrightarrow{\sigma_{X[Y, \mathcal{D}_2]}^\sharp} & A[Y, \mathcal{D}_2] & & fg(m)\mathfrak{X}_m & \mapsto & \sigma_Y^\sharp(fg(m))h_2(\tilde{\tau}(m))\mathfrak{X}_{\tilde{\tau}(m)} \end{array}$$

Hence, the varieties  $(X[Y, \mathcal{D}_1], \sigma_{X[Y, \mathcal{D}_1]})$  and  $(X[Y, \mathcal{D}_2], \sigma_{X[Y, \mathcal{D}_2]})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic.

**3.3. Galois cohomology and real torus actions.** We recall some cohomological results in view of simplifying the Altmann-Hausen presentation in the case where the real acting torus is quasi-split.

Let  $G$  be an abstract group equipped with a  $\Gamma$ -action denoted by  $\star$ . A *cocycle*  $a : \Gamma \rightarrow G$  is a map such that  $a_{id} = 1$  and  $a_\gamma(\gamma \star a_\gamma) = 1$ . Two cocycles  $a$  and  $b$  are *equivalent* if there exists  $g \in G$  such that  $b_\gamma = g^{-1}a_\gamma(\gamma \star g)$ . The set of cocycles modulo this equivalence relation is the *first pointed* set of Galois cohomology  $H^1(\Gamma, G)$ . If  $G$  is an abelian  $\Gamma$ -group, then  $H^1(\Gamma, G)$  is a group (see [15]).

In the following result (Corollary 3.8 (ii)), we see that the Altmann-Hausen presentation simplifies if a certain cohomology set is trivial. This simplification consists of choosing a pp-divisor on  $Y$  such that  $h = 1$ .

**Corollary 3.8.** *Fix a real torus  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau)$ . Let  $M := \text{Hom}_{gr}(\mathbb{G}_{m, \mathbb{C}}^n, \mathbb{G}_{m, \mathbb{C}})$ .*

(i) *Let  $(Y, \sigma_Y)$  be a normal semi-projective variety. Let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$  and  $\mathcal{D}$  be an  $\omega_N$ -pp divisor on  $Y$ . If*

$$\forall m \in \omega_N^\vee \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)),$$

*then there exists an  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on the affine variety  $X[Y, \mathcal{D}]$  such that  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau)$  acts on  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ .*

(ii) *Let  $(X, \sigma_X)$  be an affine variety endowed with an action of  $(\mathbb{T}, \tau)$  of weight cone  $\omega_M \subset M_{\mathbb{Q}}$ , and let  $\omega_N$  be the cone in  $N$  dual to  $\omega_M$ . Let  $(Y, \sigma_Y)$  be the  $\mathbb{R}$ -variety of Theorem 3.6 and let  $G := \text{Hom}_{gr}(M, \mathbb{C}(Y)^*)$  endowed with the  $\Gamma$ -action  $\gamma \star f := \sigma_Y^\sharp \circ f \circ \tilde{\tau}$ . If  $H^1(\Gamma, G) = \{1\}$ , then there exists an  $\omega_N$ -pp divisor  $\mathcal{D}$  on  $Y$  such that*

$$\forall m \in \omega_M \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)),$$

*and such that the varieties  $(X, \sigma_X)$  and  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic.*

*Proof.* (i) We specify the proof of Theorem 3.3 with  $h = 1 \in \mathbb{C}(Y)^*$  and  $\tilde{\tau} = id_M$ .

(ii) By Theorem 3.6 there exists a normal semi-projective variety  $Y$ , an  $\omega_N$ -pp-divisor  $\mathcal{D}$  on  $Y$ , an  $\mathbb{R}$ -structure  $\sigma_Y$  on  $Y$  and a monoid morphism  $h : \omega_M \cap M \rightarrow \mathbb{C}(Y)^*$  such that  $\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m)))$  and  $h(m)\sigma_Y^\sharp(h(\tilde{\tau}(m))) = 1$  for all  $m \in \omega_M \cap M$ , and such that the varieties

