



HAL
open science

On the automorphisms of Mukai varieties

Thomas Dedieu, Laurent Manivel

► **To cite this version:**

| Thomas Dedieu, Laurent Manivel. On the automorphisms of Mukai varieties. 2021. hal-03229102v1

HAL Id: hal-03229102

<https://hal.science/hal-03229102v1>

Preprint submitted on 18 May 2021 (v1), last revised 28 Jan 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE AUTOMORPHISMS OF MUKAI VARIETIES

THOMAS DEDIEU AND LAURENT MANIVEL,
with an appendix by YURI PROKHOROV

ABSTRACT. Mukai varieties are Fano varieties of Picard number one and coindex three. In genus seven to ten they are linear sections of some special homogeneous varieties. We describe the generic automorphism groups of these varieties. When they are expected to be trivial for dimensional reasons, we show they are indeed trivial, up to three interesting and unexpected exceptions in genera 7, 8, 9, and codimension 4, 3, 2 respectively. We conclude in particular that a generic prime Fano threefold of genus g has no automorphisms for $7 \leq g \leq 10$. In the Appendix by Y. Prokhorov, the latter statement is extended to $g = 12$.

CONTENTS

1. Introduction	2
2. Jordan decomposition miscellany	5
2.1. Stable subspaces	5
2.2. Jordan types of tensor products	6
2.3. General strategy	8
3. Genus 8	11
3.1. Unipotent elements	11
3.2. Semisimple elements	12
3.3. Codimension three	16
4. Genus 10	20
4.1. Unipotent elements	20
4.2. Semisimple elements	21
5. Genus 9	24
5.1. Unipotent elements	24
5.2. Semisimple elements	25
5.3. Codimension two	28
6. Genus 7	34
6.1. A brief reminder on spin modules	34
6.2. Unipotent elements	35
6.3. Semisimple elements	37
6.4. Codimension four	40
References	51

Date: May 18, 2021.

1. INTRODUCTION

The classification of Fano threefolds by Fano, Iskovskih, and finally Mori and Mukai is a milestone in the history of complex algebraic geometry [IP99]. From this classification, and the subsequent work of Mukai, a prime Fano threefold of genus $g = 7, 8, 9, 10$ turns out to be a linear section of a G -homogeneous variety $M_g \subseteq \mathbf{P}(V_{n(g)})$, for some simple algebraic group G of which $V_{n(g)}$ is an irreducible representation of dimension $n(g)$. These groups, representations and homogeneous varieties are recalled in the table below (the notation k_g is introduced above Theorem 1):

g	G	$\dim(G)$	V_g	$\dim(V_g)$	M_g	$\dim(M_g)$	k_g
7	Spin_{10}	45	Δ_+	16	\mathbf{S}_{10}	10	4
8	SL_6	35	$\wedge^2 \mathbf{C}^6$	15	$\mathrm{G}(2, 6)$	8	3
9	Sp_6	21	$\wedge^{(3)} \mathbf{C}^6$	14	$\mathrm{LG}(3, 6)$	6	2
10	G_2	14	\mathfrak{g}_2	14	G_2/P_2	5	2

The following numerical relations hold:

$$\mathrm{codim}(M_g) = g - 2, \quad \mathrm{index}(M_g) = \dim(M_g) - 2.$$

Fano varieties X with coindex $\dim(X) + 1 - \mathrm{index}(X) = 3$ are called *Mukai varieties*. They were classified by Mukai [Muk89], modulo a conjecture on the existence of smooth canonical divisors which was later proved by Mella [Mel99], see also [CLM98] for a different approach based on the theory of *extensions*. Note that the coindex is preserved by taking hyperplane sections, so it is enough to classify maximal Mukai varieties, those that are not hyperplane sections of any smooth variety. The homogeneous varieties M_g are precisely the maximal Mukai varieties of degree $2g - 2$, for $7 \leq g \leq 10$. The Mukai varieties of those degrees are thus the smooth linear sections of the minimally embedded $M_g \subseteq \mathbf{P}(V_g)$. These are the varieties we study in this paper.

Of course M_g has a big automorphism group. Its hyperplane sections also admit non trivial automorphisms. In genus $g = 7, 8, 9$ there is in fact a unique smooth hyperplane section, up to the action of G : the representations V_g (or rather their duals) are prehomogeneous, and the claim readily follows from the easy classification of the G -orbits. Moreover the smooth hyperplane section is acted on by the generic stabilizer of $\mathbf{P}(V_g^*)$, which was computed in [SK77]. In genus $g = 8$ one obtains the isotropic Grassmannian $\mathrm{LG}(2, 6)$, which is homogeneous under the action of the symplectic group Sp_6 . In genus $g = 7$, the automorphism group of the hyperplane section is not reductive and its action is only prehomogeneous. In genus $g = 9$, the automorphism

group is reductive, but too small to act on the hyperplane section with an open orbit. In genus 10, a generic element of $V_{10}^* = \mathfrak{g}_2$ is a regular semisimple element in the Lie algebra, which is stabilized by a maximal torus in G_2 ; up to the action of G_2 there is therefore a one dimensional family of hyperplane sections of G_2/P_2 , whose connected automorphism group is a two dimensional torus (see [PZ18, PZ21] for a recent study).

There are a few other small codimensional sections of M_g with non trivial automorphisms, the existence of which can be deduced from the fact that the action of $\mathrm{GL}_k \times G$ on $\mathbf{C}^k \otimes V_g$ is prehomogeneous, for some small integers $k > 1$, with non trivial generic stabilizer. This happens for $g = 7, k = 2, 3$, and for $g = 8, k = 2$ (see [BFM18] for the connection with exceptional Lie groups). The case $g = 7, k = 2$ was studied in detail in [Kuz18].

Let us compile those k -codimensional linear sections $X \subseteq M_g$ with non trivial automorphisms in the table below, with their generic connected automorphism groups (possibly up to some finite group).

g	k	$\mathrm{Aut}^0(X)$	[SK77]
7	1	$(\mathbf{G}_m \times \mathrm{Spin}_7) \times \mathbf{G}_a^8$	Proposition 31 p.121
	2	$G_2 \times \mathrm{SL}_2$	Proposition 32 p.124
	3	SL_2^2	Proposition 33 p.126
8	1	Sp_6	
	2	SL_2^3	Proposition 12 p.94
9	1	SL_3	Proposition 22 p.108
10	1	\mathbf{G}_m^2	[PZ18]

We are thus led to let $k_g = 4, 3, 2, 2$ for $g = 7, 8, 9, 10$, respectively. The first main result of this paper can then be stated as follows.

Theorem 1. *For $g = 7, 8, 9, 10$, a generic linear section X of M_g , of positive dimension, and of codimension $k \geq k_g$, has only trivial automorphisms, except for $k = k_g$ and $g = 7, 8, 9$.*

Quite surprisingly, we could not find this result in the literature for the well-studied case of Fano threefolds. For any smooth prime Fano threefold of genus $g < 12$, the automorphism group is known to be finite by [KPS18]. It is known to be trivial for a general prime Fano threefold of genus 6 [DK18, Proposition 3.21], but the corresponding statement in higher genus seems new. The case where $g = 12$, which requires a different approach, is treated in the Appendix by Yuri Prokhorov.

Corollary 2. *The automorphism group of a general prime Fano threefold of genus $g \geq 7$ is trivial.*

Our strategy to prove Theorem 1 will be to reduce this statement to the following one. Denote by \bar{G} the image of G in $\mathrm{PGL}(V_g)$.

Theorem 3. *For $g = 7, 8, 9, 10$, let $L \subseteq V_g$ be a generic linear subspace such that*

$$(\star) \quad \min(\mathrm{codim}_{V_g}(L), \dim(L)) \geq k_g.$$

Then the stabilizer of L in \bar{G} is trivial, except if equality holds in (\star) and $g = 7, 8, 9$.

In order to deduce Theorem 1 from Theorem 3, we need to prove that any automorphism of $X = M_g \cap \mathbf{P}(L)$ must be induced by an element of G stabilizing L (at least when the latter is generic). This kind of statement is at the heart of Mukai's approach to prime Fano threefolds and K3 surfaces of small genus. More precisely, each Mukai variety of genus $g = 7, 8, 9, 10$ admits a unique special vector bundle E_g of rank $r_g = 5, 2, 3, 2$, which defines its embedding into M_g , itself naturally embedded in a Grassmannian of rank r_g subspaces. For Fano threefolds, Mukai [Muk89, Theorem 0.9] proved the following statement (for genus 10, see Proposition 5.1 in op. cit. and the discussion that follows):

Proposition 4. *Let $X = M_g \cap \mathbf{P}(L)$ and $X' = M_g \cap \mathbf{P}(L')$ be smooth linear sections of M_g , of the same dimension $n \geq 2$, and suppose that $\varphi : X \rightarrow X'$ is an isomorphism. Then there exists $g \in G$ such that $L' = g(L)$ and $\varphi = g^*$.*

Proof. We just need to check that Mukai's argument extend to the case where X and X' have dimension larger than three. In fact the argument is essentially by induction on the dimension. The key point is to make use of the bundle E_g , and its restrictions F and F' to X and X' , respectively. These bundles must be stable, because they are stable on a generic hyperplane section. Moreover the restriction of F to a generic K3 section $S \subseteq X$, and the restriction of F' to $\varphi(S)$ must be isomorphic (there is a unique such bundle on a K3 surface of genus g). This isomorphism then lifts to an isomorphism between F and $\varphi^*(F')$ (this requires some cohomological vanishing that readily follows from Bott's theorem). And then the discussion of [Muk89, Section II], almost without change, shows that this isomorphism has to be induced by the linear action of some element of G . \square

The above proof doesn't work when X and X' are curves. Statements for curves analogous to Proposition 4 are derived in [CD20, §4], but Theorem 1 is well-known in this case anyway. Note by the way that Theorem 3 in the case of curve linear sections by L tells us that a general curve of genus $g = 7, 8, 9, 10$ has only finitely many (up to the action of G) models as a linear section of the Mukai variety M_g ; we refer again to [CD20] for refinements on this subject.

There are three exceptional cases in the previous statements, in genus 7, 8, 9, for which non trivial finite groups of symmetries show up unexpectedly. This is the second main result of this paper.

Theorem 5. *For $g = 7, 8, 9$, let $L \subseteq V_g$ be a generic linear subspace of codimension k_g . Denote by $\text{Aut}_G(L)$ the image in $\text{PGL}(L)$ of the stabilizer $\text{Stab}(L) \subseteq G$. Let $X = M_g \cap \mathbf{P}(L)$ be the corresponding Mukai variety.*

Then

$$\mathrm{Aut}(X) = \mathrm{Aut}_G(L) = \begin{cases} (\mathbf{Z}/2\mathbf{Z})^2 & \text{for } g = 7, \\ \mathbf{C}^* \rtimes ((\mathbf{Z}/3\mathbf{Z})^2 \rtimes \mathbf{Z}/2\mathbf{Z}) & \text{for } g = 8, \\ (\mathbf{Z}/2\mathbf{Z})^4 & \text{for } g = 9. \end{cases}$$

Moreover in all cases of the above statement we are able to identify the fixed locus of the various automorphisms. Actually the genus 8 case has already been discussed by Piontkowski and Van de Ven, who obtained by direct computations a less precise result [PV99, Theorem 4.6].

Proving these two statements will require a careful analysis, the principles of which are explained in the next section. Geometrically, we conclude that the generic codimension four sections of M_7 , and the generic codimension two sections of M_9 are stabilized by finitely many non trivial involutions that we will describe explicitly. The case of $g = 8$ is more specific and less unexpected since by duality, there is a plane cubic associated to a codimension three section of M_8 , on which the action of $\mathrm{Aut}_G(L)$ can be read off.

In a subsequent paper, we will interpret the previous Theorem in terms of θ -representations, and show that our small groups of automorphisms can be seen as traces of complex reflection groups defined as generalized Weyl groups of some graded Lie algebras.

Acknowledgements. We thank Christian Kratthenthaler for his help, and Yuri Prokhorov for his comments on the automorphisms of prime Fano three-folds, and his permission to include his Appendix on the genus twelve case.

2. JORDAN DECOMPOSITION MISCELLANY

2.1. Stable subspaces. Let $S \subseteq \bar{G}$ be the stabilizer of a generic subspace $L \subseteq V_g$. In the relevant range of dimensions, we will prove that S is trivial by proving that it contains no non trivial semisimple or unipotent element. Indeed, if $g \in G$ stabilizes L , one can use its Jordan decomposition $g = g_s g_n$ in G , and observe that since g_s and g_n are polynomials in g (once considered as elements of $\mathrm{Hom}(V_g)$), they must also stabilize L .

This reduction will allow us to treat separately unipotent and semisimple elements. We will stratify the set of those elements and control for each stratum the dimension of the stable subspaces dimension. Then a simple dimension count will imply that the generic L has no stabilizer. This dimension count will be based on a straightforward bound for the variety of m -dimensional spaces stabilized by a unipotent or semisimple endomorphism, in terms of its Jordan type.

Proposition 6. *For $g \in \mathrm{GL}(V)$, let $G_m(g) \subseteq \mathrm{G}(m, V)$ denote the variety of m -dimensional subspaces which are stabilized by g .*

- (1) If g is semisimple with eigenvalues of multiplicities e_1, \dots, e_p , the dimension of $G_m(g)$ is bounded by the maximum of the

$$f_1(e_1 - f_1) + \dots + f_p(e_p - f_p)$$

for $0 \leq f_i \leq e_i$ and $f_1 + \dots + f_p = m$.

- (2) If g is unipotent with b_k Jordan blocks of size k for $1 \leq k \leq q$, we let $\beta_p = b_q + \dots + b_p$ for all $p = 1, \dots, q$, and then the dimension of $G_m(g)$ is bounded by the maximum of the

$$\gamma_1(\beta_1 - \gamma_1) + \dots + \gamma_q(\beta_q - \gamma_q),$$

taken over the sequences $\gamma_1 \geq \dots \geq \gamma_q \geq 0$ such that $\gamma_1 + \dots + \gamma_q = m$ and $\gamma_p \leq \beta_p$ for any $p \leq q$.

Proof. If g is semisimple and its eigenspace decomposition is $V = E_1 \oplus \dots \oplus E_p$, simply observe that a subspace L stabilized by g must be of the form $L = F_1 \oplus \dots \oplus F_p$ for $F_i \subseteq E_i$.

If $g = id + X$ is unipotent, we construct a stable subspace L such that the restriction Y of X to L has c_i blocks of size i , for $1 \leq i \leq q$, as follows. We first choose a subspace L_q , of dimension c_q , transverse to $\ker(X^{q-1})$; this is possible for $c_q \leq b_q$, and then it is an open, non empty condition. Inductively, for any $1 \leq p < q$, we then choose a subspace L_p , of dimension $c_q + \dots + c_p$, such that $XL_{p+1} \subseteq L_p \subseteq \ker(X^p)$, transverse to $\ker(X^{p-1})$; the latter condition can be realized only when $c_q + \dots + c_p \leq b_q + \dots + b_p$, and then it is an open, non empty condition. Finally we let $L = L_1 + \dots + L_q$. By the transversality conditions we imposed, this is a direct sum and therefore, the dimension of L is $m = c_1 + 2c_2 + \dots + qc_q$. Moreover, by construction L is stable with the prescribed Jordan type, and every such L can be obtained like that. In terms of the dimensions x_p of $\ker(X^p)$, given by $x_p = \sum_{k=1}^q \min(k, p)b_k$, we can express the number of parameters for (L_1, \dots, L_q) as

$$\sum_{p=1}^q c_p(x_p - c_q - \dots - c_p).$$

Now, L being given, we can choose (L_1, \dots, L_q) inside L subject to the same conditions as above; the number of parameters for (L_1, \dots, L_q) is then given by the same formula, but with x_p replaced by the dimension of $\ker(Y^p)$; which is $y_p = \sum_{k=1}^q \min(k, p)c_k$. Finally, the number of parameters for L itself is the difference between these two numbers. Letting $\beta_p = b_q + \dots + b_p$ and $\gamma_p = c_q + \dots + c_p$ we get the result announced. \square

2.2. Jordan types of tensor products. In the sequel we will meet representations W defined as tensor products $U \otimes V$, and we will need to control the Jordan type of an endomorphism $Z \in \text{Hom}(W)$ defined from two endomorphisms $X \in \text{Hom}(U)$ and $Y \in \text{Hom}(V)$, as $Z = X \otimes \text{Id}_V + \text{Id}_U \otimes Y$. Let u, v denote the dimensions of U and V . Recall that a nilpotent endomorphism is called *regular* if it has only one Jordan block.

Proposition 7. *Suppose that $X \in \text{Hom}(U)$ and $Y = \text{Hom}(V)$ are regular nilpotent, and that $u \geq v$. Then Z has v Jordan blocks, of sizes $u - v + 1, u - v + 3, \dots, u + v - 3, u + v - 1$.*

Proof. Choose a basis e_1, \dots, e_u of U such that $X(e_i) = e_{i-1}$ (with $e_0 = 0$), and similarly a basis f_1, \dots, f_v of V such that $Y(f_i) = f_{i-1}$ (with $f_0 = 0$). Then $Z(e_i \otimes f_j) = e_{i-1} \otimes f_j + e_i \otimes f_{j-1}$. Let $W_\ell \subseteq W$ be generated by the vectors $e_i \otimes f_j$ for $i + j = \ell$. Then Z has degree -1 . We will prove that $Z^p : W_{\ell+1} \rightarrow W_{\ell+p+1}$ always has maximal rank; this immediately implies our claim. Moreover, it follows from the more precise statement that $Z^p : W_{\ell+1} \rightarrow W_{u+v+1-\ell}$ is an isomorphism for $p = u + v - 2\ell > 0$. In fact, this immediately follows from the hard Lefschetz theorem applied to $\mathbf{P}^{u-1} \times \mathbf{P}^{v-1}$, by identifying $W_{\ell+1}$ with its $(\ell-1)$ -th cohomology group. For a more (but not completely) elementary argument, observe that the matrix of Z^p in the canonical basis of $W_{\ell+1}$ and $W_{u+v+1-\ell}$ has for coefficients simple binomial coefficients, and that one just need to check that the determinant of this matrix is non zero; more precisely, we need that

$$\det \left(\binom{p}{u+1-i-j} \right)_{1, \ell-v+1 \leq i, j \leq \ell} \neq 0.$$

When $\ell \geq v$, this is the determinant of a triangular matrix with 1's on the antidiagonal, so it is equal to 1 up to sign. For $\ell < v$, the evaluation is more difficult, but classical. We can apply [Kr99, §2.2 and (2.17)], which gives

$$\det \left(\binom{a+b}{a-i+j} \right)_{1 \leq i, j \leq \ell} = \prod_{i=1}^{\ell} \prod_{j=1}^a \prod_{k=1}^b \frac{i+j+k-1}{i+j+k-2},$$

and let $a = u - \ell$, $b = v - \ell$ to get the result, which is certainly non zero. \square

Remark. The educated reader will have noticed that the previous identity can also be deduced from the Weyl dimension formula for the irreducible representations of GL_{a+b} .

We will also need a skew-symmetric version of Proposition 7, where we consider the action of an element $X \in \text{Hom}(U)$ on $\wedge^2 U$. We will denote the induced operator by $\wedge^2 X$.

Proposition 8. *Suppose that $X \in \text{Hom}(U)$ is regular nilpotent.*

- *If $u = 2v$ is even, $\wedge^2 X$ has v Jordan blocks, of sizes $1, 5, \dots, 2u - 3$.*
- *If $u = 2v + 1$ is odd, $\wedge^2 X$ has v Jordan blocks, of sizes $3, 7, \dots, 2u - 3$.*

Proof. We use a similar argument as in the previous statement. Again we choose a basis e_1, \dots, e_u of U such that $X(e_i) = e_{i-1}$ (with $e_0 = 0$). Then $\Lambda := \wedge^2 U \subseteq U \otimes U$ has a basis given by $e_i \wedge e_j = e_i \otimes e_j - e_j \otimes e_i$ for $i < j$, and the action of $\wedge^2 X$ is the restriction of the action of $Z = X \otimes id_U + id_U \otimes X$. In particular this action is again of degree -1 with respect to the induced graded on Λ_k and we are reduced to proving that $Z^p : \Lambda_k \rightarrow \Lambda_{2u+2-k}$ is an isomorphism for $p = 2u + 2 - 2k \geq 0$. As before this follows from the hard

Lefschetz theorem applied to $G(2, u)$, or from the more elementary check that a determinant is non zero; more precisely, we need that

$$\det \left(\binom{p}{u+1-k-i+j} - \binom{p}{u+1-i-j} \right)_{i,j \in I(k,u)} \neq 0,$$

where $I(k, u)$ is the set of integers i such that $1 \leq i < k - i \leq u$. See again [Kr99] for the computation of this determinant. In particular the result is indeed non zero. \square

Remark. To be more precise, the exact value $D(u, k)$ of the previous determinant is given by two nice formulas, depending on the parity of k .

Indeed, applying identity (3.19) of [Kr99], with $q = 1$, $A = 2u$ and $L_i = -u - i$, we get for $k = 2\ell$, after letting $v = u - \ell$,

$$D(u, 2\ell) = \frac{1}{(\ell-1)!} \prod_{i=1}^{\ell-1} \frac{(2i)!(2v-2i+1)!}{(v-i)!(v+i)!}.$$

Moreover, applying identity (3.18) of [Kr99], with $q = 1$, $A = 2u$ and $L_i = -u - i + 1$, we get for $k = 2\ell + 1$, after letting $v = u - \ell - 1$,

$$D(u, 2\ell + 1) = \prod_{i=1}^{\ell} \frac{(2i-1)!(2v-2i)!}{(v-i+1)!(v+i)!}.$$

2.3. General strategy. We shall proceed to a case by case study of the linear sections of the maximal Mukai varieties M_g , for $g = 7, 8, 9, 10$. As we already explained, we will check that a general $L \subseteq V_g$, whose dimension m belongs to the relevant range, cannot be stabilized by any non-trivial unipotent or semisimple element in \bar{G} , except in the special cases listed in Theorem 5, for which our analysis will show that there are no non-trivial unipotent elements in the stabilizer, and provide a short list of possible semisimple elements stabilizing L . The discussion of these two cases will proceed along the lines indicated in the following two paragraphs.

