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ON EVENTUAL COMPACTNESS OF COLLISIONLESS KINETIC SEMIGROUPS WITH VELOCITIES BOUNDED AWAY FROM ZERO

B. LODS AND M. MOKHTAR-KHARROUBI

ABSTRACT. In this paper, we consider the long time behaviour of collisionless kinetic equation with stochastic diffuse boundary operators for velocities bounded away from zero. We show that under suitable reasonable conditions, the semigroup is eventually compact. In particular, without any irreducibility assumption, the semigroup converges exponentially to the spectral projection associated to the zero eigenvalue as $t \rightarrow \infty$. This contrasts drastically to the case allowing arbitrarily slow velocities for which the absence of a spectral gap yields to at most algebraic rate of convergence to equilibrium. Some open questions are also mentioned.

Keywords: Kinetic equation; Boundary operators; Non-zero velocities; Convergence to equilibrium.

1. INTRODUCTION

The present paper is the third of a program initiated in [15] and pursued in [16] on the systematic study of L^1 -solutions $\psi(t)$ to the transport equation

$$\partial_t \psi(x, v, t) + v \cdot \nabla_x \psi(x, v, t) = 0, \quad (x, v) \in \Omega \times V, \quad t \geq 0 \quad (1.1a)$$

with initial data

$$\psi(x, v, 0) = \psi_0(x, v), \quad (x, v) \in \Omega \times V, \quad (1.1b)$$

under *diffuse boundary*

$$\psi|_{\Gamma_-} = H(\psi|_{\Gamma_+}), \quad (1.1c)$$

where Ω is a bounded open subset of \mathbb{R}^d and V is a given closed subset of \mathbb{R}^d (see Assumptions 1.1 for major details),

$$\Gamma_{\pm} = \{(x, v) \in \partial\Omega \times V; \pm v \cdot n(x) > 0\}$$

($n(x)$ being the outward unit normal at $x \in \partial\Omega$) and H is a linear boundary operator relating the outgoing and incoming fluxes $\psi|_{\Gamma_+}$ and $\psi|_{\Gamma_-}$ in the domain Ω .

Our main assumption on the phase space is summarized in the following

Assumption 1.1. *The phase space $\Omega \times V$ is such that*

- (1) $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is an open and bounded subset with C^1 boundary $\partial\Omega$.
- (2) V is the support of a nonnegative Borel measure \mathfrak{m} which is orthogonally invariant (i.e. invariant under the action of the orthogonal group of matrices in \mathbb{R}^d) for which there exists some $r_0 > 0$ such that

$$|v| \geq r_0 \quad \forall v \in V. \quad (1.2)$$

- (3) The measure \mathfrak{m} is absolutely continuous with respect to the Lebesgue measure over \mathbb{R}^d .

With respect to our previous contributions, the main novelty of the present paper lies in assumption (1.2) which, since V is a closed subset of \mathbb{R}^d , is equivalent to $0 \notin V$.

This corresponds to the physical situation of a gas in a vessel for which particle velocities are bounded away from zero as it occurs for instance in the study of kinetic neutron transport in nuclear reactors [18]. Heuristically, a particle starting from Ω with given velocity v will reach the boundary $\partial\Omega$ in some finite time and suffer collision with the boundary which will induce a very fast thermalization of the gas.

Our main scope for the present paper is to give a rigorous justification of this heuristic consideration and show that, under suitable assumptions on the boundary operator H , the convergence to equilibrium for solution to (1.1) is *exponential*. This will be done by a careful spectral analysis of the transport operator T_H associated to (1.1) (see Section 2 for precise functional setting and definitions) combined with some compactness properties of the C_0 -semigroup associated to (1.1). It is important to emphasize already that our approach does not resort to any kind of irreducibility properties of the semigroup. This is in contrast with the framework adopted in our previous contributions. In particular, our result covers situations more general than the mere return to equilibrium but deals rather with the general asymptotic properties of the C_0 -semigroup governing (1.1).

1.1. Related literature. Deriving the precise rate of convergence to equilibrium for linear or nonlinear kinetic equations is of course a problem of paramount importance for both theoretical and applied study of kinetic models. This problem has a long history for collisional models for which both qualitative and quantitative approaches have been proposed (see [9, 10, 18, 20]).

For collisionless kinetic equations for which thermalization is driven by boundary effects, the literature on the topic is more recent. We refer the reader to [4, 5, 12, 16] for a complete overview of the literature on the topic and mention here only the pioneering works [1, 14].

For general domains, a general theory on the existence of an invariant density and its asymptotic stability (i.e. convergence to equilibrium) has been obtained recently [15] (see also earlier one-dimensional results [19]). More precisely, whenever the C_0 -semigroup $(U_H(t))_{t \geq 0}$ associated to T_H is *irreducible* we proved in [15] that there exists a unique invariant density $\Psi_H \in \mathcal{D}(T_H)$ with

$$\Psi_H(x, v) > 0 \quad \text{for a. e. } (x, v) \in \Omega \times V, \quad \int_{\Omega \times V} \Psi_H(x, v) dx \otimes \mathbf{m}(dv) = 1$$

and

$$\lim_{t \rightarrow \infty} \|U_H(t)f - \mathbf{P}_0 f\|_{L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))} = 0, \quad \forall f \in L^1(\Omega \times V, dx \otimes \mathbf{m}(dv)) \quad (1.3)$$

where \mathbf{P}_0 denotes the ergodic projection (see (1.5) for the precise definition).

In our contribution [16], using an explicit representation of the semigroup $(U_H(t))_{t \geq 0}$ obtained recently in [3] as well as some involved tauberian approach, we obtain explicit rates of convergence to equilibrium for solutions to (1.1) under mild assumptions on the initial datum ψ_0 . The ideas introduced in [16] are applied in the present contribution to deal with non zero velocities.

In most of the existing literature, arbitrarily slow particles are taken into account. In particular, the return to equilibrium can be made arbitrarily slow. The existence of too many slow particles is the reason for the slow return to equilibrium in the case of a collisionless gas in a container

with constant wall temperature. For the specific case studied in this paper, i.e.

$$|v| \geq r_0 \quad \forall v \in V$$

the literature is scarce. We mention, for collisional linear kinetic equation, the pioneering work [13] which obtains also the eventual compactness of the semigroup governing the collisional transport equation with *absorbing* boundary conditions. For the collisionless model (1.1) studied here, we only mention that an exponential convergence to equilibrium has been obtained in [1] for a model of radiative transfer (corresponding to unitary velocities, i.e. V is the unit sphere of \mathbb{R}^d). The very elegant proof of [1] consists in reducing the problem to the study of a renewal integral equation for a scalar unknown quantity. Such a method exploits extensively several symmetry property of the domain Ω and seems to apply only for spherically symmetric domain under some isotropy of the initial condition ψ_0 in (1.1b).

1.2. Our contribution. Let us make our assumptions more precise together with our main result. With respect to our previous contribution [15], we do not consider abstract and general boundary operator here but focus our attention on the specific case of a diffuse boundary operator of the following type:

Assumption 1.2. *The boundary operator $H : L^1(\Gamma_+, d\mu_+) \rightarrow L^1(\Gamma_-, d\mu_-)$ is an isotropic diffuse operator ($d\mu_{\pm}$ are positive measures on Γ_{\pm} see Section 2), i.e. it is given by*

$$H\psi(x, v) = \int_{v' \cdot n(x) > 0} \mathbf{k}(x, |v|, |v'|) \psi(x, v') |v' \cdot n(x)| \mathbf{m}(dv'), \quad (x, v) \in \Gamma_-$$

where the kernel $\mathbf{k}(x, |v|, |v'|)$ is nonnegative and measurable with

$$\int_{v \cdot n(x) < 0} \mathbf{k}(x, |v|, |v'|) |v \cdot n(x)| \mathbf{m}(dv) = 1, \quad \forall (x, v') \in \Gamma_+. \quad (1.4)$$

We refer to Section 5 for various examples of physical interest of diffuse boundary operators covered by our results. We will often use the abuse of notation $\mathbf{k}(x, v, v') = \mathbf{k}(x, |v|, |v'|)$, keeping in mind that the kernel is isotropic with respect to each velocity variables. This isotropy is simplifying assumption but more general kernels can be handled by our approach as illustrated in [16]. We preferred here to adopt this simplified framework avoiding too technical computations.

As already said, our approach does not require any irreducibility properties, and in particular, covers situation more general than those studied usually where the existence (and uniqueness) of some normalized steady solution to (1.1) is assumed yielding to the convergence (1.3).

Besides a new simplified proof of a weak compactness result given in [15], we extend the convergence in (1.3) into two directions:

- First, we get rid of the irreducibility assumption and study the long-time asymptotics of the C_0 -semigroup $(U_H(t))_{t \geq 0}$ also in the case in which there is more than one steady solution to (1.1).
- Second, we make the convergence (1.3) *quantitative* by showing that the semigroup $(U_H(t))_{t \geq 0}$ is *eventually compact*. Besides its own interest, such a compactness result implies that the convergence in (1.3) is *exponentially fast*. Moreover, it implies that 0 is a *semi-simple* eigenvalue of T_H (this is the main tool which allows us to avoid any irreducibility assumption for the long-time asymptotics).

More precisely, our main result can be stated as follows

Theorem 1.3. *Assume that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > \frac{1}{2}$ and H satisfies Assumption 1.2 and 4.3. Then, the C_0 -semigroup $(U_H(t))_{t \geq 0}$ governing equation (1.1) is eventually compact in $L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$, i.e. there exists some $\tau_* > 0$ such that*

$$U_H(t) \text{ is a compact operator in } L^1(\Omega \times V, dx \otimes \mathbf{m}(dv)) \text{ for any } t > \tau_*.$$

Moreover, there exists $\lambda_* > 0$ such that

$$\mathfrak{S}(\mathbb{T}_H) \cap \{\lambda \in \mathbb{C} ; \operatorname{Re} \lambda > -\lambda_*\} = \{0\}$$

where 0 is an eigenvalue of \mathbb{T}_H which is a first order pole of the resolvent $\mathcal{R}(\cdot, \mathbb{T}_H)$. In particular, for any $\lambda_0 \in (0, \lambda_*)$ there is $C > 0$ such that

$$\|U_H(t)f - \mathbf{P}_0 f\|_{L^1(\Omega \times V)} \leq C \exp(-\lambda_0 t) \|f\|_{L^1(\Omega \times V)}$$

for any $t \geq 0$, and any $f \in L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$ where \mathbf{P}_0 is the spectral projection associated to the zero eigenvalue.

Remark 1.4. *Whenever the semigroup $(U_H(t))_{t \geq 0}$ is irreducible, one has*

$$\mathbf{P}_0 f = \varrho_f \Psi_H, \quad \text{with} \quad \varrho_f = \int_{\Omega \times V} f(x, v) dx \otimes \mathbf{m}(dv), \quad (1.5)$$

for any $f \in L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$ where Ψ_H is the unique positive invariant density of \mathbb{T}_H with unit mass. In this case, like in (1.3), \mathbf{P}_0 is the so-called ergodic projection of \mathbb{T}_H .