$(X, \sigma_X)$  and  $(X[Y, \mathcal{D}'], \sigma_{X[Y, \mathcal{D}']})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic. Consider  $a : \Gamma \rightarrow G$  defined by  $a_{id} : m \mapsto 1 \in \mathbb{C}(Y)^*$  and  $a_\gamma : m \mapsto h(m)$ . By construction,  $a$  is a cocycle. Since  $H^1(\Gamma, G) = \{1\}$ , the cocycle  $b : \Gamma \rightarrow G$  defined by  $b_{id} = b_\gamma : m \mapsto 1 \in \mathbb{C}(Y)^*$  is equivalent to  $a$ , so there exists  $g \in G$  such that  $h(m) = g^{-1}(m)\sigma_Y^\sharp(g(\tilde{\tau}(m)))$ . Let  $m \in \omega_M \cap M$ , then:

$$\begin{aligned} \sigma_Y^*(\mathcal{D}(m)) &= \mathcal{D}(\tilde{\tau}(m)) + \operatorname{div}_Y(h(\tilde{\tau}(m))) = \mathcal{D}(\tilde{\tau}(m)) + \sigma_Y^* \operatorname{div}_Y(g(m)) - \operatorname{div}_Y(g(\tilde{\tau}(m))) \\ &\iff \sigma_Y^*(\mathcal{D}(m) - \operatorname{div}_Y(g(m))) = \mathcal{D}(\tilde{\tau}(m)) - \operatorname{div}_Y(g(\tilde{\tau}(m))). \end{aligned}$$

So, if  $\mathcal{D}'$  is the pp-divisor defined by  $\mathcal{D}'(m) := \mathcal{D}(m) - \operatorname{div}_Y(g(m))$ , then  $\sigma_Y^*(\mathcal{D}'(m)) = \mathcal{D}'(\tilde{\tau}(m))$  and the  $M$ -graded algebras  $A[Y, \mathcal{D}]$  and  $A[Y, \mathcal{D}']$  are isomorphic with respect to  $\sigma_{X[Y, \mathcal{D}]}^\sharp$  and  $\sigma_{X[Y, \mathcal{D}']}^\sharp$  (see the diagram of Remark 3.7 with  $\mathcal{D}_1 = \mathcal{D}$ ,  $h_1 = h$ ,  $\mathcal{D}_2 = \mathcal{D}'$  and  $h_2 = 1$ ). Hence the varieties  $(X, \sigma_X)$  and  $(X[Y, \mathcal{D}'], \sigma_{X[Y, \mathcal{D}']})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic.  $\square$

**Remark 3.9.** The converse of Corollary 3.8 (ii) is false; see Example 4.4.

We provide some situations with trivial Galois cohomology set making it possible to apply Corollary 3.8 to simplify the Altmann-Hausen presentation. Let  $\mathbb{L}/\mathbb{K}$  be a quadratic extension and denote by  $\{\varphi_{id}, \varphi_\gamma\}$  the Galois group  $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$ .

**Lemma 3.10.** *Let  $M$  be a rank  $n$  lattice and let  $G_n := \operatorname{Hom}_{gr}(M, \mathbb{L}^*)$ . If  $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$  acts on  $G_n$  by  $\gamma \star f := \varphi_\gamma \circ f$ , then  $H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_n) = \{1\}$ .*

*Proof.* We prove this result by induction on  $n$ . Let  $n = 1$ , then  $G_1 \cong \mathbb{L}^*$ . The Galois group  $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$  acts on  $G_1 \cong \mathbb{L}^*$  and this action comes from the  $\Gamma$ -action on  $\mathbb{L}$  defined by  $\gamma \cdot z = \varphi_\gamma(z)$ . By Hilbert's theorem 90,  $H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_1) = \{1\}$ . Let  $n \geq 1$  and assume that the Lemma 3.10 is true for this fixed  $n$ . We have a  $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$ -equivariant short exact sequence of  $\operatorname{Gal}(\mathbb{L}/\mathbb{K})$ -groups:

$$1 \longrightarrow G_n \xrightarrow{\Psi} G_{n+1} \xrightarrow{\Psi'} G_1 \longrightarrow 1$$

with  $\Psi(f) : \mathbb{Z}^n \oplus \mathbb{Z} \rightarrow \mathbb{L}^*$ ,  $(k_1, \dots, k_n, k) \mapsto f(k_1, \dots, k_n)$  and  $\Psi'(g) : \mathbb{Z} \rightarrow \mathbb{L}^*$ ,  $k \mapsto g(0, \dots, 0, k)$ , where  $f \in G_n$  and  $g \in G_{n+1}$ . This induces an exact sequence in Galois cohomology:

$$1 \rightarrow G_n^{\operatorname{Gal}(\mathbb{L}/\mathbb{K})} \rightarrow G_{n+1}^{\operatorname{Gal}(\mathbb{L}/\mathbb{K})} \rightarrow G_1^{\operatorname{Gal}(\mathbb{L}/\mathbb{K})} \rightarrow H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_n) \rightarrow H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_{n+1}) \rightarrow H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_1)$$