For each of the special cases of Theorem 5, we provide specific representation theoretic arguments to give a definitive description of the stabilizer. These shall be introduced in due time.

2.3.1. Unipotent elements. Equivalently, we will check that a generic L of dimension m cannot be preserved by a non trivial nilpotent element in the Lie algebra \mathfrak{g} of G . For this we will use the fact that \mathfrak{g} has only finitely many nilpotent orbits \mathcal{O} , for each of which we can provide a representative X . Then X acts on V_g as a nilpotent operator, with a Jordan decomposition that we will determine; Proposition 7 and Proposition 8 will be extremely useful for that. Using Proposition 6, we will then deduce the dimension $d_m(\mathcal{O})$ of the variety of m -dimensional subspaces of V_g stabilized by X . The claim we are aiming for will then follow from the inequalities

$$(\star) \quad \dim \mathcal{O} + d_m(\mathcal{O}) < \dim G(m, V_g) \quad \forall \mathcal{O} \neq \{0\}.$$

In fact it is sufficient to prove the non-strict inequality in the above condition: to see this we consider the projection map $\pi : (L, X) \mapsto L$, defined on the incidence variety $\mathcal{I}_{\mathcal{O}} \subseteq \text{Gr}(m, V_g) \times \mathcal{O}$ parametrizing pairs (L, X) such that $X.L = L$; our claim follows from the fact that the fibres of π always have dimension at least 1, which in turn comes from the observation that if L is stabilized by some $X \in \mathcal{O}$ then it is also stabilized by all multiples λX , $\lambda \in \mathbf{C}^*$.

We shall use a Python script (see §2.3) to verify (\star) for all nilpotent orbits \mathcal{O} : for each \mathcal{O} , we exhaustively list all possible sequences (γ_i) in the notation of Proposition 6, and thus compute the maximum giving the value of $d_m(\mathcal{O})$.

In practice we proceed as follows: we give the list of all nilpotent orbits, including their dimensions and the Jordan decompositions for the actions of their members on the representation V_g , and then we use the Python toolkit contained in the file `stab-nilp.py` to verify the inequality (\star) for all of them. The data in our Python format together with the function calls for the verification for genus g is contained in the file `g**-nilp.py`, where `**` is the value of g . We have found that (\star) always holds (with equality in the sole case $g = 8$ and $m = 3$ or 12 , which is fine as well as noted above), so that the generic m -plane L in V_g has no non-trivial unipotent element in its stabilizer if $k_g \leq m \leq \dim(V_g) - k_g$.

2.3.2. Semisimple elements. In order to check that a generic L of dimension m cannot be preserved by a non trivial semisimple element of G , we will make a similar dimension count. First observe that for $g \leq 9$, the representation V_g has only weights of multiplicity one. This implies that a generic semisimple element acts on V_g with multiplicity one eigenvalues; in particular, it stabilizes only finitely many subspaces of V_g . These eigenvalues will be obtained by including our semisimple element into some maximal torus $T \subseteq G$ and making use of the weight decomposition of V_g with respect to this torus.

Positive dimensional families of stable subspaces will only occur when some of the eigenvalues will coincide; we will call such coincidences *collapsing*. Each type of collapsing will force our semisimple element to belong to a certain locally closed stratum $S \subseteq G$, and for each stratum we will use Proposition 6 in order to compute (or at least bound) the dimension $d_m(S)$ of the variety of m -dimensional subspaces of V_g stabilized by an element of S .

Each stratum S is determined by a set $\mathcal{W}(S)$ of values taken by the roots of G . The stratum S is then the disjoint union $S = \coprod_{\mathbf{w} \in \mathcal{W}(S)} \mathcal{O}_{\mathbf{w}}$ of the conjugacy classes corresponding to the values $\mathbf{w} \in \mathcal{W}(S)$. For each value of \mathbf{w} the dimension of $\mathcal{O}_{\mathbf{w}}$ may be computed by considering the adjoint action of G on its Lie algebra \mathfrak{g} , as follows.

Remark 9. *Let $\gamma \in G$. The conjugacy class of γ has codimension $\text{rk}(G) + \delta$ in G , where δ is the number of roots of G taking the value 1 on γ . Indeed,*

the tangent space at $1 \in G$ to the stabilizer of γ for the adjoint action is $\ker(\text{Ad}(\gamma) - \text{id}_{\mathfrak{g}})$.

To show that a generic m -dimensional L has trivial stabilizer, it is sufficient to prove the inequalities

$$\forall S \neq \{1\} : \quad \dim(S) + d_m(S) < \dim \text{Gr}(m, V_g).$$

In fact this criterion may be improved as follows.

Lemma 10. *The stabilizer of a generic m -dimensional L does not intersect S as soon as*

$$(U) \quad \forall \mathbf{w} \in \mathcal{W}(S) : \quad \dim(\mathcal{O}_{\mathbf{w}}) + d_m(S) < \dim \text{Gr}(m, V_g).$$

Proof. Consider the projection map $\pi : (L, \gamma) \mapsto L$ defined on the incidence variety $\mathcal{I}_S \subseteq \text{Gr}(m, V_g) \times S$ parametrizing pairs (L, γ) such that $\gamma.L = L$. For a pair (L, γ) with γ semisimple, the condition $\gamma.L = L$ is equivalent to L being a direct sum of subspaces of the eigenspaces of γ . But then the eigenvalues are irrelevant, so for each pair $(L, \gamma) \in \mathcal{I}_S$ there is in fact a whole family of pairs $(L, \gamma_{\mathbf{w}}) \in \mathcal{I}_S$ obtained by letting the eigenvalues of γ move in $\mathcal{W}(S)$. Therefore the generic fibre of π has dimension at least $\dim \mathcal{W}(S)$. The claim follows since $\dim(S) = \dim \mathcal{W}(S) + \dim(\mathcal{O}_{\mathbf{w}})$ for any $\mathbf{w} \in \mathcal{W}(S)$. \square

To proceed with this strategy, we have written an elementary piece of Python code (included in the ancillary files) to automatize the computation of the maximum in Proposition 6, and the verification of the inequalities (U) of Lemma10. In practice we also reduce the cases to be checked by using the fact that if a decomposition into eigenspaces is obtained from another one by breaking the eigenspaces into smaller pieces, then d_m will be larger for the decomposition with larger pieces.

It is also important to take into consideration the action of the Weyl group on the roots of G in order to reduce the various cases to be checked.

To structure our analysis, we shall distinguish between *degenerations*, which are the relations gotten when a root of G takes the value 1, and *collapsings* which are the other coincidences between the weights of V_g . In particular collapsings have no effect on the dimension of the conjugacy class.

We encode the decomposition of V_g into eigenspaces as a partition of $n = \dim(V_g)$, which we write as $[\mu_1^{a_1}, \dots, \mu_p^{a_p}]$ if there are a_i eigenspaces of dimension μ_i for $i = 1, \dots, p$. When listing eigenvalues, we indicate the multiplicity between parentheses. When we write “(U) holds for all m ”, we intend that it is so if $k_g \leq m \leq n - k_g$. To help locate the exceptional cases, i.e., those which may give rise to a non-trivial stabilizer for the general subspace, we indicate them with a 'P' sign together with a label including the genus.

Our Python toolkit for the semi-simple case is in the file `stab-sm spl.py`, and the specific data for the genus g is in the file `g**-sm spl.py`. At some points in analyzing the possible collapsings, it is convenient to use Macaulay2

to perform some elementary but tedious polynomial manipulations: the relevant files in these cases are `g07-collapse.m2`.

3. GENUS 8

We will start with the genus 8 case, which is the easiest one since it only involves the projective linear group and its familiar action on the Grassmannian $M_8 = G(2, 6)$, embedded in $\mathbf{P}(V_8) = \mathbf{P}(\wedge^2 \mathbf{C}^6)$ by the usual Plücker embedding. As outlined in our general strategy, we will analyse the possibility for a given unipotent or semisimple element of $G = \mathrm{PSL}_6$, to stabilize a generic subspace L of $\wedge^2 \mathbf{C}^6$. Once this is done, we will conclude that the stabilizer S_L of a generic L of dimension 4 to 11 must be trivial. Moreover, if L has dimension 3 or 12, its stabilizer can only contain involutions and order three elements of a very specific type, which will allow us to determine completely the structure of S_L and $\mathrm{Aut}_G(L)$. This will be the conclusion of a lengthy and laborious analysis that the reader may easily skip, in case she is ready to trust the authors.

3.1. Unipotent elements. There are eleven nilpotent orbits in \mathfrak{sl}_6 , corresponding to the eleven partitions $\pi = (\pi_1 \geq \dots \geq \pi_m)$ of six. A representative of \mathcal{O}_π is obtained by choosing a splitting $\mathbf{C}^6 = U_1 \oplus \dots \oplus U_m$, where U_i has dimension π_i , and letting $X_\pi = X_1 + \dots + X_m$, with X_i a regular nilpotent element in $\mathfrak{sl}(U_i) \subseteq \mathfrak{sl}_6$. The Jordan type of the action of X_π on $\wedge^2 \mathbf{C}^6$ can then be obtained by decomposing

$$\wedge^2 \mathbf{C}^6 = \left(\bigoplus_{i=1}^m \wedge^2 U_i \right) \oplus \left(\bigoplus_{j < k} U_j \otimes U_k \right)$$

and applying Proposition 7 and Proposition 8. The result is the following:

<i>Partition</i>	<i>Dimension</i>	<i>Jordan type</i>
6	30	9, 5, 1
5, 1	28	7, 5, 3
4, 2	26	5 ² , 3, 1 ²
4, 1 ²	24	5, 4 ² , 1 ²
3 ²	24	5, 3 ³ , 1
3, 2, 1	22	4, 3 ² , 2 ² , 1
3, 1 ³	18	3 ⁴ , 1 ³
2 ³	18	3 ³ , 1 ⁶
2 ² , 1 ²	16	3, 2 ⁴ , 1 ⁴
2, 1 ⁴	10	2 ⁴ , 1 ⁷
1 ⁶	0	1 ¹⁶

Arguing as indicated in §2.3.1, we conclude that for $m = 3, \dots, 12$, the general m -dimensional linear subspace $L \subseteq \wedge^2 \mathbf{C}^6$ has no nilpotent element in its stabilizer.

Remark. By semi-continuity, one could argue that it suffices to exhibit a single L in each relevant dimension, whose stabilizer is made of semisimple

elements only. An example of such an L can be generated by decomposable tensors $e_p \wedge e_q$, for a set I of pairs (p, q) . If X stabilizes L , then for each (p, q) in I , $X(e_p \wedge e_q) = Xe_p \wedge e_q + e_p \wedge Xe_q$ must be a combination of the $e_r \wedge e_s$, for (r, s) in I . As a consequence, Xe_p must be a linear combination of the e_r 's such that (r, q) belongs to I . In other words, $X_{rp} = 0$ as soon as there exists a $q \neq r, p$ such that $(p, q) \in I$ but $(r, q) \notin I$. We can then look for configurations I such that for any pair $p \neq r$, there exists q satisfying this property; then any X in the stabilizer of L will have to be diagonal in our fixed basis.

A direct verification shows that we can choose

$$I = \{(13), (16), (25), (34), (45)\},$$

and its unions with (26) and (56). Since obviously the complements of these sets also satisfy the required property, we get a suitable L for each dimension between 5 and 10.

3.2. Semisimple elements. Let g be a semisimple element in GL_6 , with eigenvalues t_1, \dots, t_6 . The codimension in GL_6 of the orbit of g is

$$\mathrm{codim}(\mathcal{O}_g) = \sum_s n_s(g)^2,$$

where the $n_s(g)$ are the multiplicities of the eigenvalues. The Weyl group is the symmetric group \mathfrak{S}_6 , acting by permutations on the t_i 's. The eigenvalues of the action of g on $\wedge^2 \mathbf{C}^6$ are the $t_i t_j$ for $1 \leq i < j \leq 6$, each with multiplicity 1. These eigenvalues are not always distinct, and we shall discuss their possible collapsings as explained in §2.3.2.

3.2.1. Regular case. Assume g is regular, i.e., its eigenvalues t_i are pairwise distinct. Then the conjugacy class of g has dimension 30. The eigenvalue $t_i t_j$ can coincide with $t_k t_l$ only if the pairs (i, j) and (k, l) are disjoint. As a consequence, each eigenspace E_μ for the action on $\wedge^2 \mathbf{C}^6$ has dimension at most three. One verifies (using our automatized procedure) that (U) holds unless $m = 3$ and there are at least three 3-dimensional E_μ 's.

Let us decide whether the latter possibility can indeed happen. Up to acting with the Weyl group, we may assume that a first triple collapsing involved is

$$12 = 34 = 56, \text{ i.e., } t_1 t_2 = t_3 t_4 = t_5 t_6.$$

Again up to the Weyl action, the second one is necessarily either (i) $13 = 2* = **$, or (ii) $13 = 5* = **$ (i.e., either $t_1 t_3 = t_2 t_a = t_b t_c$, or $t_1 t_3 = t_5 t_a = t_b t_c$). In case (i), we may end up with either $13 = 24 = **$ or $13 = 25 = **$. In case (ii), we may end up with either $13 = 52 = **$ which has already been found, since $13 = 56 = **$ is impossible as 56 is already involved in the first triple collapsing. The two possibilities left are thus $13 = 24 = 56$ which in fact is impossible, and $13 = 25 = 46$. The upshot is that the only possibility to have two triple collapsings is

$$12 = 34 = 56 \text{ and } 13 = 25 = 46.$$

One finds (for instance using Macaulay2, see the ancillary files listed in §2.3) that the only possibility with all $t_i \neq 0$ and pairwise distinct is

$$(t_1, t_4, t_5) = (a, aj, aj^2) \text{ and } (t_2, t_3, t_6) = (b, bj^2, bj)$$

with $a, b \in \mathbf{C}^*$ and j a primitive cubic root of 1. This involves exactly three triple collapsings, the third one being $16 = 24 = 35$. Pg8.1

3.2.2. *Subregular case.* By this we mean that the eigenvalues of g have multiplicities at most equal to 2. We shall examine successively the three possibilities $[2, 1^4]$, $[2^2, 1^2]$ and $[3^2]$.

A) $t_1(2), t_3, t_4, t_5, t_6(1)$. By this we mean that g has the double eigenvalue t_1 , and the four simple eigenvalues t_3, t_4, t_5, t_6 , all five pairwise distinct. Then the conjugacy class of g has dimension 28. The eigenvalues for the representation are

$$\begin{array}{llll} (i) & t_1^2(1) & (ii) & t_1t_3(2) & (iii) & t_3t_4, t_3t_5, t_3t_6(1) \\ & & & t_1t_4(2) & & t_4t_5, t_4t_6(1) \\ & & & t_1t_5(2) & & t_5t_6(1) \\ & & & t_1t_6(2) & & \end{array}$$

The possible collapsings are the following. The eigenvalue t_1^2 is necessarily distinct from those of type (ii), but may equal some of type (iii). Eigenvalues of type (ii) are pairwise distinct, and each may equal at most one of type (iii) (for instance, t_1t_3 may equal only one among t_4t_5, t_4t_6, t_5t_6). Eigenvalues of type (iii) may collapse at worst in pairs.

It follows that the eigenspaces in $\wedge^2 \mathbf{C}^6$ always have dimension at most 3, hence (U) holds in all cases (including $m = 3$, since conjugacy classes now have dimension only 28).

B) $t_1, t_3(2), t_5, t_6(1)$. Then the conjugacy class of g has dimension 26 and the eigenvalues in the representation are

$$\begin{array}{llll} (i) & t_1^2(1) & (ii) & t_1t_3(4) & (iii) & t_1t_5(2) & (iv) & t_5t_6(1) \\ & t_3^2(1) & & & & t_1t_6(2) \\ & & & & & t_3t_5(2) \\ & & & & & t_3t_6(2) \end{array}$$

The possible collapsings are the following. The eigenvalue t_1^2 may equal t_3^2 , is different from t_1t_3 , may equal at most one type (iii) eigenvalue, or the type (iv) eigenvalue. The eigenvalue t_1t_3 may only collapse with t_5t_6 . Type (iii) eigenvalues may at most collapse in pairs, and all are distinct from t_5t_6 .

Assume the type (iii) eigenvalues collapse in two pairs. Up to the Weyl action, we may assume that $15 = 36$ and $16 = 35$. Since t_1, t_3, t_5, t_6 are pairwise distinct and nonzero, one finds that necessarily $t_3 = -t_1$ and $t_6 = -t_5$. In this case we find the eigenvalues

$$\begin{array}{llll} t_1^2(2) & -t_1^2(4) & t_1t_5(4) & -t_5^2(1) \\ & & -t_1t_5(4) & \end{array}$$

with the only possible further collapsing $t_1^2 = -t_5^2$, and (U) always holds.

If $t_1 t_3 = t_5 t_6$, there can be at most one collapsing among type (iii) eigenvalues (if there are two, we must be in another regularity stratum), say $t_1 t_5 = t_3 t_6$, and then $t_6 = -t_1$ and $t_5 = -t_3$, so we find the eigenvalues

$$\begin{array}{lll} t_1^2 (1) & t_1 t_3 (5) & -t_1 t_3 (4) \\ t_3^2 (1) & & -t_1^2 (2) \\ & & -t_3^2 (2) \end{array}$$

with the only possible further collapsing $t_1^2 = -t_3^2$, and (U) always holds.

In the other cases we have smaller eigenspaces, hence (U) holds in all these cases as well.

C) $t_1, t_2, t_3 (2)$. Then the conjugacy class of g has dimension 24 and the eigenvalues in the representation are

$$\begin{array}{ll} (i) & t_1^2 (1) \\ & t_2^2 (1) \\ & t_3^2 (1) \\ (ii) & t_1 t_2 (4) \\ & t_1 t_3 (4) \\ & t_2 t_3 (4) \end{array}$$

The possible collapsings are the following. There can be one simple collapsing between type (i) eigenvalues. Type (ii) eigenvalues are always pairwise distinct. Each type (i) eigenvalue may equal exactly one type (ii) eigenvalue.

If a collapsing between type (i) eigenvalues occurs, say $t_1^2 = t_2^2$, then the only other possible collapsing is $t_3^2 = t_1 t_2$, in which case the eigenspaces give a partition $[5, 4^2, 2]$, and (U) holds for all m (in the relevant range).

If two collapsings between type (i) and (ii) eigenvalues occur, say $t_1^2 = t_2 t_3$ and $t_2^2 = t_1 t_3$, then one finds that necessarily

$$(t_1, t_2, t_3) = (a, aj, aj^2)$$

for some $a \in \mathbf{C}^*$ and j a primitive cubic root of unity, so that the third relation $t_3^2 = t_1 t_2$ also holds, and we have three 5-dimensional eigenspaces. (U) holds only for $m \geq 4$, whereas for $m = 3$ we find a 36-dimensional family of stable 3-spaces by considering sums of three lines in the three 5-dimensional eigenspaces. Pg8.2

3.2.3. *Penultimate cases.* We now consider the cases in which g has one eigenvalue of multiplicity at least 3, and in total at least three pairwise distinct eigenvalues, which amount to the following partitions of 6: $[3, 1^3]$, $[3, 2, 1]$, and $[4, 1, 1]$.

A) $t_1 (3), t_2, t_3, t_4 (1)$. Then the conjugacy class of g has dimension 24. The eigenvalues for the representation are

$$\begin{array}{lll} (i) & t_1^2 (3) & (ii) & t_1 t_2 (3) & (iii) & t_2 t_3 (1) \\ & & & t_1 t_3 (3) & & t_2 t_4 (1) \\ & & & t_1 t_4 (3) & & t_3 t_4 (1) \end{array}$$

Type (ii) eigenvalues are necessarily pairwise distinct, and so are those of type (iii). The eigenvalue t_1^2 must be distinct from those of type (ii), and may equal at most one of type (iii). Each eigenvalue of type (ii) may coincide

with only one eigenvalue of type (iii). So at most the eigenspaces give the partition $[4^3, 3]$, and (U) holds in all cases.

B) $t_1(3), t_2(2), t_3(1)$. Then the conjugacy class of g has dimension 22. The eigenvalues for the representation are

$$\begin{array}{ll} (i) & t_1^2(3) \\ & t_2^2(1) \end{array} \quad \begin{array}{ll} (ii) & t_1t_2(6) \\ & t_1t_3(3) \\ & t_2t_3(2) \end{array}$$

Type (ii) eigenvalues are necessarily pairwise distinct. Each eigenvalue of type (i) may coincide with only one eigenvalue of type (i), and t_1^2 may equal t_2^2 . If $t_1^2 = t_2^2$, then no further collapsing is possible, and the eigenspaces give the partition $[6, 4, 3, 2]$. In this case (U) holds for all m . Otherwise, it may happen at worst that $t_1^2 = t_2t_3$ and $t_2^2 = t_1t_3$, in which case we would have the partition $[6, 5, 4]$, and we find that (U) holds in all cases.

C) $t_1(4), t_2, t_3(1)$. Then the conjugacy class of g has dimension 18. The eigenvalues for the representation are

$$\begin{array}{ll} (i) & t_1^2(6) \\ & t_2t_3(1) \end{array} \quad \begin{array}{ll} (ii) & t_1t_2, t_1t_3(4) \\ & t_2t_3(1) \end{array}$$

The only possible collapsing is $t_1^2 = t_2t_3$, so at worst we get the partition $[7, 4^2]$, and (U) holds in all cases.

3.2.4. *Remaining cases.* Eventually, we consider the cases in which g has two distinct eigenvalues t_1, t_2 of multiplicities μ_1, μ_2 respectively. Then the eigenvalues for the representation are

$$t_1^2 \binom{\mu_1}{2} \quad t_2^2 \binom{\mu_2}{2} \quad t_1t_2(\mu_1\mu_2)$$

with the only possible collapsing $t_1^2 = t_2^2$.

