Model for ancient Greek and Roman coinage production

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Abstract

Coinage production of ancient powers such as Athens and Rome is usually inferred from die statistics of monetary issues. The present work applies a Kaplan-Meier analysis of resistance to failure to 29 sets of well-documented monetary issues. The failure rate function assumes a U-shaped form known in reliability engineering literature as the 'bathtub curve'. With the geometric distribution of die failure being demonstrably violated for a large fraction of the data sets, the die distribution of each data set was instead fitted by a mixture of two Weibull distributions corresponding to two failure regimes. Dies can be divided into bad dies, failing early for various reasons, and good dies, failing late by fatigue. The dual populations reflect the efforts of the smiths at the time to produce bronze dies that would meet two conflicting needs: the reduction of premature die failure (= infant mortality) and the limitation of ductile deformation during minting. The variable proportions of the two populations suggest that not all workshops had fully mastered die technology. Because of the dichotomy induced by contrasting mechanical properties, corrections for missing dies based on singletons and causes of die failure must be carefully assessed for each data set.

1. Introduction

Quantifying monetary production in ancient societies that left little or no minting accounts, or imprecise and biased citations, is crucial to understanding ancient economies and how fast societies adopted minted bullion as a mean of payment (= monetization). A common strategy is a three-step process. First, the number of original dies, one of the two metallic pieces used to strike a coin, one for each side of the coin, is determined from the corpus of coins available for a single issue. Generally, facing-up (obverse) dies are more robust that facing-down (reverse) dies. Second, a correction is made to account for the missing dies, i.e., those that are not present in the corpus. More than 20 statistical methods have been proposed for this task, the results of which are generally considered unproblematic as long as the ratio of number of coins/number of dies (n/d) is higher than 3 (Callataÿ, 1995). This is the case for most ancient Greek coinages, for which (n/d) commonly exceeds 10. Third, the original number of dies is multiplied by what is considered the average production of a die, which is a much more contentious issue. Whenever comparisons between the number of dies and the number of coins found in hoards are possible, relatively simple first-order rules hold up: dies used to mint large silver coins were more productive than dies used for smaller denominations, while dies used for striking bronze coins were wearing out or failing faster than those used to strike silver coins (Sellwood, 1963; Faucher, 2009, 2011, 2013). This is the case of coins bearing the name of Alexander the Great; the cistophori minted in multiple localities by the Attalids, kings of Pergamon (282-128 B.C.E.) (Callataÿ, 2013; Meadow, 2013); and the various issues of the Roman Republic coinage.

Now, what is the average production of a die? Explicitly written sources relevant to this question are nearly non-existent (Kinns, 1983). In contrast, a dozen favorable cases exist for which extrapolated volumes of struck coins can be placed into precisely dated contexts (Callataÿ, 1995). The results provided by this rare evidence appear coherent. Attempts of striking coins under conditions mimicking those of ancient mints also have been made but so far reliable results are few (Sellwood, 1963; Faucher, 2009, 2011, 2013). While these experiments are of interest in terms of metallurgy and thus relevant to some extent to the issues discussed below, in particular that of the most defective dies, they may not inform on highly productive dies (Buttrey and Cooper, 1994). An alternative promising approach is finite element modeling of minting, which has the advantage of restoring the distribution of stress and strain during coin striking (Brekelmans et al., 1988; Alexandrino et al., 2018,2019).

The monetary flux out of a particular mint is reflected by the production of its issues. Production is modulated not only by the number of active dies but also by their effective yield. The average production of a population of dies clearly depends on how these dies behave once put into production. . It seems pointless to spend resources crafting dies that would fail before they met with some sort of specifications. Although the cost of producing dies is unknown, it is clear that over time their designs became increasingly sophisticated and that the technology was improved to enhance productivity, which must have come with a cost. Questions are: how important is premature failure, also known as burn-out or infant mortality? In other words, many dies with a small production, while prominent in die studies, had a small contribution to the entire volume. What about the average die with an average production? Experiments are useful (Sellwood, 1963; Faucher et al., 2009, 2011, 2013) but of very limited extent, which render them somewhat unreliable for deriving average die productivities. A critical matter is whether the overall volume of a given issue may be dominated by particularly sturdy dies with a very large production. These questions have been variously addressed in the past. It was first common practice to represent a given die distribution by the symmetric normal approximation to a binomial distribution (Good, 1953; Good and Toulmin, 1956; Carter and Moore, 1980) (Fig. 1, curve A). This symmetrical model was, however, shown to be unacceptable for a number of reasons and, in the 1980s, the negative binomial distribution, a variant of the Poisson distribution, with a negative asymmetric curve (Fig. 1, curve B), then a gamma distribution, became the favored representations (Carter, 1983;

Esty and Carter, 1992). Around the same time, Callataÿ (1987), after scrutinizing hundreds of data sets from ancient Greece and Rome, focused on those with large numbers of infrequent dies (singletons, doubletons, etc.), and proposed a combination of a negative binomial distribution accounting for infant mortality with a binomial distribution accounting for the surviving specimens (Fig. 1, curve C). Callataÿ's (1987) point was particularly important because the correction for missing dies developed by Esty (1984) and Carter (1983, 1992) critically depends on singleton frequencies.

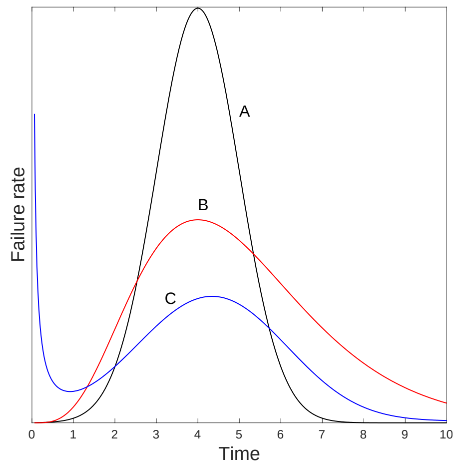

Figure 1. Different models used to account for the frequency of dies (modified from Callataÿ, 1987). (A) Normal distribution, (B) negative binomial or Poisson distribution (Esty and Carter, 1992), and (C) mixture of two distributions involving infant mortality and metal fatigue (Callataÿ, 1980, 1987).

More recently, Esty (2011) assessed that both models (B), the negative binomial distribution of Carter (1980), and (C), the mixed distribution hypothesized by Callataÿ (1987), were flawed. He concluded that die statistics is best represented by the geometrical model, i.e., a constant-failure distribution indicative of a Poisson process, and derived simple analytical formulas based on die counts, singletons, and sample size to estimate the number of missing dies. The present work reassesses Esty's (2011) assumptions and their relevance to the statistical parameters derived from the 608 data sets gathered by Callataÿ (1997, 2003) for the following reasons:

1. Although these data sets represent some of the best-known samples and provide a glimpse of original die distribution, a perspective based on other mints, in particular those of the Roman Republic, would be useful. The denarii of Crepusius can be considered a sample of high quality for Roman Republican coinage because the proportion of singletons is low (Buttrey, 1976) and their coverage, i.e., the proportion of non-singletons, is high (Esty, 1986). The (n/d) ratio (number of coins/number of dies) of this data set remains in the low range of most Greek data sets.
2. It has been noticed that even for the best-documented samples with (n/d) > 10 and coverage > 99% for which formulas postulate that essentially all the produced dies are known, new dies continue to

appear, enlarging the sample. Based on ten die studies, Callataÿ (1993) concluded that Carter's (1983) formulas based on the (n/d) ratio, long dominant among numismatists for estimating the number of unobserved dies and relying on model (B) of Fig. 1, overestimates the original number of dies when the n/d ratio is < 3 and underestimates it after that point.