By induction,  $H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_n) = \{1\}$  and  $H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_1) = \{1\}$ . Therefore,

$$H^1(\operatorname{Gal}(\mathbb{L}/\mathbb{K}), G_{n+1}) = \{1\}.$$

$\square$

**Lemma 3.11.** *Let  $M := M_0 \oplus M_0$ , where  $M_0 \cong \mathbb{Z}^n$ , and let  $\rho := \Gamma \rightarrow \operatorname{GL}(M)$  be the representation which exchanges the two factors. Let  $G = \operatorname{Hom}_{gr}(M, \mathbb{L}^*)$  be endowed with the  $\Gamma$ -action defined by  $\gamma \star f := \varphi_\gamma \circ f \circ \rho(\gamma^{-1})$ , Then  $H^1(\Gamma, G) = \{1\}$ .*

*Proof.* A cocycle is uniquely determined by a homomorphism  $h = a_\gamma : M \rightarrow \mathbb{L}^*$  which satisfies  $h(m, m')\varphi_\gamma(h(m', m)) = 1$  (the constant homomorphism) for every  $(m, m') \in M = M_0 \oplus M_0$ . In particular, we have  $\varphi_\gamma(h(-m', 0))h(0, -m') = 1$ , hence  $\varphi_\gamma(h(-m', 0)) = h(0, m')$ . Now let  $f : M \rightarrow \mathbb{L}^*$  be the homomorphism defined by  $f(m, m') = h(-m, 0)$ . Then, we have:

$$f^{-1}(m, m')(\gamma \star f)(m, m') = h(m, 0)\varphi_\gamma(h(-m', 0)) = h(m, 0)h(0, m') = h(m, m').$$

Hence  $a$  is equivalent to the cocycle  $\Gamma \rightarrow G$ ,  $\gamma \mapsto 1$ , and so  $H^1(\Gamma, G) = \{1\}$ .  $\square$

**Lemma 3.12.** *Let  $M := M_1 \oplus M_0 \oplus M_0$ , where  $M_1 \cong \mathbb{Z}^p$  and  $M_0 \cong \mathbb{Z}^q$ , and let  $\rho := \rho_1 \times \rho_0 : \Gamma \rightarrow \operatorname{GL}(M)$  be the representation such that  $\rho_1$  is the identity on  $M_1$  and  $\rho_0$  exchanges the two factors on  $M_0 \oplus M_0$ . Let  $G = \operatorname{Hom}_{gr}(M, \mathbb{L}^*)$  be endowed with the  $\Gamma$ -action defined by  $\gamma \star f := \varphi_\gamma \circ f \circ \rho(\gamma^{-1})$ . Then  $H^1(\Gamma, G) = \{1\}$ .*

*Proof.* Denote  $G_1 := \operatorname{Hom}_{gr}(M_1, \mathbb{L}^*)$  and  $G_0 := \operatorname{Hom}_{gr}(M_0 \oplus M_0, \mathbb{L}^*)$ . We have a  $\Gamma$ -equivariant short exact sequence of groups:

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_0 \longrightarrow 1$$

We obtain an exact sequence in Galois cohomology:

$$1 \longrightarrow G_1^\Gamma \longrightarrow G^\Gamma \longrightarrow G_0^\Gamma \longrightarrow H^1(\Gamma, G_1) \longrightarrow H^1(\Gamma, G) \longrightarrow H^1(\Gamma, G_0)$$

By Lemma 3.10 and Lemma 3.11, we have  $H^1(\Gamma, G_1) = \{1\}$  and  $H^1(\Gamma, G_0) = \{1\}$ . Hence

$$H^1(\Gamma, G) = \{1\}.$$

□

A consequence of the Lemma 3.12 is the following proposition:

**Proposition 3.13.** *Fix a real torus  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau)$  where  $\tau = \tau_0^p \times \tau_2^q$  and  $n = p + 2q$ . Let  $M := \text{Hom}_{gr}(\mathbb{G}_{m, \mathbb{C}}^n, \mathbb{G}_{m, \mathbb{C}})$ .*

(i) *Let  $(Y, \sigma_Y)$  be a normal semi-projective variety. Let  $\omega_N$  be a pointed cone in  $N_{\mathbb{Q}}$  and  $\mathcal{D}$  be an  $\omega_N$ -pp divisor on  $Y$ . If*

$$\forall m \in \omega_N^\vee \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)), \quad (3)$$

*then there exists an  $\mathbb{R}$ -structure  $\sigma_{X[Y, \mathcal{D}]}$  on the affine variety  $X[Y, \mathcal{D}]$  such that  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau)$  acts on  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$ .*