A) If $(\mu_1, \mu_2) = (5, 1)$, then the conjugacy class of g has dimension 10, and the only possible partition is $[10, 5]$. (U) holds for all m .

B) If $(\mu_1, \mu_2) = (4, 2)$, then the conjugacy class of g has dimension 16, and at worst we get the partition $[8, 7]$. (U) holds for all m .

C) If $(\mu_1, \mu_2) = (3, 3)$, then the conjugacy class of g has dimension 18.

If $t_1^2 \neq t_2^2$ we get the partition $[9, 3^2]$. We obtain a 36-dimensional family of pairs (L, g) with $g.L = L$ by considering those L entirely contained in the 9-dimensional eigenspace. Pg8.3

If $t_1^2 = t_2^2$ we get the partition $[9, 6]$. We obtain a 37-dimensional family of pairs (L, g) with $g.L = L$ by considering sums of a 2-plane in the 9-dimensional eigenspace and a 1-plane in the 6-dimensional one. Pg8.4

Conclusion. We have found that if L is a generic k -plane with $3 < k < 12$, then its stabilizer is trivial, whereas if L is generic of dimension 3 or 12, then its stabilizer may contain only elements as described in the four cases g8.1, g8.2, g8.3, g8.4 (see below for an explicit description).

3.3. Codimension three. We consider in this section a general three-dimensional subspace $L \subseteq \wedge^2 \mathbf{C}^6$ (later on we shall consider L^\perp which has the dual size). By the previous analysis, the stabilizer S_L of L in PSL_6 (not SL_6) is made of semisimple elements, and it can contain

- (1) at most a one dimensional family (not necessarily connected a priori) of elements with two eigenvalues of multiplicity three, such that if A and B denote the two eigenspaces, then $L \subseteq A \otimes B$ (case g8.3);
- (2) at most a one dimensional family of involutions with two eigenspaces E, F of dimension three, such that L is the sum of a line $L_1 \subseteq \wedge^2 E \oplus \wedge^2 F$ and a plane $L_2 \subseteq E \otimes F$ (case g8.4);
- (3) at most a one dimensional family of elements with eigenvalues $a, ja, j^2a, a^{-1}, ja^{-1}, j^2a^{-1}$ for some $a \in \mathbf{C}^*$, with $a^6 \neq 1$; then the induced action on $\wedge^2 \mathbf{C}^6$ has the eigenvalues $1, j, j^2$ with multiplicity three, and L is the sum of three lines in those eigenspaces (case g8.1);
- (4) a finite number of elements with eigenvalues $1, j, j^2$, of multiplicity two; then the induced action on $\wedge^2 \mathbf{C}^6$ has the eigenvalues $1, j, j^2$ with multiplicity five, and L is the sum of three lines in those eigenspaces (case g8.2).

We will call these elements as elements of type (1) to (4).

Type (1). Let us first explain the origin of the first family.

Lemma 11. *There exists a unique pair (A, B) of transverse three-planes in \mathbf{C}^6 such that $L \subseteq A \otimes B \subseteq \wedge^2 \mathbf{C}^6$.*

Proof. Observe that once we know that the pair (A, B) does exist, the elements $g_s = s\mathrm{Id}_A + s^{-1}\mathrm{Id}_B$ belong to the connected component S_L^0 of S_L . Since we know that this connected component is at most one dimensional, it must coincide with the set of those elements. In particular the pair (A, B) must be unique.

In order to prove the existence of the pair (A, B) , we use the following approach. Let $U \subseteq \mathrm{G}(3, 6) \times \mathrm{G}(3, 6)$ be the open subset of pairs of transverse planes (A, B) , and $Z \rightarrow U$ be the relative Grassmannian with fiber $\mathrm{G}(3, A \otimes B)$. The dimension of Z is 36. We need to prove that the natural map $\pi : Z \rightarrow \mathrm{G}(3, \wedge^2 \mathbf{C}^6)$ that forgets the pair (A, B) is dominant, hence generically finite since the dimensions coincide. For this we will prove that the differential of π is an isomorphism at the general point $z = (A, B, L)$ of Z . Recall that the tangent space to a Grassmannian is the bundle of morphisms from the tautological to the quotient vector bundle. We readily deduce that the tangent space of Z at z fits into the relative exact sequence

$$0 \rightarrow \mathrm{Hom}(L, A \otimes B/L) \rightarrow T_z Z \rightarrow \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(B, A) \rightarrow 0.$$

Moreover $\wedge^2 \mathbf{C}^6 = \wedge^2 A \oplus A \otimes B \oplus \wedge^2 B$, so

$$T_L \mathrm{G}(3, \wedge^2 \mathbf{C}^6) \simeq \mathrm{Hom}(L, \wedge^2 A \oplus (A \otimes B/L) \oplus \wedge^2 B).$$

We are therefore reduced to showing that the morphisms

$$\sigma : \text{Hom}(A, B) \longrightarrow \text{Hom}(L, \wedge^2 B), \quad \tau : \text{Hom}(B, A) \longrightarrow \text{Hom}(L, \wedge^2 B)$$

are isomorphisms, where σ is defined by sending $u \in \text{Hom}(A, B)$ to the composition

$$L \hookrightarrow A \otimes B \xrightarrow{u \otimes \text{Id}_B} B \otimes B \longrightarrow \wedge^2 B,$$

and τ is defined similarly. The following Lemma therefore concludes the proof of the previous one. \square

Lemma 12. *Let a_1, a_2, a_3 be some basis of A , and b_1, b_2, b_3 some basis of B . Consider the 3-space L generated by*

$$\begin{aligned} p &= xa_1 \otimes b_1 + ya_2 \otimes b_2 + za_3 \otimes b_3, \\ q &= za_1 \otimes b_2 + xa_2 \otimes b_3 + ya_3 \otimes b_1, \\ r &= ya_1 \otimes b_3 + za_2 \otimes b_1 + xa_3 \otimes b_2, \end{aligned}$$

for $[x, y, z]$ in \mathbf{P}^2 such that x^3, y^3, z^3 are pairwise distinct. Then π is étale at $z = (A, B, L)$.

In particular, L is generic in $\text{G}(3, \wedge^2 \mathbf{C}^6)$. As a consequence it is also generic in $\text{G}(3, A \otimes B)$.

Proof. We make an explicit computation. The map that we must check to be an isomorphism sends $u \in \text{Hom}(A, B)$ to the morphism from L to $\wedge^2 B$ defined by

$$\begin{aligned} p &\mapsto xu(a_1) \wedge b_1 + yu(a_2) \wedge b_2 + zu(a_3) \wedge b_3, \\ q &\mapsto zu(a_1) \wedge b_2 + xu(a_2) \wedge b_3 + yu(a_3) \wedge b_1, \\ r &\mapsto yu(a_1) \wedge b_3 + zu(a_2) \wedge b_1 + xu(a_3) \wedge b_2. \end{aligned}$$

Denote $u(a_i) = \sum u_{ij} b_j$ and suppose that u is mapped to the zero morphism. Then we get nine equations on the u_{ij} 's, which split into three subsystems of size three. For instance, the three equations involving u_{11}, u_{23}, u_{32} are

$$yu_{23} - zu_{32} = zu_{11} - yu_{32} = yu_{11} - zu_{23} = 0,$$

and this system is invertible if and only if $y^3 - z^3 \neq 0$. This implies the claim. \square

Type (2). Now consider an element h of type (2) in S_L . For $g_s = s\text{Id}_A + s^{-1}\text{Id}_B$, belonging to the connected component of S_L , the product $g_s h$ must again be of type (2), in particular it must be an involution. We deduce that h must exchange A and B .

We claim that such elements do exist. In particular the natural map $S_L \rightarrow \mathfrak{S}_2$, the permutation group of the pair A, B , is surjective. In order to see this, we may suppose that L is given in the normal form of Lemma 12. Then we can exhibit the following type (2) transformations preserving L : just fix an integer k (modulo 3), choose ζ some root of unity, and let $h(a_i) = \zeta^i b_{i+k}$ (where indices are computed modulo 3).

Type (3). Consider an element g in S_L whose eigenvalues are $a, ja, j^2a, a^{-1}, ja^{-1}, j^2a^{-1}$ for some $a \in \mathbf{C}^*$. Denote by $e_1, e_2, e_3, f_1, f_2, f_3$ a basis of eigenvectors for these eigenvalues. The action of g on $\wedge^2 \mathbf{C}^6$ admits the eigenvalues $1, j, j^2$, each with multiplicity 3 (except for special values of a), and L must be a direct sum of lines L_1, L'_1, L''_1 contained in the associated eigenspaces, that is

$$L_1 \subseteq \langle e_1 \wedge f_1, e_2 \wedge f_3, e_3 \wedge f_2 \rangle, \quad L'_1 \subseteq \langle e_1 \wedge f_2, e_2 \wedge f_1, e_3 \wedge f_3 \rangle,$$

$$L''_1 \subseteq \langle e_1 \wedge f_3, e_2 \wedge f_2, e_3 \wedge f_1 \rangle.$$

In particular L is contained in $\langle e_1, e_2, e_3 \rangle \otimes \langle f_1, f_2, f_3 \rangle$, and by the previous Lemma we may suppose that $A = \langle e_1, e_2, e_3 \rangle$ and $B = \langle f_1, f_2, f_3 \rangle$. In particular the cube of g acts on A and B by homotheties, and belongs to the connected component of S_L .

Type (4). Finally, consider an element h of type (4). Then $g_s h g_s^{-1}$ is also of type (4). Since we have only finitely many such elements in S_L , and s varies in a connected set, we conclude that necessarily, $g_s h g_s^{-1} = h$. In particular the eigenspaces C_1, C_2, C_3 of h are direct sums of their intersections with A and B . We claim that each of them must be the sum of a line in A and a line in B . Indeed, if this were not the case, we would be able to deduce that, up to some permutation of indices, C_1 is contained in A , C_3 is contained in B , and C_2 is the sum of a line $a \subseteq A$ with a line $b \subseteq B$. Recall that L must be the direct sum of lines $L_1 \subseteq \wedge^2 C_1 \oplus C_2 \otimes C_3$, $L'_1 \subseteq \wedge^2 C_2 \oplus C_1 \otimes C_3$, $L''_1 \subseteq \wedge^2 C_3 \oplus C_1 \otimes C_2$. Since we also know that $L \subseteq A \otimes B$, we would deduce that in fact $L_1 = a \otimes C_3$, $L'_1 = C_1 \otimes b$ and $L''_1 = a \otimes b \oplus C_1 \otimes C_3$. Counting dimensions, we would conclude that L cannot be general.

So the conclusion is that h preserves A and B , and its restriction to each of these has eigenvalues $1, j, j^2$. In particular its cube has to belong to S_L^0 .

Synthesis. By the previous analysis, the action of S_L on $\mathbf{P}(L)$ induces an injective morphism of S_L/S_L^0 into $\mathrm{PSL}(L)$. The induced action on $\mathbf{P}(L)$ preserves the genus one curve \mathcal{C} cut out on $\mathbf{P}(L)$ by the Pfaffian hypersurface.

Let $T_L \subseteq S_L$ denote the subgroup of elements sending A and B to themselves, hence of type (1), (3) or (4). We have seen that the image of T_L in $\mathrm{PSL}(L)$ consists of regular semisimple elements whose cubes are all trivial; this implies that they act on \mathcal{C} (which is a general curve of genus 1) by translation by some 3-torsion point.

Elements of type (2), that is, in $S_L - T_L$, induce involutions of \mathcal{C} that must be point reflections across an inflection point. Indeed, recall that an element of type (2) in S_L has eigenspaces E, F of dimension three, such that L is the sum of a line $L_1 \subseteq \wedge^2 E \oplus \wedge^2 F$ and a plane $L_2 \subseteq E \otimes F$. In this situation, L_1 is generated by a two-form λ of rank four, hence degenerate, and moreover $L_1 \wedge L_2 \wedge L_2 = 0$. Therefore, if μ, ν are two-forms that generate L_2 , the Pfaffian of a general element of L writes

$$\mathrm{Pf}(x\lambda + y\mu + z\nu) = 3x^2\lambda^2 \wedge (y\mu + z\nu) + \mathrm{Pf}(y\mu + z\nu).$$

This shows that $p = [1, 0, 0] = [L_1]$ is an inflection point of \mathcal{C} . Moreover, the line $x = 0$ cuts (in general) the curve \mathcal{C} at three points q_1, q_2, q_3 , such that the tangent line to \mathcal{C} at each q_i passes through p , which means that the degree zero divisors $p - q_i$ are 2-torsion. Therefore q_1, q_2, q_3 are fixed points of the point reflection across p . The upshot is that the involution of $\mathbf{P}(L)$ associated to the decomposition $L = L_1 \oplus L_2$, once restricted to \mathcal{C} , is just the symmetry with respect to the inflection point $p = [L_1]$.

So far, we have proved that the image S_L/S_L^0 injects in the subgroup $(\mathbf{Z}/3\mathbf{Z})^2 \rtimes (\mathbf{Z}/2\mathbf{Z})$ of automorphisms of \mathcal{C} of translations by an element of 3-torsion and point reflections across an inflection point. Let us show that this injection is in fact an isomorphism. In our analysis of elements of type (2) above, we have already seen that the image of S_L/S_L^0 indeed contains elements of order 2. It will thus be sufficient to prove that all order 3 translations are in the image.

To do so we may assume that L is given in the normal form of Lemma 12. Then the curve \mathcal{C} is defined by the cubic polynomial

$$\text{Pf}(up + vq + wr) = \begin{vmatrix} ux & vz & wy \\ wz & uy & vx \\ vy & wx & uz \end{vmatrix} = xyz(u^3 + v^3 + w^3) - (x^3 + y^3 + z^3)uvw,$$

and its group $(\mathbf{Z}/3\mathbf{Z})^2$ of translations by 3-torsion points is generated by the two transformations

$$s : (u : v : w) \mapsto (u : jv : jw) \quad \text{and} \quad t : (u : v : w) \mapsto (w : u : v)$$

(we leave this as an entertaining exercise; hints may be found in [Do12, §3.1]). The translation s is realized by the type (3) element g acting diagonally on the basis $(a_1, a_2, a_3, b_1, b_2, b_3)$ with eigenvalues $\alpha, j\alpha, j^2\alpha, \alpha^{-1}, j\alpha^{-1}, j^2\alpha^{-1}$ for some $\alpha \in \mathbf{C}^*$. The translation t is realized by the type (4) element g sending a_i to a_{i+1} and b_i to b_{i-1} for $i = 1, 2, 3$, with indices computed modulo 3.

We have thus proved that the image of S_L/S_L^0 contains a set of generators of $(\mathbf{Z}/3\mathbf{Z})^2 \rtimes (\mathbf{Z}/2\mathbf{Z})$, and in conclusion we have proved the following.

Proposition 13. $\text{Aut}_G(L) \simeq S_L/S_L^0 \simeq (\mathbf{Z}/3\mathbf{Z})^2 \rtimes (\mathbf{Z}/2\mathbf{Z})$.

Note in particular that $\text{Aut}_G(L)$ contains eight elements of order three and nine involutions.

On the dual size, observe that the connected component S_L^0 does not act trivially on $\mathbf{P}(L^\perp)$. Indeed, L^\perp is the sum of $\wedge^2 A^\vee$, $\wedge^2 B^\vee$, and a dimension three subspace of $A^\vee \otimes B^\vee$, and a non trivial element $g_s = s\text{Id}_A + s^{-1}\text{Id}_B$ of S_L^0 acts on these pieces with distinct eigenvalues (by abuse of notation, we denote by A^\vee the subspace B^\perp of $(\mathbf{C}^6)^\vee$, and similarly A^\perp by B^\vee). We get

$$\text{Aut}_G(L^\perp) \simeq S_L \simeq \mathbf{C}^* \times ((\mathbf{Z}/3\mathbf{Z})^2 \rtimes (\mathbf{Z}/2\mathbf{Z})).$$

That this is indeed a semi-direct product comes from the fact that we have from the above analysis an explicit splitting of the exact sequence

$$0 \rightarrow \mathbf{C}^* \rightarrow \text{Aut}_G(L^\perp) \rightarrow ((\mathbf{Z}/3\mathbf{Z})^2 \rtimes (\mathbf{Z}/2\mathbf{Z})) \rightarrow 0.$$

The corresponding codimension three section X of $G(2,6)$ is a Fano five-fold of index 3 with automorphism group $\text{Aut}(X) = \text{Aut}_G(L^\perp)$. The involutions in this automorphism group fix the intersection of X with their eigenspaces, that is, the intersection of $G(2,6)$ with a general hyperplane in $\mathbf{P}(\wedge^2 E \oplus \wedge^2 F)^\vee$, and a general codimension two subspace in $\mathbf{P}(E \otimes F)^\vee$. The former is the union of two skew lines, and the latter is a del Pezzo surface of degree six.

The order three elements of the automorphism group are of the form $\text{Id}_P + j\text{Id}_Q + j^2\text{Id}_R$, where P, Q, R are transverse planes in \mathbf{C}^6 . The eigenspaces of the induced action on $\wedge^2 \mathbf{C}^6$ are $\wedge^2 P \oplus Q \otimes R$ and its two siblings. Each of them intersect L^\perp along a generic hyperplane. Since

$$\mathbf{P}(\wedge^2 P \oplus Q \otimes R) \cap G(2,6) = \mathbf{P}(\wedge^2 P) \cup \mathbf{P}(Q) \times \mathbf{P}(R),$$

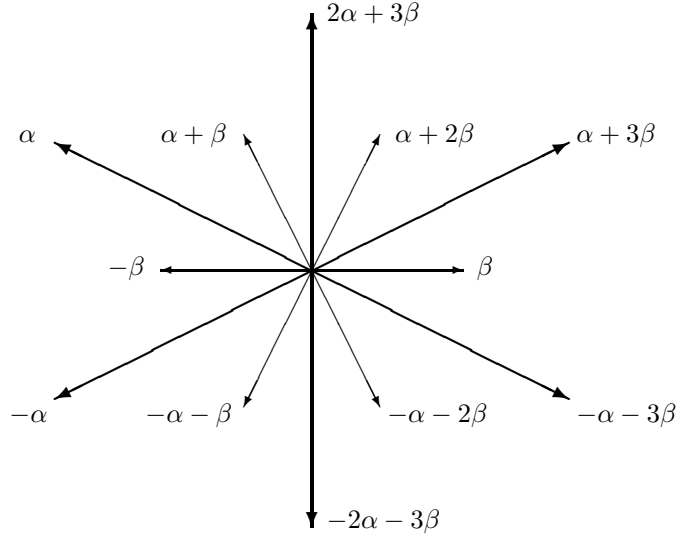
a generic hyperplane section gives a conic. We conclude that the fixed loci of the order three automorphisms of X are unions of three conics.

Remark. As we already mentioned in the Introduction, codimension three sections of $G(2,6)$ were considered before by Piontkowski and Van de Ven [PV99]. Through direct computations they identified the connected component of the automorphism group, and they proved that the quotient embeds in the group of projective transformations of the associated plane cubic curve. Our result, which follows from a completely different proof, is more precise since we completely identify this quotient, as well as the geometric nature of its eighteen elements.

4. GENUS 10

We proceed with the case of genus 10, which is rather straightforward because the relevant Lie group G_2 has only rank two. Indeed, this implies that the number of unipotent orbits, and the number of cases to be discussed for semisimple elements, are relatively small.

4.1. Unipotent elements. Recall that the root system of \mathfrak{g}_2 is as follows, where we denote by (α, β) a pair of simple roots, with α long and β short:



According to [CMcG93, p.128], there are four non trivial nilpotent orbits in \mathfrak{g}_2 . They admit the following representatives, where as usual we have decomposed \mathfrak{g}_2 into a Cartan subalgebra and root spaces, and X_γ denotes a generator of the root space \mathfrak{g}_γ .

Orbit	Dimension	Representative	Jordan type
\mathcal{O}_{reg}	12	$X_\alpha + X_\beta$	12, 2
\mathcal{O}_{subreg}	10	$X_{\alpha+2\beta} + X_\beta$	5, 3 ³
\mathcal{O}_{short}	8	X_β	4 ² , 3, 1 ³
\mathcal{O}_{min}	6	X_α	3, 2 ⁴ , 1 ³

The determination of the Jordan types follows from explicit computations done with the help of [Wil03]. Using Proposition 6 we can then exclude the possibility for a unipotent element of G_2 to stabilize a generic subspace L of \mathfrak{g}_2 of dimension 2 to 12.

4.2. Semisimple elements. Suppose g is a semisimple element of G_2 , say an element of our fixed maximal torus. Then its eigenvalues in the adjoint representation on \mathfrak{g}_2 are 1 with multiplicity 2, the rank of G_2 , and the values taken by the roots, in other words:

$$\begin{aligned}
 & 1 \quad (2) \quad \alpha^2\beta^3 \quad (1) \\
 & \alpha, \alpha\beta, \alpha\beta^2, \alpha\beta^3 \quad (1) \\
 & \beta^{-1}, \beta \quad (1) \\
 & \alpha^{-1}\beta^{-3}, \alpha^{-1}\beta^{-2}, \alpha^{-1}\beta^{-1}, \alpha^{-1} \quad (1) \\
 & \alpha^{-2}\beta^{-3} \quad (1)
 \end{aligned}$$

By Lemma 9, the conjugacy class of g has dimension $\dim(G_2) - \delta$, with δ the multiplicity of the eigenvalue 1 (recall that $\dim(G_2) = 14$).

The Weyl group is isomorphic to the dihedral group of order 12, generated by the rotation of order $\pi/3$ and the reflection across the x -axis. It acts as such on the roots, pictured as above.

Degenerations occur if one root takes the value 1 on g , and collapsings occur when two roots take the same value. In the generic case, $\alpha, \beta \neq 1$ and the roots take pairwise distinct values, there is only one double eigenvalue (namely 1), and the conjugacy class of g has dimension 12, so that (U) always holds in dimensions 2 to 12.