The proof of the above result is based upon suitable compactness properties of some boundary operators which have been studied already in our contributions [15, 16] and made precise here in the situation considered here. We recall that these operators, already studied in [15], are the fundamental bricks on which the resolvent of \mathbb{T}_H is constructed, in particular, for $\lambda > 0$, it is known that the resolvent $\mathcal{R}(\lambda, \mathbb{T}_H)$ is given by

$$\mathcal{R}(\lambda, \mathbb{T}_H) = \mathcal{R}(\lambda, \mathbb{T}_0) + \sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda$$

where the operators $\Xi_\lambda, M_\lambda, G_\lambda$ are precisely defined in Section 2 while $\mathcal{R}(\lambda, \mathbb{T}_0)$ is the resolvent of the transport operator associated to absorbing boundary conditions (corresponding to $H = 0$).

Under the assumption $0 \notin V$ (and in contrast with what happens in the general case $0 \in V$), the spectrum of \mathbb{T}_0 is empty and the various operators are defined and bounded for any $\lambda \in \mathbb{C}$ and depend on λ in an analytic way. Moreover,

$$(M_\lambda H)^2 \text{ is a weakly compact operator in } L^1(\Gamma_+, d\mu_+)$$

We give here a new simplified proof of this weak-compactness property which was obtained in [15, Theorem 5.1] by highly technical means. The simplified proof presented here is based on an important change of variables for boundary operators introduced in [16]. Such compactness induces naturally a complete picture of the asymptotic spectrum of the generator \mathbb{T}_H : the spectrum $\mathfrak{S}(\mathbb{T}_H)$ of \mathbb{T}_H in $L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$ consists of isolated eigenvalues with finite algebraic multiplicities and there is $\lambda_* > 0$ such that

$$\mathfrak{S}(\mathbb{T}_H) \cap \{\lambda \in \mathbb{C} ; \operatorname{Re} \lambda > -\lambda_*\} = \{0\}.$$

Moreover, using a suitable change of variable introduced in [16], one can also prove an explicit decay of $(M_\lambda H)^2$ of the form

$$\left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L^1(\Gamma_+, d\mu_+))} \leq \frac{C}{|\lambda|} \quad \forall \lambda \in \mathbb{C} \quad \operatorname{Re} \lambda > 0. \quad (1.6)$$

This allows to transfer the compactness of the $(M_\lambda H)^2$ into some compactness of the semigroup $U_H(t)$ for t large enough. Indeed, thanks to a representation of the semigroup $(U_H(t))_{t \geq 0}$ as a series of operators, reminiscent of Dyson-Phillips expansion series and derived in [3],

$$U_H(t)f = \sum_{n=0}^{\infty} U_n(t)f, \quad t > 0, \quad f \in L^1(\Omega \times V, dx \otimes \mathbf{m}(dv))$$

our assumption $0 \notin V$ implies that, for any $N > 0$, there is $\tau_N > 0$ such that

$$U_n(t) = 0 \quad \forall t > \tau_N, \quad n < N,$$

i.e. the first terms of the representation series vanish for t large enough. We wish to emphasize here that such a representation series is a very natural representation of the solution to (1.1) which consists in following the trajectories of particles inside the domain Ω and for which change of velocities occur *only* due to the interaction of with the boundary $\partial\Omega$. Roughly speaking, for each $n \in \mathbb{N}$, the term $U_n(t)$ takes into account the n -th rebound on the particles on $\partial\Omega$.

From the above considerations, we can deduce by complex Laplace inversion formula [2] that

$$U_H(t)f = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} H (M_{\varepsilon+i\eta} H)^n G_{\varepsilon+i\eta} f d\eta, \quad \varepsilon > 0$$

where, thanks to the estimate (1.6), the convergence actually holds in *operator norm* yielding the compactness of $U_H(t)$ for $t > \tau_N$ if N is large enough.

We believe that the approach adopted here is robust enough to be applied also to more general problems (including collisional models with general boundary conditions) as well as the study of (1.1) in more general $L^p(\Omega \times V, dx \otimes \mathbf{m}(dv))$, $1 \leq p < \infty$. Moreover, even though our analysis is restricted, for technical reasons, to the case of a diffuse boundary operator satisfying Assumption 1.2, we are convinced that our method could also be adapted to deal with more general partly diffusive boundary operators (of Maxwell-type) as those considered in [15, 4] (see Appendix A for partial results in that direction).

1.3. Organization of the paper. After this Introduction, Section 2 presents several technical known results and the functional setting introduced in [15]. In Section 3, we recall the fundamental change of variable obtained in [16] as well as the compactness of $(M_\lambda H)^2$ together with the full proof of Estimate (1.6). In Section 4 we apply this compactness estimates to derive the eventual compactness of the semigroup $(U_H(t))_{t \geq 0}$ (Theorem 4.11) yielding to our main result Theorem 1.3. Section 5 exhibits several examples of applications of our results as well as some open problems and conjectures about related questions. The paper ends with two Appendices. Appendix A gives a description of the asymptotic spectrum of T_H in the more general case of *partly* diffuse boundary operators and discusses in an informal way the *quasi-compactness* of $(U_H(t))_{t \geq 0}$. Appendix B gives a short proof of the weak compactness of $HM_\lambda H$.

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2. PRELIMINARY RESULTS

We collect here several preliminary and known results scattered in the literature. Notice that, in this Section, we will make no use of our fundamental assumption $0 \notin V$. In particular, the results quoted in this Section remains valid in the general case in which $0 \in V$. We will see in the subsequent Sections that several of the results presented here can be drastically improved under (1.2).

2.1. Functional setting. Let us now introduce the *travel time* of particles in Ω , defined as:

Definition 2.1. For any $(x, v) \in \bar{\Omega} \times V$, define

$$t_{\pm}(x, v) = \inf\{s > 0; x \pm sv \notin \Omega\}.$$

To avoid confusion, we will set $\tau_{\pm}(x, v) := t_{\pm}(x, v)$ if $(x, v) \in \partial\Omega \times V$.

Under the assumption (1.2), the travel time is actually bounded, since

$$t_{\pm}(x, v) \leq \frac{D}{|v|} \leq \frac{D}{r_0}, \quad \forall v \in V \tag{2.1}$$

where D denotes the diameter of Ω , $D = \sup\{|x - y|, x, y \in \bar{\Omega}\}$.

In order to exploit this local nature of the boundary conditions, we introduce the following notations. For any $x \in \partial\Omega$, we define

$$\Gamma_{\pm}(x) = \{v \in V; \pm v \cdot n(x) > 0\}, \quad \Gamma_0(x) = \{v \in V; v \cdot n(x) = 0\}$$

and we define the measure $\mu_x(dv)$ on $\Gamma_{\pm}(x)$ given by

$$\mu_x(dv) = |v \cdot n(x)| \mathbf{m}(dv).$$

We introduce the partial Sobolev space $W_1 = \{\psi \in X; v \cdot \nabla_x \psi \in X\}$. It is known [6, 7] that any $\psi \in W_1$ admits traces $\psi|_{\Gamma_{\pm}}$ on Γ_{\pm} such that

$$\psi|_{\Gamma_{\pm}} \in L^1_{\text{loc}}(\Gamma_{\pm}; d\mu_{\pm}(x, v)) \quad \text{where} \quad d\mu_{\pm}(x, v) = |v \cdot n(x)| \pi(dx) \otimes \mathbf{m}(dv),$$

denotes the “natural” measure on Γ_{\pm} . Notice that, since $d\mu_+$ and $d\mu_-$ share the same expression, we will often simply denote it by

$$d\mu(x, v) = |v \cdot n(x)| \pi(dx) \otimes \mathbf{m}(dv),$$

the fact that it acts on Γ_- or Γ_+ being clear from the context. Note that

$$\partial\Omega \times V := \Gamma_- \cup \Gamma_+ \cup \Gamma_0,$$

where

$$\Gamma_0 := \{(x, v) \in \partial\Omega \times V; v \cdot n(x) = 0\}.$$

We introduce the set

$$W = \{\psi \in W_1; \psi|_{\Gamma_{\pm}} \in L^1_{\pm}\}.$$

One can show [6, 7] that $W = \{\psi \in W_1; \psi|_{\Gamma_+} \in L^1_+\} = \{\psi \in W_1; \psi|_{\Gamma_-} \in L^1_-\}$. Then, the trace operators B^\pm :

$$\begin{cases} B^\pm : W_1 \subset X \rightarrow L^1_{\text{loc}}(\Gamma_\pm; d\mu_\pm) \\ \psi \mapsto B^\pm \psi = \psi|_{\Gamma_\pm}, \end{cases}$$

are such that $B^\pm(W) \subseteq L^1_\pm$. Let us define the *maximal transport operator* T_{\max} as follows:

$$\begin{cases} T_{\max} : \mathcal{D}(T_{\max}) \subset X \rightarrow X \\ \psi \mapsto T_{\max} \psi(x, v) = -v \cdot \nabla_x \psi(x, v), \end{cases}$$

with domain $\mathcal{D}(T_{\max}) = W_1$. Now, for any *bounded boundary operator* $H \in \mathcal{B}(L^1_+, L^1_-)$, define T_H as

$$T_H \varphi = T_{\max} \varphi \quad \text{for any } \varphi \in \mathcal{D}(T_H) := \{\psi \in W; \psi|_{\Gamma_-} = H(\psi|_{\Gamma_+})\}.$$

In particular, the transport operator with absorbing conditions (i.e. corresponding to $H = 0$) will be denoted by T_0 .

2.2. About the resolvent of T_H . For any $\lambda \in \mathbb{C}$ such that $\text{Re} \lambda > 0$, define

$$\begin{cases} M_\lambda : L^1_- \rightarrow L^1_+ \\ u \mapsto M_\lambda u(x, v) = u(x - \tau_-(x, v)v, v) e^{-\lambda \tau_-(x, v)}, \quad (x, v) \in \Gamma_+; \end{cases}$$

$$\begin{cases} \Xi_\lambda : L^1_- \rightarrow X \\ u \mapsto \Xi_\lambda u(x, v) = u(x - t_-(x, v)v, v) e^{-\lambda t_-(x, v)} \mathbf{1}_{\{t_-(x, v) < \infty\}}, \quad (x, v) \in \Omega \times V; \end{cases}$$

$$\begin{cases} G_\lambda : X \rightarrow L^1_+ \\ \varphi \mapsto G_\lambda \varphi(x, v) = \int_0^{\tau_-(x, v)} \varphi(x - sv, v) e^{-\lambda s} ds, \quad (x, v) \in \Gamma_+; \end{cases}$$

and

$$\begin{cases} R_\lambda : X \rightarrow X \\ \varphi \mapsto R_\lambda \varphi(x, v) = \int_0^{t_-(x, v)} \varphi(x - tv, v) e^{-\lambda t} dt, \quad (x, v) \in \Omega \times V; \end{cases}$$

where $\mathbf{1}_E$ denotes the characteristic function of the measurable set E . The interest of these operators is related to the resolution of the boundary value problem:

$$\begin{cases} (\lambda - T_{\max})f = g, \\ B^- f = u, \end{cases} \quad (2.2)$$

where $\lambda > 0$, $g \in X$ and u is a given function over Γ_- . Such a boundary value problem, with $u \in L^1_-$ and $g \in X$ can be uniquely solved and its unique solution $f \in \mathcal{D}(T_{\max})$ is given by

$$f = R_\lambda g + \Xi_\lambda u$$

with $B^+ f \in L^1_+$ and

$$\|B^+ f\|_{L^1_+} + \lambda \|f\|_X \leq \|u\|_{L^1_-} + \|g\|_X.$$

We refer to [15] for more details. In particular, for any $\lambda > 0$

$$\|\Xi_\lambda\|_{\mathcal{B}(L^1_-, X)} \leq \lambda^{-1} \quad \|R_\lambda\|_{\mathcal{B}(X)} \leq \lambda^{-1}. \quad (2.3)$$

Moreover, one has the obvious estimates

$$\|M_\lambda\|_{\mathcal{B}(L^1_-, L^1_+)} \leq 1, \quad \|G_\lambda\|_{\mathcal{B}(X, L^1_+)} \leq 1$$

for any $\lambda > 0$. Actually, for $\lambda = 0$, we can extend the definition of these operators in an obvious way and, in contrast with what happens in the general case in which $0 \in V$ (see [16, Section 2.4]), the fact that velocities are bounded away from zero implies here that all the resulting operators remain bounded for $\lambda = 0$. Indeed, when $0 \in V$, the operators Ξ_0 and R_0 are not necessarily bounded (the estimates (2.3) clearly deteriorate when $\lambda \rightarrow 0$), see [16, Section 2.4] for a thorough description of these operators. We will see in Section 4 that the situation is much more favourable whenever $0 \notin V$.