(ii) *Let  $(X, \sigma_X)$  be an affine variety endowed with an action of  $(\mathbb{T}, \tau)$  of weight cone  $\omega_M \subset M_{\mathbb{Q}}$ , and let  $\omega_N$  be the cone in  $N$  dual to  $\omega_M$ . There exists a semi-projective variety  $(Y, \sigma_Y)$  and an  $\omega_N$ -pp divisor  $\mathcal{D}$  on  $Y$  such that*

$$\forall m \in \omega_M \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)),$$

*and such that the varieties  $(X, \sigma_X)$  and  $(X[Y, \mathcal{D}'], \sigma_{X[Y, \mathcal{D}]})$  are  $(\mathbb{T}, \tau)$ -equivariantly isomorphic.*

For a complexity one quasi-split  $(\mathbb{T}, \tau)$ -action on an affine  $\mathbb{R}$ -variety  $(X, \sigma)$  (i.e.  $\text{Dim}(\mathbb{T}) = \text{Dim}(X) - 1$ ), we recover the real version of Langlois' result in [12] about quasi-split torus actions on varieties of complexity one over an arbitrary field.

**3.4. Some one-to-one correspondence.** The two main results (Theorem 3.3 and 3.6) establish correspondences between real affine varieties endowed with a real torus action and triples  $(Y, \mathcal{D}, h)$ . In this section, we focus on this correspondence. In general, there is no one-to-one correspondence between  $(\mathbb{T}, \tau)$ -varieties and triple  $(Y, \mathcal{D}, h)$ . However, Altmann and Hausen define the notion of minimal pp-divisor in [2, Section 8] that leads us to the following result:

**Theorem 3.14.** *Let  $(\mathbb{T}, \tau)$  be a real torus and let  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m, \mathbb{C}})$ . Let  $\omega_N \subset N_{\mathbb{Q}}$  (resp.  $\omega'_N \subset N_{\mathbb{Q}}$ ) be a pointed cone, let  $(Y, \sigma_Y)$  (resp.  $(Y', \sigma'_Y)$ ) be a normal semi projective variety,  $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \omega_N)$  (resp.  $\mathcal{D}' \in \text{PPDiv}_{\mathbb{Q}}(Y', \omega'_N)$ ) be a minimal pp-divisor and  $h : \omega_N^\vee \cap M \rightarrow \mathbb{C}(Y)$  be a monoid morphism such that:*

$$\forall m \in \omega_N^\vee \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}(m)) + \text{div}_Y(h(\tilde{\tau}(m))) \text{ and } h(m)\sigma_Y^\sharp(h(\tilde{\tau}(m))) = 1,$$

*(resp.  $h' : \omega_N'^\vee \cap M \rightarrow \mathbb{C}(Y')$ ). The affine  $\mathbb{R}$ -varieties  $(X[Y, \mathcal{D}], \sigma_{X[Y, \mathcal{D}]})$  and  $(X[Y', \mathcal{D}'], \sigma_{X[Y', \mathcal{D}]})$  are  $(\mathbb{T}, \tau)$ -isomorphic if and only if the following holds:*

- (i) *there exists an isomorphism  $\psi : (Y, \sigma_Y) \rightarrow (Y', \sigma'_Y)$ ;*
- (ii) *there exists a lattice automorphism  $L : N \rightarrow N$  such that  $L \circ \hat{\tau} = \hat{\tau} \circ L$ ;*
- (iii) *there exists a monoid morphism  $g : M \rightarrow \mathbb{C}(Y)$ ;*
- (iv) *for all  $m \in \omega_M \cap M$ ,  $\psi^*(\mathcal{D}'(m)) = \mathcal{D}(L^*(m)) + \text{div}_Y(g(m))$ ;*
- (v) *for all  $m \in \omega_M \cap M$ ,  $\frac{\sigma_Y^\sharp(g(\tilde{\tau}(m)))}{g(m)} = \frac{\psi^\sharp(h(m))}{h'(L^*(m))}$  (i.e. the cocycles defined by  $h' \circ L^*$  and  $\psi^\sharp \circ h$  are equivalent).*

*Proof.* By [2, Theorem 8.8], the affine  $\mathbb{C}$ -varieties  $X[Y, \mathcal{D}]$  and  $X[Y', \mathcal{D}']$  are  $\mathbb{T}$ -isomorphic if and only if the following holds:

- (i) there exists an isomorphism  $\psi : Y' \rightarrow Y$ ;

- (ii) there exists a lattice automorphism  $L : N \rightarrow N$ ;
- (iii) there exists a monoid morphism  $g : M \rightarrow \mathbb{C}(Y')$ ;
- (iv) for all  $m \in \omega_M \cap M$ ,  $\psi^*(\mathcal{D}'(m)) = \mathcal{D}(L^*(m)) + \text{div}_Y(g(m))$ ;