4.2.1. *Degenerate case.* In this case, we assume that some root takes the value 1. Up to the action of the Weyl group we may suppose that this root is either α or β . We treat the two cases separately.

A) $\alpha = 1$. Then we find the eigenvalues

$$\beta^{-3}(2) \quad \beta^{-2}(1) \quad \beta^{-1}(2) \quad 1(4) \quad \beta(2) \quad \beta^2(1) \quad \beta^3(2)$$

In the generic case, the conjugacy class has dimension 10. Collapsings occur if β is a primitive root of 1 of order $o = 2, 3, 4, 5, 6$.

- i) if $o = 2$, the conjugacy class has dimension 8, and the partition is $[8, 6]$;
- ii) if $o = 3$, the conjugacy class has dimension 6, and the partition is $[8, 3, 3]$;
- iii) if $o = 4, 5, 6$, the conjugacy class has the generic dimension 10, and the partition is $[4^3, 2]$, $[4, 3^3, 2^2]$, and $[4^2, 2^2, 1^2]$ respectively.

In all cases (U) holds for all m .

B) $\beta = 1$. Then we find the eigenvalues

$$\alpha^{-2}(1) \quad \alpha^{-1}(4) \quad 1(4) \quad \alpha(4) \quad \alpha^2(1) .$$

In the generic case, the conjugacy class has dimension 10. Collapsings occur if α is a primitive root of 1 of order $o = 2, 3, 4$.

- i) if $o = 2$, the conjugacy class has dimension 8, and the partition is $[8, 6]$;
- ii) if $o = 3$, the conjugacy class has the generic dimension 10, and the partition is $[5, 5, 4]$;
- iii) if $o = 4$, the conjugacy class has the generic dimension 10, and the partition is $[4^3, 2]$.

In all cases (U) holds for all m .

4.2.2. *Nondegenerate cases.* It remains to consider those cases in which two roots coincide. Up to the action of the Weyl group, this reduces to a short list of possibilities.

If two short roots collapse, we may assume that β collapses with $\beta^{-1}, \alpha\beta, \alpha\beta^2$. In the latter two cases we have respectively $\alpha, \alpha\beta$ which take the value 1, and these possibilities have already been investigated.

If a short root collapses with a long root, we may assume that β collapses with $\alpha, \alpha^2\beta^3, \alpha\beta^3$. We discard the latter possibility, which corresponds to the degeneration $\alpha\beta^2 = 1$ given by a short root. We also discard the second one which is equivalent, up to the Weyl action, to $\beta = \beta^{-1}$.

If two long roots collapse, we may assume that α collapses with $\alpha^{-1}, \alpha^2\beta^3, \alpha\beta^3$. We discard the collapsing with $\alpha^2\beta^3$ which coincides with a degeneration given by a long root. We are thus left with the following list: a) $\beta = \beta^{-1}$; b) $\beta = \alpha$; c) $\alpha = \alpha^{-1}$; d) $\alpha = \alpha\beta^3$.

From now on we exclude $\alpha = 1$ or $\beta = 1$, which have already been considered. We record once and for all that for the partitions $[5, 5, 4]$ and $[4^3, 2]$, (U) holds for all m unconditionally, since conjugacy classes have dimension at most 12.

A) $\alpha^2 = 1$. We thus have $\alpha = -1$, which gives the eigenvalues

$$\begin{array}{cccccc} -\beta^{-3}(1) & -\beta^{-2}(1) & -\beta^{-1}(1) & -1(2) & -\beta(1) & -\beta^2(1) & -\beta^3(1) \\ \beta^{-3}(1) & & \beta^{-1}(1) & 1(2) & \beta(1) & & \beta^3(1) \end{array} \cdot$$

In the generic case, the conjugacy class of g has dimension 12. Further collapsings occur if β is a 2, 3, 4, 5, 6-th root of 1 or -1 .

- i) if $\beta = -1$, we find the two eigenvalues $1(6)$ and $-1(8)$, and the conjugacy class has dimension 8;
- ii) if $\beta^2 = -1$, we find the eigenvalues

$$\begin{array}{cc} -1(2) & -\beta(4) \\ 1(4) & \beta(4) \end{array} \cdot$$

and the conjugacy class has dimension 10.

In the other cases, the collapsings don't go over the latter partition $[4^3, 2]$, hence (U) holds for all m .

B) $\beta^2 = 1$, that is $\beta = -1$, which gives the eigenvalues

$$\begin{array}{cccccc} & \alpha^{-1}(2) & 1(2) & \alpha(2) & & \\ -\alpha^{-2}(1) & -\alpha^{-1}(2) & -1(2) & -\alpha(2) & -\alpha^2(1) & \end{array} \cdot$$

Further collapsings occur if α is a 2, 3, 4-th root of 1 or a 2, 3-rd root of -1 . The case $\alpha = -1$ has already been considered.

- i) if $\alpha^2 = -1$, we find the eigenvalues

$$\begin{array}{cc} 1(4) & \alpha(4) \\ -1(2) & -\alpha(4) \end{array} \cdot$$

and the conjugacy class has dimension 10.

In the other cases, the collapsings don't go over the above partition $[4^3, 2]$, hence (U) always holds.

C) $\beta^3 = 1$, that is $\beta = j$, a primitive cubic root of 1. We get the eigenvalues

$$\begin{array}{cccccc} \alpha^{-2}(1) & \alpha^{-1}(2) & 1(2) & \alpha(2) & \alpha^2(1) & \\ & j\alpha^{-1}(1) & j(1) & j\alpha(1) & & \\ & j^2\alpha^{-1}(1) & j^2(1) & j^2\alpha(1) & & \end{array}$$

The condition giving the largest number of collapsings is $\alpha^3 = 1$, in which case we obtain ($\alpha \neq 1$)

$$1(4) \quad j(5) \quad j^2(5) \cdot$$

So we have at most $[5^2, 4]$ and (U) always holds.

D) $\alpha = \beta$. We find the eigenvalues

$$\begin{array}{cccccccccccc} \alpha^{-5} & \alpha^{-4} & \alpha^{-3} & \alpha^{-2} & \alpha^{-1} & 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \\ (1) & (1) & (1) & (1) & (2) & (2) & (2) & (1) & (1) & (1) & (1) \end{array}$$

Collapsings occur for α a root of 1 of order $o \leq 10$. The cases $o = 2, 3$ have been considered already, and for $o \geq 4$ we get at most the partition $[4^3, 2]$ (for $o = 4$), hence (U) holds in all cases.

Conclusion. We conclude from this analysis that no non trivial semisimple element of G_2 can stabilize a generic subspace of \mathfrak{g}_2 of any dimension from 2 to 12. Since this was also the case for unipotent elements, the stabilizer of a generic subspace must be trivial in this range of dimensions.

5. GENUS 9

5.1. Unipotent elements. Nilpotent orbits in \mathfrak{sp}_{2n} are parametrized by partitions $\pi = (\pi_1, \dots, \pi_k)$ of $2n$ whose odd parts have even multiplicities. According to [CMcG93, p.77], one obtains a representative of the corresponding orbit as follows. Observe that \mathfrak{sl}_r embeds in \mathfrak{sp}_{2r} , so counting odd parts with half their multiplicities, we get embeddings

$$(\times_{\pi_i \text{ even}} \mathfrak{so}_{\pi_i}) \times (\times_{\pi_j \text{ odd}} \mathfrak{sl}_{\pi_j}) \subseteq \mathfrak{sp}_{2n}.$$

Adding regular nilpotent elements of each factor, we get a representative X_π of the nilpotent orbit \mathcal{O}_π .

In order to make concrete computations in \mathfrak{sp}_6 , we first choose a basis $e_1, e_2, e_3, e_{-2}, e_{-1}$ of \mathbf{C}^6 in which the invariant skew-symmetric form writes $\omega = e_1^* \wedge e_{-1}^* + e_2^* \wedge e_{-2}^* + e_3^* \wedge e_{-3}^*$. The 14-dimensional module

$$\wedge^{(3)} \mathbf{C}^6 = \ker(\wedge^3 \mathbf{C}^6 \xrightarrow{\omega} \mathbf{C}^6)$$

admits the basis consisting of the eight vectors $f_{\pm\pm\pm} = e_{\pm 1} \wedge e_{\pm 2} \wedge e_{\pm 3}$, and the six vectors $g_{\pm k}$, where $k = 1, 2, 3$, given by $g_1 = e_1 \wedge (e_2 \wedge e_{-2} - e_3 \wedge e_{-3})$, and so on. Our goal is to determine the Jordan type of the action on $\wedge^{(3)} \mathbf{C}^6$ of a member X_π of each of the eight nilpotent orbits \mathcal{O}_π of \mathfrak{sp}_6 . Following the above mentioned rule, we provide below an explicit representative X_π , in matrix form in our fixed basis, of \mathcal{O}_π . The induced action in our preferred basis of $\wedge^{(3)} \mathbf{C}^6$ is then easily computed, and in particular its Jordan type is

readily obtained.

$$\begin{aligned}
X_{[6]} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & X_{[42]} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
X_{[41^2]} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & X_{[3^2]} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
X_{[2^3]} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & X_{[2^2 1^2]} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
X_{[21^4]} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, & X_{[1^6]} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

We finally obtain the following Jordan types (the dimension is computed by using the rule given in §6.2 below):

Orbit	Dimension	Jordan type
$\mathcal{O}_{[6]}$	18	10, 4
$\mathcal{O}_{[42]}$	16	6, 4 ²
$\mathcal{O}_{[41^2]}$	14	4, 3 ² , 2 ²
$\mathcal{O}_{[3^2]}$	14	5, 3 ² , 1 ³
$\mathcal{O}_{[2^3]}$	12	4, 2 ⁵
$\mathcal{O}_{[2^2 1^2]}$	10	3 ² , 2 ² , 1 ⁴
$\mathcal{O}_{[21^4]}$	6	2 ⁵ , 1 ⁴
$\mathcal{O}_{[1^6]}$	0	1 ¹⁴

5.2. Semisimple elements. If g is a semisimple element of Sp_6 , with eigenvalues t_1, t_2, t_3 and their inverses, then the eigenvalues of the induced action on $\wedge^{(3)}\mathbf{C}^6$ are the same six eigenvalues (type I), plus the eight products

$t_1^{\pm 1}t_2^{\pm 1}t_3^{\pm 1}$ (type II). The action of the Weyl group is generated by the permutations of $\{1, 2, 3\}$ and all possible exchanges between t_i and its inverse.

The roots are $t_i^{\pm 1}t_j^{\pm 1}$ for all $1 \leq i < j \leq 3$ and $t_i^{\pm 2}$ for $1 \leq i \leq 3$, in the multiplicative notation. This gives the following degeneration stratification:

- A) $t_2 = t_3$;
- B) $t_3^2 = 1$;
- C) $t_1 = t_2 = t_3$;
- D) $t_2 = t_3$ and $t_1^2 = 1$;
- E) $t_2 = t_3$ and $t_3^2 = 1$;
- F) $t_2^2 = t_3^2 = 1$;
- G) $t_1 = t_2$ and $t_1^2 = t_3^2 = 1$;

(The case $t_1 = t_2 = t_3$ and $t_1^2 = 1$ is trivial, because then $g = \pm 1$ belongs to the centre of Sp_6).

5.2.1. *Regular case.* In this case we assume that no root takes the value 1 on g , so that $t_1, t_2, t_3, t_1^{-1}, t_2^{-1}, t_3^{-1}$ are pairwise distinct, and not equal to ± 1 . Then the conjugacy class of g has dimension 18.

Collapsings among type II are of the following kinds:

- a) $t_1t_2t_3 = t_1^{-1}t_2^{-1}t_3$, if $t_1^2t_2^2 = 1$;
- b) $t_1t_2t_3 = t_1^{-1}t_2^{-1}t_3^{-1}$, provided $t_1^2t_2^2t_3^2 = 1$.

It follows that type II eigenvalues can at most collapse in pairs in this regularity stratum. Since on the other hand type I eigenvalues are pairwise distinct, in the present case all eigenspaces in the representation have dimension at most 3. Then (U) holds in all cases: if one is laborious enough, it is possible to check all cases one by one, and we have implemented this in Python.

5.2.2. $t_2 = t_3$. In this case we have the eigenvalues

$$t_1^{\pm 1} (3) \quad t_2^{\pm 1} (2) \quad t_1^{\pm 1}t_2^{\pm 2} (1).$$

We assume that $t_1^{\pm 1}, t_2^{\pm 1}$ are 4 pairwise distinct values not equal to ± 1 (the case when they coincide is a further degeneration class), hence the conjugacy class of g has dimension 16.

Therefore, collapsings must involve the eigenvalues $t_1^{\pm 1}t_2^{\pm 2}$, and can be of the following kinds (as always, up to the Weyl action): a) $t_1^2 = t_2^2$; b) $t_2^3 = t_1$; c) $t_2^4 = 1$; d) $t_1^2 = t_2^4$.

Assume $t_1^2 = t_2^2$, hence $t_2 = -t_1$. Then the eigenvalues are

$$t_1^{\pm 1} (4) \quad -t_1^{\pm 1} (2) \quad t_1^{\pm 3} (1).$$

The only further collapsings in this degeneration class are if $t_1^4 = -1$ or $t_1^6 = 1$ (note that if $t_1 = i$, then $t_2 = t_1^{-1}$), and they give the partitions $[4^2, 3^2]$ and $[4^2, 2^3]$ respectively. (U) holds in all cases.

Assume $t_1 = t_2^3$. Then the eigenvalues are

$$t_2^{\pm 3} (3) \quad t_2^{\pm 1} (3) \quad t_2^{\pm 5} (1).$$

It is excluded in this case that t_2 be a root of 1 of order 2, 3, 4, 6 so the possible collapsings happen when it is a root of order 8 or 10. They give the partitions $[4^2, 3^2]$ and $[3^4, 2]$, and (U) holds in all cases.

Assume $t_2 = i$. Then we have the eigenvalues

$$t_1^{\pm 1}(3) \quad \pm i(2) \quad -t_1^{\pm 1}(2),$$

and no further collapsing is possible. We thus have the partition $[3^2, 2^4]$, which gives (U) in all cases.

Eventually, if $t_1^2 = t_2^4$ we get the eigenvalues

$$t_1^{\pm 1}(3) \quad t_2^{\pm 1}(2) \quad t_1 t_2^{-2}(2) \quad (t_1 t_2^2)^{\pm 1}(1)$$

and no new further collapsing is possible.

5.2.3. $t_3^2 = 1$. In this case we have the eigenvalues

$$t_1^{\pm 1}, t_2^{\pm 1}(1) \quad t_3(2) \quad t_3 t_1^{\pm 1} t_2^{\pm 1}(2)$$

and the conjugacy class of g has dimension 16.

Type I eigenvalues are pairwise distinct. Up to the Weyl action, the only possible collapsing between types I and II is $t_1^2 = t_3 t_2$, and the only possible one in type II is $t_1^2 = t_2^2$.

Assume $t_1^2 = t_3 t_2$. Then also $t_1^{-2} = t_3 t_2^{-1}$, and the eigenvalues are

$$t_1^{\pm 1}(3), \quad t_2^{\pm 1}(1) \quad t_3(2) \quad t_3 t_1 t_2, t_3 t_1^{-1} t_2^{-1}(2).$$

A further collapsing happens only if either $t_2 = t_3 t_1^{-1} t_2^{-1}$ ($\Leftrightarrow t_1^5 = t_3$), or $(t_1 t_2)^2 = 1$ ($\Leftrightarrow t_1^6 = 1$). They give the partitions $[3^4, 2]$ and $[4, 3^2, 2, 1^2]$ respectively, and (U) holds in all cases.

Assume $t_1^2 = t_2^2$. Possible combinations with collapsings between types I and II have been considered above, so the only new possibility of a further collapsing is that simultaneously $t_1^2 = t_2^{-2}$, which is possible only if $t_2 = \pm i$ and $t_1 = -t_2$, which takes us to another stratum.

5.2.4. $t_1 = t_2 = t_3$. We call t the common value, which in this stratum is assumed to verify $t^2 \neq 1$. In this case the conjugacy class of g has dimension 12 because of the 6 new relations $t_i/t_j = 1$, $i \neq j$, and the eigenvalues are

$$t^{\pm 1}(6) \quad t^{\pm 3}(1).$$

Collapsings occur if either $t^4 = 1$ or $t^6 = 1$, in which cases we get the partitions $[7^2]$ and $[6^2, 2]$ respectively. In the latter case (U) holds for all m , and for all $m > 2$ in the former.

With $t = i$, we obtain for $m = 2$ a 12-dimensional family of pairs (L, g) with $g.L = L$ by considering sums of two lines in the two eigenspaces.

5.2.5. $t_2 = t_3$ and $t_1^2 = 1$. In this case we have the eigenvalues

$$t_1(6) \quad t_2^{\pm 1}(2) \quad t_1 t_2^{\pm 2}(2)$$

with $t_2 \neq \pm 1$, and the conjugacy class of g has dimension 14.

The possible collapsings are $t_2 = t_1 t_2^{-2}$ ($\Leftrightarrow t_2^3 = t_1$), and $t_2^4 = 1$, which cannot happen simultaneously. They give the partitions $[6, 4^2]$ and $[6, 4, 2^2]$ respectively, and (U) holds in all cases.

5.2.6. $t_2 = t_3$ and $t_3^2 = 1$. In this case we have the eigenvalues

$$t_1^{\pm 1}(5) \quad t_2(4)$$

with $t_1 \neq \pm 1$, and the conjugacy class of g has dimension 12 (6 relations $t_2/t_3, t_2^2/t_3^2$ and their inverses). No collapsing is possible, and we have the only partition $[5^2, 4]$, for which (U) holds for all m .

5.2.7. $t_2^2 = t_3^2 = 1$. We may assume that $t_2 = 1, t_3 = -1$, and $t_1 \neq \pm 1$. We have the eigenvalues

$$t_1^{\pm 1}(1) \quad \pm 1(2) \quad -t_1^{\pm 1}(4)$$

and the conjugacy class of g has dimension 14.

The only possible collapsing happens when $t_1^2 = -1$, in which case we have the partition $[5^2, 2^2]$, and (U) holds in all cases.

5.2.8. $t_1 = t_2$ and $t_1^2 = t_3^2 = 1$. In this case we may assume that $t_1 = t_2 = -1$ and $t_3 = 1$. We have the four relations $t_1/t_2, t_1 t_2, t_1^2, t_2^2, t_3^2$ and their inverses, so the conjugacy class of g has dimension 8. (For verification: all roots take the value 1 except $t_1/t_3, t_2/t_3, t_1 t_3, t_2 t_3$ and their inverses, so the conjugacy class of g has dimension 8 indeed).

The eigenvalues in the representation are

$$-1(4) \quad 1(10).$$

(U) holds for $m > 2$, but we find a 16-dimensional family of 2-planes fixed by g by considering those 2-planes inside the 10-dimensional eigenspace.

Pg9.2

Conclusion. We have found that if P is a generic k -plane with $2 < k < 12$, then its stabilizer is trivial, whereas if P is generic of dimension 2 or 12, then its stabilizer may contain only elements as described in the two cases g9.1 and g9.2, and only finitely many of them.

5.3. Codimension two. Let $P \subseteq \wedge^{(3)}\mathbf{C}^6 \subseteq \wedge^3\mathbf{C}^6$ be a generic plane. It follows from the analysis above that the stabilizer of P is finite, and its non-trivial elements may only be of the two following kinds (up to sign):

- I) involutions $\text{Id}_A - \text{Id}_{A^\perp}$ with A a non-degenerate plane, provided $P \subseteq A \otimes \wedge^{(2)}A^\perp$ (case g9.2); following our usual notation, $\wedge^{(2)}A^\perp$ is the kernel of the contraction by ω on $\wedge^2 A^\perp$;

- II) anti-involutions $i(\text{Id}_E - \text{Id}_F)$ with E, F transverse Lagrangian subspaces, provided P is the sum of two lines in $\wedge^3 E \oplus (E \otimes \wedge^2 F)$ and $(\wedge^2 E \otimes F) \oplus \wedge^3 F$ respectively (case g9.1); beware that the two latter spaces are not entirely contained in $\wedge^{(3)}\mathbf{C}^6$.

Proposition 14 below tells us that there are indeed elements of type I, and that it is always possible to decompose A^\perp as the sum of two orthogonal non-degenerate planes A_2, A_3 in such a way that $P \subseteq A \otimes A_2 \otimes A_3 \subseteq A \otimes \wedge^{(2)}A^\perp$. Elements of type II are taken care of in Proposition 15, and the stabilizer of P is completely described in Corollary 16.

Proposition 14. *Let $P \subseteq \wedge^{(3)}\mathbf{C}^6 \subseteq \wedge^3\mathbf{C}^6$ be a generic plane.*

- (1) *There exists a unique triple (A_1, A_2, A_3) of non-degenerate, pairwise orthogonal planes in \mathbf{C}^6 , such that $P \subseteq A_1 \otimes A_2 \otimes A_3 \subseteq \wedge^{(3)}\mathbf{C}^6$.*
- (2) *The three planes A_1, A_2, A_3 are the only non-degenerate 2-planes $A \subseteq \mathbf{C}^6$ such that $P \subseteq A \otimes \wedge^{(2)}A^\perp$.*

Proof. The proof partly relies on the fact that the Lagrangian Grassmannian $\text{LG}(3, 6) \subseteq \mathbf{P}(\wedge^{(3)}\mathbf{C}^6)$ is a variety with *one apparent double point*. In other words, given a general point x in $\mathbf{P}(\wedge^{(3)}\mathbf{C}^6)$, there exists a unique bisecant to $\text{LG}(3, 6)$ passing through x [CMR04, Example 2.9].