3. The master variable of existing minting models is the time elapsed since start of production. Although some texts have carefully dealt with this variable, it depends on a number of assumptions (Carter and Carter, 1983), such as the number of anvils, human error, and work scheduling, all factors difficult to verify.
4. Esty assumed the materiality of the geometric/exponential distribution and this assumption should be assessed.

Even with extremely well-documented samples, singletons continue to be surprisingly numerous, which is a strong indication of high infant mortality. As shown in Fig. 2, the proportion of singletons remains large regardless of data set quality: up to 40% for a ratio (n/d) of 5, up to 30% for a ratio (n/d) of 10, and up to 15% for a ratio (n/d) of 15. The example of a single issue of drachms of the Euboean League (Callataÿ 1997: n° 147) with (n/d) > 40 and nevertheless counting more than 10% of singletons (3 out of 28) is a strong caveat that a purely statistical approach to die studies is inadequate. This is the basis of the present work, which revisits the data from the combined perspective of die survival and mechanical properties of dies. The purpose of this approach is to assess the minting process using principles of reliability engineering (Billinton and Allan, 1992; Nash, 2016) to derive both the lifetime of dies and the survival rate of coins from die statistics of large issues. We will apply the theory to the die productivity of 23 issues from the Archaic and Classical Greece compiled by Callataÿ (1997, 2003), one Alexander issue from Damascus (Glenn, 2018), three Roman Republican issues abundantly discussed in the literature (e.g., Buttrey, 1976; Carter and Ross, 1992), and the Yehud issue (Callataÿ, in press) to ensure the validity and general value of the results. The 29 data sets cover a broad range of (n/d) ratios, including data sets with rather low (n/d) values, such as those of the Yehud coinage characterized by an extremely high proportion of singletons (Callataÿ, in press), as well as data sets from the numbered Roman Republican issues of Crepusius (RRC 361, Buttrey, 1976) and Censorinus (RRC 360, Debernardi et al., 2020) as an additional reference.

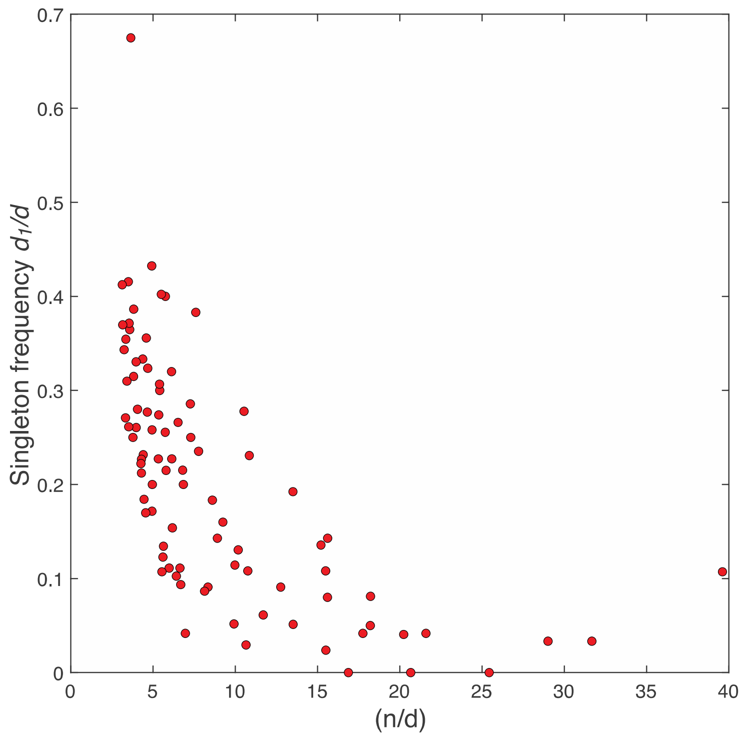

Figure 2. Ratios of number of coins to obverse dies (n/d on the *x*-axis) and frequency d_1/d of singleton dies (on the *y*-axis). Redrawn from Callataÿ (2021).

Survival theory is a largely descriptive approach that uses the statistics of failure, though not of failure *time* because this variable is not available, but of the number of coins (yield) struck by dies used for a particular issue. The distribution of the yields among different dies or groups of dies, and in particular their scatter, offer an under-used source of information on the minting process. Survival theory is widely used in a variety of fields from engineering to medicine to identify the factors causing failure or death in order to control them. Although closely connected, the concepts ruling the mechanical properties of metals and alloys (Meyers and Chawla, 2008) are not identical and the survival models therefore will be properly set apart from each other.

2. A die survival theory

The tenet intrinsic to the present work is the significance of the die multiplicity scale: singletons signal dies with a smaller production than doubletons, which themselves have a smaller production than tripletons, etc. It therefore makes no difference how the coins are dispatched once produced and we can assume that the production is immediately stored and mixed in a vault where it will never be spent. Time is a variable that has pervaded publications trying to support statistical models (e.g., Carter and Moore, 1980; Carter, 1983; Callataÿ, 1987) and the difficulties of parameterizing such a vision have been reviewed multiple times (e.g., Buttrey, 1994). Time will therefore not be considered an objective control variable of coin production and will not be used for the present purpose.

It will further be assumed that dies are either functional or irreparably damaged. For the purpose of illustration, we will first assume that all the coins of a given issue have been recovered. d_i is the number of dies, often improperly called die frequency, observed i times. Here i will be referred to as die multiplicity and is clearly a function of how many specimens in total have been recovered. For example, d_1 is the number of singletons, d_2 the number of doubletons, d_3 the number of tripletons, etc. Tables of d_i vs i are die histograms in which the *a priori* bin width is unity. $d = \Sigma_i\, d_i$ is the total number of dies in the population, and $f_i\,(i) = d_i\,/d$ the frequency of dies with multiplicity i. The subscript i refers to numbers and fractions of dies. $F_i(i) = \sum_{j=0}^{j=i} f_i\;(j)$ is the cumulated fraction of dies summed over the bins 1 to i, while $R\,(i) = 1 - F_i(i)$ is the fraction of dies surviving at that point.

Likewise, $n_i = i\; d_i = i f_i\,(i)\, d$ is the number of coins in the i-th bin, $n = \Sigma_i\; i\; d_i = \Sigma_i\; n_i$ the total number of coins in the sample, and $f_k\;(i) = n_i/n$ the coin frequency in bin i. Note that the subscript k refers to numbers and fractions of coins, not dies. The bin width is now variable and equal to n_i. The total production of singletons, doubletons, and tripletons will therefore be $1d_1 + 2d_2 + 3d_3 = n_1 + n_2 + n_3$.

What about failure frequency? The cumulated fraction of coins produced by the dies that struck 1, 2, or 3 coins is $(n_1 + n_2 + n_3))/n = f_k\,(1) + f_k\,(2) + f_k\,(3)$. Let us now define r_i as the number of surviving dies after the i-th failure, e.g.:

$$d = d_1 + d_2 + r_2 = d_1 + d_2 + d_3 + r_3 \qquad (1)$$

However straightforward the relationships

$$d_i = r_{i-1} - r_i \qquad (2a)$$

$$f_i\,(i) = F\,(i) - F\,(i-1) \qquad (2b)$$

may look, they show that d_i has the significance of a number of failed dies at multiplicity i.

The standard ratio known as (n/d) (total number of coins/total number of dies), a characteristic index, is not homologous to a mean productivity, but has the dimension of multiplicity i. (n/d) is actually the average weighted values of multiplicity since

$$(\mathrm{n/d}) = \sum f_i(i)\; i = S_i \qquad (3)$$

where S_i is the surface area beneath the histogram of die frequencies vs multiplicity.