The morphisms  $\psi$  and  $L$  induces a  $\mathbb{T}$ -equivariant isomorphism of graded algebras:

$$\Psi : A[Y, \mathcal{D}] \rightarrow A[Y', \mathcal{D}'], f\mathfrak{X}_m \mapsto \psi^\sharp(f)g(m)\mathfrak{X}_{L^*(m)}.$$

Therefore the diagram

$$\begin{array}{ccc} A[Y, \mathcal{D}] & \xrightarrow{\Psi} & A[Y', \mathcal{D}'] \\ \sigma_Y^\sharp \downarrow & & \sigma_{Y'}^\sharp \downarrow \\ A[Y, \mathcal{D}] & \xrightarrow{\Psi} & A[Y', \mathcal{D}'] \end{array}$$

$$\text{commutes if and only for all } m \in \omega_M \cap M, \frac{\sigma_Y^\sharp(g(\tilde{\tau}(m)))}{g(m)} = \frac{\psi^\sharp(h(m))}{h'(L^*(m))}. \quad \square$$

#### 4. EXAMPLES

**4.1. Split real torus actions on normal affine  $\mathbb{R}$ -varieties.** We describe actions of the real split torus  $\mathbb{G}_{m, \mathbb{R}}^n$  on affine  $\mathbb{R}$ -varieties. By definition, a  $\mathbb{G}_{m, \mathbb{R}}^n$ -action on an  $\mathbb{R}$ -variety  $(X, \sigma_X)$  is an action of the real torus  $(\mathbb{G}_{m, \mathbb{C}}^n, \tau_0^{\times n})$  on  $(X, \sigma_X)$ . Fix a real torus  $\mathbb{G}_{m, \mathbb{R}}^n = (\mathbb{G}_{m, \mathbb{C}}^n, \tau_0^{\times n})$  and let  $M := \text{Hom}_{gr}(\mathbb{G}_{m, \mathbb{C}}^n, \mathbb{G}_{m, \mathbb{C}})$ . The condition (3) of Proposition 3.13 becomes:

$$\forall m \in \omega_M \cap M, \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(m).$$

**Example 4.1.** We pursue Example 1.22. In the case of a  $\mathbb{G}_{m, \mathbb{R}}$ -action, the sequence obtained from the inclusion  $(X, \sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$  of Proposition 3.1 does not always have a  $\Gamma$ -equivariant section. Indeed, consider the affine variety  $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$ , where  $\sigma(x, y) = (\bar{y}, \bar{x})$ . Note that the torus  $(\mathbb{G}_{m, \mathbb{C}}, \tau_0)$  acts on  $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$  by  $t \cdot (x, y) = (tx, ty)$ . Then, we have the following split short exact sequence

$$1 \longrightarrow \mathbb{G}_{m, \mathbb{C}} \longrightarrow \mathbb{G}_{m, \mathbb{C}}^2 \longrightarrow \mathbb{G}_{m, \mathbb{C}} \longrightarrow 1$$

with  $\mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{G}_{m, \mathbb{C}}^2, t \mapsto (t, t)$  and  $\mathbb{G}_{m, \mathbb{C}}^2 \rightarrow \mathbb{G}_{m, \mathbb{C}}, (s, t) \mapsto s/t$ . We obtain the diagrams of Remark 1.8 with  $M' = \mathbb{Z}^2$ ,  $M = \mathbb{Z}$ ,  $M_Y = \mathbb{Z}$ , and with the following lattice homomorphisms:

$$F^* := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad P^* := \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \tilde{\tau}' := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \tilde{\tau}_Y := [-1]; \quad \tilde{\tau} = [1].$$

We can show that there is no  $\Gamma$ -equivariant section  $s^* : M \rightarrow \mathbb{Z}^2$ . Indeed, note that if such a section exists, we obtain  $\mathbb{G}_{m, \mathbb{R}} \times \mathbb{S}^1 \cong R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m, \mathbb{C}})$ , which is false (see Proposition 1.7). Let

$$s^* := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be a section. Then, an Altmann-Hausen presentation of the  $\mathbb{G}_{m, \mathbb{R}}$ -action on  $(\mathbb{A}_{\mathbb{C}}^2, \sigma)$  is:

- $\mathbb{P}^1 = \mathbb{A}_{\mathbb{C}, v}^1 \cup \{\infty\}$ ;
- $\sigma_Y : v \mapsto \bar{v}^{-1}$ ;
- $\mathcal{D} := [1, +\infty[\otimes\{\infty\}$ ;
- $h := w = x/y$ .