Let us apply this observation to $x = [p]$, for p a general point of P . This means that we can write p in the form $p = u_1 \wedge u_2 \wedge u_3 + v_1 \wedge v_2 \wedge v_3$, where $U = \langle u_1, u_2, u_3 \rangle$ and $V = \langle v_1, v_2, v_3 \rangle$ are Lagrangian subspaces of \mathbf{C}^6 , in general position, and uniquely defined by p . Now consider another general point p' of P , again with its two associated Lagrangian subspaces U', V' of \mathbf{C}^6 . Under the generality assumption, we can describe U' and V' as the graphs of two isomorphisms α and β from U to V . Moreover $\alpha \circ \beta^{-1}$ is in general semisimple; let f_1, f_2, f_3 be a basis of eigenvectors in V , with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Let $e_i = \beta^{-1}(f_i)$. Since $u_1 \wedge u_2 \wedge u_3$ is a multiple of $e_1 \wedge e_2 \wedge e_3$ and $v_1 \wedge v_2 \wedge v_3$ is a multiple of $f_1 \wedge f_2 \wedge f_3$, we conclude that there exist scalars a, b, a', b' such that

$$\begin{aligned} p &= ae_1 \wedge e_2 \wedge e_3 + bf_1 \wedge f_2 \wedge f_3, \\ p' &= a'(e_1 + f_1) \wedge (e_2 + f_2) \wedge (e_3 + f_3) + \\ &\quad + b'(e_1 + \lambda_1 f_1) \wedge (e_2 + \lambda_2 f_2) \wedge (e_3 + \lambda_3 f_3). \end{aligned}$$

Letting $A_i = \langle e_i, f_i \rangle$, we deduce that $P \subseteq A_1 \otimes A_2 \otimes A_3$. Note that the isotropy of U, V, U', V' implies that A_1, A_2, A_3 are pairwise orthogonal.

Now observe that the pairwise orthogonal triples (A_1, A_2, A_3) are parametrized by a variety X of dimension 12, so that the relative Grassmannian Y of planes in $A_1 \otimes A_2 \otimes A_3$ has dimension $12 + 12 = 24$. This is also the dimension of $\mathbf{G}(2, \wedge^{(3)}\mathbf{C}^6)$, and since the previous observations imply that the natural map $\pi : Y \rightarrow \mathbf{G}(2, \wedge^{(3)}\mathbf{C}^6)$ is dominant, it must be generically finite. As a consequence, the triple (A_1, A_2, A_3) that we have constructed from two general points p, p' of P cannot change when we vary p and p' , so it must be canonically defined by P . We conclude that π is in fact birational, which implies assertion (1).

Let us turn to assertion (2). We first make some points for future use in the proof. Let $A \subseteq \mathbf{C}^6$ a non-degenerate 2-plane. Observe that $\mathbf{P}(A \otimes \wedge^{(2)} A^\perp)$ contains the Segre product $\mathbf{P}(A) \times \mathbf{Q}^3(A^\perp)$, where $\mathbf{Q}^3(A^\perp)$ is the hyperplane section of $\mathbf{G}(2, A^\perp)$ defined by ω , that is the intersection $\mathbf{G}(2, A^\perp) \cap \mathbf{P}(\wedge^{(2)} A^\perp)$ inside $\wedge^2 A^\perp$. Moreover, this Segre product is contained in $\mathbf{LG}(3, 6)$. We shall use the fact that the Segre product $\mathbf{P}^1 \times \mathbf{Q}^3 \subseteq \mathbf{P}^9$ is a variety with one apparent double point as well (see e.g. [CMR04, Example 2.4]).

Let $C_P \subseteq \mathbf{LG}(3, 6)$ be the curve described by the points $x, x' \in \mathbf{LG}(3, 6)$ such that $[p] \in \langle x, x' \rangle$ when $[p]$ varies in $\mathbf{P}(P)$. Note that $P \subseteq A \otimes \wedge^{(2)} A^\perp$ if and only if $C_P \subseteq \mathbf{P}(A \otimes \wedge^{(2)} A^\perp)$: the if part follows from the fact that $\mathbf{P}(P)$ is contained in the span of C_P , and the only if part from the fact that $\mathbf{P}(A) \times \mathbf{Q}^3(A^\perp)$ is a variety with one apparent double point inside its span, so that for general $[p] \in \mathbf{P}(P)$ the two points $x, x' \in \mathbf{LG}(3, 6)$ necessarily lie on $\mathbf{P}(A) \times \mathbf{Q}^3(A^\perp)$.

Note that a point $x \in \mathbf{LG}(3, 6)$ lies in $\mathbf{P}(A \otimes \wedge^{(2)} A^\perp)$ if and only if the corresponding Lagrangian 3-plane $\Pi_x \subseteq \mathbf{C}^6$ intersects A non-trivially: the only if part is tautological, and conversely if there exists a non-zero $a \in \Pi_x \cap A$, then $\Pi_x \subseteq a^\perp$, hence the intersection of $\Pi_x \cap A^\perp$ is a plane, and therefore x belongs to $\mathbf{P}(A) \times \mathbf{Q}^3(A^\perp)$. Moreover, when this holds the intersection $\Pi_x \cap A$ is necessarily a line, since A is non-degenerate and Π_x is isotropic.

Now let us eventually prove assertion (2). Let $A \subseteq \mathbf{C}^6$ a non-degenerate 2-plane such that $P \subseteq A \otimes \wedge^{(2)} A^\perp$. Let x be a general point on C_P . We have seen in the proof of assertion (1) that $C_P \subseteq \mathbf{P}(A_1) \times \mathbf{P}(A_2) \times \mathbf{P}(A_3)$, so the Lagrangian 3-plane Π_x is the direct sum of three lines $L_1(x) \subseteq A_1$, $L_2(x) \subseteq A_2$, $L_3(x) \subseteq A_3$. Our assumption that $P \subseteq A \otimes \wedge^{(2)} A^\perp$ implies that Π_x and A intersect along a line $L_A(x)$.

Consider the family of automorphisms $\sigma = s_1 \text{Id}_{A_1} + s_2 \text{Id}_{A_2} + s_3 \text{Id}_{A_3}$, $s_i \in \mathbf{C}^*$ for $i = 1, 2, 3$. For all such σ , the plane $\sigma(A)$ intersects Π_x along the line $\sigma(L_A(x))$ for all $x \in C_P$ (indeed σ leaves the lines $L_i(x)$ fixed, hence also $\Pi_x = L_1(x) + L_2(x) + L_3(x)$), and therefore C_P is contained in $\mathbf{P}(\sigma(A) \otimes \wedge^{(2)} \sigma(A)^\perp)$. Considering general such σ 's, we thus obtain a family of non-degenerate planes $\sigma(A)$ such that $P \subseteq \sigma(A) \otimes \wedge^{(2)} \sigma(A)^\perp$. Such planes being only finitely many by our analysis in §5.2, we must have $\sigma(A) = A$ for all σ . This implies that A is the sum of two lines in A_i and A_j respectively. Since A_i and A_j are orthogonal whereas A is non-degenerate, we must have $i = j$ and assertion (2) is proved. \square

Proposition 15. *Let $P \subseteq \wedge^{(3)} \mathbf{C}^6 \subseteq \wedge^3 \mathbf{C}^6$ be a generic plane, and let A_1, A_2, A_3 be as in Proposition 14. Then there exists exactly twelve pairs (E, F) of transverse Lagrangian subspaces of \mathbf{C}^6 , such that E and F both meet all three A_i 's non trivially, and P meets non trivially $\wedge^3 E \oplus (E \otimes \wedge^2 F)$ and $(\wedge^2 E \otimes F) \oplus \wedge^3 F$.*

Proof. Suppose that we have decomposed each A_i into the direct sum of two lines, $A_i = E_i \oplus F_i$. There is an induced decomposition $A_1 \otimes A_2 \otimes A_3 = A_E \oplus A_F$, with

$$\begin{aligned} A_E &= (E_1 \otimes E_2 \otimes E_3) \oplus (E_1 \otimes F_2 \otimes F_3) \oplus (F_1 \otimes E_2 \otimes F_3) \oplus (F_1 \otimes F_2 \otimes E_3), \\ A_F &= (F_1 \otimes F_2 \otimes F_3) \oplus (F_1 \otimes E_2 \otimes E_3) \oplus (E_1 \otimes F_2 \otimes E_3) \oplus (E_1 \otimes E_2 \otimes F_3). \end{aligned}$$

There are 6 parameters for the six lines E_i, F_i , and then 3 + 3 parameters for choosing a line in A_E and a line in A_F ; taking their direct sum, this gives a family of planes in $A_1 \otimes A_2 \otimes A_3$ with twelve parameters; since 12 is also the dimension of $\mathbf{G}(2, A_1 \otimes A_2 \otimes A_3)$, we can expect that a generic plane P can be obtained in this way. In order to check that this guess is correct, we compute the generic rank of the differential of the following map η . Let Q_i denote the complement of the diagonal in $\mathbf{P}(A_i) \times \mathbf{P}(A_i)$. Over $Q = Q_1 \times Q_2 \times Q_3$, there are two rank four vector bundles \mathcal{A}_E and \mathcal{A}_F defined by the formulas above; they are both sub-bundles of the trivial vector bundle with fiber $A_1 \otimes A_2 \otimes A_3$, and the direct sum map induces the morphism

$$\eta : Z = \mathbf{P}(\mathcal{A}_E) \times_Q \mathbf{P}(\mathcal{A}_F) \longrightarrow \mathbf{G}(2, A_1 \otimes A_2 \otimes A_3)$$

that we claim is dominant. To check this, we fix a basis α_i, α_{-i} of A_i . Local coordinates on an open subset of Q are obtained by considering in A_i the lines E_i and F_i generated by $e_i = \alpha_i + x_i \alpha_{-i}$ and $f_i = \alpha_{-i} + y_i \alpha_i$. We then get local relative coordinates on Z by considering lines generated by $d = e_1 e_2 e_3 + p_1 e_1 f_2 f_3 + p_2 f_1 e_2 f_3 + p_3 f_1 f_2 e_3$ and $d' = f_1 f_2 f_3 + q_1 f_1 e_2 e_3 + q_2 e_1 f_2 e_3 + q_3 e_1 e_2 f_3$ (for brevity we omit the tensor product signs). At first order, we compute that

$$\begin{aligned} d &= \alpha_1 \alpha_2 \alpha_3 + x_1 \alpha_{-1} \alpha_2 \alpha_3 + x_2 \alpha_1 \alpha_{-2} \alpha_3 + x_3 \alpha_1 \alpha_2 \alpha_{-3} + \\ &\quad + p_1 \alpha_1 \alpha_{-2} \alpha_{-3} + p_2 \alpha_{-1} \alpha_2 \alpha_{-3} + p_3 \alpha_{-1} \alpha_{-2} \alpha_3, \\ d' &= \alpha_{-1} \alpha_{-2} \alpha_{-3} + y_1 \alpha_1 \alpha_{-2} \alpha_{-3} + y_2 \alpha_{-1} \alpha_2 \alpha_{-3} + y_3 \alpha_{-1} \alpha_{-2} \alpha_3 + \\ &\quad + q_1 \alpha_{-1} \alpha_2 \alpha_3 + q_2 \alpha_1 \alpha_{-2} \alpha_3 + q_3 \alpha_1 \alpha_2 \alpha_{-3}, \end{aligned}$$

which implies our claim. Note that $E = E_1 \oplus E_2 \oplus E_3$ and $F = F_1 \oplus F_2 \oplus F_3$ are Lagrangian subspaces of \mathbf{C}^6 and that the symplectic automorphism $s = i(\text{Id}_E - \text{Id}_F)$ leaves P invariant.

Now suppose that there is another decomposition $A_i = E'_i \oplus F'_i$ compatible with P , hence two other Lagrangian subspaces E' and F' of \mathbf{C}^6 such that the symplectic automorphism $t = i(\text{Id}_{E'} - \text{Id}_{F'})$ also leaves P invariant. Then also $u = st$ leaves P invariant, hence we must have $u^2 = \pm \text{Id}$ by the analysis of §5.1 and 5.2.

If $u^2 = \text{Id}$, then $st = ts$ and therefore, the decompositions $A_i = E_i \oplus F_i = E'_i \oplus F'_i$, which are given by the eigenspaces of s and t , must be the same. This only leaves the possibility to exchange E_i with F_i . The eight possible permutations give four pairs of Lagrangian spaces (E, F) .

If $u^2 = -\text{Id}$, then $st = -ts$, so that for all $j = 1, 2, 3$ the action of t on A_j exchanges the eigenspaces of s . We can thus find generators e_j of E_j

and f_j of F_j such that $t(e_j) = f_j$ and $t(f_j) = -e_j$. The eigenspaces of t and u acting on A_j are then the lines generated by $e_j \pm im_j$ and $e_j \pm m_j$, respectively, so that the associated pairs of Lagrangian spaces are

$$\begin{aligned} E' &= \langle e_1 + if_1, e_2 + if_2, e_3 + if_3 \rangle, & F' &= \langle e_1 - if_1, e_2 - if_2, e_3 - if_3 \rangle, \\ E'' &= \langle e_1 + f_1, e_2 + f_2, e_3 + f_3 \rangle, & F'' &= \langle e_1 - f_1, e_2 - f_2, e_3 - f_3 \rangle. \end{aligned}$$

Moreover, in this case the plane P is generated by two vectors of the form

$$\begin{aligned} p &= xe_1 \wedge e_2 \wedge e_3 + y_1e_1 \wedge f_2 \wedge f_3 + y_2f_1 \wedge e_2 \wedge f_3 + y_3f_1 \wedge f_2 \wedge e_3, \\ p' &= xf_1 \wedge f_2 \wedge f_3 + y_1f_1 \wedge e_2 \wedge e_3 + y_2e_1 \wedge f_2 \wedge e_3 + y_3e_1 \wedge e_2 \wedge f_3. \end{aligned}$$

Indeed, P contains a vector as p above, and since it is stable under t it also contains $t(p) = p'$, which is linearly independent from p . Similarly, $u(p) = ip'$.

Eventually, note that there can be no other decomposition of the A_i 's compatible with P , since t and u are the only anti-involutions up to sign, that exchange the eigenspaces of s .

Conversely, we claim that the pairs (E', F') and (E'', F'') as above, and correspondingly t and u indeed exist. To see this, recall that we can always generate P , up to sign, by two vectors

$$\begin{aligned} p &= xe_1 \wedge e_2 \wedge e_3 + y_1e_1 \wedge f_2 \wedge f_3 + y_2f_1 \wedge e_2 \wedge f_3 + y_3f_1 \wedge f_2 \wedge e_3, \\ p' &= x'f_1 \wedge f_2 \wedge f_3 + y'_1f_1 \wedge e_2 \wedge e_3 + y'_2e_1 \wedge f_2 \wedge e_3 + y'_3e_1 \wedge e_2 \wedge f_3. \end{aligned}$$

Multiplying f_i by some scalar τ_i , we can always reduce to the case where $x' = x$ and $y'_i = y_i$ for each i , if the coefficients x, x', y_i, y'_i are non zero, a condition which holds by genericity of P . Then P is preserved by t and u , and the rest follows. \square

Corollary 16. *The stabilizer in PSP_6 of a generic plane $P \subseteq \wedge^{(3)}\mathbf{C}^6$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^4$.*

Proof. We have established in §5.1 and 5.2 that the elements of Sp_6 that stabilize such a generic plane P are involutions of two possible types I and II as described at the beginning of §5.3. If (A_1, A_2, A_3) is the unique triple of non degenerate, pairwise orthogonal planes in \mathbf{C}^6 such that $P \subseteq A_1 \otimes A_2 \otimes A_3$, the stabilizer contains the involutions $\pm \mathrm{Id}_{A_1} \pm \mathrm{Id}_{A_2} \pm \mathrm{Id}_{A_3}$. These generate in PSP_6 a copy of $(\mathbf{Z}/2\mathbf{Z})^2$. Moreover, with the previous notations, the three involutions defined in PSP_6 by $s, t, u = st$, generate another copy of $(\mathbf{Z}/2\mathbf{Z})^2$. Note that s and t are only defined up to sign, this ambiguity being absorbed by the first three involutions. All these elements in PSP_6 commute one with another, although their representatives in Sp_6 may anticommute. We thus get a copy of $(\mathbf{Z}/2\mathbf{Z})^4$ inside the stabilizer of P .

Let us prove that this $(\mathbf{Z}/2\mathbf{Z})^4$ is indeed the whole stabilizer of P . Consider an element r of the latter. If r is of type I, i.e., an involution $\pm(\mathrm{Id}_A - \mathrm{Id}_{A^\perp})$ for some non degenerate plane A of \mathbf{C}^6 , then by Proposition 14 r is one of $\pm \mathrm{Id}_{A_1} \pm \mathrm{Id}_{A_2} \pm \mathrm{Id}_{A_3}$ and belongs to our $(\mathbf{Z}/2\mathbf{Z})^4$. So suppose that r is of type II, i.e., an anti-involution $i(\mathrm{Id}_E - \mathrm{Id}_F)$ associated to a pair (E, F) of transverse Lagrangian subspaces of \mathbf{C}^6 . Let $a \neq \pm \mathrm{Id}$ be one

of the involutions $\pm \text{Id}_{A_1} \pm \text{Id}_{A_2} \pm \text{Id}_{A_3}$ that stabilize P . Consider $u = ra$. We know that $u^2 = \pm \text{Id}$, so $ra = ar$ or $ra = -ar$. In the latter case, r permutes the eigenspaces of a , which are two of dimensions 2 and 4, a contradiction. So a and r commute, and therefore E, F are as in Proposition 15. \square

Note that the first copy of $(\mathbf{Z}/2\mathbf{Z})^2$ considered in this proof acts trivially on $\mathbf{P}(P)$ since $P \subseteq A_1 \otimes A_2 \otimes A_3$, so $\text{Aut}_G(L) \simeq (\mathbf{Z}/2\mathbf{Z})^2$ is generated by the three involutions s, t, u only.

On the dual size, we get $\text{Aut}_G(L^\perp) = (\mathbf{Z}/2\mathbf{Z})^4$, and this is the automorphism group of the general codimension two linear section X of $\text{LG}(3, 6)$. This completes the proof of our main Theorem in genus 9. We have shown more precisely that $\text{Aut}(X)$ consists of two different types of involutions: three of type I (coming from elements of Sp_6 with eigenspaces of dimensions 2 and 4), and 12 of type II (with two eigenspaces of dimensions 3).

Proposition 17. *The fixed locus in X of an involution of type I is a Del Pezzo surface of degree four (not in its anticanonical embedding). The fixed locus of an involution of type II is the disjoint union of two Veronese surfaces.*

Proof. Suppose that X is defined by P^\perp , for P a generic plane of $\wedge^{\langle 3 \rangle} \mathbf{C}^6$. If s is an involution of type I, there exists a non isotropic plane $A \subseteq \mathbf{C}^6$ such that $P \subseteq A \otimes \wedge^{\langle 2 \rangle} A^\perp$, and $s = \text{Id}_A - \text{Id}_{A^\perp}$. We have seen that the eigenspace decomposition of the induced action is

$$\wedge^{\langle 3 \rangle} \mathbf{C}^6 = A \otimes \wedge^{\langle 2 \rangle} A^\perp \oplus K,$$

where $K \subseteq \wedge^2 A \otimes A^\perp \oplus \wedge^3 A^\perp$ is the kernel of the contraction map to A^\perp by ω . We deduce the eigenspace decomposition of P^\perp as $P_+ \oplus P_-$, where P_+ is the orthogonal to P in $A \otimes \wedge^{\langle 2 \rangle} A^\perp$ and $P_- \simeq K^*$. The fixed locus of s is then the union of the intersections of $\text{LG}(3, 6)$ with $\mathbf{P}(P_+)$ and $\mathbf{P}(P_-)$. It is easy to check that the latter is empty: the intersection of $\mathbf{P}(\wedge^2 A \otimes A^\perp \oplus \wedge^3 A^\perp)$ with $\text{G}(3, 6)$ consists in the three-planes that either contain A , or are contained in A^\perp , and none of those is isotropic. The former is a general codimension two linear section of the intersection of $\mathbf{P}(A \otimes \wedge^{\langle 2 \rangle} A^\perp)$ with $\text{G}(3, 6)$, which consists in those three planes that are generated by a line of A and an isotropic plane in A^\perp . Geometrically, this is a codimension two linear section of $\mathbf{P}^1 \times \mathbf{Q}^3$, hence a Del Pezzo surface of degree four.

Now consider the case where s is an involution of type II; then there exist two transverse Lagrangian subspaces E and F of \mathbf{C}^6 such that $s = i(\text{Id}_E - \text{Id}_F)$. The induced eigenspace decomposition of $\wedge^{\langle 3 \rangle} \mathbf{C}^6$ is into seven-dimensional isotropic spaces

$$U_+ = \wedge^3 E \oplus K_+, \quad \text{and} \quad U_- = \wedge^3 F \oplus K_-,$$

where $K_+ \subseteq E \otimes \wedge^2 F$ (resp. $K_- \subseteq F \otimes \wedge^2 E$) is the kernel of the contraction map to F (resp. E) by ω . Moreover, P is generated by two generic lines in U_+ and U_- , and dually, P^\perp is the direct sum of a general hyperplane in U_+

with a general hyperplane in U_- . Note that E and F are in duality through ω , and that once we have identified F with E^* , we get K_+ as the image of the natural (Koszul type) map from $S^2E \otimes \wedge^3E^*$ to $E \otimes \wedge^2E^*$. It is then easy to check that the intersection of $\mathbf{P}(U_+) \simeq \mathbf{P}(\mathbf{C} \oplus S^2E)$ with $\mathrm{LG}(3,6)$ is a cone over a Veronese surface. Cutting with a general hyperplane we get the Veronese surface back, and our second claim follows. \square

The intersection of $\mathbf{P}(U_+)$ with $\mathbf{P}(A_1) \times \mathbf{P}(A_2) \times \mathbf{P}(A_3)$ consists of the four points $\langle L_1, L_2, L_3 \rangle$ plus $\langle L_1, M_2, M_3 \rangle$ and its permutations. In general these points do not belong to the curve C_P , which must be preserved by any automorphism stabilizing P . Note that C_P is by definition a double cover of $\mathbf{P}(P)$, the branch locus being the intersection of $\mathbf{P}(P)$ with the tangent variety to $\mathrm{LG}(3,6)$. This tangent variety being a quartic hypersurface (see, e.g., [LM02, Proposition 6.4]), C_P is in fact an elliptic curve. We conclude that the type II involutions must restrict on C_P to translations by two-torsion points. This is very similar to what we observed in genus 8.