3. The failure probability function

A number of useful parameters widely used in reliability engineering literature can be retrieved from the multiplicity histogram, including the failure probability function, the mean time to failure, and the total number of specimens consistent with the histogram (Bracquemond and Gaudoin, 2003; Rausand and Høyland, 2003; Nash, 2016).

The function noted $z(i)$ provides the estimate of the probability of die failure at each stroke. It is the ratio of the number of retired dies to the number of surviving dies times the number of strokes. This function is closely related to the Kaplan-Meier survival estimate widely used in medical studies (Goel et al., 2010). Taking tripletons as an example, d_3 dies, out of a total of d_2, fail after 3 d_3 blows:

$$z(3) = \frac{No\ of\ failed\ dies}{No\ of\ surviving\ dies \times number\ of\ strokes} = \frac{d_3}{r_2 \times 3 d_3} = \frac{1}{3 r_2} \qquad (4)$$

From this equation, we can retrieve several equivalent expressions, including the standard definition of the failure function:

$$z(i) = \frac{1}{i r_{i-1}} = \frac{1}{i d R(i-1)} = \frac{1}{i d\left(1 - F_i(i) + f_i(i)\right)} \qquad (5)$$

A continuous approximation for $z(i)$ is

$$z(i) = \frac{d_i}{r(i) \times (i d_i)} = \frac{r_{i-1} - r_i}{r_i \times (i d_i)} = \frac{r_{i-1}/d - r_i/d}{r_i/d \times (i d_i)} \qquad (6)$$

and

$$z(i) = \frac{f_i(i)}{\left(1 - F_i(i)\right) \times (i d_i)} \approx -\frac{d \ln\left(1 - F_i(i)\right)}{dk} \qquad (7)$$

where k is the cumulated number of strokes. As a result, the failure probability function $z(i)$ can be retrieved from the slope of the relationship between $\ln\left(1 - F_i(i)\right)$ and the number of coins struck until i multiplicity $\sum_{l=1}^{l=i} n_l$. The failure probability of the geometric distribution and of its continuous equivalent, the exponential distribution (random failure), is constant.

The expected value of coin production per die (yield) can be defined as the weighted average of strokes per die until failure. If all the coins of the issue are available, the number of coins struck until the d_i dies of multiplicity i fail is simply the sum of all the coins struck up to multiplicity of $l = i$. The mean number Y of coins struck until failure (apparent average die productivity) therefore is:

$$\mathrm{Y} = \sum_i f_k(i) i = S_k \qquad (8)$$

where S_k is the surface area beneath the histogram of coin frequencies vs multiplicity i.

4. Results

Table 1 lists the basic data of the 29 issues targeted in the present study. The histogram of coin distributions among the classes of increasing multiplicity (Fig. 3) shows that not all data sets present a single peak. The abscissa in Fig. 3 are the weights $f_i(i)$ and $f_k(i)$ used to calculate (n/d) and Υ, respectively, and are shifted with respect to one another. $\Upsilon = S_k$ (tan field) is shifted with respect to (n/d) $= S_i$ (blue field) towards higher values and the field is larger. Note that, because of early failure, there is little correlation between the number of coins and the number of dies.

Plots of the fraction of failed dies vs the fractional output, or coins struck (Fig. 4), show a strong deviation from the diagonal line of constant failure probability (exponential distribution of the number of coins between successive failures). Nevertheless, the semi-log plot of the die survivor function (1-$F_i(i)$) *vs* k, which is the fraction of preserved coins struck ranked by increasing multiplicity (Eqn. 7) (Fig. 5a), has a sideways sigmoid form. This shape is common to all the data sets. We chose to display this plot against the fraction of preserved coins struck rather than their actual number so as to work with a common scale. The slope <1 of the logarithmic plot of $\ln\left(1 - F_i(i)\right)$ (log of log) vs ln k (the cumulated number of coins) at low multiplicity (Fig. 5b) shows that the observed distribution of dies clearly deviates from the geometric distribution. This observation is remarkable since this distribution plays a central role in die studies (Esty and Carter 1992; Esty, 2011; Callataÿ, in press) .

The failure functions $z(i)$ were calculated in two different ways: (1) from Eqn. 5 (blue bars in Fig. 6) and (2) from the slope of the natural logarithm of the die survivor function of Fig. 2 (Eqn. 7, red lines in Fig. 6). The two estimates are consistent with each other. At low values of i, $z(i)$ decreases, passes by a minimum, and then increases for the most productive dies, which is a nearly systematic feature of the present hazard curves. An exception is the case of Censorinus denarii, which have rather small (n/d) ratios (< 3.5). For the samples with higher (n/d) ratios, such as Syracuse tetradrachms, drachms from the Euboean League, and Bruttium denarii (Fig. 4), the negative dz/dk edge is more prominent.

The apparent average productivity Υ of the dies exceeds (n/d) by a factor of 1.3 to 3.4, with a value of 7 for the 209-Drachms set (Table 1). This factor is unrelated to the number of dies and the number of coins, which demonstrates the quality of the data.

Table 1: Characteristics of the 29 data sets used in this work. Drachm is abbreviated as ‘dr’, stater as ‘stat’, and ‘den’ as denarii.

Present notation	set#	*d*	*n*	(*n*/*d*)	d_1	Υ	ref.
Numismatic notation		*o*	*n*	(n/d)	o^1		

1997-147-Dr-Euboea	21	28	1128	40.3	3	68.7	1
68-Tetradr-Messana	10	30	950	31.7	1	52.6	1
44-Didr-Gela	8	30	870	29.0	1	47.3	1
24-Didr-Tarent	6	37	839	22.7	0	29.4	1
1997-12-Bruttium	24	47	1000	21.3	2	27.9	2
Tetradr-Syracuse	22	75	1544	20.6	3	32.9	1
95-Tetradr-Syracuse	13	37	674	18.2	3	29.8	1
15-Didr-Velia	3	20	364	18.2	1	27.8	1
99-Decadr-Syracuse	14	24	424	17.7	1	31.7	1
19-Didr-Tarent	4	23	388	16.9	0	21.2	1
180-Hemidr-Pharsalos	15	28	437	15.6	4	26.3	1
93-Tetradr-Syracuse	12	63	978	15.5	3	21.3	1
21-Didr-Tarent	5	59	897	15.2	8	29.4	1
26-Didr-Tarent	7	49	573	11.7	3	29.3	1
66-Tetradr-Messana	9	58	575	9.9	3	13.2	1
Crepusius den	27	408	3810	9.3	26	13.8	3
6-Stat-Metapontum	1	100	924	9.2	16	15.9	1
255-Didr-Pixodaros	20	60	516	8.6	11	30.8	1
Yehud-Obv	25	206	1768	8.6	75	67.9	4
Tetradr-Alex-Damas	23	59	493	8.4	1	12.3	5
Yehud-Rev	26	281	1923	6.8	92	37.0	4
90-Tetradr-Syracuse	11	139	924	6.6	13	9.6	1
Bursio-Rev den	27	405	2359	5.8	72	9.9	7
198-Stat-Corinth	17	227	1302	5.7	58	13.2	1
Bursio-Obv den	28	460	2359	5.1	77	8.3	7
215-6thStat-Mytilene	19	177	779	4.4	41	8.7	1
197-Stat-Corinth	16	112	444	4.0	37	9.4	1
209-Dr-Sinope	18	163	594	3.6	110	28.2	1
Censorinus den	29	419	1418	3.4	120	5.3	6

References (1) Callataÿ (2003) (2) Callataÿ (1997) (3) Buttrey (1976) revised by Richard Schaefer (4) Callataÿ (in press). (5) Glenn (2018) (6) Debernardi et al. (2020) (7) De Ruyter (1996).