We can complement the above result with the following whose proof can be extracted from [15, Proposition 2.6]:

Proposition 2.2. *Introduce the half-plane*

$$\mathbb{C}_+ = \{z \in \mathbb{C} ; \operatorname{Re} z > 0\}.$$

For any $\lambda \in \mathbb{C}_+$ one has $r_\sigma(M_\lambda H) < 1$ and

$$\mathcal{R}(\lambda, T_H) = R_\lambda + \Xi_\lambda H \mathcal{R}(1, M_\lambda H) G_\lambda = \mathcal{R}(\lambda, T_0) + \sum_{n=0}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda \quad (2.4)$$

where the series converges in $\mathcal{B}(X)$.

3. GENERAL PROPERTIES OF THE BOUNDARY OPERATOR H

3.1. Useful change of variables from [16]. We begin this section with a very useful change of variables, derived in our previous contribution [16, Section 6], which can be formulated as follows

Proposition 3.1. *Assume that $\partial\Omega$ satisfies Assumption 1.1. For any $x \in \partial\Omega$, we set*

$$\mathbb{S}_+(x) = \left\{ \sigma \in \mathbb{S}^{d-1} ; \sigma \cdot n(x) > 0 \right\} = \Gamma_+(x) \cap \mathbb{S}^{d-1}.$$

Then, for any nonnegative measurable mapping $g : \mathbb{S}^{d-1} \mapsto \mathbb{R}$, one has,

$$\int_{\mathbb{S}_+(x)} g(\sigma) |\sigma \cdot n(x)| d\sigma = \int_{\partial\Omega} g\left(\frac{x-y}{|x-y|}\right) \mathcal{J}(x, y) \pi(dy),$$

and

$$\mathcal{J}(x, y) = \mathbf{1}_{\Sigma_+(x)}(y) \frac{|(x-y) \cdot n(x)|}{|x-y|^{d+1}} |(x-y) \cdot n(y)|, \quad \forall y \in \Sigma_+(x) \quad (3.1)$$

with

$$\Sigma_+(x) = \{y \in \partial\Omega :]x, y[\subset \Omega ; (x-y) \cdot n(x) > 0 ; n(x-y) \cdot n(y) < 0\}$$

where $]x, y[= \{tx + (1-t)y ; 0 < t < 1\}$ is the open segment joining x and y .

It is easy to deduce from the above expression of $\mathcal{J}(x, y)$, that $\mathcal{J}(x, y) \leq |x-y|^{1-d}$ for any $(x, y) \in \partial\Omega \times \partial\Omega$. Whenever the boundary $\partial\Omega$ is more regular than the mere class \mathcal{C}^1 one can strengthen this estimate to get the following

Lemma 3.2. [16, Lemma 6.5] *Assume that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$, $\alpha \in (0, 1)$ then, there exists a positive constant $C_\Omega > 0$ such that*

$$|(x - y) \cdot n(x)| \leq C_\Omega |x - y|^{1+\alpha}, \quad \forall x, y \in \partial\Omega.$$

Consequently, with the notations of Lemma 3.1, there is a positive constant $C > 0$ such that

$$\mathcal{J}(x, y) \leq \frac{C}{|x - y|^{d-1-2\alpha}}, \quad \forall x, y \in \partial\Omega, x \neq y.$$

We recall then the following generalization of the polar decomposition theorem (see [22, Lemma 6.13, p.113]):

Lemma 3.3. *Let \mathbf{m}_0 be the image of the measure \mathbf{m} under the transformation $v \in \mathbb{R}^d \mapsto |v| \in [0, \infty)$, i.e. $\mathbf{m}_0(I) = \mathbf{m}(\{v \in \mathbb{R}^d ; |v| \in I\})$ for any Borel subset $I \subset \mathbb{R}^+$. Then, for any $\psi \in L^1(\mathbb{R}^d, \mathbf{m})$ it holds*

$$\int_{\mathbb{R}^d} \psi(v) \mathbf{m}(dv) = \frac{1}{|\mathbb{S}^{d-1}|} \int_0^\infty \mathbf{m}_0(d\rho) \int_{\mathbb{S}^{d-1}} \psi(\rho\sigma) d\sigma$$

where $d\sigma$ denotes the Lebesgue measure on \mathbb{S}^{d-1} with surface $|\mathbb{S}^{d-1}|$.

Remark 3.4. *Notice that, under the assumption $0 \notin V$, one sees that the measure \mathbf{m}_0 is supported on $[r_0, \infty)$ where r_0 is defined in (1.2).*

We can deduce from the above change of variables the following useful expression for $\text{HM}_\lambda \text{H}$ (see [16, Proposition 6.8]). Recall that H is assumed to satisfy Assumption 1.2.

Proposition 3.5. *For any $\lambda \in \overline{\mathbb{C}}_+$, it holds*

$$\text{HM}_\lambda \text{H}\varphi(x, v) = \int_{\Gamma_+} \mathcal{J}_\lambda(x, v, y, w) \varphi(y, w) |w \cdot n(y)| \mathbf{m}(dw) \pi(dy) \quad (3.2)$$

where

$$\mathcal{J}_\lambda(x, v, y, w) = \mathcal{J}(x, y) \int_0^\infty \rho \mathbf{k}(x, |v|, \rho) \mathbf{k}(y, \rho, |w|) \exp\left(-\lambda \frac{|x - y|}{\rho}\right) \frac{\mathbf{m}_0(d\rho)}{|\mathbb{S}^{d-1}|} \quad (3.3)$$

for any $(x, v) \in \Gamma_-$, $(y, w) \in \Gamma_+$.

3.2. Weak-compactness. In [15, Section 5], we derived in a broad generality the weak-compactness of $\text{HM}_0 \text{H}$ for a general class of diffuse boundary operator H (see [15, Theorem 5.1] for a precise statement). For a given $x \in \partial\Omega$, we introduce the bounded operator

$$\text{H}(x) \in \mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x)))$$

with kernel $\mathbf{k}(x, \cdot, \cdot)$. We introduce the following definition

Definition 3.6. *We say that the family*

$$\text{H}(x) \in \mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x))), \quad x \in \partial\Omega$$

is collectively weakly compact if, for any $x \in \partial\Omega$, $\text{H}(x)$ is weakly-compact and

$$\lim_{m \rightarrow \infty} \sup_{x \in \partial\Omega} \sup_{v' \in \Gamma_+(x)} \int_{S_m(x, v')} \mathbf{k}(x, v, v') \mu_x(dv) = 0$$

where, for any $m \in \mathbb{N}$ and any $(x, v') \in \Gamma_+$

$$S_m(x, v') = \{v \in \Gamma_-(x) ; |v| \geq m\} \cup \{v \in \Gamma_-(x) ; \mathbf{k}(x, v, v') \geq m\}.$$

We recall a key weak compactness result from [15] which holds for Ω of class \mathcal{C}^1 . The proof established therein is very long and highly technical but, thanks to Proposition 3.5, we are able to provide a new and much shorter proof for Ω of class $\mathcal{C}^{1,\alpha}$ ($\alpha > 0$), see Appendix B:

Theorem 3.7. *Under Assumption 1.2, assume that the family*

$$\mathbf{H}(x) \in \mathcal{B}(L^1(\Gamma_+(x)), L^1(\Gamma_-(x))), \quad x \in \partial\Omega$$

is collectively weakly compact. Then, $\mathbf{HM}_0\mathbf{H} : L_+^1 \rightarrow L_-^1$ is weakly-compact. In particular, $\mathbf{HM}_0 \in \mathcal{B}(L_+^1)$ is power-compact.

4. MAIN RESULTS

4.1. Fine properties of \mathbf{T}_H . We recall now that our main assumption about the velocity space V is that

$$0 \notin V$$

and therefore (1.2) holds true. In this case, one sees that the measure m_0 appearing in Lemma 3.3 is supported on a subset of $[r_0, \infty)$. In this case, as already mentioned,

$$t_-(x, v) \leq \frac{D}{r_0}, \quad \forall (x, v) \in \overline{\Omega} \times V$$

and this results readily in the following properties of the operators introduced in Section 2.2

Lemma 4.1. *The mappings*

$$\begin{aligned} \lambda \in \mathbb{C} &\longmapsto \Xi_\lambda \in \mathcal{B}(L_-^1, X), & \lambda \in \mathbb{C} &\longmapsto M_\lambda \in \mathcal{B}(L_-^1, L_+^1) \\ \lambda \in \mathbb{C} &\longmapsto G_\lambda \in \mathcal{B}(X, L_+^1), & \lambda \in \mathbb{C} &\longmapsto R_\lambda \in \mathcal{B}(X) \end{aligned}$$

are all well-defined and analytic (i.e. there are entire mappings). In particular, $\mathfrak{S}(\mathbf{T}_0) = \emptyset$.

Proof. The proof of the result is straightforward. For instance, one can check that

$$\|M_\lambda\|_{\mathcal{B}(L_-^1, L_+^1)} \leq \exp((\operatorname{Re}\lambda)^- D r_0^{-1}) \|M_0\|_{\mathcal{B}(L_-^1, L_+^1)} = \exp((\operatorname{Re}\lambda)^- D r_0^{-1})$$

where $(\operatorname{Re}\lambda)^- = \max(0, -\operatorname{Re}\lambda)$ is the negative part of $\operatorname{Re}\lambda$. One argues in the same way for the other operators. As far as analyticity is concerned, let us for instance focus on Ξ_λ . For any $f \in L_-^1$ and $g \in X^*$ (the dual of X) the mapping

$$\lambda \in \overline{\mathbb{C}}_+ \mapsto \langle g, \Xi_\lambda f \rangle \in \mathbb{C}$$

is analytic (where $\langle \cdot, \cdot \rangle$ is the duality bracket between X^* and X). This proves that

$$\lambda \in \overline{\mathbb{C}}_+ \mapsto \Xi_\lambda \in \mathcal{B}(L_-^1, X)$$

is analytic (see [2, Proposition A.3, Appendix A]). One argues in the same way for the other operators. \square

A first result about the spectrum of \mathbf{T}_H is the following

Lemma 4.2. *Let $\lambda \in \mathbb{C}$. Then, $\lambda \in \mathfrak{S}(\mathbf{T}_H)$ if and only if $1 \in \mathfrak{S}(M_\lambda H)$. In particular $\mathfrak{S}(\mathbf{T}_H) = \mathfrak{S}_p(\mathbf{T}_H)$.*