6. GENUS 7

Now comes the hardest case. A first mild difficulty is that it involves a spin module for a Lie group of rank five, with 16 unipotent orbits and an important number of cases to consider for semisimple elements. From our analysis of unipotent and semisimple elements, we will conclude that in the critical codimension four we need to consider certain special involutions, related to splitting of spin modules under restriction to smaller Lie groups. These splittings are well known as subrepresentations, but we will need to be very specific about the corresponding subspaces of the spin module in order to be able to understand the generic automorphism groups.

6.1. A brief reminder on spin modules. Suppose that $V = V_{2n}$ is a complex vector space of dimension $2n$, endowed with a non degenerate quadratic form Q . Chose a decomposition $V = E \oplus F$ into maximal isotropic subspaces. So Q vanishes on E and F , and defines a perfect duality between E and F .

Let $\Delta := \wedge^*E$ denote the exterior algebra of E . It admits a natural action of E defined by the wedge product, and also a natural action by contraction with Q . Explicitly, if $x = a + \alpha$ with $a \in E$ and $\alpha \in F$, we have for example in degree one

$$x.e = a \wedge e + Q(\alpha, e)1.$$

If $x' = a' + \alpha'$ is another vector, its action on $x.e$ is given by

$$x'.(x.e) = a' \wedge a \wedge e + Q(\alpha', a)e - Q(\alpha', e)a + Q(\alpha, e)a'$$

since $\alpha'.1 = 0$. This is only partially skew-symmetric in x and x' , and

$$x'.(x.e) + x.(x'.e) = \left(Q(\alpha', a) + Q(\alpha, a') \right) e = Q(x, x')e.$$

This formula extends if we replace e by any element in Δ . Thus it extends to an action of the Clifford algebra $Cl(V, Q)$ on Δ , and then of the spin group

$\text{Spin}(V, Q)$ by restriction. In the sequel we will only need the infinitesimal action of the Lie algebra of the spin group, which is simply $\wedge^2 V$. This action is readily deduced from the previous formulas, through the identifications

$$x \wedge x' = \frac{1}{2}(x.x' - x'.x) = x.x' - \frac{1}{2}Q(x, x')1$$

as operators on Δ . Since the action of such an operator on Δ preserves its $\mathbf{Z}/2\mathbf{Z}$ -grading, the spin module Δ actually splits into the direct sum of the two half-spin representations

$$\Delta_+ = \wedge^+ E, \quad \Delta_- = \wedge^- E,$$

which are both irreducible, of the same dimension 2^{n-1} .

Note that this construction is not canonical, in the sense that it relies on the initial choice of the decomposition $V = E \oplus F$, while the half-spin representations don't (at least up to isomorphism; they are, in particular, indistinguishable). This is a major source of complications, as we will see in the sequel.

In the rest of this section, we shall write Δ for either one of the half-spin representations Δ_{\pm} of Spin_{10} , and reserve the notation Δ_{\pm} for certain subpieces of our half-spin representation of Spin_{10} . These subpieces will be defined by restricting the representation to a copy of $\text{Spin}_4 \times \text{Spin}_6$. Suppose for instance that \mathbf{C}^{10} has been split into the direct sum $U \oplus U^{\perp}$ for some non degenerate four-plane U , and that $E = E' \oplus E''$ has been chosen to be the sum of an isotropic plane $E' \subseteq U$ and an isotropic three-plane $E'' \subseteq U^{\perp}$. Then the formulas

$$\wedge^+ E = \wedge^+ E' \otimes \wedge^+ E'' \oplus \wedge^- E' \otimes \wedge^- E'', \quad \wedge^- E = \wedge^+ E' \otimes \wedge^- E'' \oplus \wedge^- E' \otimes \wedge^+ E''$$

show that the spin representations indeed split into pieces of the same dimension. This is actually a general fact, independent of the possibility to split E compatibly with U , as we shall see later on.

6.2. Unipotent elements. There are 16 unipotent orbits in Spin_{10} , corresponding to the 16 nilpotent orbits in \mathfrak{so}_{10} . Classically, they are indexed by partitions of 10 in which even parts have even multiplicities. If π is such a partition, π^* is the dual partition, and $n_-(\pi)$ denotes the number of odd parts, then the codimension of the corresponding orbit \mathcal{O}_{π} is given by

$$\text{codim}(\mathcal{O}_{\pi}) = \frac{1}{2} \left(\sum_i (\pi_i^*)^2 - n_-(\pi) \right).$$

(Note that in \mathfrak{sp}_{2n} , a similar formula holds, but with a plus sign; we have used this above in the genus 9 case). We deduce that the dimensions are

the following:

<i>Partition</i>	<i>Dimension</i>	<i>Jordan type</i>
9, 1	40	5, 11
7, 3	38	2, 6, 8
5, 5	36	1, 3, 5, 7
7, 1 ³	36	1 ² , 7 ²
5, 3, 1 ²	34	3 ² , 5 ²
5, 2 ² , 1	32	3, 4 ² , 5
4 ² , 1 ²	32	1 ³ , 4 ² , 5
3 ³ , 1	30	2 ⁴ , 4 ²
5, 1 ⁵	28	4 ⁴
3 ² , 2 ²	28	1 ² , 2 ² , 3 ² , 4
3 ² , 1 ⁴	26	1 ⁴ , 3 ⁴
3, 2 ² , 1 ³	24	1 ² , 2 ⁴ , 3 ²
2 ⁴ , 1 ²	20	1 ⁵ , 2 ⁴ , 3
3, 1 ⁷	16	2 ⁸
2 ² , 1 ⁶	14	1 ⁸ , 2 ⁴
1 ¹⁰	0	1 ¹⁶

A representative X_π of each nilpotent orbit \mathcal{O}_π can be obtained as follows (see [CMcG93, p.77]). Denote by (μ_1, \dots, μ_k) the odd parts of π . Denote its even parts by (ν_1, \dots, ν_k) , counted with half their multiplicities (always even). Consider the natural embeddings

$$\mathfrak{so}_{\mu_1} \times \cdots \times \mathfrak{so}_{\mu_k} \times \mathfrak{sl}_{\nu_1} \times \cdots \times \mathfrak{sl}_{\nu_l} \subseteq \mathfrak{so}_{\mu_1} \times \cdots \times \mathfrak{so}_{\mu_k} \times \mathfrak{so}_{2\nu_1} \times \cdots \times \mathfrak{so}_{2\nu_l} \subseteq \mathfrak{so}_{10}.$$

Choose regular nilpotent elements X_i in \mathfrak{so}_{μ_i} , Y_j in \mathfrak{sl}_{ν_j} . Then one can let

$$X_\pi = X_1 + \cdots + X_k + Y_1 + \cdots + Y_l \in \mathcal{O}_\pi.$$

The restriction of a half-spin representation of \mathfrak{so}_{10} to $\mathfrak{so}_{\mu_1} \times \cdots \times \mathfrak{so}_{\mu_k} \times \mathfrak{so}_{2\nu_1} \times \cdots \times \mathfrak{so}_{2\nu_l}$ will split into a sum of tensor products of spin representations for the different factors; this decomposition can easily be obtained inductively, since a half-spin representation of \mathfrak{so}_{2m} will restrict to $\Delta_a \otimes \Delta_b$ on $\mathfrak{so}_{2a+1} \times \mathfrak{so}_{2b+1}$ for $m = a + b + 1$, and to $\Delta_a^+ \otimes \Delta_b^+ \oplus \Delta_a^- \otimes \Delta_b^-$ on $\mathfrak{so}_{2a} \times \mathfrak{so}_{2b}$ for $m = a + b$; while a spin representation of \mathfrak{so}_{2m+1} will restrict to $\Delta_a^+ \otimes \Delta_b \oplus \Delta_a^- \otimes \Delta_b$ on $\mathfrak{so}_{2a} \times \mathfrak{so}_{2b+1}$ for $m = a + b$. Moreover, restricting a half-spin representation of \mathfrak{so}_{2a} to \mathfrak{sl}_a yields the even (or odd) exterior algebra of the natural representation.

In order to compute the Jordan type of the action of each X_π on a half-spin representation of \mathfrak{so}_{10} , it is therefore enough to know the Jordan type of the action on a spin representation of a regular nilpotent element of \mathfrak{so}_{2m+1} for $m \leq 4$, and the Jordan type of the action of a regular nilpotent element of \mathfrak{sl}_n on the exterior algebra of the natural representation, for $n \leq 4$. These

are given as follows:

\mathfrak{so}_3	2	\mathfrak{sl}_2	1^2	2
\mathfrak{so}_5	4	\mathfrak{sl}_3	1, 3	1, 3
\mathfrak{so}_7	1, 7	\mathfrak{sl}_4	$1^3, 5$	4^2
\mathfrak{so}_9	5, 11			

One finally deduces the Jordan type of X_π on a half-spin representation by using Proposition 7. The results are given in the table above.

Arguing as indicated in §2.3.1, we conclude that for $m = 4, \dots, 12$, the general m -dimensional linear subspace $L \subseteq \Delta_+$ has no unipotent element in its stabilizer.

6.3. Semisimple elements. Now let g be a semisimple element in SO_{10} , that we may suppose to belong to a standard maximal torus. If $\varepsilon_1, \dots, \varepsilon_5$ are the diagonal characters of this torus, the characters of the half-spin representation are the

$$\frac{1}{2}(\pm\varepsilon_1 \pm \dots \pm \varepsilon_5),$$

with an even number of plus signs. In other words, let $t_1 = s_1^2, \dots, t_5 = s_5^2$ and their inverses denote the eigenvalues of g , with $V_t \subseteq \mathbf{C}^{10}$ the eigenspace corresponding to the eigenvalue t . Then the eigenvalues of the induced action on $\mathbf{P}(\Delta)$ must be (up to some scalar) the $s_1^{\pm 1} \dots s_5^{\pm 1}$, again with an even number of plus signs; up to some scalar, these are the same as 1, the ten products $t_i t_j$ (that we call eigenvalues of type I), and the five products $t_p t_q t_r t_s$ (type II).

The roots are $t_i^{\pm 1} t_j^{\pm 1}$ for all $1 \leq i < j \leq 5$. The Weyl action is generated by the permutations of the t_i 's, and the exchanges of two t_i 's with their inverses. In particular, a relation of the form $t_i t_j = 1$ can always be replaced by $t_i = t_j$ since we have the freedom of exchanging t_j with t_j^{-1} ; such a relation will therefore take us to another type.

6.3.1. Regular case. This means that we consider those semisimple $g \in \mathrm{SO}_{10}$ such that the t_i 's are pairwise distinct. Then the conjugacy class of g has dimension 40.

We may assume that no eigenvalue of type I equals 1, because a relation $t_1 t_2 = 1$ may be replaced by $t_1 = t_2$ by acting with the Weyl group, and this takes us to another regularity class. The ten eigenvalues of type I can at worst collapse in pairs, because of the regularity assumption on g . Type II eigenvalues are pairwise distinct. It is possible to pair a type I eigenvalue with one of type II, but then these two eigenvalues cannot be paired with anything else. It is also possible to pair a type II eigenvalue with 1, but up to the Weyl action this is equivalent to a collapsing between two type I eigenvalues.

The upshot is that the eigenvalues for the action on Δ can collapse at most in pairs, and we can make at most 7 pairs (by pairing each type II with one type I, and making two pairs of pure type I). One checks that

(U) holds in all cases: if one is laborious, it is possible to check all cases one by one, and we have implemented this in Python.

6.3.2. *Subregular case.* This means that only two eigenvalues coincide; suppose this is $t_4 = t_5$. For the action on Δ , the eigenvalues are

$$\begin{array}{ll} 1 & (1) \quad t_1 t_2, t_1 t_3, t_2 t_3 & (1) \quad t_1 t_2 t_3 t_4 & (2) \\ & t_1 t_4, t_2 t_4, t_3 t_4 & (2) \quad t_1 t_2 t_4^2 & (1) \\ & & t_4^2 & (1) \quad t_1 t_3 t_4^2 & (1) \\ & & & t_2 t_3 t_4^2 & (1) \end{array}$$

For generic values of the t_i 's, the conjugacy class of g has dimension 38.

Among type I, $t_1 t_2, t_1 t_3, t_2 t_3$ are pairwise distinct, and so are $t_1 t_4, t_2 t_4, t_3 t_4, t_4^2$. The type II eigenvalues are pairwise distinct, as always in a given degeneration stratum. Moreover $t_1 t_2 t_3 t_4$ may be paired only with 1 and t_4^2 . It follows that the eigenspaces have dimension at most 4 at any rate. A direct verification then shows that (U) holds in all cases, except if we have a partition containing either $[4^2, 3^2]$ or $[4^3]$.

It is not possible that $t_1 t_2, t_1 t_3, t_2 t_3$ collapse all at the same time with $t_3 t_4, t_2 t_4, t_1 t_4$ respectively without violating our subregularity assumption, but it is possible that $t_1 t_2, t_1 t_3, t_1 t_2 t_3 t_4$ collapse with $t_3 t_4, t_2 t_4, t_4^2$, iff t_2, t_3, t_4 equal $-1, 1, -t_1$ respectively (up to exchanging t_2 and t_3). This may conveniently be checked using Macaulay2, see the ancillary files listed in §2.3. In the latter case, the eigenvalues are (writing t for t_1)

$$\pm 1 \ (1) \quad \pm t, \pm t^2 \ (3) \quad \pm t^3 \ (1)$$

(note that we automatically have the fourth collapsing $t_1 t_4 = t_2 t_3 t_4^2$). The special values $t = \pm i$ are forbidden by the subregularity assumption (otherwise $t_1 t_4 = 1$), so we obtain at most the partition $[3^4, 2^2]$, and (U) always holds.

The other possibilities amount to those investigated above by taking the Weyl action into account. For instance, if $t_1 t_4, t_2 t_4$ collapse with $t_2 t_3 t_4^2, t_1 t_3 t_4^2$, we have $t_1 = t_2 t_3 t_4$ and $t_2 = t_1 t_3 t_4$, and those relations may be changed into $t_1 t_4 = t_2 t_3$ and $t_2 t_4 = t_1 t_3$ by exchanging t_4, t_5 with their inverses.

6.3.3. $t_3 = t_4 = t_5$. In this case we assume that no root other than $t_3/t_4, t_3/t_5, t_4/t_5$ and their inverses take the value 1, so the conjugacy class of g has dimension 34. The eigenvalues of the action of g on Δ are

$$\begin{array}{ll} 1 & (1) \quad t_1 t_2 & (1) \quad t_1 t_2 t_3^2 & (3) \\ & t_1 t_3 & (3) \quad t_1 t_3^3 & (1) \\ & t_2 t_3 & (3) \quad t_2 t_3^3 & (1) \\ & & t_3^2 & (3) \end{array}$$

The eigenvalues with multiplicity 3 may collapse with at most one other, the latter necessarily with multiplicity 1. Thus we cannot get any eigenspace of dimension larger than 4, and this is enough for (U) to hold in all cases.

6.3.4. $t_2 = t_3$ and $t_4 = t_5$. The conjugacy class of g has dimension at most 36, and may be strictly smaller if t_2^2 or t_4^2 take the value 1. The eigenvalues of the action on Δ are

$$\begin{array}{l} 1 \quad (1) \quad t_1 t_2 \quad (2) \quad t_1 t_2^2 t_4 \quad (2) \\ \quad \quad \quad t_1 t_4 \quad (2) \quad t_1 t_2 t_4^2 \quad (2) \\ \quad \quad \quad t_2 t_4 \quad (4) \quad t_2^2 t_4^2 \quad (1) \\ \quad \quad \quad t_2^2 \quad (1) \\ \quad \quad \quad t_4^2 \quad (1) \end{array}$$

We make the following observations:

- i) $t_1 t_2$ may collapse only with a) t_4^2 , b) $t_1 t_2 t_4^2$, c) $t_2^2 t_4^2$; b) happens iff $t_4^2 = 1$ and excludes a) and c); a) and c) may happen at the same time, iff $t_2^2 = 1$. A strictly analogous observation holds for all the eigenvalues with multiplicity 2.
- ii) $t_2 t_4$ may not collapse with any other eigenvalue.

It follows that the eigenspaces in Δ have at most dimension 4. One verifies then that (U) holds for all cases, except for $m = 4$ if the partition is $[4^4]$ and the conjugacy class has dimension 36. But if the partition is $[4^4]$, necessarily either t_2^2 or t_4^2 equals 1, hence the conjugacy class of g has dimension at most 34 and (U) holds in this case as well.

6.3.5. $t_1 = t_2$ and $t_3 = t_4 = t_5$. For generic values of t_1 and t_3 , exactly 8 roots take the value 1, hence the dimension of the conjugacy class is at most 32. The eigenvalues of the action on Δ are

$$\begin{array}{l} 1 \quad (1) \quad t_1^2 \quad (1) \quad t_1^2 t_3^2 \quad (3) \\ \quad \quad \quad t_1 t_3 \quad (6) \quad t_1 t_3^3 \quad (2) \\ \quad \quad \quad t_3^2 \quad (3) \end{array}$$

We observe that:

- i) $t_1 t_3$ may only collapse with $t_1 t_3^3$, iff $t_3^2 = 1$;
- ii) t_3^2 may only collapse with $1, t_1^2, t_1^2 t_3^2$, and a similar statement holds for $t_1^2 t_3^2$.

Let us first consider the main degeneration stratum for this case, i.e., we assume that neither t_1^2 nor t_3^2 equals 1. Then $t_1 t_3$ cannot collapse, $t_3^2, t_1^2 t_3^2$ may only collapse with $t_1^2, 1$ respectively, and the two latter are mutually exclusive, and $t_1 t_3^3$ may only collapse with 1 or t_1^2 . So at most we have the partition $[6, 4, 3^2]$, and (U) holds in all cases.

Now assume that $t_1^2 = 1$. Then the conjugacy class of g has dimension at most 30 (we have the two extra relations $t_1 t_2 = t_1^{-1} t_2^{-1} = 1$). In this case the eigenvalues become

$$1 \quad (2) \quad t_1 t_3 \quad (6) \quad t_3^2 \quad (6) \quad t_1 t_3^3 \quad (2)$$

If $t_3^2 \neq 1$, the only possible further collapsing is $1 = t_1 t_3^3$, which gives the partition $[6^2, 4]$, and then (U) holds in all cases. If $t_3^2 = 1$, we obtain the partition $[8^2]$ and the conjugacy class has dimension 24 (6 new relations $t_3 t_4, t_3 t_5, t_3 t_4$ and their inverses). In this case we find a 48-dimensional

family of pairs (L, g) with $g.L = L$ by considering those L that are the sums of two 2-planes in each of the two 8-dimensional eigenspaces. Pg7.1

Eventually, let us check the case when $t_3^2 = 1$ but $t_1^2 \neq 1$. Then the conjugacy class has dimension 26 and the eigenvalues are

$$1 \ (4) \quad t_1 t_3 \ (8) \quad t_1^2 \ (4).$$

There is no possible further collapsing, and (U) holds in all cases.

6.3.6. $t_2 = t_3 = t_4 = t_5$. The eigenvalues on Δ are

$$1 \ (1) \quad t_1 t_2 \ (4) \quad t_1 t_2^3 \ (4) \\ t_2^2 \ (6) \quad t_2^4 \ (1)$$

Let us first consider the case $t_2^2 \neq 1$. Then the only possible collapsings are $t_1 t_2 = t_2^4$, iff $t_1 = t_2^3$, and $t_1 t_2^3 = 1$, iff $t_1^{-1} = t_2^3$. So at most we have the partition $[6, 5^2]$, and the conjugacy class of g has dimension 28, so that (U) holds in all cases.

If $t_2^2 = 1$ the eigenvalues become

$$1 \ (8) \quad t_1 t_2 \ (8)$$

and there is no further possible collapsing. In this case the conjugacy class has dimension 16, and (U) holds in all cases.

6.3.7. $t_1 = t_2 = t_3 = t_4 = t_5$. The eigenvalues on Δ are

$$1 \ (1) \quad t^2 \ (10) \quad t^4 \ (5)$$

If $t^2 = 1$ the action of g on Δ is trivial so we discard this case. Thus $t^2 \neq 1$, the conjugacy class of g has dimension 20, and the only possible collapsing is $t^4 = 1$, which gives the partition $[10, 6]$. (U) holds in all cases.

Conclusion. The stabilizer of a generic subspace $P \subseteq \Delta$ of dimension 5 to 11 is trivial. If P has dimension 4 or 12, non-trivial elements in its stabilizer must be of the kind described in case g7.1 above.

6.4. Codimension four. By the previous study, a general 4-plane $P \subseteq \Delta$ may only be stabilized by a finite number of involutions $\pm t_U$, with $t_U = \text{Id}_U - \text{Id}_{U^\perp}$ in SO_{10} for some non-degenerate four-plane $U \subseteq \mathbf{C}^{10}$. The restriction of the half-spin representation Δ to $\mathfrak{so}(U) \times \mathfrak{so}(U^\perp)$ decomposes into the direct sum of two eight-dimensional sub-representations Δ_+ and Δ_- (recall our notation convention at the end of subsection 6.1), and P is stabilized by the induced action of t_U if and only if it is the direct sum of two 2-planes $P_+ \subseteq \Delta_+$ and $P_- \subseteq \Delta_-$.

We shall prove (see Theorem 26) that there exist exactly three non degenerate four-planes $U, V, W \subseteq \mathbf{C}^{10}$ satisfying the above conditions, hence P is only stabilized by the three corresponding involutions t_U, t_V, t_W . We will see that U, V, W must be in very special relative position: their pairwise intersections will be non degenerate planes A, B, C .