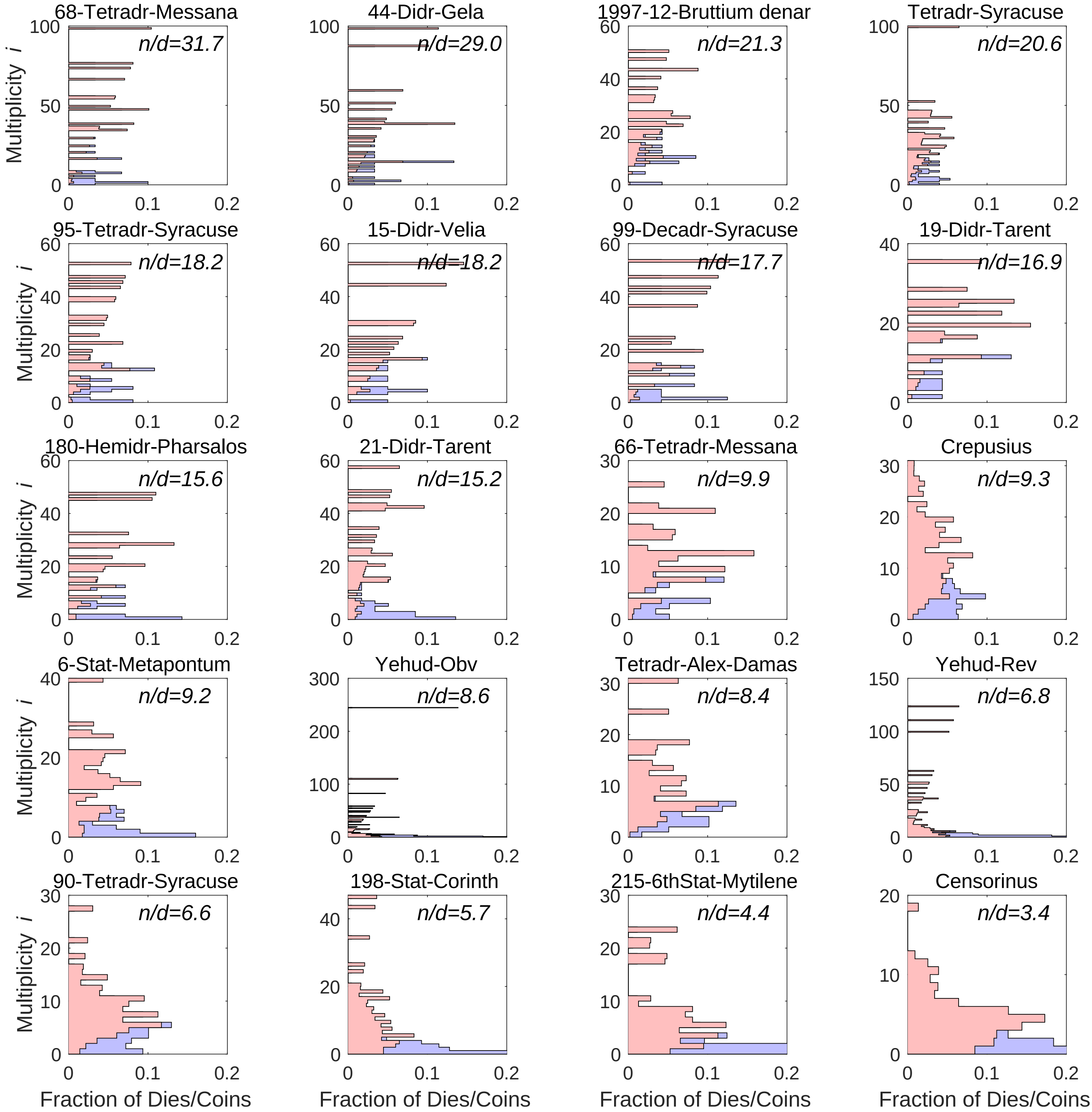

Figure 3. Example of histograms showing die fractions (in blue) and coin fractions (in tan) (x-coordinate) as a function of the coins struck (y-coordinate). (n/d) is the surface area of the blue field, while Y is the surface area of the tan field. Brown-shaded areas represent overlap of the blue and tan fields. This plot shows that, in general, the largest number of coins is not necessarily produced by the most abundant dies. Sixteen samples out of 29 were selected for this plot to present a printable overview of shape variability.

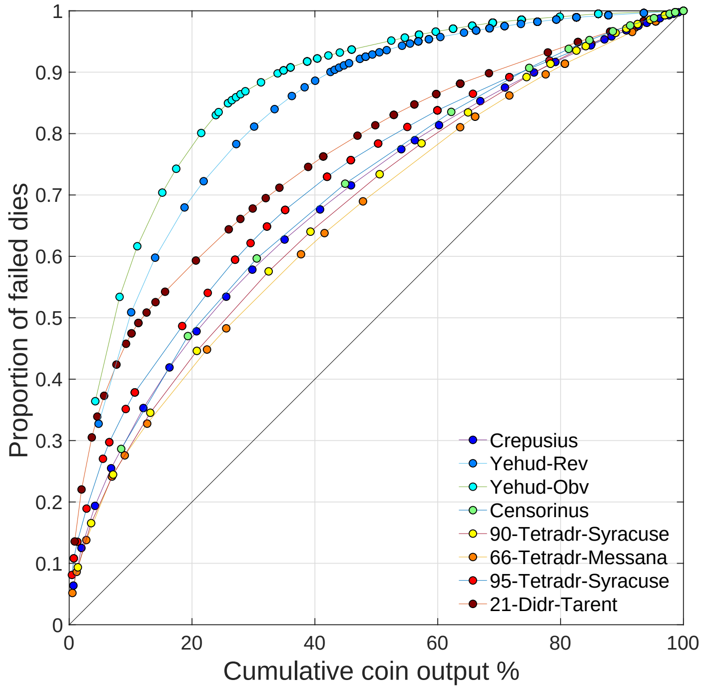

Figure 4. Plot of the fraction of failed dies vs the fractional output (coins struck) for eight data sets. The diagonal line shows the relationship expected for a regime of constant failure probability per blow (homogeneous Poisson process).