Proof. We first notice that, thanks to Lemma 4.1, it is straightforward that, if $1 \notin \mathfrak{G}(M_\lambda H)$ then $(\lambda - T_H)$ is invertible with

$$\mathcal{R}(\lambda, T_H) = \mathcal{R}(\lambda, T_0) + \Xi_\lambda H \mathcal{R}(1, M_\lambda H) G_\lambda.$$

This proves that, if $\lambda \in \mathfrak{G}(T_H)$ then $1 \in \mathfrak{G}(M_\lambda H)$. Conversely, assume that $1 \in \mathfrak{G}(M_\lambda H)$. Since

$$|M_\lambda \varphi| \leq M_{\operatorname{Re} \lambda} |\varphi| \leq \begin{cases} M_0 |\varphi| & \text{if } \operatorname{Re} \lambda \geq 0 \\ \exp(-\operatorname{Re} \lambda D r_0^{-1}) |\varphi| & \text{if } \operatorname{Re} \lambda < 0. \end{cases} \quad (4.1)$$

Because $HM_0H \in \mathcal{B}(L_+^1, L_-^1)$ is weakly-compact, so is $HM_\lambda H$ ($\lambda \in \mathbb{C}$) by a domination argument. Thus, for $\lambda \in \mathbb{C}$, $(M_\lambda H)^2 \in \mathcal{B}(L_+^1)$ is weakly-compact and $(M_\lambda H)^4$ by the Dunford-Pettis property and therefore $\mathfrak{G}(M_\lambda H) = \mathfrak{G}_p(M_\lambda H)$. Let then $\psi \in L_+^1$ be such that $\psi = M_\lambda H \psi$, setting $u = H\psi$ and $\varphi = \Xi_\lambda u$ one sees that $\varphi \neq 0$, $\varphi \in \mathcal{D}(T_{\max})$ with $T_{\max} \varphi = \lambda \Xi_\lambda u = \lambda \varphi$ and, by construction, $B^- \varphi = u$, $B^+ \varphi = \psi = HB^- u$ which implies $\varphi \in \mathcal{D}(T_H)$. This proves that $\lambda \in \mathfrak{G}_p(T_H)$. \square

4.2. Useful decay estimates. For this section, we make the following assumption

Assumption 4.3. Assume that m_0 is given by*

$$m_0(d\rho) = |\mathbb{S}^{d-1}| \rho^{d-1} \varpi(\rho) d\rho$$

for some positive $\varpi(\rho) > 0$ with

$$\lim_{\rho \rightarrow \infty} \rho^{d+2} \mathbf{k}(y, |v|, \rho) \mathbf{k}(y, \rho, |w|) \varpi(\rho) = 0, \quad \forall (x, v) \in \Gamma_-, (y, w) \in \Gamma_+; \quad (4.2)$$

$$\sup_{(y, w) \in \Gamma_+} \mathbf{k}(y, r_0, |w|) < \infty, \quad (4.3)$$

$$\sup_{(y, w) \in \Gamma_+} \int_{r_0}^{\infty} \rho^{d+1} (\rho \mathbf{k}(y, \rho, |w|) |\varpi'(\rho)| + \rho \varpi(\rho) |\partial_\rho \mathbf{k}(y, \rho, |w|)| + \mathbf{k}(y, \rho, |w|) \varpi(\rho)) d\rho < \infty; \quad (4.4)$$

and

$$\sup_{x \in \partial\Omega} \sup_{(y, w) \in \Gamma_+} \int_{r_0}^{\infty} \rho^{d+2} \varpi(\rho) \mathbf{k}(y, \rho, |w|) d\rho \int_{\Gamma_-(x)} |\partial_\rho \mathbf{k}(x, |v|, \rho)| \mu_x(dv) < \infty. \quad (4.5)$$

We can then prove the following:

Lemma 4.4. Under Assumption 4.3 and if $\partial\Omega$ is of class $\mathcal{C}^{1, \alpha}$ with $\alpha > \frac{1}{2}$, then for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, it holds

$$\sup_{(y, w) \in \Gamma_+} \int_{\Gamma_+} |\mathcal{I}_\lambda(x, v, y, w)| d\mu_-(x, v) \leq \frac{C}{|\lambda|} \exp(Dr_0^{-1}(\operatorname{Re} \lambda)^-)$$

for some positive $C > 0$ where $(\operatorname{Re} \lambda)^- = -\min(0, \operatorname{Re} \lambda)$ denotes the negative part of $\operatorname{Re} \lambda$.

Proof. A more general proof has been given in [16, Proposition 6.8] to get a decay of order $1/|\lambda|$. We resume the proof here to emphasize the difference and the emergence of the additional exponential term. From (3.3) and Lemma 3.2, one has for all $(x, v) \in \Gamma_-, (y, w) \in \Gamma_+$

$$|\mathcal{I}_\lambda(x, v, y, w)| \leq \frac{C_\Omega}{|x - y|^{d-1-2\alpha}} \left| \int_{r_0}^{\infty} \rho \mathbf{k}(x, |v|, \rho) \mathbf{k}(y, \rho, |w|) \exp(-\lambda|x - y|\rho^{-1}) \frac{m_0(d\rho)}{|\mathbb{S}^{d-1}|} \right|.$$

*Notice that this amounts to a measure m which is absolutely continuous with respect to the Lebesgue measure over \mathbb{R}^d , namely $m(dv) = \varpi(|v|)dv$.

for some positive constant C_Ω . We compute this last integral as follows:

$$\begin{aligned} & \int_{r_0}^{\infty} \varrho \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) \frac{\mathbf{m}_0(d\varrho)}{|\mathbb{S}^{d-1}|} \\ &= \frac{1}{\lambda|x-y|} \int_{r_0}^{\infty} \varrho^{d+2} \varpi(\varrho) \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \left(\frac{\lambda|x-y|}{\varrho^2} \exp(-\lambda|x-y|\varrho^{-1}) \right) d\varrho \end{aligned}$$

which, after integration by parts and using (4.2) yields

$$\begin{aligned} & \int_{r_0}^{\infty} \varrho \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) \frac{\mathbf{m}_0(d\varrho)}{|\mathbb{S}^{d-1}|} \\ &= -\frac{1}{\lambda|x-y|} \int_{r_0}^{\infty} \frac{d}{d\varrho} \left[\varrho^{d+2} \varpi(\varrho) \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \right] \exp(-\lambda|x-y|\varrho^{-1}) d\varrho \\ &\quad - \frac{1}{\lambda|x-y|} \left(r_0^{d+2} \varpi(r_0) \mathbf{k}(x, |v|, r_0) \mathbf{k}(y, r_0, |w|) \exp(-\lambda|x-y|r_0^{-1}) \right). \end{aligned}$$

This results in the following estimate for the kernel $\mathcal{J}_\lambda(x, v, y, w)$:

$$|\mathcal{J}_\lambda(x, v, y, w)| \leq \frac{C_\Omega}{|\lambda||x-y|^{d-2\alpha}} (|I_1(\lambda, x, y, v, w)| + I_2(\lambda, x, v, y, w))$$

with

$$I_1(\lambda, x, v, y, w) = \int_{r_0}^{\infty} \frac{d}{d\varrho} \left[\varrho^{d+2} \varpi(\varrho) \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \right] \exp(-\lambda|x-y|\varrho^{-1}) d\varrho$$

and

$$I_2(\lambda, x, v, y, w) = \left(r_0^{d+2} \varpi(r_0) \mathbf{k}(x, |v|, r_0) \mathbf{k}(y, r_0, |w|) \exp(-\operatorname{Re}\lambda|x-y|r_0^{-1}) \right)$$

for any $\lambda \neq 0$, $(x, v) \in \Gamma_-$, $(y, w) \in \Gamma_+$. Notice that, for any $(y, w) \in \Gamma_+$ and $x \in \partial\Omega$

$$\int_{\Gamma_-(x)} I_2(\lambda, x, v, y, w) |v \cdot n(x)| \mathbf{m}(dv) = r_0^{d+2} \varpi(r_0) \exp(-\operatorname{Re}\lambda|x-y|r_0^{-1}) \mathbf{k}(y, r_0, |w|)$$

using the normalization (1.4). Thus

$$\int_{\Gamma_-(x)} I_2(\lambda, x, v, y, w) |v \cdot n(x)| \mathbf{m}(dv) \leq C \exp((\operatorname{Re}\lambda)^- D r_0^{-1}) \mathbf{k}(y, r_0, |w|)$$

for some positive constant $C > 0$ depending only on r_0 . Using (4.3) we get then

$$\sup_{(y,w) \in \Gamma_+} \int_{\Gamma_-(x)} I_2(\lambda, x, v, y, w) |v \cdot n(x)| d\mu_-(x, v) \leq C \|\mathbf{k}(\cdot, r_0, \cdot)\|_{L^\infty(\Gamma_+)} \exp((\operatorname{Re}\lambda)^- D r_0^{-1}). \quad (4.6)$$

Distributing the derivative with respect to ϱ thanks to Leibniz rule, one writes

$$I_1(\lambda, x, v, y, w) = \sum_{j=1}^4 I_{1,j}(\lambda, x, v, y, w)$$

where

$$\left\{ \begin{array}{l} I_{1,1}(\lambda, x, v, y, w) = \int_{r_0}^{\infty} \varrho^{d+2} \varpi(\varrho) \mathbf{k}(x, |v|, \varrho) \partial_{\varrho} \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) d\varrho \\ I_{1,2}(\lambda, x, v, y, w) = \int_{r_0}^{\infty} \varrho^{d+2} \varpi(\varrho) \partial_{\varrho} \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) d\varrho \\ I_{1,3}(\lambda, x, v, y, w) = \int_{r_0}^{\infty} \varrho^{d+2} \varpi'(\varrho) \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) d\varrho \\ I_{1,4}(\lambda, x, v, y, w) = (d+2) \int_{r_0}^{\infty} \varrho^{d+1} \varpi(\varrho) \mathbf{k}(x, |v|, \varrho) \mathbf{k}(y, \varrho, |w|) \exp(-\lambda|x-y|\varrho^{-1}) d\varrho. \end{array} \right.$$

Using the normalisation condition (1.4), one has

$$\begin{aligned} \int_{\Gamma_{-}(x)} |I_{1,1}(\lambda, x, v, y, w)| |v \cdot n(x)| \mathbf{m}(dv) \\ \leq \int_{r_0}^{\infty} \varrho^{d+2} \varpi(\varrho) |\partial_{\varrho} \mathbf{k}(y, \varrho, |w|)| \exp((\operatorname{Re} \lambda)^{-} |x-y|\varrho^{-1}) d\varrho \\ \leq \exp((\operatorname{Re} \lambda)^{-} D r_0^{-1}) \int_{r_0}^{\infty} \varrho^{d+2} \varpi(\varrho) |\partial_{\varrho} \mathbf{k}(y, \varrho, |w|)| d\varrho. \end{aligned}$$

Thus, assumption (4.4) yields

$$\sup_{(y,w) \in \Gamma_{+}} \int_{\Gamma_{-}(x)} |I_{1,1}(\lambda, x, v, y, w)| |v \cdot n(x)| \mathbf{m}(dv) \leq C \exp((\operatorname{Re} \lambda)^{-} D r_0^{-1}).$$

In the same way, one sees easily that (4.4) implies that

$$\begin{aligned} \sup_{(y,w) \in \Gamma_{+}} \int_{\Gamma_{-}(x)} (|I_{1,3}(\lambda, x, v, y, w)| + |I_{1,4}(\lambda, x, v, y, w)|) |v \cdot n(x)| \mathbf{m}(dv) \\ \leq C \exp((\operatorname{Re} \lambda)^{-} D r_0^{-1}). \end{aligned}$$