This will be the conclusion of a rather long detour, along which we will need to understand spin modules and their splittings under restrictions to

such subalgebras of \mathfrak{so}_{10} as $\mathfrak{so}(U) \times \mathfrak{so}(U^\perp)$. The whole point of §6.4.1 to 6.4.3 is to obtain an *explicit* description of these splittings (which is achieved in Theorem 25), so as to be able to compute the rank of a suitable differential which is the keystone of the proof of Theorem 26.

6.4.1. *How to split a spin module in eight dimensions.* As a warm-up, let $V_4 \subseteq \mathbf{C}^8$ be a non degenerate four-dimensional subspace, and $V_4' = V_4^\perp$ its orthogonal with respect to the quadratic form Q . The restriction of the spin representation Δ to $\mathfrak{so}(V_4) \times \mathfrak{so}(V_4')$ splits into two four-dimensional submodules. How can we identify them concretely?

Recall that the half-spin representations can be defined as $\Delta_+ = \wedge^+ E$ and $\Delta_- = \wedge^- E$, once a splitting $\mathbf{C}^8 = E \oplus F$ into transverse Lagrangian subspaces has been fixed. This clearly implies that there is a natural equivariant map from $\mathbf{C}^8 \otimes \Delta_\pm$ to Δ_\mp , defined by wedge products and contractions by vectors of E and F in \mathbf{C}^8 . Iterating, one obtains natural morphisms from $\wedge^{2p} \mathbf{C}^8 \otimes \Delta_\pm$ to Δ_\pm , for any p .

In particular, in $\wedge^4 \mathbf{C}^8$ we can consider the Plücker line associated to V_4 . This line induces an endomorphism ψ_{V_4} of Δ_\pm , well defined up to scalars. Being canonically defined by V_4 , the eigenspace decomposition of this endomorphism must be compatible with the structure of Δ_\pm as a module over $\mathfrak{so}(V_4) \times \mathfrak{so}(V_4')$. So if ψ_{V_4} is not a homothety, this endomorphism has no other choice than to admit two four dimensional eigenspaces: the two submodules of the decomposition. This is indeed what will happen, and this yields an efficient method in order to locate concretely these submodules (the existence of which we know a priori only by abstract arguments).

Now suppose that V_4 is transverse to F , so that it can be defined as the graph of a morphism $\Gamma \in \text{Hom}(E, F)$. If V_4 is also transverse to E , this morphism is an isomorphism. In this case, a crucial observation is that one can define a canonical element of $\wedge^4 E_4$, up to sign, by letting

$$\gamma = \frac{v_1 \wedge v_2 \wedge v_3 \wedge v_4}{\det Q(v_i, \Gamma(v_j))^{1/2}}$$

with (v_1, v_2, v_3, v_4) any basis of E_4 .

Proposition 18. *As representations of $\mathfrak{so}(V_4) \times \mathfrak{so}(V_4')$, the half-spin representations Δ_+ and Δ_- split into $\Delta_+ = \delta_+ \oplus \delta_-$ and $\Delta_- = \delta'_+ \oplus \delta'_-$, where*

$$\delta_\pm = \langle 1 \pm \gamma, \theta \pm \Gamma(\theta). \gamma, \theta \in \wedge^2 E \rangle, \quad \delta'_\pm = \langle e \pm \Gamma(e). \gamma, e \in E \rangle.$$

Proof. An explicit computation shows that ψ_{V_4} acts, as expected, as homotheties on each of these subspaces, with opposite factors. (And by equivariance, this computations needs to be done only for one specific V_4 .) \square

Observe that since γ is only defined up to sign, the two modules δ_+ and δ_- , as well as δ'_+ and δ'_- , are in fact indistinguishable, as must be the rule for half-spin representations.

6.4.2. *How to split a spin module in ten dimensions.* Let now $V_4 \subseteq \mathbf{C}^{10}$ be a non degenerate four-dimensional subspace, and $V_6 = V_4^\perp$ its orthogonal with respect to the quadratic form Q . The restriction of a half-spin representation Δ to $\mathfrak{so}(V_4) \times \mathfrak{so}(V_6)$ splits into two eight-dimensional submodules. We want to identify them concretely.

Suppose that Δ has been constructed as $\wedge^+ E$, where $\mathbf{C}^{10} = E \oplus F$ is a fixed splitting into Lagrangian spaces. When V_4 is transverse to both E and F , its two projections are isomorphisms onto subspaces $E_4 \subseteq E$ and $F_4 \subseteq F$, and V_4 can be defined as the graph of an isomorphism $\Gamma \in \text{Hom}(E_4, F_4)$. Moreover, since E and F are in perfect duality through the quadratic form Q , the hyperplanes E_4 and F_4 are orthogonal to lines $F_1 \subseteq F$ and $E_1 \subseteq E$, such that in general, $E = E_1 \oplus E_4$ and $F = F_1 \oplus F_4$. Observe that this yields a splitting

$$\Delta = \wedge^+ E = \wedge^+ E_4 \oplus E_1 \otimes \wedge^- E_4.$$

Proposition 19. *As a representation of $\mathfrak{so}(V_4) \times \mathfrak{so}(V_6)$, the half-spin representations Δ splits into $\Delta = \delta_+^8 \oplus \delta_-^8$, where*

$$\delta_+^8 = \delta_+ \oplus E_1 \otimes \delta'_-, \quad \delta_-^8 = \delta_- \oplus E_1 \otimes \delta'_+$$

with $\delta_\pm^{(l)}$ defined as in Proposition 18.

Proof. Exactly as for Proposition 18, this only requires the computation of how ψ_{V_4} acts, and only for one specific V_4 . We leave this to the reader. \square

6.4.3. *How to split a spin module from a triple of four planes.* Now consider the following situation: three orthogonal, non degenerate planes $A, B, C \subseteq \mathbf{C}^{10}$ are given, and we want to describe a simultaneous splitting with respect to the three four planes U, V, W that are sums of two of those planes.

Lemma 20. *There exists a unique decomposition of Δ into the direct sum of four-dimensional subspaces $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ such that*

$$\begin{aligned} \Delta|_{\mathfrak{so}(U) \times \mathfrak{so}(U^\perp)} &= (\Delta_1 \oplus \Delta_2) \oplus (\Delta_3 \oplus \Delta_4), \\ \Delta|_{\mathfrak{so}(V) \times \mathfrak{so}(V^\perp)} &= (\Delta_1 \oplus \Delta_3) \oplus (\Delta_2 \oplus \Delta_4), \\ \Delta|_{\mathfrak{so}(W) \times \mathfrak{so}(W^\perp)} &= (\Delta_1 \oplus \Delta_4) \oplus (\Delta_2 \oplus \Delta_3). \end{aligned}$$

Proof. The four dimensional space U defines a line in $\wedge^4 \mathbf{C}^{10}$, hence an endomorphism ψ_U of Δ , up to a scalar. This operator has two eigenspaces of dimension eight, corresponding to two opposite eigenvalues, which are nothing else than the two components of the restriction of Δ to $\mathfrak{so}(U) \times \mathfrak{so}(U^\perp)$.

If $U = A \oplus B$, we also have two associated operators ψ_A and ψ_B defined by the Plücker lines of A and B in $\wedge^2 \mathbf{C}^{10}$. The orthogonality of A and B implies that ψ_U is proportional to $\psi_A \psi_B = \psi_B \psi_A$. Moreover ψ_A and ψ_B also have two eigenspaces of dimension eight, corresponding to two opposite eigenvalues, which are the two components of the restriction of Δ to $\mathfrak{so}(A) \times \mathfrak{so}(A^\perp)$ and $\mathfrak{so}(B) \times \mathfrak{so}(B^\perp)$. We can normalize them, up to a sign, so that the two eigenvalues are ± 1 , and then normalize ψ_U as $\psi_A \psi_B$.

If $V = A \oplus C$ and $W = B \oplus C$, with the same normalizations we get that $\psi_V = \psi_A \psi_C$ and $\psi_W = \psi_B \psi_C$. But then, since $\psi_C^2 = 1$ we deduce that $\psi_W = \psi_U \psi_V$. This implies the claim after simultaneous diagonalization. \square

Our next goal will be to identify the eigenspaces of ψ_A . For this we first observe that A admits a basis (x, x') of isotropic vectors. We denote their decompositions on E and F as $x = a + \alpha$ and $x' = a' + \alpha'$. By the isotropy condition $Q(a, \alpha) = Q(a', \alpha') = 0$. We let $E_2 = \langle a, a' \rangle$ and $E_3 = \langle \alpha, \alpha' \rangle^\perp$. In general, $E = E_2 \oplus E_3$.

By the same type of computations as before, we check the following:

Lemma 21. *Consider the following two elements of $\wedge^+ E$:*

$$\Phi = 1 + \frac{a \wedge a'}{Q(a', \alpha)}, \quad \Phi' = 1 - \frac{a \wedge a'}{Q(a, \alpha')}.$$

The two eigenspaces of ψ_A are

$$\Delta_+^A = \Phi \otimes \wedge^+ E_3 \oplus a \otimes \wedge^- E_3, \quad \Delta_-^A = \Phi' \otimes \wedge^+ E_3 \oplus a' \otimes \wedge^- E_3.$$

Note that Φ, Φ' are canonically defined up to permutation: indeed the isotropic basis (x, x') is unique up to scalars and order, and Φ, Φ' remain unchanged when we multiply x, x' by non zero scalars. Moreover they are not proportional, since $Q(a', \alpha) + Q(a, \alpha') = Q(x, x')$ cannot be zero, the plane A being non degenerate.

The main consequence we will derive from this statement is that in degree four, that is, when we project to $\wedge^4 E$, all the elements of Δ_+^A are projected to four-forms that are divisible by a .

In order to diagonalize simultaneously the three operators ψ_A, ψ_B, ψ_C , we need to compute the triple intersections like $\Delta_+^A \cap \Delta_+^B \cap \Delta_+^C$. From the previous observation, we know that projecting this space to $\wedge^4 E$, we should obtain four forms that are divisible by a, b, c . In general these three vectors are independant and we conclude that $\Delta_+^A \cap \Delta_+^B \cap \Delta_+^C$ must be a two-dimensional space of spinors of the form

$$\Omega_0(d) + \Omega_2(d) + a \wedge b \wedge c \wedge d, \quad d \in E / \langle a, b, c \rangle,$$

that we are going to describe explicitly.

Consider the vectors

$$\begin{aligned} K &= Q(\alpha, b)c + Q(\beta, c)a + Q(\gamma, a)b, \\ L &= Q(\alpha, b)\gamma + Q(\beta, c)\alpha + Q(\gamma, a)\beta. \end{aligned}$$

For any $d \in E$, define

$$M(d) = Q(\alpha, d)b \wedge c + Q(\beta, d)c \wedge a + Q(\gamma, d)a \wedge b.$$

Lemma 22. *The two quantities*

$$\Omega_0(d) = Q(L, d) \quad \text{and} \quad \Omega_2(d) = K \wedge d + M(d)$$

vanish when $d \in \langle a, b, c \rangle$.

Proof. The expression of $\Omega_0(d)$ is symmetric under the cyclic permutation of (a, b, c) and (α, β, γ) , so it suffices to check that $\Omega_0(a) = 0$. This immediately follows from $Q(\alpha, a) = 0$ and $Q(\alpha, b) + Q(\beta, a) = Q(x, y) = 0$.

For the same symmetry reasons, we only need to check that $\Omega_2(a) = 0$. But since $Q(\alpha, a) = 0$, we have $M(a) = Q(\beta, a)c \wedge a + Q(\gamma, a)a \wedge b = a \wedge K$, and we are done. \square

Lemma 23. *The contraction of K and $M(d)$ by α yields*

$$\begin{aligned}\alpha.K &= Q(\alpha, K) = 0, \\ \alpha.M(d) &= Q(\alpha, d)K - \Omega_0(d)a.\end{aligned}$$

Proof. Since $Q(\alpha, c) + Q(\gamma, a) = 0$, the contraction $\alpha.(b \wedge c) = K - Q(\beta, c)a$. Using that $Q(\alpha, c) + Q(\gamma, a) = 0$, we deduce that

$$\alpha.M(d) = Q(\alpha, d)K - \left(Q(\alpha, b)Q(\gamma, d) + Q(\beta, c)Q(\alpha, d) + Q(\gamma, a)Q(\beta, d) \right)a,$$

as claimed. \square

By symmetry we also have $Q(\beta, K) = Q(\gamma, K) = 0$. With $Q(\alpha, K) = 0$ this characterizes K up to scalar. Note that the fact that $\langle a, b, c \rangle \cap \langle \alpha, \beta, \gamma \rangle^\perp$ is non zero is a consequence of the fact that the matrix of the scalar products of a, b, c with α, β, γ is skew-symmetric. Of course a similar discussion also applies to L .

Proposition 24. *For any $d \in E$, consider in Δ_+ the spinor*

$$\Omega(d) = a \wedge b \wedge c \wedge d + \Omega_2(d) + \Omega_0(d).$$

Then $\psi_A(\Omega(d)) = \frac{1}{2}Q(x, x')\Omega(d)$.

Proof. Recall that $\psi_A = x \wedge x'$. It decomposes as $\psi_A = \psi_2 + \psi_0 + \psi_{-2}$ with $\psi_2 = a \wedge a'$, $\psi_0 = \frac{1}{2}(a.\alpha' + \alpha.a' - a'.\alpha - \alpha'.a)$ and $\psi_{-2} = \alpha \wedge \alpha'$. Note that

$$\psi_0 = \frac{1}{2}(Q(\alpha, a') - Q(\alpha', a)) + \psi'_0$$

with $\psi'_0 = a.\alpha' - a'.\alpha$. So the identity $\psi_A(\Omega(d)) = \frac{1}{2}Q(x, x')\Omega(d)$ is equivalent, since $Q(x, x') = Q(\alpha, a') + Q(\alpha', a)$, to

$$\psi'_A(\Omega(d)) = Q(\alpha', a)\Omega(d)$$

for $\psi'_A = \psi_2 + \psi'_0 + \psi_{-2}$. Decomposing into graded pieces, this is equivalent to the three homogeneous identities

$$(1) \quad \psi'_0(a \wedge b \wedge c \wedge d) + \psi_2(\Omega_2(d)) = Q(\alpha', a)a \wedge b \wedge c \wedge d,$$

$$(2) \quad \psi_{-2}(a \wedge b \wedge c \wedge d) + \psi'_0(\Omega_2(d)) + \Omega_0(d)a \wedge a' = Q(\alpha', a)\Omega_2(d),$$

$$(3) \quad \psi_{-2}(\Omega_2(d)) + \psi'_0(\Omega_0(d)) = Q(\alpha', a)\Omega_0(d).$$

Let us start with the last one. Since $\Omega_0(d)$ has degree zero, it is killed by ψ'_0 , so we need to prove that

$$(4) \quad \psi_{-2}(\Omega_2(d)) = Q(\alpha', a)\Omega_0(d).$$

But by Lemma 23, the contraction

$$(5) \quad \alpha.\Omega_2(d) = \alpha.M(d) - Q(\alpha, d)K = -\Omega_0(d)a,$$

and then by skew-symmetry we can compute that

$$\psi_{-2}(\Omega_2(d)) = -\alpha'.(\alpha.(\Omega_2(d))) = \alpha'.(\Omega_0(d)a) = Q(\alpha', a)\Omega_0(d),$$

as required.

In order to prove the identity (2), first observe that since $Q(\alpha, a) = 0$, we have $\alpha.(a \wedge b \wedge c \wedge d) = -a \wedge (\alpha.(b \wedge c \wedge d))$. Therefore,

$$(6) \quad \psi_{-2}(a \wedge b \wedge c \wedge d) = Q(\alpha', a)\alpha.(b \wedge c \wedge d) - a \wedge (\alpha'.\alpha.(b \wedge c \wedge d)).$$

Moreover, a direct computation shows that

$$(7) \quad \alpha.(b \wedge c \wedge d) = \Omega_2(d) - a \wedge R(d),$$

where $R(d) = Q(\gamma, d)b - Q(\beta, d)c + Q(\beta, c)d$. In particular, contracting with α' and wedging with a yields

$$a \wedge (\alpha'.\alpha.(b \wedge c \wedge d)) = a \wedge (\alpha'.\Omega_2(d)) - Q(\alpha', a)a \wedge R(d).$$

So the terms involving $R(d)$ cancel out and we get

$$(8) \quad \psi_{-2}(a \wedge b \wedge c \wedge d) = Q(\alpha', a)\Omega_2(d) - a \wedge (\alpha'.\Omega_2(d)).$$

On the other hand, the identity (5) gives

$$(9) \quad \psi'_0(\Omega_2(d)) = a \wedge (\alpha'.\Omega_2(d)) + \Omega_0(d)a \wedge a'.$$

Adding the two previous identities (8) and (9) yields (2).

Finally, let us prove identity (1). We have

$$a \wedge (\alpha'.(a \wedge b \wedge c \wedge d)) = Q(\alpha', a)a \wedge b \wedge c \wedge d,$$

and since $Q(\alpha, a) = 0$, identity (7) yields

$$a' \wedge (\alpha.(a \wedge b \wedge c \wedge d)) = -a' \wedge a \wedge (\alpha.(b \wedge c \wedge d)) = -a' \wedge a \wedge \Omega_2(d).$$

Subtracting the latter identity to the previous one, we deduce that

$$\psi'_0(a \wedge b \wedge c \wedge d) = Q(\alpha', a)a \wedge b \wedge c \wedge d - a \wedge a' \wedge \Omega_2(d),$$

as required. \square

Denote by $\Delta(a, b, c)$ the subspace of Δ generated by the spinors of the form $\Omega(d)$. By Lemma 22 this is a two-dimensional subspace at most, and generically (for example, as soon as $a \wedge b \wedge c \neq 0$) its dimension is indeed two.

We also define the subspace $\Delta(a', b, c)$ of Δ , by permuting a and a' , and consistently x and x' . By cyclic symmetry on a, b, c , we thus define eight subspaces $\Delta(a^{(i)}, b^{(j)}, c^{(k)})$ of Δ , where the prime between parentheses means that the prime can be present or not.

Example. Suppose that A is generated by $x = e_1 + f_2$ and $x' = e_2 + f_1$, that B is generated by $y = e_2 - f_1$ and $y' = e_1 - f_2$, and C by $z = e_3 + f_4$ and $z' = e_4 + f_3$. So $a = e_1$, $b = e_2$, $c = e_3$, while $\alpha = f_2$, $\beta = -f_1$, $\gamma = f_4$.

Then $K = e_3$, $L = f_4$ and $M(d) = d_2e_{23} + d_1e_{13} + d_4e_{12}$, so $\Omega_0(d) = d_4$ and $\Omega_2(d) = d_4(e_{12} + e_{34}) + d_5e_{35}$. We deduce that

$$\Delta(a, b, c) = \langle e_{1234} + e_{12} + e_{34} + 1, e_{1235} + e_{35} \rangle.$$

Similarly, $a' = e_2$, $b' = e_1$, $c' = e_4$, while $\alpha' = f_1$, $\beta' = -f_2$, $\gamma' = f_3$, thus

$$\Delta(a', b', c') = \langle e_{1234} - e_{12} - e_{34} + 1, e_{1245} - e_{45} \rangle.$$

By Lemma 24, $\Delta(a, b, c)$ is made of eigenvectors of ψ_A , and by cyclic symmetry it is also made of eigenvectors of ψ_B and ψ_C . Note that we have proved more precisely that

$$\psi_A(\Omega) = x \wedge x'.(\Omega) = \frac{1}{2}Q(x, x')\Omega \quad \forall \Omega \in \Delta(a, b, c).$$

Permuting a and a' , and consistently x and x' , we can also assert that

$$\psi_A(\Omega) = -x' \wedge x.(\Omega) = -\frac{1}{2}Q(x', x)\Omega \quad \forall \Omega \in \Delta(a', b, c).$$

We deduce immediately:

Theorem 25. *In general, the eight terms decomposition*

$$\Delta = \bigoplus \Delta(a^{(i)}, b^{(i)}, c^{(i)}),$$

where the parentheses mean that the primes can be present or not, is a minimal common eigenspace decomposition for the three operators ψ_A, ψ_B, ψ_C .

We can immediately deduce the minimal common eigenspace decomposition of ψ_U, ψ_V, ψ_W . It is actually coarser than the previous one, being given by the four 4-dimensional spaces

$$\Delta_0 = \Delta(a, b, c) \oplus \Delta(a', b', c'),$$

$$\Delta_1 = \Delta(a', b, c) \oplus \Delta(a, b', c'),$$

$$\Delta_2 = \Delta(a, b', c) \oplus \Delta(a', b, c'),$$

$$\Delta_3 = \Delta(a, b, c') \oplus \Delta(a', b', c).$$

6.4.4. *Conclusion of the proof.* We are now set to deduce the following result from the above analysis.

Theorem 26. *Let $P \subseteq \Delta$ be a general 4-dimensional subspace. There exists a triple (A, B, C) of mutually orthogonal non-isotropic planes in \mathbf{C}^{10} such that P is stabilized by the three involutions t_U, t_V, t_W associated to the four-planes $U = A \oplus B$, $V = A \oplus C$, $W = B \oplus C$, and only by those involutions (and their opposites).*

As a direct corollary we obtain that the stabilizer of P in Spin_{10} is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$, which proves our main Theorem 5 in genus 7. Recall that $t_U = \text{Id}_U - \text{Id}_{U^\perp}$.