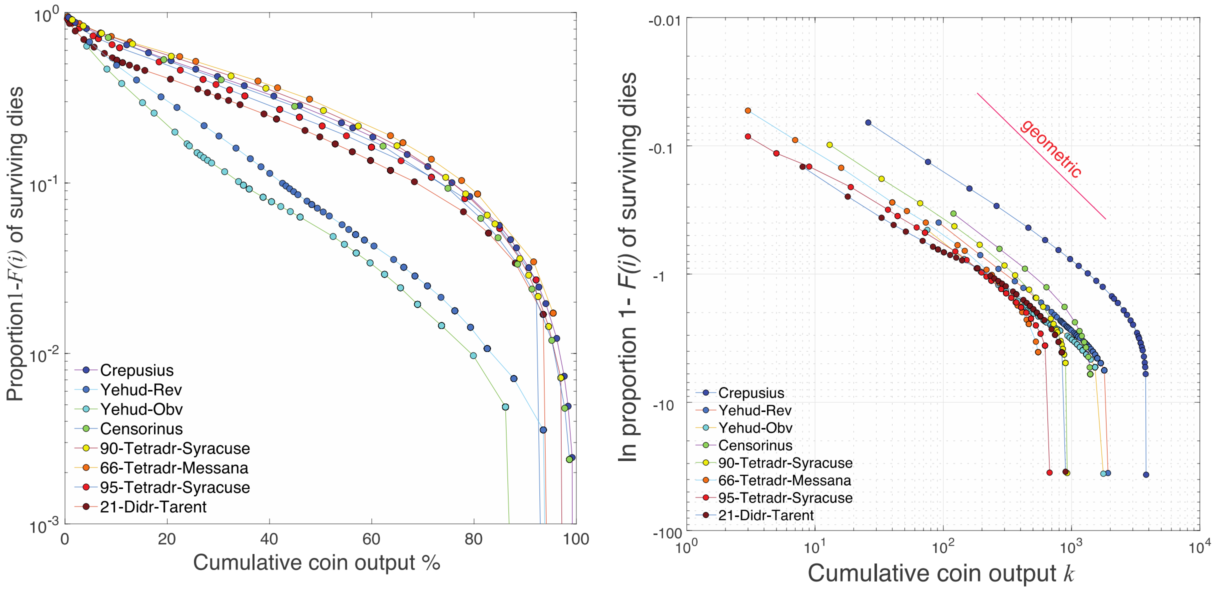

Figure 5. (a) (left-hand side panel) Die survivor function (proportion of surviving dies after k blows vs the fractional coin output recovered) on a semi-logarithmic scale for eight data sets. The thin lines are the cubic smoothing splines run through the points and used to calculate $z(k)$. The slopes of the curves are the negative of the failure function $z(k)$, which is the apparent probability of failure per stroke. The upturning segment represents early failure. The drooping tail suggests deviation from the random failure regime and indicates metal fatigue which depleted the class of dies with high multiplicity. The sigmoidal shape of the curves demonstrates significant deviations from the geometric distribution and

its continuous equivalent, the exponential distribution, at low and high multiplicity. (b) (right-hand side panel) Log-log plot of $\ln(1 - F_i(i))$ (log of log) vs k (the cumulated number of coins). A single geometric distribution would give a straight-line with a slope of -1 (red line). At low multiplicity, the linear alignments emphasize a non-geometric distribution of dies with prominent infant mortality. Breaks in the slope emphasize the presence of more than one sub-population.

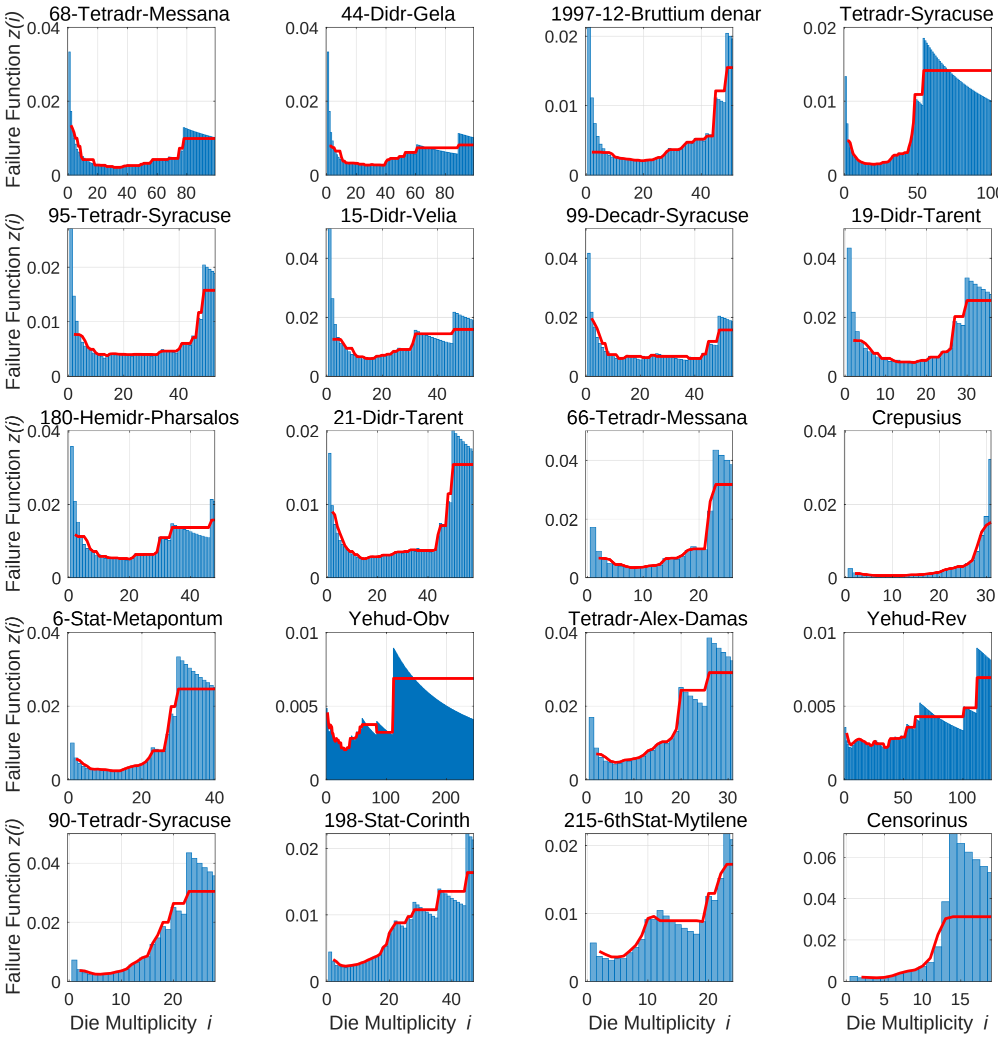

Figure 6. Selection of plots of the failure function vs die multiplicity for 20 of the 29 data sets targeted in the present study. The failure functions $z(i)$ were calculated in two different ways: (1) from Eqn. 5 (blue bars) and (2) from the slope of the natural logarithm of the die survivor function of Fig. 2 (Eqn. 7). The U-shape of these curves is typical of bathtub functions known from survival studies in mechanical and electrical engineering. A plateau at intermediate die multiplicity indicates a constant failure rate, which in turn indicates that the geometric distribution is a locally suitable approximation. In most cases, the failure rate at low and high multiplicities is much higher, which indicates that the frequency histogram deviates from the geometric distribution. Some plots are very asymmetric, which reflects a good mastery of infant mortality by the mint workers.

5. Discussion

The variety of properties necessary to describe the mechanical behavior of metals and alloys is large. S*trength*, refers to resistance to reversible deformation (elasticity), while *hardness* measures the resistance to localized deformation and is usually measured by applying stress with a sharp object. A material can be *ductile* (with reference to irreversible plastic deformation without failure beyond the *yield* point) or *brittle* (fragile). *Toughness* relates to the energy required to break a particular material. Because we ignore so much of the actual minting conditions in ancient mints (metal temperature, striking pace, working position, blow strength, etc.), the values of these critical properties are still the subject of many conjectures (Carter and Carter, 1983; Selldon, 1963; Faucher, 2011, 2013). We here propose an indirect way of inferring die lifetime not through these properties but trough a survival analysis much reminiscent of the Kaplan-Meier handling of patient survival during therapy or reliability assessment in engineering.

Coins are manufactured from a blank disk of silver, the flan, held between two bronze dies hit with a hammer at relatively low temperature. The metallurgist's ultimate task is to maximize plastic extrusion of the flan into an incuse with a variable amount of detail (heads, animals, etc.), while maintaining the bronze dies in the elastic regime in order to maximize their lifetime and ensure that the multiple blows they will sustain do not distort the engraving too quickly. The usual connection between lifetime and metal properties may be seen through the S-N Wöhler curves (measured stress vs the number of bending cycles to failure) and the standard Manson-Coffin model. This model describes failure as resulting from the combined elastic behavior at low stress and plastic behavior at high stress (Meyers and Chawla, 2009). Fatigue studies (Davis, 2001) suggest that, depending on material properties, high-tin bronzes typical of dies (Malkmus, 2008; Gitler and Ponting 2006; Blet-Lemarquand and Duval, 2012) may reach their plastic regime, and therefore remain undeformed, for a maximum of about 10^5 pressure cycles. Such estimates are marginally consistent with accepted values of die productivity derived from experiments (Selldon, 1963; Faucher, 2011, 2013), which ranges from 10,000 to 30,000, sometimes even more (Callataÿ, 2000).