Finally, one checks easily that (4.5) implies

$$\sup_{x \in \partial \Omega} \sup_{(y,w) \in \Gamma_{+}} \int_{\Gamma_{-}(x)} |I_{1,2}(\lambda, x, v, y, w)| |v \cdot n(x)| \mathbf{m}(dv) \leq C \exp((\operatorname{Re} \lambda)^{-} D r_0^{-1}).$$

Combining all these estimates, we finally obtain that there exists some positive constant C (depending only on r_0) such that

$$\begin{aligned} \int_{\Gamma_{-}(x)} |\mathcal{I}_{\lambda}(x, v, y, w)| |v \cdot n(x)| \mathbf{m}(dv) \\ \leq \frac{C}{|\lambda| |x-y|^{d-2\alpha}} \exp((\operatorname{Re} \lambda)^{-} D r_0^{-1}) \quad \forall x \in \partial \Omega, \quad \forall (y, w) \in \Gamma_{+}. \end{aligned}$$

We get the result since, for $\alpha > \frac{1}{2}$,

$$\sup_{y \in \partial \Omega} \int_{\partial \Omega} \frac{\pi(dx)}{|x-y|^{d-2\alpha}} < \infty$$

being the kernel $|x - y|^{2\alpha-d}$ of order strictly less than $d - 1$ (see [11, Prop. 3.11]). \square

The above, combined with Proposition 3.5 yields the following

Lemma 4.5. *Assume that Assumption 4.3 are in force and $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $\alpha > \frac{1}{2}$. There exists a positive constant C such that*

$$\|(M_\lambda H)^2\|_{\mathcal{B}(L_+^1)} \leq \frac{C}{|\lambda|} \exp(2r_0^{-1}(\operatorname{Re}\lambda)^{-D})$$

holds for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

Proof. It is clear from Proposition 3.5 that, for any $\psi \in L_+^1$,

$$\begin{aligned} \|(M_\lambda H)^2 \psi\|_{L_+^1} &\leq \|M_\lambda\|_{\mathcal{B}(L_-^1, L_+^1)} \|HM_\lambda H\psi\|_{L_-^1} \\ &\leq \|M_\lambda\|_{\mathcal{B}(L_-^1, L_+^1)} \int_{\Gamma_+} |\psi(y, w)| d\mu_+(y, w) \int_{\Gamma_-} |\mathcal{J}_\lambda(x, v, y, w)| d\mu_-(x, v) \end{aligned}$$

so that, using that $\|M_\lambda\|_{\mathcal{B}(L_-^1, L_+^1)} \leq \exp((\operatorname{Re}\lambda)^{-D}r_0^{-1})$ we get

$$\|(M_\lambda H)^2 \psi\|_{L_+^1} \leq \exp((\operatorname{Re}\lambda)^{-D}r_0^{-1}) \sup_{(y,w) \in \Gamma_+} \int_{\Gamma_-} |\mathcal{J}_\lambda(x, v, y, w)| d\mu_-(x, v)$$

and we conclude then with Lemma 4.4. \square

We also establish here a simple consequence of Lemma 4.5:

Lemma 4.6. *Assume that Assumption 4.3 are in force and $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $\alpha > \frac{1}{2}$. For any $N \geq 2$, there exists some positive constant $C_N > 0$ depending on N and such that, for any $\lambda \in \mathbb{C}_+$ it holds*

$$\left\| \sum_{n=N}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda \right\|_{\mathcal{B}(X)} \leq C_N |\lambda|^{-\lfloor \frac{N}{2} \rfloor} \frac{1}{\operatorname{Re}\lambda (1 - \exp(-Dr_0^{-1}\operatorname{Re}\lambda))} \quad (4.7)$$

where $\lfloor \frac{N}{2} \rfloor$ denotes the integer part of $\frac{N}{2}$. In particular, for any $N \geq 4$,

$$\int_{-\infty}^{\infty} \left\| \sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} H (M_{\varepsilon+i\eta} H)^n G_{\varepsilon+i\eta} \right\|_{\mathcal{B}(X)} d\eta < \infty, \quad \forall \varepsilon > 0. \quad (4.8)$$

Proof. Since $r_\sigma(M_\lambda H) < 1$ for any $\operatorname{Re} \lambda > 0$, one has

$$\sum_{n=N}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda = \Xi_\lambda H (M_\lambda H)^N \mathcal{R}(1, M_\lambda H) G_\lambda.$$

One notices that, for any $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > 0$, one has

$$\|\Xi_\lambda H\|_{\mathcal{B}(L_+^1, X)} \leq \frac{1}{\operatorname{Re}\lambda}, \quad \|G_\lambda\|_{\mathcal{B}(X, L_+^1)} \leq 1$$

so that, for any $N \geq 2$

$$\left\| \sum_{n=N}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda \right\|_{\mathcal{B}(X)} \leq \frac{1}{\operatorname{Re}\lambda} \left\| (M_\lambda H)^N \right\|_{\mathcal{B}(L_+^1)} \|\mathcal{R}(1, M_\lambda H)\|_{\mathcal{B}(L_+^1)}.$$

Since, for $\operatorname{Re}\lambda > 0$,

$$\|\mathcal{R}(1, M_\lambda H)\|_{\mathcal{B}(L_+^1)} \leq \frac{1}{1 - \|M_\lambda H\|_{\mathcal{B}(L_+^1)}} \leq \frac{1}{1 - \exp(-Dr_0^{-1}\operatorname{Re}\lambda)}$$

one deduces that

$$\left\| \sum_{n=N}^{\infty} \Xi_\lambda H (M_\lambda H)^n G_\lambda \right\|_{\mathcal{B}(X)} \leq \frac{1}{\operatorname{Re}\lambda (1 - \exp(-Dr_0^{-1}\operatorname{Re}\lambda))} \|(M_\lambda H)^N\|_{\mathcal{B}(X)}.$$

Now, since $\|M_\lambda H\|_{\mathcal{B}(L_+^1)} \leq 1$, one deduces easily from Lemma 4.5 that

$$\|(M_\lambda H)^N\|_{\mathcal{B}(L_+^1)} \leq \left(\frac{C}{|\lambda|}\right)^{\lfloor \frac{N}{2} \rfloor}$$

from which (4.7) follows. One deduces then, for any $\varepsilon > 0$ that

$$\left\| \sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} H (M_{\varepsilon+i\eta} H)^n G_{\varepsilon+i\eta} \right\|_{\mathcal{B}(X)} \leq \frac{C_N}{\varepsilon (1 - \exp(-Dr_0^{-1}\varepsilon))} |\varepsilon + i\eta|^{-\lfloor \frac{N}{2} \rfloor}$$

and, for $N \geq 4$, (4.8) follows since $\lfloor \frac{N}{2} \rfloor > 1$. \square

4.3. Semigroup decay. We aim now to prove that the semigroup $(U_H(t))_{t \geq 0}$ generated by T_H converges exponentially fast to equilibrium. We will use here the following representation of the semigroup in terms of a Dyson-Phillips obtained in [3]. First, recall the definition of the C_0 -semigroup generated by T_0 :

$$U_0(t)f(x, v) = f(x - tv, v)\chi_{\{t < t_-(x, v)\}}, \quad f \in X, \quad t \geq 0.$$

We begin with the following definition where $\mathcal{D}_0 = \{f \in \mathcal{D}(T_{\max}) ; B^- f = 0 = B^+ f\}$:

Definition 4.7. Let $t \geq 0$, $k \geq 1$ and $f \in \mathcal{D}_0$ be given. For $(x, v) \in \bar{\Omega} \times V$ with $t_-(x, v) < t$, there exists a unique $y \in \partial\Omega$ with $(y, v) \in \Gamma_-$ and a unique $0 < s < \min(t, \tau_+(y, v))$ such that $x = y + sv$ and then one sets

$$[U_k(t)f](x, v) = [HB^+ U_{k-1}(t-s)f](y, v).$$

We set $[U_k(t)f](x, v) = 0$ if $t_-(x, v) \geq t$ and $U_k(0)f = 0$.

Then, one has the following extracted from [3]:

Theorem 4.8. For any $k \geq 1$, $f \in \mathcal{D}_0$ one has $U_k(t)f \in X$ for any $t \geq 0$ with

$$\|U_k(t)f\|_X \leq \|f\|_X.$$

In particular, $U_k(t)$ can be extended to be a bounded linear operator, still denoted $U_k(t) \in \mathcal{B}(X)$ with

$$\|U_k(t)\|_{\mathcal{B}(X)} \leq 1 \quad \forall t \geq 0, k \geq 1.$$

Moreover, the following holds for any $k \geq 1$

(1) $(U_k(t))_{t \geq 0}$ is a strongly continuous family of $\mathcal{B}(X)$.

(2) For any $f \in X$ and $\lambda > 0$, setting

$$\mathcal{L}_k(\lambda)f = \int_0^\infty \exp(-\lambda t)U_k(t)f dt$$

one has, for $k \geq 1$,

$$\mathcal{L}_k(\lambda)f \in \mathcal{D}(\mathbb{T}_{\max}) \quad \text{with} \quad \mathbb{T}_{\max}\mathcal{L}_k(\lambda)f = \lambda \mathcal{L}_k(\lambda)f$$

and $\mathbb{B}^\pm \mathcal{L}_k(\lambda)f \in L_\pm^1$ with

$$\mathbb{B}^- \mathcal{L}_k(\lambda)f = \mathbb{H}\mathbb{B}^+ \mathcal{L}_{k-1}(\lambda)f \quad \mathbb{B}^+ \mathcal{L}_k(\lambda)f = (\mathbb{M}_\lambda \mathbb{H})^k \mathbb{G}_\lambda f.$$

(3) For any $f \in X$, the series $\sum_{k=0}^\infty U_k(t)f$ is strongly convergent and it holds

$$U_{\mathbb{H}}(t)f = \sum_{k=0}^\infty U_k(t)f$$

Remark 4.9. One sees from the point (2) together with [15, Theorem 2.4] that, for any $k \geq 1$,

$$\mathcal{L}_k(\lambda)f = \Xi_\lambda \mathbb{H}\mathbb{B}^+ \mathcal{L}_{k-1}(\lambda)f.$$

Since $\mathcal{L}_0(\lambda)f = \mathbb{R}_\lambda f$ we deduce that, for any $k \geq 1$,

$$\mathcal{L}_k(\lambda) = \Xi_\lambda \mathbb{H} (\mathbb{M}_\lambda \mathbb{H})^{k-1} \mathbb{G}_\lambda.$$

In particular, one sees that, in the representation series (2.4) that, for any $n \geq 0$

$$\Xi_\lambda \mathbb{H} (\mathbb{M}_\lambda \mathbb{H})^n \mathbb{G}_\lambda f = \int_0^\infty \exp(-\lambda t)U_{n+1}(t)f dt \quad (4.9)$$

for any $\lambda > 0$ which is of course coherent with the above point (3) and the representation of the resolvent of $\mathbb{T}_{\mathbb{H}}$.