Now we give a  $(\Gamma \times \mathbb{G}_{m, \mathbb{C}})$ -equivariant inclusion  $(X, \sigma) \hookrightarrow (\mathbb{A}_{\mathbb{C}}^n, \sigma')$  which admits a  $\Gamma$ -equivariant section. First, note that  $\mathbb{A}_{\mathbb{C}}^2 \cong \text{Spec}(\mathbb{C}[x, y, z]/(x + y - z)) \subset \mathbb{A}_{\mathbb{C}}^3$ , where the closed embedding is given by:

$$\mathbb{A}_{\mathbb{C}}^2 \rightarrow \mathbb{A}_{\mathbb{C}}^3, (x, y) \mapsto (x, y, x + y).$$

Consider the action of  $\mathbb{G}_{m, \mathbb{C}}$  on  $\mathbb{A}_{\mathbb{C}}^3$  defined by  $t \cdot (x, y, z) = (tx, ty, tz)$ . This action is obtained from the inclusion:

$$\mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbb{G}_{m, \mathbb{C}}^3, t \mapsto (t, t, t).$$

Consider the  $\mathbb{R}$ -group structure on  $\mathbb{G}_{m,\mathbb{C}}^3$  defined by  $\sigma'(t_1, t_2, t_3) = (\overline{t_2}, \overline{t_1}, \overline{t_3})$ . The closed immersion  $\mathbb{A}_{\mathbb{C}}^2 \cong \text{Spec}(\mathbb{C}[x, y, z]/(x+y-z)) \subset \mathbb{A}_{\mathbb{C}}^3$  is  $(\Gamma \times \mathbb{G}_{m,\mathbb{C}})$ -equivariant. We obtain the diagrams of Remark 1.8 with  $M' = \mathbb{Z}^3$ ,  $M = \mathbb{Z}$ ,  $M_Y = \mathbb{Z}^2$ , and with the following lattice homomorphisms:

$$F^* := \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}; \quad P^* := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}; \quad \tilde{\tau}' := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \tilde{\tau}_Y := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad s^* := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The section  $s^*$  is  $\Gamma$ -equivariant. An Altmann-Hausen presentation of the  $\mathbb{G}_{m,\mathbb{R}}$ -action on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$  is given by:

- $Y := \mathbb{P}^2 = U_1 \cup U_2 \cup U_3$ , where

$$U_1 = \text{Spec}(\mathbb{C}[v_1, w_1]), \quad U_2 = \text{Spec}(\mathbb{C}[v_2, w_2]), \quad \text{and} \quad U_3 = \text{Spec}(\mathbb{C}[v_3, w_3]),$$

with gluing morphism obtained from  $(v_1 = x/z, w_1 = y/z)$ ,  $(v_2 = y/x, w_2 = z/x)$  and  $(v_3 = z/y, w_3 = x/y)$ ;

- $\sigma_Y$  is the  $\mathbb{R}$ -structure exchanging  $x$  and  $y$  and fixing  $z$ ;
- $\mathcal{D} := [1, +\infty[\otimes D$ , where  $D|_{U_1} = 0$ ,  $D|_{U_2} = \{w_2 = 0\}$  and  $D|_{U_3} = \{v_3 = 0\}$ ; and
- $h := 1$ .

We deduce from this an Altmann-Hausen presentation of the  $\mathbb{G}_{m,\mathbb{R}}$ -action on  $(\text{Spec}(\mathbb{C}[x, y, z]/(x+y-z)), \sigma')$ :

- $Y_X := U_{1X} \cup U_{2X} \cup U_{3X} \cong \mathbb{P}^1$ , where

$$U_{1X} = \text{Spec}(\mathbb{C}[v_1, w_1]/(v_1 + w_1 - 1)), \quad U_{2X} = \text{Spec}(\mathbb{C}[v_2, w_2]/(v_2 - w_2 + 1)),$$

$$\text{and} \quad U_{3X} = \text{Spec}(\mathbb{C}[v_3, w_3]/(v_3 - w_3 - 1));$$

- $\sigma_{Y_X} = \sigma_Y|_X$ ;
- $\mathcal{D}_X := [1, +\infty[\otimes D$ , where  $D|_{U_1} = 0$ ,  $D|_{U_2} = \{(1, 0)\}$  and  $D|_{U_3} = \{(0, 1)\}$ ; and
- $h_X := 1$ .