The sigmoidal shape of the curves shown in Fig. 5a, as opposed to the straight-line expected from a geometric distribution (the slope $\neq -1$ in Fig. 5b), and the variability of the failure rate function $z(k)$ with output (Fig. 6) unambiguously show that failure probability per blow varies with the cumulated number of coins. For the geometric distribution, the failure rate function $z(k)$ is constant. $z(k)$ variations therefore measure the deviation of the frequency histograms from the geometric distribution used for many die studies (Esty, 1986, 2066, 2011) at low and high multiplicity. The continuous equivalent of the geometric distribution is the exponential distribution, which presents the same properties.

The failure rate function $z(k)$ usually shows a U-shaped form (Fig. 6), known as the 'bathtub curve', which supports the supposition that most dies are used up or at least employed until near failure. A strong assumption will be made here: dies fail because of metal failure and not because of human error, such as excessive blows or coin misalignment with the blow direction. Failure theory is well advanced in reliability engineering (see a variety of examples in Nash, 2016). A typical bathtub curve can be seen as representing two superimposed regimes (Nash, 2016): (1) an early regime of rapid failure (low production, here translating into low die multiplicity) due to defective dies manufactured with carbon segregation, bubbles, inclusions, and cracks initially present in the metal ('infant mortality' or 'burn-in'), and (2) a metal fatigue regime in which some dies fail prematurely because of structural damage, such as build-up of dislocation walls and crack growth caused by repeated blows. The detrimental infant mortality regime can be nearly suppressed, which causes a strong asymmetry of the curve (e.g., the Censorinus data set) and reveals the remarkable talent of the metallurgists.

The fundamental principle of die studies is that classes with the smallest multiplicity are those most likely to be depleted by early die failure. The less-preferred alternative would be that die counts in preserved samples do not faithfully represent the corresponding distributions in the original populations. The slope of the arrays in Fig. 5 is equal to minus the probability failure function $z(i)$. The steep slope at the upturning edge of the curve at low multiplicity to the left shows that many dies fail after a short period of activity. In contrast, the steep slope at the down-turning edge at high multiplicity to the right indicates failure by metal fatigue.

With the geometric distribution having been discounted by the present analysis, alternative probability distributions must be sought. The constant slopes in Figure 5b, in which $\ln\left(1 - F_i(i)\right)$ is plotted vs ln k, each on a log scale, suggests that a Weibull distribution should be a good representation, at least at low multiplicity. Reliability engineering studies offer multiple examples of such behavior: steel rods, electrical insulation, airplane components, etc. (Nash, 2016). Related studies exist in medicine when the survival of patients under treatment is compared with that of patients receiving a placebo (Kaplan-Meier analysis, see Goel et al., 2010). In the latter case, patients dropping out of the trial or known to have survived until the end of it must be included in the statistics. These cases are said to *censor* the trial and techniques exist to handle them. For minting, censoring should be applied to dies decommissioned before failure, e.g., dies set aside for any reason but failure, and to dies surviving, if any, at the end of the minting operation, but the relevant data are missing. It should therefore be born in mind that this is a limitation to applying reliability concepts to die studies.

When failure probability is not constant, like in the present case, the most widely used failure probability function is indeed the continuous Weibull distribution with cumulative function (c.d.f):

$$F(k) = 1 - e^{-\left(\frac{k}{\lambda}\right)^{\beta}} \quad (9)$$

It can be checked that β = 1 gives the exponential distribution and that a plot of ln [$\ln\left(1 - F_i(i)\right)$] vs ln *k* gives a straight line with a slope of –β. The Weibull point distribution function (p.d.f.) is

$$f(k) = \frac{\beta}{\lambda}\left(\frac{k}{\lambda}\right)^{\beta-1} e^{-\left(\frac{k}{\lambda}\right)^{\beta}} \quad (10)$$

where λ is the scale parameter and β the shape parameter or Weibull modulus, which varies inversely with the spread of the failure range (Meyers and Chawla, 2008). A discrete Weibull mass distribution function can also be used after λ^{β} is replaced by the new parameter –1/ln *q*:

$$f(k) = q^{k^{\beta}} - q^{(k+1)^{\beta}} \quad (11)$$

(Nakagawa and Osaki, 1975). Again, the geometric distribution is obtained for β = 1. At this stage, however, the discussion will be developed using the continuous Weibull p.d.f. The hazard function of the Weibull distribution is

$$z(k) = \frac{f(k)}{1-F(k)} = \frac{\beta}{\lambda}\left(\frac{k}{\lambda}\right)^{\beta-1} \quad (12)$$

When β < 1, the slope *dz*/*dk* is negative and positive otherwise.

In metallurgical terms, Weibull analysis provides an estimate of the distribution of microcrack length within a given object. If β is large, failure occurs over a narrow range of blows because cracks will go off nearly simultaneously. If β is small, cracks will go off and failure spread over a much larger number of blows. This is, for example, the case of brick. We therefore tested the statement that the die histograms (in blue, Fig. 3) are a mixture of two continuous Weibull p.d.f. The test was made on the cumulative distribution function, which has a non-decreasing, much smoother shape than the point density function. The function was fitted to the observed cumulated fraction of failed dies as a function of the number of coins struck using the expression:

$$F(k) = \omega F_1(k) + (1-\omega)F_2(k) \quad (13)$$

where $F_1(k;\ \beta_1, \lambda_1)$ and $F_2(k;\ \beta_2, \lambda_2)$ are two Weibull c.d.f. (Eqn. 9) and ω is a number such as 0 ≤ ω ≤ 1. This approach is in line with Callataÿ's (1987, 2000, in press) suggestion of a mixed distribution controlled by infant mortality and metal fatigue, but with negative binomial p.d.f.'s replaced by two Weibull p.d.f.'s. The results listed in Table 2 have been obtained using the multi-dimensional 'Levenberg-Marquardt' algorithm, also known as damped least-squares, implemented by Matlab (e.g., https://en.wikipedia.org/wiki/Levenberg-Marquardt_algorithm). Some of the fits are shown in Fig. 7.

The fit is, in general, remarkable but with occasional deviations at high multiplicity. The shape factors form two groups well centered around the values $\beta_1 = 0.79$ and $\beta_2 = 2.4$, which suggests a relatively stable die technology. The scale parameter λ_1 is variable but does not correlate with the number of coins n, while $\lambda_2 \sim 0.6\, n$. The metal fatigue population is reduced in some data sets, e.g., Yehud obverses, 26 and 209, which may reveal poorer control of die technology. Another remarkable observation is the relatively small range of ω, β_1, and β_2, which implies that the dies seem to all have a similar proportion of mishap and wear out following the same law. The Weibull modulus is invariant upon multiplication of k by a constant and therefore does not depend on the number of coins in the sample. The value β_2 of the modulus is at least one order of magnitude lower than the values determined for modern steel and iron (Ono, 2019) and more in the range of modern ceramics and pottery (Meyers and Chawla, 2008; Ono, 2019). In general, ductile materials, such as copper and steel, have β values between 10 and 100, while lower values are associated with brittle metals and alloys. What is striking about these results is the large proportion, typically 75%, of dies classified as defective. Some workshops, for which the production is accounted for by a single population of dies with infant-mortality characteristics, may not have achieved full control of the art of producing super-productive dies or failed to hire dependable workers.