The exact expression of the iterated $U_k(t)$ allows to prove the following which is the crucial point for our analysis here, namely, under the assumption

$$|v| \geq r_0, \quad \forall v \in V$$

each term of the above series is vanishing for large time:

Lemma 4.10. Let $(U_k(t))_{k \geq 0, t \geq 0}$ be the family of operators defined in Definition 4.7. Then, under assumption (1.2), for any $n \geq 0$,

$$U_n(t) \equiv 0 \quad \forall t \geq \tau_n := \frac{(n+1)D}{r_0}.$$

Proof. Once noticed that, for $t \geq \tau_0$, $U_0(t) = 0$, the proof is a simple induction using the definition 4.7. Indeed, assuming $U_{k-1}(t) = 0$ for $t \geq \tau_{k-1} = k\tau_0$, one recalls that

$$U_k(t)f(x, v) = [\mathbb{H}\mathbb{B}^+ U_{k-1}(t-s)f](y, v), \quad (x, v) \in \Omega \times V, \quad y = x - t_-(x, v)v$$

with $s = t_-(x, v)$, we get that, if $t-s \geq \tau_{k-1}$ then $U_{k-1}(t-s)f = 0$. Being $s = t_-(x, v) \leq \tau_0$, we have that, as soon as $t \geq \tau_{k-1} + \tau_0$, it holds $t-s \geq \tau_{k-1}$ and $U_k(t)f(x, v) = 0$ for any (x, v) , i.e. $U_k(t) = 0$ for $t \geq \tau_k = \tau_{k-1} + \tau_0 = k\tau_0$. \square

We are in position to prove the main result of this paper

Theorem 4.11. *Assume that Assumption 4.3 and (1.2) are in force and $\partial\Omega$ is of class $C^{1,\alpha}$ with $\alpha > \frac{1}{2}$. Then,*

$$U_{\mathbf{H}}(t) \text{ is compact for any } t \geq \frac{5D}{r_0}$$

where we recall that $D = \text{diam}(\Omega)$ and $r_0 := \inf\{|v| ; v \in V\}$.

Proof. From Lemma 4.10 and Theorem 4.8, for any $N \geq 0$ and any $f \in X$, one has

$$U_{\mathbf{H}}(t)f = \sum_{n=N+1}^{\infty} U_n(t)f \quad \forall t \geq \tau_N$$

Moreover, according to the Laplace inversion formula [2, Proposition 3.12.1], for any $\varepsilon > 0$ and any $t \geq 0$ one has

$$\begin{aligned} U_{\mathbf{H}}(t)f &= \lim_{\ell \rightarrow \infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} \exp((\varepsilon + i\eta)t) \mathcal{R}(\varepsilon + i\eta, \mathbf{T}_{\mathbf{H}}) f \, d\eta, \\ &= \lim_{\ell \rightarrow \infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} \exp((\varepsilon + i\eta)t) \sum_{n=0}^{\infty} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} f \, d\eta, \quad \forall f \in \mathcal{D}(\mathbf{T}_{\mathbf{H}}). \end{aligned}$$

Then, one deduces easily from (4.9) that, for any $f \in \mathcal{D}(\mathbf{T}_{\mathbf{H}})$ it holds, for $\varepsilon > 0$,

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \frac{1}{2\pi} \int_{-\ell}^{\ell} \exp((\varepsilon + i\eta)t) \sum_{n=0}^{N-1} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} f \, d\eta \\ = \sum_{n=0}^{N-1} U_n(t)f = 0, \quad \text{if } t \geq \tau_N. \end{aligned}$$

Therefore, for any $f \in \mathcal{D}(\mathbf{T}_{\mathbf{H}})$ and any $t \geq \tau_N$,

$$\begin{aligned} U_{\mathbf{H}}(t)f &= \sum_{n=N}^{\infty} U_{n+1}(t)f \\ &= \frac{1}{2\pi} \lim_{\ell \rightarrow \infty} \int_{-\ell}^{\ell} \exp((\varepsilon + i\eta)t) \left(\sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} f \right) d\eta \quad \varepsilon > 0 \end{aligned} \tag{4.10}$$

where the convergence holds in X . Recall that $r_{\sigma}(\mathbf{M}_{\varepsilon+i\eta} \mathbf{H}) < 1$ for any $\eta \in \mathbb{R}$ and therefore

$$\sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} = \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^N \mathcal{R}(1, \mathbf{M}_{\varepsilon+i\eta} \mathbf{H}) \mathbf{G}_{\varepsilon+i\eta}$$

is a compact operator for any $N \geq 4$. Consequently, for any $\ell \in \mathbb{R}$,

$$\frac{1}{2\pi} \int_{-\ell}^{\ell} \left(\sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} \right) d\eta$$

is a compact operator as soon as $N \geq 4$. Since moreover, Lemma 4.6 implies that the integral

$$\int_{-\infty}^{\infty} \left\| \sum_{n=N}^{\infty} \Xi_{\varepsilon+i\eta} \mathbf{H} (\mathbf{M}_{\varepsilon+i\eta} \mathbf{H})^n \mathbf{G}_{\varepsilon+i\eta} \right\|_{\mathcal{B}(X)} d\eta < \infty$$

one sees that the convergence in (4.10) actually holds in *operator norm* and, as such, $U_H(t)$ is the limit of compact operators which proves the compactness of $U_H(t)$ for any $t \geq \tau_N$ and $N \geq 4$. \square

The role of the zero eigenvalue of T_H can be made more precise here and the asymptotic behaviour of $(U_H(t))_{t \geq 0}$ follows, yielding a full proof of Theorem 1.3 in the Introduction:

Corollary 4.12. *Assume that Assumption 4.3 and (1.2) are in force and $\partial\Omega$ is of class $C^{1,\alpha}$ with $\alpha > \frac{1}{2}$. Then,*

0 is a simple pole of the resolvent of T_H

and, for any $a \in (0, \lambda_*)$, there exists a positive constant $C_a > 0$ such that, for any $f \in X$, it holds

$$\|U_H(t)f - \mathbf{P}_0 f\|_X \leq C_a \exp(-at) \|f\|_X \quad \forall t \geq 0$$

where \mathbf{P}_0 denotes the spectral projection associated to the zero eigenvalue.

Proof. Recall that, since $(U_H(t))_{t \geq 0}$ is a positive C_0 -semigroup in the L^1 -space X , its type $\omega_0(U_H)$ coincide with the spectral bound $s(T_H)$ of its generator. Because $\|U_H(t)\|_{\mathcal{B}(X)} = 1$, one has

$$\omega_0(U_H) = 0 = s(T_H).$$

Due to the eventual compactness of $(U_H(t))$, its essential type $\omega_{\text{ess}}(U_H)$ is such that

$$-\infty = \omega_{\text{ess}}(U_H) < \omega_0(U_H) = 0 = s(T_H).$$

In particular, 0 is an isolated eigenvalue of T_H with finite algebraic multiplicity and there is $\lambda_* > 0$ such that

$$\mathfrak{S}(T_H) \cap \{\lambda \in \mathbb{C} ; \operatorname{Re} \lambda \geq -\lambda_*\} = \{0\}. \quad (4.11)$$

Moreover (see [8, Theorem 9.11]), for any $a \in (0, \lambda_*)$, there is $C_a > 0$ such that

$$\|U_H(t)(\mathbf{I} - \mathbf{P}_0)f\|_X = \|U_H(t)f - \exp(tN_0)\mathbf{P}_0 f\|_X \leq C_a \exp(-at) \|f\|_X \quad (4.12)$$

for any $t \geq 0$ and any $f \in X$ where \mathbf{P}_0 is the spectral projection associated to the zero eigenvalue and $N_0 = T_H \mathbf{P}_0$ is a nilpotent bounded operator. Precisely, if m denotes the order of the pole of the resolvent $\mathcal{R}(\cdot, T_H)$, one has $N_0^m = 0$, $N_0^j \neq 0$ with $j < m$ and consequently,

$$\exp(tN_0) = \sum_{k=0}^{m-1} \frac{t^k}{k!} N_0^k.$$

Since the semigroup $(U_H(t))_{t \geq 0}$ is bounded, we deduce that the mapping

$$t \geq 0 \mapsto \left\| \sum_{k=0}^{m-1} \frac{t^k}{k!} N_0^k \mathbf{P}_0 \right\|_{\mathcal{B}(X)}$$

is bounded. The only way for this to be true is that

$$N_0^k \mathbf{P}_0 = 0 \quad \forall k = 1, \dots, m-1$$

which, since $N_0 = T_H \mathbf{P}_0$, implies in particular that $T_H \mathbf{P}_0^2 = 0$. Because \mathbf{P}_0 is a projection, one has

$$N_0 = 0,$$

i.e. $m = 1$ which proves the first part of the result. The second part has been established in (4.12) (see also [8, Theorem 9.11]). \square

Remark 4.13. Notice that, since 0 is a simple pole of the resolvent $\mathcal{R}(\cdot, \mathbb{T}_H)$, its geometrical and algebraic multiplicity (as an eigenvalue of \mathbb{T}_H) coincide, i.e.

$$\dim \text{Ker}(\mathbb{T}_H) = \dim \text{Range}(\mathbf{P}_0) = n \in \mathbb{N}$$

and

$$X = \text{Ker}(\mathbb{T}_H) \oplus \text{Range}(\mathbb{T}_H)$$

where the range of \mathbb{T}_H is closed.

Remark 4.14. Whenever the C_0 -semigroup $(U_H(t))_{t \geq 0}$ is irreducible, the expression of the spectral projection is more explicit. We recall here that, if one assumes, besides Assumption 1.2, that

$$k(x, v, v') > 0 \quad \text{for } \mu_x\text{-a.e. } v \in \Gamma_-(x), v' \in \Gamma_+(x). \quad (4.13)$$

then (see [15, Section 4]) then operator $\mathbb{M}_0 H$ is irreducible as well as the C_0 -semigroup $(U_H(t))_{t \geq 0}$. The semigroup admits a unique invariant density $\Psi_H \in \mathcal{D}(\mathbb{T}_H)$ with

$$\Psi_H(x, v) > 0 \quad \text{for a. e. } (x, v) \in \Omega \times \mathbb{R}^d, \quad \|\Psi_H\|_X = 1,$$

and

$$\text{Ker}(\mathbb{T}_H) = \text{Span}(\Psi_H).$$

In this case, the projection \mathbf{P}_0 is given by (1.5), i.e.

$$\mathbf{P}_0 f = \varrho_f \Psi_H, \quad \text{with } \varrho_f = \int_{\Omega \times V} f(x, v) dx \otimes \mathbf{m}(dv).$$

More generally, such an expression of \mathbf{P}_0 is true if $\dim \text{Ker}(\mathbb{T}_H) = 1$ (independently of the irreducible assumption).

5. EXAMPLES AND OPEN PROBLEMS

In this Section, we briefly illustrate the main results established so far for several examples of particular relevance. We also propose several open problems we believe of interest for the study of linear transport equation.

We begin with the following example:

Example 5.1. We consider the case in which

$$\mathbf{k}(x, v, v') = \gamma^{-1}(x) \mathbf{G}(x, v)$$

where $\mathbf{G} : \partial\Omega \times V \rightarrow \mathbb{R}^+$ is a measurable and nonnegative mapping such that

- (i) $\mathbf{G}(x, \cdot)$ is radially symmetric for π -almost every $x \in \partial\Omega$;
- (ii) $\mathbf{G}(\cdot, v) \in L^\infty(\partial\Omega)$ for almost every $v \in V$;
- (iii) The mapping $x \in \partial\Omega \mapsto \mathbf{G}(x, \cdot) \in L^1(V, |v| \mathbf{m}(dv))$ is piecewise continuous,
- (iv) The mapping $x \in \partial\Omega \mapsto \gamma(x)$ is bounded away from zero where

$$\gamma(x) := \int_{\Gamma_-(x)} \mathbf{G}(x, v) |v \cdot n(x)| \mathbf{m}(dv) \quad \forall x \in \partial\Omega,$$

i.e. there exist $\gamma_0 > 0$ such that $\gamma(x) \geq \gamma_0$ for π -almost every $x \in \partial\Omega$.