**4.2. Weil restriction actions on normal affine  $\mathbb{R}$ -varieties.** By definition, a  $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ -action on a real algebraic variety  $(X, \sigma_X)$  is an action of the real torus  $(\mathbb{G}_{m,\mathbb{C}}, \tau_2)$  on  $(X, \sigma_X)$ . Fix a real torus  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ . Let  $M := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^2, \mathbb{G}_{m,\mathbb{C}})$  and  $N$  its dual lattice. The condition (3) of Proposition 3.13 becomes:

$$\forall m \in \omega_M \cap M, \quad \sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(\tilde{\tau}_2(m)).$$

**Example 4.2.** The sequences of Example 3.2 admit a  $\Gamma$ -equivariant section  $s : N' \rightarrow N$  defined by

$$s := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

An Altmann-Hausen presentation of the  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ -action on  $(\mathbb{A}_{\mathbb{C}}^3, \sigma')$  is given by

$$((Y, \sigma_Y), \mathcal{D}, h),$$

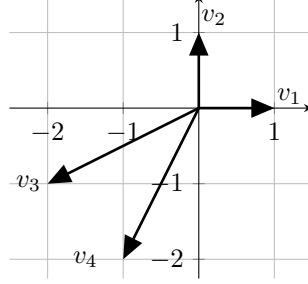
where (see Example 3.2):

- $Y := \mathbb{P}^1 = \mathbb{A}_{\mathbb{C}}^1 \cup \{\infty\}$ ;
- $\sigma_Y$  is the complex conjugation on the coordinates;
- $\mathcal{D} := \Delta \otimes \{\infty\}$ ; and
- $h := 1$ .

**Example 4.3.** We give an Altmann-Hausen presentation of the  $\mathbb{G}_{m,\mathbb{C}}^2$ -action on  $\mathbb{A}_{\mathbb{C}}^4$  introduced in Example 1.14. Using toric downgrading results of [2, §11], we obtain the presentation of the  $\mathbb{G}_{m,\mathbb{C}}^2$ -action on  $X$ . Consider the immersion  $\mathbb{T} := \mathbb{G}_{m,\mathbb{C}}^2 \hookrightarrow \mathbb{G}_{m,\mathbb{C}}^4$ ,  $(s, t) \mapsto (s, t, st^2, s^2t)$ . We denote  $\mathbb{T}_Y := \mathbb{G}_{m,\mathbb{C}}^4/\mathbb{T}$ ,  $M := \text{Hom}_{gr}(\mathbb{T}, \mathbb{G}_{m,\mathbb{C}})$ ,  $M' := \text{Hom}_{gr}(\mathbb{G}_{m,\mathbb{C}}^4, \mathbb{G}_{m,\mathbb{C}})$  and  $M_Y := \text{Hom}_{gr}(\mathbb{T}_Y, \mathbb{G}_{m,\mathbb{C}})$ . Then, we have the split short exact sequences of Remark 1.8 with:

$$F^* := \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}; \quad P^* := \begin{bmatrix} -1 & -2 \\ -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \tilde{\tau}' := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad \tilde{\tau}_Y = \tilde{\tau} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The section  $s : N' \rightarrow N$  defined by  $s := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$  is  $\Gamma$ -equivariant. Let  $Y$  be the toric variety defined by the following fan obtained from  $P$  (see [2, §11] for details):



Since this fan is stable under the lattice involution  $\hat{\tau}_Y$ , the  $\mathbb{R}$ -group structure  $\tau_Y$  extends to an  $\mathbb{R}$ -structure  $\sigma_Y$  on  $Y$ . Let  $\mathcal{D}$  be the divisor defined in Example 2.6, where  $D_1, \dots, D_4$  are the toric divisors obtained from the rays  $v_1, \dots, v_4$  respectively.

An Altmann-Hausen presentation of the  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ -action on  $(\mathbb{A}_{\mathbb{C}}^4, \sigma')$  is given by  $((Y, \sigma_Y), \mathcal{D}, h = 1)$ . An Altmann-Hausen presentation of the  $(\mathbb{G}_{m,\mathbb{C}}^2, \tau_2)$ -action on  $(X, \sigma)$  is given by  $((Y_X, \sigma_{Y_X}), \mathcal{D}_X, h_X)$ , where:

- $Y_X$  is the closure of the image of  $\mathbb{G}_{m,\mathbb{C}}^4 \cap X$  in  $Y$ ;
- $\mathcal{D}_X := \Delta_3 \otimes D_3 \cap Y_X + \Delta_4 \otimes D_4 \cap Y_X$ ;
- $\sigma_{Y_X} = \sigma_Y|_{Y_X}$ ; and
- $h_X := 1$ .

**4.3. Circles actions on normal affine  $\mathbb{R}$ -varieties.** By definition, a  $\mathbb{S}^1$ -action on a algebraic  $\mathbb{R}$ -variety  $(X, \sigma_X)$  is an action of the torus  $(\mathbb{G}_{m,\mathbb{C}}, \tau_1)$  on  $(X, \sigma_X)$ . Note that  $\mathbb{G}_{m,\mathbb{C}}$  acts on  $X$  and the algebra  $\mathbb{C}[X]$  is graded by  $M \cong \mathbb{Z}$ . By [6, Lemma 1.7],  $\mathbb{C}[X]_m \neq 0$  for all  $m \in M$ , so the weight cone of this action is  $\omega_M := M_{\mathbb{Q}}$ .