Let us finally return to the models of Fig. 1 by drawing the distribution of coin frequency of a typical data set with n = 1000 coins, 54% of dies belonging to the population of infant mortality, β_1=0.7 and β_2=2.45, and λ_1 = 80 and λ_2 = 600 (Fig. 8). By typical is meant that these values represent a behavior common to the variety of cases depicted in Fig. 7. With cumulative coin output replacing time, it is clear that Callataÿ's (1987) model C is the closest to observation, although, as a result of the pervasive infant mortality, with a less pronounced hump. Expressing this model as a function of die multiplicity would require assumptions on the distribution of lifetimes, which is beyond the scope of this work.

The contrast between the two failure regimes provides a response to the smith dilemma: how to make dies that do not crack early but nevertheless resist deformation after thousands of blows? Tin-rich bronzes with Sn contents up to 20% are used for dies (Malkmus, 2008; Gitler and Ponting 2006; Blet-Lemarquand-Duval, 2012). All dies are melted so as to homogenize the alloy and remove bubbles, inclusions, and defects, which, after human error, are probably the main causes of early failure and infant mortality. Upon cooling, in addition to the ductile Cu-Sn solid solution, known as α-phase, high-tin bronzes crystallize a brittle component called δ phase (Saunders and Miodownik, 1990). If a quenched bronze cast is tempered, i.e., reheated, the two phases separate by spinodal decomposition, a process of phase separation by uphill diffusion, resulting in a hardened, cohesive alloy (Cribb and Ratka, 2002). Mao et al. (2009) showed that α dendrites substantially reinforce the strength of grain boundaries with the best result obtained for alloys with a peritectic composition of 22% Sn. While this truly magic proportion clearly was known to ancient metallurgists around the Mediterranean, how well tempering, which would have strongly affected die lifetimes, was understood is uncertain but was a critical factor of mint productivity.

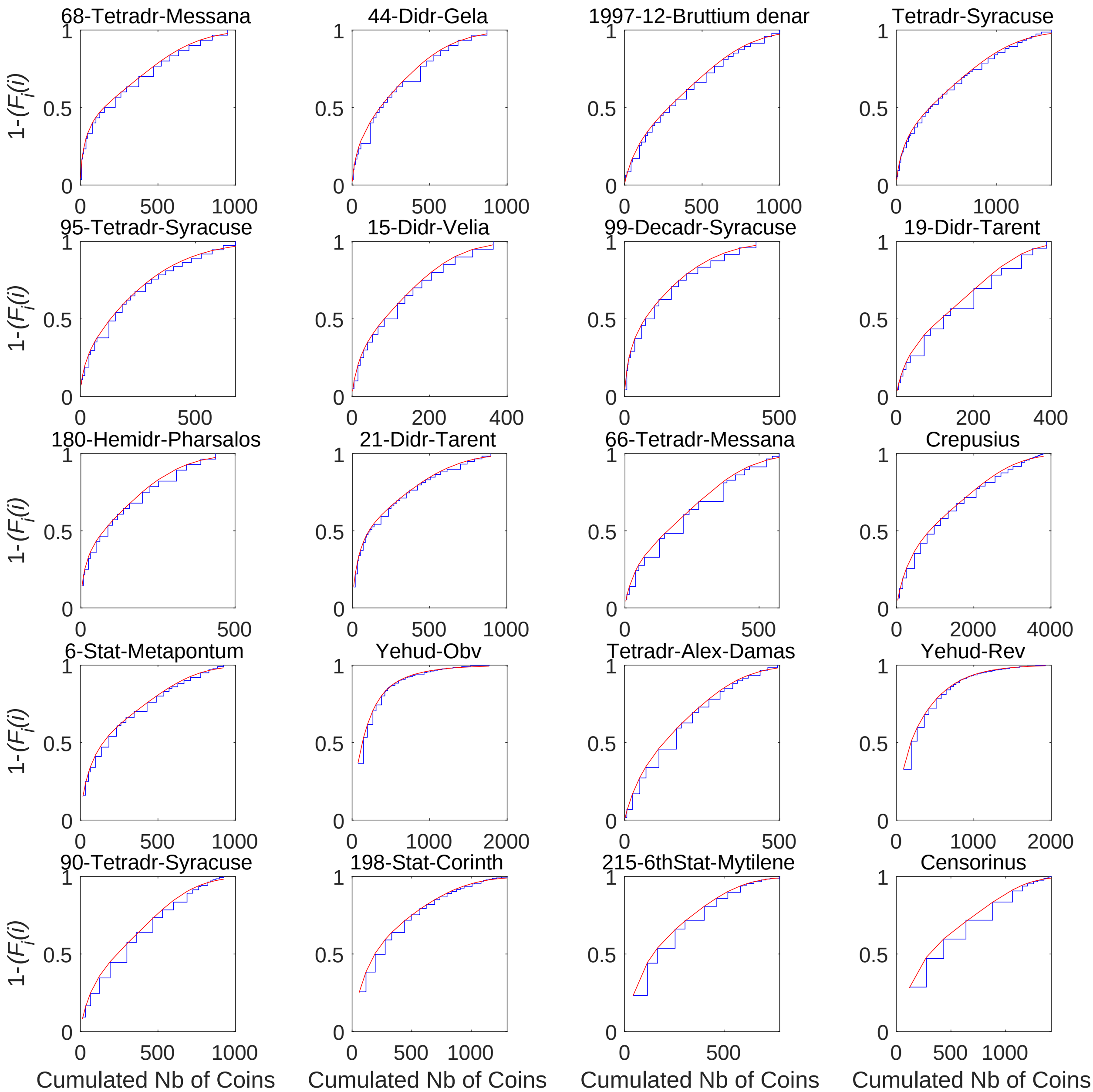

Figure 7. Examples of fits of the mixed Weibull distribution (Eqn. 13) to the observed cumulated fractions of failed dies vs the cumulated coin production. Blue: observed fractions; red: fitted distributions.

Table 2: Results of fitting Eqn. 13 to the observed cumulated fractions of failed dies. ω is the fraction of the first population, λ_1 its scale factor of population, and β_1 its shape factor, with similar notation for population 2. *n* is the number of coins in the issue and *res* the mean squared deviation between the fitted values and the data. The Weibull modulus β is invariant upon multiplication of *k* by a constant. This is not the case of λ, and therefore no statistics on λ_1 and λ_2 are given. The samples have been ordered by increasing values of β_1 to emphasize the data sets with $\beta_1 \sim 1$ for which Esty's (2011) singleton-based correction will remain accurate.

	ω	λ_1	β_1	λ_2	β_2	*n*	*res*†
1997-147-Dr-Euboea	0.57	68.6	0.521	610	1.72	1128	0.011
180-Hemidr-Pharsalos	0.58	34.4	0.594	245	1.97	437	0.0087
68-Tetradr-Messana	0.56	56.5	0.617	562	2.14	950	0.0069
95-Tetradr-Syracuse	0.49	57.9	0.62	336	1.56†	674	0.0095
21-Didr-Tarent	0.63	61.6	0.687	516	2.04	897	0.0048
44-Didr-Gela	0.59	106.8	0.688	485	2.01	870	0.0105
99-Decadr-Syracuse	0.53	28.3	0.697	220	1.66	424	0.0083
Tetradr-Syracuse	0.59	186.3	0.703	914	2.27	1544	0.0074
93-Tetradr-Syracuse	0.61	138.6	0.719	639	2.43	978	0.0089
6-Stat-Metapontum	0.71	115.6	0.722	593	2.73	924	0.0081
Yehud-Obv	0.94	200.7	0.729	253	2.06	1768	0.0033
15-Didr-Velia	0.56	45.1	0.734	210	2.17	364	0.0072
255-Didr-Pixodaros	0.61	44.6	0.743	231	1.8	516	0.0044
66-Tetradr-Messana	0.54	79.9	0.766	358	2.45	575	0.0095
19-Didr-Tarent	0.58	63.9	0.800	261	2.89	388	0.0097
90-Tetradr-Syracuse	0.57	132.4	0.812	570	2.59	924	0.0095
26-Didr-Tarent	0.87	90.5	0.814	370	3.35	573	0.0058
Crepusius	0.65	581.6	0.822	2443	2.76	3810	0.0098
1997-12-Bruttium-den	0.59	181.2	0.827	636	2.54	1000	0.0111
198-Stat-Corinth	0.73	167.9	0.828	762	2.48	1302	0.0046
24-Didr-Tarent	0.54	129.3	0.855	540	2.68	839	0.0060
215-6thStat-Mytilene	0.71	120.4	0.891	474	2.79	779	0.0037
Tetradr-Alexander-Damascus	0.61	88.8	0.909	303	2.48	493	0.0073
197-Stat-Corinth	0.73	65.1	0.915	263	2.63	444	0.0035
Censorinus	0.70	243	0.957	967	3.42	1418	0.0064
Bursio-Rev den	0.72	626	0.862	116	5.23	2359	0.0306
Bursio-Obv den	0.74	833	0.882	124	5.61	2359	0.0188
Yehud-Rev	0.83	73.2	1.047	451	1.11†	1923	0.0017
209-Dr-Sinope	0.86	81	1.345	316	2.74	594	0.0006
average	0.63		0.79		2.43		
std dev.	0.13		0.16		0.46		