In that case, it is easy to show that the associated boundary operator \mathbf{H} is satisfying Assumption 1.2 and, whenever

$$\mathbf{m}(dv) = \varpi(|v|)dv$$

for some radially symmetric and nonnegative function $\varpi(|v|)$, one checks without difficulty that Assumption 4.3 are met if

$$\lim_{\varrho \rightarrow \infty} \varrho^{d+2} \mathbf{G}(y, \varrho) \varpi(\varrho) = 0, \quad \forall y \in \partial\Omega$$

$$\sup_{y \in \partial\Omega} \int_{r_0}^{\infty} \left(\mathbf{G}(y, \varrho) \left(|\varpi'(\varrho)| + \frac{\varpi(\varrho)}{\varrho} \right) + |\partial_{\varrho} \mathbf{G}(y, \varrho)| \varpi(\varrho) \right) \varrho^{d+2} d\varrho < \infty. \quad (5.1)$$

Under such assumption, the existence of an invariant density $\Psi_{\mathbf{H}}$ has been derived in [15] and, for $\partial\Omega$ of class $\mathcal{C}^{1,\alpha}$ ($\alpha > \frac{1}{2}$), the conclusions of Theorem 4.11 and Corollary 4.12 hold true. Notice that, in this case, the zero eigenvalue is simple.

Example 5.2. A more specific case can be considered here which corresponds to the previous Example with

$$\mathbf{G}(x, v) = \mathcal{M}_{\theta(x)}(v), \quad \mathcal{M}_{\theta}(v) = (2\pi\theta)^{-d/2} \exp\left(-\frac{|v|^2}{2\theta}\right), \quad x \in \partial\Omega, \quad v \in \mathbb{R}^d.$$

Then,

$$\gamma(x) = \kappa_d \sqrt{\theta(x)} \int_{\mathbb{R}^d} |w| \mathcal{M}_1(w) dw, \quad x \in \partial\Omega$$

for some positive constant κ_d depending only on the dimension. Assuming the mapping $\theta : \partial\Omega \mapsto \mathbb{R}^+$ to be continuous and bounded from below by some positive constant,

$$\inf_{x \in \partial\Omega} \theta(x) = \theta_0 > 0.$$

Then, for the special choice

$$\varpi(\varrho) = \begin{cases} \varrho^m, & m \geq 0 \\ \exp(\alpha \varrho^s), & \alpha > 0, \quad s \in (0, 2), \\ \exp(\beta \varrho^2), & \beta \in (0, \frac{1}{2\theta_{\infty}}) \end{cases}$$

where $\theta_{\infty} = \sup_{x \in \partial\Omega} \theta(x)$, one sees that Assumption 4.3 are met (see (5.1)). Therefore, for $\partial\Omega$ of class $\mathcal{C}^{1,\alpha}$ ($\alpha > \frac{1}{2}$), the conclusions of Theorem 4.11 and Corollary 4.12 hold true.

We conclude this Section with the following open problems. The first one regards the role of the regularity of $\partial\Omega$

Open Problem 1. We may wonder if the assumption that $\partial\Omega$ is of class $\mathcal{C}^{1,\alpha}$ with $\alpha > \frac{1}{2}$ is really necessary. Such an assumption plays a role only in the proof of Lemma 4.4 thanks to Lemma 3.2 but seems only technical and, under the mere assumption $\partial\Omega$ of class \mathcal{C}^1 , we infer that $U_{\mathbf{H}}(t)$ is compact for t large enough.

A second open problem regards the case in which the 0 is not a simple eigenvalue

Open Problem 2. If 0 is not a simple eigenvalue, i.e. if

$$\dim \text{Ker}(\mathbf{T}_{\mathbf{H}}) = \dim(\text{Range } \mathbf{P}_0) = n > 1,$$

then one may wonder what is exactly the form of the spectral projection \mathbf{P}_0 . We conjecture that, in this case, there exist exactly n distinct nonnegative eigenfunctions Ψ_1, \dots, Ψ_n with pairwise disjoint supports associated to the zero eigenvalue of $\mathbf{T}_{\mathbf{H}}$.

APPENDIX A. THE CASE OF PARTLY DIFFUSE BOUNDARY CONDITIONS

In this appendix, we provide some insights about the generalisation of the results obtained so far to the general case of partly diffuse boundary operators as introduced in our first contribution [15]. We describe the asymptotic spectrum of the generator and give a conjecture on the *quasi-compactness* of the semigroup. We begin with recalling the definition from [15] adapted to our context:

Definition A.1. We shall say that a boundary operator $H \in \mathcal{B}(L_+^1, L_-^1)$ is stochastic partly diffuse if it writes

$$H\psi(x, v) = \alpha(x) R\psi(x, v) + (1 - \alpha(x)) K\psi(x, v), \quad (x, v) \in \Gamma_-, \psi \in L_+^1 \quad (\text{A.1})$$

where $\alpha(\cdot) : \partial\Omega \rightarrow [0, 1]$ is measurable, $K \in \mathcal{B}(L_+^1, L_-^1)$ is a stochastic diffuse boundary operator satisfying Assumption 1.2 and R is a reflection operator

$$R(\varphi)(x, v) = \varphi(x, \mathcal{V}(x, v)) \quad \forall (x, v) \in \Gamma_-, \varphi \in L_+^1$$

where $\mathcal{V} : x \in \partial\Omega \mapsto \mathcal{V}(x, \cdot)$ is a field of bijective bi-measurable and μ_x -preserving mappings

$$\mathcal{V}(x, \cdot) : \Gamma_-(x) \cup \Gamma_0(x) \rightarrow \Gamma_+(x) \cup \Gamma_0(x)$$

such that

- i) $|\mathcal{V}(x, v)| = |v|$ for any $(x, v) \in \Gamma_-$.
- ii) If $(x, v) \in \Gamma_0$ then $(x, \mathcal{V}(x, v)) \in \Gamma_0$, i.e. $\mathcal{V}(x, \cdot)$ maps $\Gamma_0(x)$ in $\Gamma_0(x)$.
- iii) The mapping

$$(x, v) \in \Gamma_- \mapsto (x, \mathcal{V}(x, v)) \in \Gamma_+$$

is a C^1 diffeomorphism.

It has been shown in [15] that

$$(M_0 H)^2 = (M_0(\beta K))^2 + (M_0(\alpha R))^2 + M_0(\alpha R)M_0(\beta K) + M_0(\beta K)M_0(\alpha R)$$

where $\beta(x) = 1 - \alpha(x)$, $x \in \partial\Omega$. Setting

$$\beta_\infty := \operatorname{ess\,sup}_{x \in \partial\Omega} \beta(x)$$

one has $(M_0(\beta K))^2$ is weakly compact and

$$\left\| (M_0(\alpha R))^2 + M_0(\alpha R)M_0(\beta K) + M_0(\beta K)M_0(\alpha R) \right\|_{\mathcal{B}(L_+^1)} \leq (1 + \operatorname{osc}(\beta))^2 - \beta_\infty^2$$

where $\operatorname{osc}(\beta) = \operatorname{ess\,sup}_{x \in \partial\Omega} \beta(x) - \operatorname{ess\,inf}_{x \in \partial\Omega} \beta(x)$ is the oscillation of $\beta(\cdot)$. As in [15, Theorem 5.6], we assume that

$$\mathbf{c}_\beta := (1 + \operatorname{osc}(\beta))^2 - \beta_\infty^2 < 1. \quad (\text{A.2})$$

We set

$$\lambda_\beta := -\frac{r_0}{2D} \log \mathbf{c}_\beta > 0$$

and has the following

Lemma A.2. Assume that H is a partly diffuse operator in the sense of the above Definition A.1 satisfying (A.2). Then, there is a discrete set $\Theta \subset \mathbb{C}$ such that, for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\lambda_\beta$ the following alternative holds:

- i) either 1 is the resolvent set of $M_\lambda H$
- ii) or $1 \in \mathfrak{S}_p(M_\lambda H)$ and then $\lambda \in \Theta$.

Proof. Notice that

$$\begin{aligned} (M_\lambda H)^2 &= (M_\lambda(\beta K))^2 + (M_\lambda(\alpha R))^2 + M_\lambda(\alpha R)M_\lambda(\beta K) + M_\lambda(\beta K)M_\lambda(\alpha R) \\ &=: (M_\lambda(\beta K))^2 + L_\lambda \end{aligned} \quad (\text{A.3})$$

where $(M_\lambda(\beta K))^2$ is a weakly-compact operator (by a simple domination argument). Invoking (4.1), one sees that, for $\operatorname{Re}\lambda \geq 0$, it holds

$$\|L_\lambda\|_{\mathcal{B}(L_+^1)} \leq \|L_0\|_{\mathcal{B}(L_+^1)} = \left\| (M_0(\alpha R))^2 + M_0(\alpha R)M_0(\beta K) + M_0(\beta K)M_0(\alpha R) \right\|_{\mathcal{B}(L_+^1)} < 1$$

whereas, for $\operatorname{Re}\lambda < 0$

$$\|L_\lambda\|_{\mathcal{B}(L_+^1)} \leq \exp\left(-2\frac{D\operatorname{Re}\lambda}{r_0}\right) \|L_0\|_{\mathcal{B}(L_+^1)} \leq \exp\left(-2\frac{D\operatorname{Re}\lambda}{r_0}\right) c_\beta < 1$$

as soon as $\operatorname{Re}\lambda > \frac{r_0}{2D} \log c_\beta$. Consequently, $r_{\text{ess}}\left((M_\lambda H)^2\right) < 1$ for any $\operatorname{Re}\lambda > -\lambda_\beta$. From the spectral mapping theorem, we deduce then that

$$r_{\text{ess}}(M_\lambda H) < 1 \quad \forall \operatorname{Re}\lambda > -\lambda_\beta.$$

As a consequence, for $\operatorname{Re}\lambda > -\lambda_\beta$,

$$1 \in \mathfrak{G}(M_\lambda H) \iff 1 \in \mathfrak{G}_p(M_\lambda H)$$

and in particular, if $1 \in \mathfrak{G}(M_\lambda H)$ then $1 \in \mathfrak{G}_p\left((M_\lambda H)^2\right)$. Let us therefore investigate the spectral problem

$$g - (M_\lambda H)^2 g = h$$

which, thanks to (A.3) is equivalent to $g - L_\lambda g - (M_\lambda(\beta K))^2 g = h$, i.e.

$$g - \mathcal{R}(1, L_\lambda) (M_\lambda(\beta K))^2 g = \mathcal{R}(1, L_\lambda) h.$$

Since $(M_\lambda(\beta K))^2$ is weakly-compact we deduce from the analytic Fredholm alternative that the set

$$\Theta := \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > -\lambda_\beta \text{ and } 1 \in \mathfrak{G}(\mathcal{R}(1, L_\lambda) (M_\lambda(\beta K))^2)\}$$

is discrete. This in particular implies that the set

$$\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > -\lambda_\beta \text{ and } 1 \in \mathfrak{G}_p\left((M_\lambda H)^2\right)\}$$

is discrete. If now $\operatorname{Re}\lambda > -\lambda_\beta$ and $\lambda \in \mathbb{C} \setminus \Theta$, then 1 belongs to the resolvent set of $\mathcal{R}(1, L_\lambda) (M_\lambda(\beta K))^2$ which implies that 1 is the resolvent set of $(M_\lambda H)^2$. This proves the Lemma. \square

This leads then to the following

Proposition A.3. *Assume that H is a partly diffuse operator in the sense of the above Definition A.1 which satisfies (A.2). Setting*

$$\lambda_\beta := -\frac{r_0}{2D} \log c_\beta > 0,$$

for any $\eta \in (0, \lambda_\beta)$, $\mathfrak{G}(T_H) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -\eta\}$ consists at most in a finite number of eigenvalues of T_H with finite algebraic multiplicities.