In this case, the pair  $(\mathcal{D}, h)$  on the quotient  $(Y, \sigma_Y)$  mentioned in Theorem 3.3 (ii) consists of a proper polyhedral divisor  $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$  and a  $\Gamma$ -invariant rational function  $h$  on  $Y$  such that  $\sigma_Y^*(\mathcal{D}(m)) = \mathcal{D}(-m) + \text{div}_Y(h^{-m})$  for all  $m \in M$  (we recover [6, Theorem 2.7]).

In the case of  $\mathbb{S}^1$ -actions, we do not have  $\text{H}^1(\Gamma, \mathbb{C}(Y)^*) = \{1\}$ . Indeed, in contrast to the split case, we cannot apply Hilbert's theorem 90 because the action defined on  $\mathbb{C}(Y)^*$  does not extend to an action on the field  $\mathbb{C}(Y)$  (see the proof of Lemma 3.10).

Let  $Y = \text{Spec}(\mathbb{C})$  endowed with the complex conjugation, and let  $G := \text{Hom}_{gr}(\mathbb{Z}, \mathbb{C}(Y)^*) \cong \text{Hom}_{gr}(\mathbb{Z}, \mathbb{C}^*) \cong \mathbb{C}^*$ . The  $\Gamma$ -action on  $G \cong \mathbb{C}^*$  is given by  $\gamma \star z := \bar{z}^{-1}$ . A cocycle is thus a complex number  $z \in \mathbb{C}^*$  such that  $z\bar{z}^{-1} = 1$ , that is a real number. This cocycle is equivalent to  $1 \in \mathbb{C}^*$  if there exists  $w \in G \cong \mathbb{C}^*$  such that  $z = w^{-1}\bar{w}^{-1} = |w|^{-2} > 0$ . Then,  $\text{H}^1(\Gamma, \mathbb{C}^*) \cong \{\pm 1\}$ .

**Example 4.4.** (See [6, Proposition 3.1]). There are only two  $\mathbb{R}$ -forms of  $\mathbb{G}_{m,\mathbb{C}}$  compatible with an  $\mathbb{S}^1$ -action: the real circle  $X_1 = \mathbb{S}^1$  and  $X_{-1} = \text{Spec}(\mathbb{R}[x, y]/(x^2 + y^2 + 1))$ . An  $\mathbb{R}$ -structure associated to  $X_1$  is  $\sigma_1 := \tau_1 : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}, z \mapsto \bar{z}^{-1}$  and an  $\mathbb{R}$ -structure associated to  $X_{-1}$  is  $\sigma_{-1} : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}, z \mapsto -\bar{z}^{-1}$ . Consider the action by translation  $\mu : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{G}_{m,\mathbb{C}}, (t, x) \mapsto tx$ . The varieties  $X_1$  and  $X_{-1}$  are endowed with an  $\mathbb{S}^1$ -action since the following diagram commutes for  $i \in \{-1, 1\}$ :

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{\mu} & \mathbb{G}_{m,\mathbb{C}} \\ \tau_1 \times \sigma_i \downarrow & & \downarrow \sigma_i \\ \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} & \xrightarrow{\mu} & \mathbb{G}_{m,\mathbb{C}} \end{array}$$

The  $\mathbb{R}$ -variety  $(Y = \text{Spec}(\mathbb{C}), \sigma_Y)$ , where  $\sigma_Y$  is the complex conjugation, is a real Altmann-Hausen quotient of both  $X_1$  and  $X_{-1}$ .

A pair  $(\mathcal{D}_1, h_1)$  on  $Y$  consists of the trivial divisor and the real number  $h_1 = 1 \in \mathbb{C}(Y)^* = \mathbb{C}^*$ . Note however that  $H^1(\Gamma, \mathbb{C}^*) \neq \{1\}$ .

A pair  $(\mathcal{D}_{-1}, h_{-1})$  on  $Y$  consists of the trivial divisor and the real number  $h_{-1} = -1 \in \mathbb{C}(Y)^* = \mathbb{C}^*$ . Note that we cannot find a complex number  $g \in \mathbb{C}(Y)^*$  such that the cocycle  $h_{-1}$  satisfy  $h_{-1} = g\bar{g}$ , so we cannot find a presentation where the  $\Gamma$ -invariant rational function  $h$  mentioned in Theorem 3.6 is equal to 1.

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