* Not included in the statistics † (Mean squared error)$^{1/2}$

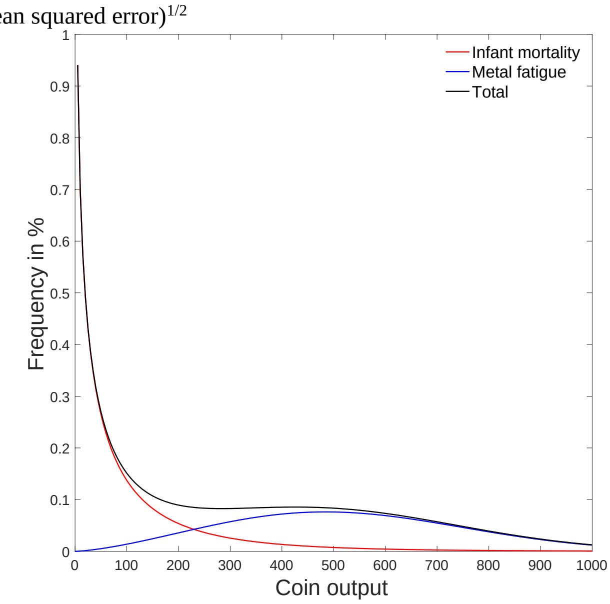

Figure 8. Typical frequency distribution versus output k for a data set with n = 1000 coins. 54% of dies belong to the population of infant mortality, $\beta_1 = 0.7$ and $\beta_2 = 2.45$, and $\lambda_1 = 80$ and $\lambda_2 = 600$. The mixture of the two populations, infant mortality and metal fatigue, makes Callataÿ's (1987) model C (Fig. 1) the best analog for the actual data.

We arrive at the following findings:

- The double die population hypothesis put forward by Callataÿ (1987) is confirmed for 23 out of 29 data sets. The populations of dies used to mint a particular issue are therefore intrinsically heterogeneous: the two regimes of infant mortality and metal fatigue should be viewed as reflecting mechanically distinct populations. The present model allows this dichotomy, which has been discussed previously in qualitative terms only (Callataÿ 1995, 2008, 2011), to now be handled in quantitative terms.
- Semi-log plots of surviving dies vs cumulated output and failure rate plots demonstrate that almost every population participating in the mixture deviates from a simple geometric distribution.
- Singleton abundances are not in general dependable estimates relevant to whole coinage issues, in particular not to the most productive part of a given coin population. The statistical properties of the metal fatigue groups are not revealed by the abundance of singletons, doubletons, or any member of the infant mortality group. For many data sets, this conclusion likely will have strong impact on the correction for missing dies using the theories and formulas developed by Esty (1984). These formulas can, however, still be used by restricting the calculations to the sub-population corresponding to infant mortality (Fig. 5b) whenever the β_1 value is close to unity, thereby hinting at a nearly geometric distribution, such as for the reverse Yehud and the Censorinus coinages. Likewise, keeping in mind that the geometric assumption is not in general

appropriate, large data sets with fewer singletons and therefore high coverage, such as the Crespusius and Censorinus coinages, and set #90 of the Syracuse tetradrachms, should still provide adequate results for the missing die corrections. For the population as a whole, we do not at this time have any better suggestions as to how to account for the missing dies in a more robust way.

- The die distributions among different multiplicities reflect efforts to accommodate two mutually conflicting needs: the reduction of infant mortality due to brittle failure of the bronze alloy used for dies, and the limitation of metal deformation during minting owing to alloy ductility.

These results have implications for the commonplace numismatic strategy, which consists in three steps: (1) estimate the original number of dies and their multiplicity, (2) use the number of singletons and the formulas developed by Esty (1984) and Carter (1983, 1992) to assess the number of missing dies, and (3) multiply this number by the average number of coins a die is supposed to strike. The present results indicate that, even for well-documented samples, large proportions of singletons inevitably overestimate the number of struck coins. The infant-mortality population has little relevance to the remainder of coin production and assigning large values (e.g., 20,000 coins per die, as commonly assumed for large Greek silver coinages) to singletons that actually failed early on may lead to erroneous results. The option of discarding from steps 2 and 3 singletons for any sample with (n/d) > 7 (Fig. 2) is not justified and physically wrong. Such limitations may not significantly affect the best-documented samples (Table 1) such as the drachms of the Euboean League, for which production would be reduced by some 10% (25 x 20,000 instead of 28 x 20,000 coins). Consequences, however, may be much more dramatic for other data sets. For the Yehud coinage, tiny silver coins are characterized by a large proportion of singletons (36%, or 75 out of 208) despite an (n/d) ratio of 8.8.

Understanding actual coinage production is key to understanding the strength of economy and its resilience to changing financial situations through war and trade (Patterson, 1972). The main limitation to furthering the understanding of monetary production remains the estimated average value of coin production, e.g., 20,000 coins per obverse die for large silver coins. The number 20,000 is widely used for Greek tetradrachms. The generally low ratios (n/d) obtained for Roman Republican denarii issues, despite their long-time circulation, have been assigned, at least in part, to a lower productivity than their Greek counterparts. In addition, die productivity also depends on coin weight and human metallurgical expertise and this must be kept in mind when dealing with different coinages such a Greek, Roman denarii, or Yehud. The respective roles of natural alloy failure and human error in determining this number is unclear. Estimates are independent of the coins themselves and is derived from cross-checking various kinds of evidence, such as survival rates in the long range, historically favorable circumstances allowing to guess daily productivity, or the epigraphic record of Delphi for the Amphictionic coinage (Kinns, 1983).

6. Conclusions

This study has for the first time extracted the failure features of dies in view of a future, better, though yet-to-be-formulated die estimator than that currently in use of Esty (2011), which, as demonstrated here, can be applied only to some sub-populations, not data sets as a whole. It has further been shown that the Weibull distribution is a better fit than the geometric distribution, especially for Greek coinage, where, possibly, smiths did not yet fully master tempering, a technique which, in contrast, seems to have been skillfully operated for Roman Republic coinage, one to two hundred years later. The scarcity of die studies on Roman Republic silver coinage (Buttrey, 1976; De Ruyter, 1996; Debernardi, 2020) compared to the wealth of studies on its Greek counterpart and the near-absence of literary sources do not allow us to develop this particular point of ancient metallurgy, but hopefully the present work will foster numismatic interest for this period.

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