Proof. Recall that, for $\operatorname{Re}\lambda > -\lambda_\beta$,

$$\lambda \in \mathfrak{S}(\mathsf{T}_H) \iff 1 \in \mathfrak{S}(M_\lambda H) \iff 1 \in \mathfrak{S}_p(M_\lambda H)$$

and

$$\lambda \in \mathfrak{S}_p(\mathsf{T}_H) \iff 1 \in \mathfrak{S}_p(M_\lambda H).$$

Therefore, from the previous Lemma, $\mathfrak{S}(\mathsf{T}_H) \cap \{\lambda \in \mathbb{C} ; \operatorname{Re}\lambda > -\lambda_\beta\}$ consists at most in a discrete set of eigenvalues with finite algebraic multiplicity. Now, if $1 \in \mathfrak{S}_p(M_\lambda H)$ then $1 \in \mathfrak{S}\left((M_\lambda H)^2\right)$. Since, for any $\eta \in (0, \lambda_\beta)$,

$$\sup_{\operatorname{Re}\lambda \geq -\eta} \left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L_+^1)} \longrightarrow 0 \quad \text{as} \quad |\operatorname{Im}\lambda| \rightarrow \infty$$

one sees that, for $\eta \in (0, \lambda_\beta)$, the set $\{\lambda \in \mathbb{C} ; \operatorname{Re}\lambda \geq -\eta\} \cap \{\lambda \in \mathbb{C} ; 1 \in \mathfrak{S}_p(M_\lambda H)\}$ is at most finite which proves the result. \square

We can complement the above with the following

Lemma A.4. *Under the Assumption of Proposition A.3, for any $\eta \in (0, \lambda_\beta)$, there is $M > 0$ such that*

$$\sup \left\{ \|\mathcal{R}(\lambda, \mathsf{T}_H)\|_{\mathcal{B}(X)} ; \operatorname{Re}\lambda \geq -\eta, \quad |\operatorname{Im}\lambda| \geq M \right\} < \infty.$$

Proof. For $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > -\lambda_\beta$, one has

$$\mathcal{R}(\lambda, \mathsf{T}_H) = \mathcal{R}(\lambda, \mathsf{T}_0) + \Xi_\lambda H \mathcal{R}(1, M_\lambda H) G_\lambda.$$

One observes that, for any $\eta \in (0, \lambda_\beta)$ and any $\operatorname{Re}\lambda \geq -\eta$, it holds

$$\|\mathcal{R}(\lambda, \mathsf{T}_0)\|_{\mathcal{B}(X)} \leq \|\mathcal{R}(-\eta, \mathsf{T}_0)\|_{\mathcal{B}(X)}.$$

Since $\lim_{|\operatorname{Im}\lambda| \rightarrow \infty} \left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L_+^1)} = 0$ uniformly on $\{\lambda \in \mathbb{C} ; \operatorname{Re}\lambda \geq -\eta\}$, for any $c < 1$, there is $M > 0$ such that

$$\left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L_+^1)} \leq c < 1, \quad \forall \lambda \in \Delta_{M,\eta}$$

where we set $\Delta_{M,\eta} := \{\lambda \in \mathbb{C} ; \operatorname{Re}\lambda \geq -\eta ; |\operatorname{Im}\lambda| \geq M\}$. In particular, $r_\sigma(M_\lambda H) < 1$ for any $\lambda \in \Delta_{M,\eta}$ and

$$\mathcal{R}(1, M_\lambda H) = \sum_{n=0}^{\infty} (M_\lambda H)^n \quad \lambda \in \Delta_{M,\eta}$$

Writing $n = 2k + s$ with $k \in \mathbb{N}$ and $s \in \{0, 1\}$, one sees that, for $\lambda \in \Delta_{M,\eta}$,

$$\begin{aligned} \|\mathcal{R}(1, M_\lambda H)\|_{\mathcal{B}(L_+^1)} &\leq \sum_{k \in \mathbb{N}, s=0,1} \left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L_+^1)}^k \|M_\lambda H\|_{\mathcal{B}(L_+^1)}^s \\ &\leq \frac{\max\left(1, \|M_\lambda H\|_{\mathcal{B}(L_+^1)}\right)}{1 - \left\| (M_\lambda H)^2 \right\|_{\mathcal{B}(L_+^1)}} \leq \frac{1}{1-c} \max\left(1, \|M_\lambda H\|_{\mathcal{B}(L_+^1)}\right) \end{aligned}$$

Therefore

$$\|\mathcal{R}(1, M_\lambda H)\|_{\mathcal{B}(L_+^1)} \leq \frac{1}{1-c} \max\left(1, \|M_{-\eta} H\|_{\mathcal{B}(L_+^1)}\right)$$

for any $\lambda \in \Delta_{M,\eta}$ which achieves the proof. \square

Arguing for instance as in [17], for any $f \in \mathcal{D}(\mathbb{T}_H^2)$, one can write $U_H(t)f$ as the inverse Laplace transform of $\mathcal{R}(\lambda, \mathbb{T}_H)$ and, moving suitably the integration path and picking up the residues at 0, one can prove that there is $\eta > 0$ and $C_f \geq 0$ such that

$$\|U_H(t)f - \mathbf{P}_0 f\|_{\mathcal{B}(X)} \leq C_f \exp(-\eta t) \quad t \geq 0$$

where C_f actually depends on $f, \mathbb{T}_H f$ and $\mathbb{T}_H^2 f$. This leads to the following conjecture

Conjecture A.5. *We conjecture that, under the Assumption of Proposition A.3, the C_0 -semigroup $(U_H(t))_{t \geq 0}$ admits a positive spectral gap $\lambda_0 \in (0, \lambda_\beta)$ such that*

$$\|U_H(t) - \mathbf{P}_0\|_{\mathcal{B}(X)} = O(\exp(-\lambda_0 t)), \quad t \geq 0.$$

APPENDIX B. PROOF OF THEOREM 3.7

We give here a simple proof of Theorem 3.7 in the case in which Ω is of class $\mathcal{C}^{1,\alpha}$ with $\alpha > 0$. We actually prove that

$$\text{HM}_\lambda \text{H} : L_+^1 \rightarrow L_-^1 \text{ is weakly-compact for any } \text{Re} \lambda \geq 0. \quad (\text{B.1})$$

As in the proof of [15, Theorem 5.1] by approximation and domination arguments, to prove the result, we can restrict ourselves without loss of generality to the case in which

$$V := \{v \in \mathbb{R}^d ; r_0 \leq |v| \leq R_0\}, \quad \text{H}\varphi(x, v) = \int_{\Gamma_+(x)} \varphi(x, v') \boldsymbol{\mu}_x(dv'), \quad \varphi \in L_+^1,$$

where $R_0 > 0$. This of course corresponds to the case $\mathbf{k}(x, v, v') \equiv 1$. In such a case, Proposition 3.5 asserts that

$$\text{HM}_\lambda \text{H}\varphi(x, v) = \int_{\Gamma_+} \mathcal{J}_\lambda(x, v, y, w) \varphi(y, w) |w \cdot n(y)| \mathbf{m}(dw) \pi(dy)$$

with

$$\mathcal{J}_\lambda(x, v, y, w) = \mathcal{J}(x, y) \int_{r_0}^{R_0} \varrho \exp\left(-\lambda \frac{|x-y|}{\varrho}\right) \frac{\mathbf{m}_0(d\varrho)}{|\mathbb{S}^{d-1}|}$$

for any $(x, v) \in \Gamma_-, (y, w) \in \Gamma_+$. Thus,

$$|\mathcal{J}_\lambda(x, v, y, w)| \leq \mathcal{J}(x, y) \int_{r_0}^{R_0} \varrho \frac{\mathbf{m}_0(d\varrho)}{|\mathbb{S}^{d-1}|} \leq C_0 \mathcal{J}(x, y)$$

since $\mathbf{m}_0([r_0, R_0]) < \infty$. By a domination argument, it is enough to prove the weak compactness of the operator $\mathbf{K} \in \mathcal{B}(L_+^1, L^1(\partial\Omega))$ given by

$$\mathbf{K}\varphi(x) = \int_{\Gamma_+(x)} \mathcal{J}(x, y) \varphi(y, w) |w \cdot n(y)| \mathbf{m}(dw) \pi(dy), \quad \varphi \in L_+^1, \quad (x, v) \in \Gamma_-.$$

This operator can be written as

$$\mathbf{K} = \mathcal{J}_0 \mathcal{P}$$

where $\mathcal{P} \in \mathcal{B}(L_+^1, L^1(\partial\Omega))$ is the projection operator

$$\mathcal{P}\varphi(x) = \int_{\Gamma_+(x)} \varphi(x, w) \boldsymbol{\mu}_x(dw), \quad x \in \partial\Omega, \quad \varphi \in L_+^1$$

and $\mathcal{J}_0 \in \mathcal{B}(L^1(\partial\Omega))$ is given by

$$\mathcal{J}_0\psi(x) = \int_{\partial\Omega} \mathcal{J}(x, y) \psi(y) \pi(dy), \quad x \in \partial\Omega, \quad \psi \in L^1(\partial\Omega).$$

Let us now show that $\mathcal{J}_0 \in \mathcal{B}(L^1(\partial\Omega))$ is weakly compact which will give the result. Again, using Lemma 3.2 together with a domination argument, it is enough to prove the weak compactness of the operator $\mathcal{J}_1 \in \mathcal{B}(L^1(\partial\Omega))$ given by

$$\mathcal{J}_1\psi(x) = \int_{\partial\Omega} |x-y|^{1+2\alpha-d}\psi(y)\pi(dy), \quad x \in \partial\Omega, \quad \psi \in L^1(\partial\Omega).$$

We note that its kernel is of order strictly less than $d-1$ since $\alpha > 0$. This is done by an approximation argument introducing, for any $\varepsilon > 0$,

$$\mathcal{J}_1^\varepsilon\psi(x) = \int_{\partial\Omega} \mathbf{1}_{|x-y|\geq\varepsilon}|x-y|^{1+2\alpha-d}\psi(y)\pi(dy), \quad x \in \partial\Omega, \quad \psi \in L^1(\partial\Omega).$$

For any $\varepsilon > 0$, $\mathcal{J}_1^\varepsilon$ has a bounded kernel and is clearly weakly compact while

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{J}_1^\varepsilon - \mathcal{J}_1\|_{\mathcal{B}(L^1(\partial\Omega))} = 0$$

because the kernel of $\mathcal{J}_1^\varepsilon - \mathcal{J}_1$ is supported on $\{|x-y| < \varepsilon\}$ (see [11, Proposition 3.11 & Exercise 1, page 121-123]). This proves (B.1) and achieves the proof of Theorem 3.7.

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