Proof. The fact that a triple (A, B, C) as in the statement should exist is indicated by the following dimension count. Let $\mathcal{U} \subseteq \mathbf{G}(2, 10)^3$ be the variety of orthogonal triples (A, B, C) of non isotropic planes; it has dimension 36 (16 parameters for a non isotropic plane A , then 12 for a non isotropic plane B orthogonal to A , finally 8 for a non isotropic plane C orthogonal to both A and B). Each $(A, B, C) \in \mathcal{U}$ decomposes the half-spin representation Δ into four 4-dimensional subspaces $\Delta_0 \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_3$, and by generality P is stabilized by the three involutions associated to U, V, W if and only if it is the direct sums of four lines contained in these four subspaces. This gives 4×3 additional parameters, hence in total $36 + 12 = 48$ parameters, which is exactly the dimension of $\mathbf{G}(4, \Delta_+)$.

To confirm this dimension count, we shall compute the rank of a suitable differential. This is to do so that we need the explicit identification of the Δ_i 's carried out in §6.4.3. Let us consider the map

$$\varphi : (A, B, C) \in \mathcal{U} \longmapsto (\Delta_0, \Delta_1, \Delta_2, \Delta_3) \in \mathbf{G}(4, \Delta_+)^4$$

(strictly speaking, we should mod out the image by the symmetric group \mathfrak{S}_4 , since the four components of Δ_+ are not well-defined individually; this doesn't affect our local computations, so we will avoid doing this). Let $\mathcal{G} \subseteq \mathcal{U} \times \mathbf{G}(4, \Delta_+)$ be the set of those (A, B, C, P) such that $P = P_0 \oplus P_1 \oplus P_2 \oplus P_3$, where each P_i is a line in Δ_i , with $(\Delta_i) = \varphi(A, B, C)$. This defines a fiber bundle \mathcal{G} over \mathcal{U} , with first and second projections p and q respectively; The total space \mathcal{G} is smooth and irreducible, of dimension 48. Since $48 = \dim \mathbf{G}(4, \Delta)$, the second projection $q : \mathcal{G} \rightarrow \mathbf{G}(4, \Delta_+)$ is dominant by Lemma 27 below, which proves the existence of (A, B, C) as asserted in the statement.

At this point we know that there exists one suitable triple (A, B, C) , and possibly only finitely many others. There remains to check that there is no other involution stabilizing P than the three provided by the triple (A, B, C) . So suppose that $R \subseteq \mathbf{C}^{10}$ is another four-plane such that t_R stabilizes P . Then $t_U t_R$ must be of the same type (up to scalar), in particular it must be an involution (up to scalar) and there must exist κ such that $t_R t_U = \kappa t_U t_R$. But then t_R induces an isomorphism between the eigenspaces of t_U with eigenvalues λ and $\kappa\lambda$. Since t_U has only two eigenspaces and these have different dimensions, this implies that $\kappa = 1$, and t_R commutes with t_U . Similarly, it commutes with t_V and t_W .

Recall that the common eigenspaces decomposition of t_U, t_V, t_W is $A \oplus B \oplus C \oplus D$, where D is the orthogonal to $A \oplus B \oplus C$. The involution t_R can be diagonalized accordingly. Let us denote by e_0, \dots, e_9 a compatible basis, and by $\varepsilon_0, \dots, \varepsilon_9$ the corresponding eigenvalues of t_R . We can express the fact that t_R has eigenspaces of dimensions 4 and 6 by imposing that $\text{tr}(t_R) = 2\theta$, with $\theta = \pm 1$. Similarly, we need $\text{tr}(t_U t_R) = 2\theta'$, with $\theta' = \pm 1$. But then $\varepsilon_0 + \dots + \varepsilon_3 = \theta + \theta'$ and $\varepsilon_4 + \dots + \varepsilon_9 = \theta - \theta'$. So up to a global change of signs, and to reordering, we must have $(\varepsilon_0, \dots, \varepsilon_3) = (1, 1, 1, -1)$ and $(\varepsilon_4, \dots, \varepsilon_9) = (1, 1, 1, -1, -1, -1)$, or

$(\varepsilon_0, \dots, \varepsilon_3) = (1, 1, -1, -1)$ and $(\varepsilon_4, \dots, \varepsilon_9) = (1, 1, 1, 1, -1, -1)$. Performing the same analysis with V and W , we conclude that there are (up to sign) only two possibilities for $(\varepsilon_0, \dots, \varepsilon_9)$, namely $(1, 1, 1, -1, 1, -1, 1, 1, -1, -1)$ or $(1, -1, 1, -1, 1, -1, 1, 1, 1, -1)$. Note that in the first case, we have split B and C into the sum of two orthogonal lines, and D into the sum of two orthogonal planes; there are $1 + 1 + 4 = 6$ parameters for such splittings. In the second case, we split A , B and C into the sum of two orthogonal lines, and D into the sum of a line and its orthogonal hyperplane; there are $1 + 1 + 1 + 3 = 6$ parameters for such splittings.

Then the choice of P is more restricted: the action of t_R splits each Δ_i into two 2-planes, and P is stabilized by t_R as well if and only if it is the sum of four lines chosen inside four of the resulting eight two-planes. The number of parameters therefore drops from 12 to 4, and since $4 + 6 < 12$, we can conclude that P cannot be generic if it is stabilized by our extra t_R . This ends the proof of Theorem 26. \square

The two following Lemmas 27 and 28 contain the computation of the differential map that is used above in the proof of Theorem 26.

Lemma 27. *The second projection map $q : \mathcal{G} \rightarrow \mathbf{G}(4, \Delta_+)$ is generically étale.*

Proof. Consider the tangent exact sequence of the fiber bundle $p : \mathcal{G} \rightarrow \mathcal{U}$,

$$0 \longrightarrow T^v \mathcal{G} \longrightarrow T \mathcal{G} \longrightarrow p^* T \mathcal{U} \longrightarrow 0.$$

Let us split $\Delta_i = P_i \oplus Q_i$ for some Q_i , $i = 0, \dots, 3$. At the point (A, B, C, P) of \mathcal{G} , we have

$$T_{(A,B,C,P)}^v \mathcal{G} = \bigoplus_{i=0}^3 \text{Hom}(P_i, Q_i),$$

while the tangent space to the Grassmannian at P is

$$T_P \mathbf{G}(4, \Delta_+) = \text{Hom}(P, \Delta_+/P) \simeq \bigoplus_{i,j} \text{Hom}(P_i, Q_j).$$

Moreover, since P_i can move freely inside Δ_i , the restriction of dq to $\text{Hom}(P_i, Q_i) \subseteq T_{P_*}^v \mathcal{G}$, followed by projection to the same factor maps inside $T_P \mathbf{G}(4, \Delta_+)$, is just the identity map. This implies that dq descends to a morphism

$$d\bar{q} : T_{(A,B,C)} \mathcal{U} \longrightarrow \bigoplus_{i \neq j} \text{Hom}(P_i, Q_j),$$

and that dq is an isomorphism if and only if $d\bar{q}$ is. This morphism is easy to describe in terms of the differential of φ ,

$$d\varphi : T_{(A,B,C)} \mathcal{U} \longrightarrow \bigoplus_{i \neq j} \text{Hom}(\Delta_i, \Delta_j).$$

Indeed the (i, j) -component $q_{i,j}$ of $d\bar{q}$ must be given, in terms of the (i, j) -component $\varphi_{i,j}$ of $d\varphi$, simply by the composition

$$P_i \hookrightarrow \Delta_i \xrightarrow{\varphi_{i,j}} \Delta_j \longrightarrow Q_j.$$

So suppose that $d\bar{q}$ is never injective. In particular there exists $X \in T_{(A,B,C)}$ such that $d\bar{q}(X) = 0$. By the previous observation, this means that $\phi_{i,j}(X) \in \text{Hom}(\Delta_i, \Delta_j)$ sends P_i to P_j . But the lines P_0, \dots, P_3 can be chosen arbitrarily, so we get a contradiction as soon as the $\phi_{i,j}(X)$ are not all zero, which is granted by Lemma 28 below. \square

Lemma 28. *The map $\varphi : \mathcal{U} \longrightarrow \text{G}(4, \Delta_+)^4$ is an immersion.*

(Actually we only need that is an immersion at some point, but since \mathcal{U} is homogeneous under the action of Spin_{10} , this is not a weaker statement.)

Proof. In order to check this claim, we first need to describe the tangent space to \mathcal{U} at (A, B, C) . Recall that $T_A \text{G}(2, V_{10})$ is naturally isomorphic with $\text{Hom}(A, B \oplus C \oplus D)$. Note that A is self-dual since it is non degenerate with respect to Q . We claim that

$$T_{(A,B,C)}\mathcal{U} \simeq (A \otimes B) \oplus (B \otimes C) \oplus (C \otimes A) \oplus (A \oplus B \oplus C) \otimes D,$$

and leave this as an exercise.

Since φ is equivariant, the kernel of $d\varphi$ must be a submodule for the group $H = \text{SO}(A) \times \text{SO}(B) \times \text{SO}(C) \times \text{SO}(D)$. Beware that although D is an irreducible module for the action of $\text{SO}(D)$, the action of $\text{SO}(A)$ (resp. $\text{SO}(B), \text{SO}(C)$) splits A (resp. B, C) into the direct sum of its two isotropic lines, generated by x and x' (resp. y, y' , z and z'), the difference being of course that $\dim(D) > 2$. The upshot is that H -submodule of $T_{(A,B,C)}\mathcal{U}$ must be generated by vectors of type $x^{(i)} \otimes y^{(j)} \in A \otimes B$ (and the likes in $A \otimes C$ and $B \otimes C$), and of type $x^{(i)} \otimes u, y^{(j)} \otimes u, z^{(k)} \otimes u$, with $u \in D$. It is therefore enough to check that $d\varphi$ does not vanish on such vectors, and by symmetry it is enough to do this for $x \otimes y$ and $x \otimes u$.

Note that the infinitesimal variation of the orthogonal triple (A, B, C) defined by $x \otimes y$ is can be described by

$$u \in A \mapsto Q(x, u)y, \quad v \in B \mapsto -Q(y, v)x, \quad w \in C \mapsto 0.$$

This means that $x, y, z, z' \mapsto 0$, while $x' \mapsto Q(x, x')y$ and $y' \mapsto -Q(y, y')x$. Projecting to E , we deduce that $a' \mapsto Q(x, x')b$, $b' \mapsto -Q(y, y')a$, and $a, b, c, c' \mapsto 0$.

We claim that the induced variation of $\Delta_0 = \Delta(a, b, c) \oplus \Delta(a', b', c')$ is non trivial. In fact, we claim that the variation of its projection to $\wedge^4 E$ is already non trivial. To see this, recall that the projection of $\Delta(a, b, c)$ to $\wedge^4 E$ is the two-dimensional space of tensors of the form $a \wedge b \wedge c \wedge d$, $d \in E$, and similarly for $\Delta(a', b', c')$. Adding these two spaces, we get a hyperplane in $\wedge^4 E \simeq E^\vee$, or equivalently, a line in E . It is easy to see that this line L can simply be obtained as the intersection of the three-planes $\langle a, b, c \rangle$ and $\langle a', b', c' \rangle$. Concretely, it can be deduced from the dependence

relation that must exist between the six vectors a, a', b, b', c, c' of E : if we write this relation as

$$\lambda a + \mu b + \nu c = \lambda' a' + \mu' b' + \nu' c',$$

the line L is generated by the vector ℓ that appears on both sides. Our infinitesimal variation sends a' to $a' + Q(x, x')b$ and b' to $b' - Q(y, y')a$, and does not change the other vectors. Rewriting the previous relation as

$$\begin{aligned} (\lambda - \mu'Q(y, y'))a + (\mu + \lambda'Q(x, x'))b + \nu c = \\ \lambda'(a' + Q(x, x')b) + \mu'(b' - Q(y, y')a) + \nu'c', \end{aligned}$$

we see that ℓ is changed into $\ell + (\lambda'Q(x, x')b - \mu'Q(y, y')a)$. In particular, in general the line L generated by ℓ moves non trivially, which is what we wanted to prove.

The case of vectors of type $x \otimes u$, $u \in D$, will be left to the reader. \square

Now both Lemmas 28 and 27 are proved, and the proof of Theorem 26 is thus complete. As we have already mentioned, the latter implies that $\text{Aut}(X) = (\mathbf{Z}/2\mathbf{Z})^2$, for $X = \mathbf{S}_{10} \cap \mathbf{P}(P^\perp)$, the general Mukai sixfold of genus 7.

Proposition 29. *The fixed locus of any of the three involutions in the automorphism group of a general six-dimensional Mukai variety $X = \mathbf{S}_{10} \cap P^\perp$, is the disjoint union of two surface rational quartic scrolls.*

Proof. Each automorphism is an involution $t = t_U$ associated to some four dimensional subspace $U \subseteq \mathbf{C}^{10}$, on restriction to which the quadratic form remains non-degenerate. Moreover the action of t_U on Δ splits it into two eigenspaces Δ_+ and Δ_- , which as $\mathfrak{so}(U) \times \mathfrak{so}(U^\perp)$ modules are tensor products of spin-modules. More concretely, recall that $\mathfrak{so}(U) \simeq \mathfrak{so}_4 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2 = \mathfrak{sl}(S) \times \mathfrak{sl}(T)$, with $U \simeq S \otimes T$ and S, T the two spin two-dimensional modules. Moreover, another exceptional isomorphism yields $\mathfrak{so}(U^\perp) \simeq \mathfrak{so}_6 \simeq \mathfrak{sl}_4 = \mathfrak{sl}(R)$, with $U^\perp \simeq \wedge^2 R$ and R, R^\vee the two spin four-dimensional modules. In particular,

$$\Delta_+ \simeq R \otimes S, \quad \text{and} \quad \Delta_- \simeq R^\vee \otimes T$$

(up to the exchange of S and T). The fixed locus of the action of t_U on X is the union of its intersections with $\mathbf{P}(\Delta_+)$ and $\mathbf{P}(\Delta_-)$. Since P is the direct sum of the two-dimensional spaces $P_+ \subseteq \Delta_+$ and $P_- \subseteq \Delta_-$, we get two disjoint subvarieties $\mathbf{S}_\pm = \mathbf{S}_{10}^\pm \cap \mathbf{P}(P_\pm^\perp)$, where $\mathbf{S}_{10}^\pm = \mathbf{S}_{10} \cap \mathbf{P}(\Delta_\pm)$.

There remains to identify these subvarieties. For that we just need to remember that $\mathbf{S}_{10} \subseteq \mathbf{P}(\Delta)$ is cut-out by quadrics, and that the quadrics vanishing on \mathbf{S}_{10} are parametrized by \mathbf{C}^{10} (see e.g. [KW12, 5.1]). In fact, given a vector $v \in \mathbf{C}^{10}$, the Clifford multiplication by v sends Δ_+ to $\Delta_- \simeq \Delta_+^\vee$ and we can let $q_v(\delta) = \langle v, \delta, \delta \rangle$ for any spinor $\delta \in \Delta$. Now $S_+ \subseteq \mathbf{P}(\Delta_+)$ is cut out by the restriction of those quadrics to Δ_+ . The decomposition $\mathbf{C}^{10} = U \oplus U^\perp$ gives two types of quadrics. For $v \in U \simeq S \otimes T$, the Clifford action of v on $\Delta_+ = R \otimes S$ maps it to $R \otimes T \simeq R \otimes T^\vee$, and therefore the

quadric q_v vanishes identically on Δ_+ . But for $v \in U^\perp \simeq \wedge^2 R$, the Clifford action of v on Δ_+ sends it to its dual, and the quadric q_v is non zero. So $S_+ \subseteq \mathbf{P}(\Delta_+) \simeq \mathbf{P}(R \otimes S)$ is the locus cut-out by the quadrics parametrized by $\wedge^2 R \simeq \wedge^2 R^\vee$, and this locus is just $\mathbf{P}(R) \times \mathbf{P}(S) \simeq \mathbf{P}^3 \times \mathbf{P}^1$. Cutting it by $\mathbf{P}(P_+)$, a generic two-codimensional subspace, we get an irreducible surface which is a rational quartic scroll. \square

Remark. In genus 8 and 9 we have been able to understand part of the exceptional automorphism group as acting on some auxiliary elliptic curve, either by pointwise symmetries or translations by torsion points. In genus 7 there is an associated abelian surface [GSW13, Theorem 9.5], but its construction is more involved and we have no geometric interpretation yet of the exceptional automorphism group as acting on this surface.

REFERENCES

- [BFM18] Bai C., Fu B., Manivel L., *On Fano complete intersections in rational homogeneous varieties*, arXiv:1808.01549, to appear in Math. Zeitschrift.
- [CD20] Ciliberto C., Dedieu Th., *K3 curves with index $k > 1$* , arXiv:2012.10642, to appear in Boll. Unione Mat. Ital.
- [CLM98] Ciliberto C., Lopez A. F., Miranda R., *Classification of varieties with canonical curve section via Gaussian maps on canonical curves*, Amer. J. Math. **120** (1998), no. 1, 1–21.
- [CMR04] Ciliberto C., Mella M., Russo F., *Varieties with one apparent double point*, J. Algebraic Geom. **13** (2004), 475–512.
- [CMcG93] Collingwood D., McGovern W., *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand 1993.
- [DK18] Debarre O., Kuznetsov A., *Gushel-Mukai varieties: classification and birationalities*, Algebr. Geom. **5** (2018), 15–76.
- [Do12] Dolgachev, I., *Classical algebraic geometry, A modern view*, Cambridge University Press 2012.
- [GSW13] Gruson L., Sam S., Weyman J., *Moduli of abelian varieties, Vinberg θ -groups, and free resolutions*, in Commutative algebra, 419–469, Springer 2013.
- [IP99] Iskovskikh V., Prokhorov Y., *Fano varieties*, Algebraic geometry V, 1–247, Encyclopaedia Math. Sci. **47**, Springer 1999.
- [KW12] Kraskiewicz W., Weyman J., *Geometry of orbit closures for the representations associated to gradings of Lie algebras of types E_6 , F_4 and G_2* , arXiv:1201.1102.
- [Kr99] Krattenthaler C., *Advanced determinant calculus*, in The Andrews Festschrift, Sém. Lothar. Combin. **42** (1999), Art. B42q.
- [Kuz18] Kuznetsov A., *On linear sections of the spinor tenfold I*, Izv. Math. **82** (2018), no. 4, 694–751.
- [KPS18] Kuznetsov A., Prokhorov Y., Shramov C., *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*, Jpn. J. Math. **13** (2018), 109–185.
- [LM02] Landsberg J.M., Manivel L., *Construction and classification of complex simple Lie algebras via projective geometry*, Selecta Math. **8** (2002), 137–159.
- [Mel99] Mella M., *Existence of good divisors on Mukai varieties*, J. Algebraic Geom. **8** (1999), no. 2, 197–206.
- [Muk89] Mukai S., *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10* , in Algebraic geometry and commutative algebra, Vol. I, 357–377, Kinokuniya 1988.
- [Muk89] Mukai S., *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000–3002.

- [PV99] Piontkowski J., Van de Ven A., *The automorphism group of linear sections of the Grassmannians $G(1, N)$* , Doc. Math. **4** (1999), 623–664.
- [PZ18] Prokhorov Y., Zaidenberg M., *Fano-Mukai fourfolds of genus 10 as compactifications of \mathbf{C}^4* , Eur. J. Math. **4** (2018), 1197–1263.
- [PZ21] Prokhorov Y., Zaidenberg M., *Fano-Mukai fourfolds of genus 10 and their automorphism groups*, arXiv:2103.12167.
- [SK77] Sato M., Kimura T., *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.
- [Wil03] Wildberger N. J., *A combinatorial construction of G_2* , J. Lie Theory **13** (2003), no. 1, 155–165.

INSTITUT DE MATHÉMATIQUES DE TOULOUSE ; UMR 5219, UPS, F-31062 TOULOUSE CEDEX 9, FRANCE

Email address: thomas.dedieu@math.univ-toulouse.fr,
manivel@math.cnrs.fr

We acknowledge support from ANR project FanoHK, grant ANR-20-CE40-0023.

APPENDIX A. AUTOMORPHISM GROUPS OF PRIME FANO THREEFOLDS OF GENUS TWELVE

by YURI PROKHOROV

Theorem A.1. *The automorphism group of a general (in the moduli sense) prime Fano threefold of genus 12 is trivial.*

Proof. For a prime Fano threefold X we denote by $F_1(X)$ the Hilbert scheme of lines, i.e. curves in X with Hilbert polynomial $h_1(t) = t + 1$. It is known that $F_1(X)$ is of pure dimension 1 (see e.g. [KPS18]).

Claim A.1.1. *For any prime Fano threefold $X = X_{22} \subseteq \mathbf{P}^{13}$ the natural homomorphism*

$$\Psi : \text{Aut}(X) \rightarrow \text{Aut}(F_1(X))$$

is injective.

Proof. Assume that Ψ is not injective. Take a non-trivial element $\varphi \in \text{Ker}(\Psi)$. Thus φ acts trivially on $F_1(X)$. Fix a line $l \subseteq X$. Apply the double projection [IP99, Theorem 4.3.3, Theorem 4.3.7] from l . It birationally transforms the pair $(X \supset l)$ to $(Y \supset \Gamma)$, where $Y = Y_5 \subseteq \mathbf{P}^6$ is a smooth quintic del Pezzo threefold and Γ is a rational normal quintic curve in Y . This transformation is φ -equivariant. A general line $l' \subseteq X$ is mapped to a line $m' \subseteq Y$ meeting Γ at one point. The set of lines in Y passing through any point $y \in Y$ is finite (see e.g. [KPS18, Corollary 5.1.5]). Since $\dim F_1(X) = 1$, the automorphism φ acts trivially on Γ . Thus the fixed point locus Y^φ contains the hyperplane section $S := Y \cap \langle \Gamma \rangle$.

Assume that φ is an element of finite order. Then its fixed point locus Y^φ is smooth. Hence Y^φ contains no one dimensional components (because

$\rho(Y) = 1$) and S is a smooth del Pezzo surface. In particular, $\chi_{\text{top}}(Y^\varphi) \geq 7$. This contradicts the topological Lefschetz fixed point formula:

$$\chi_{\text{top}}(Y^\varphi) = \chi_{\text{top}}(Y) = 4.$$

Therefore φ is an element of infinite order. Any line on Y meets S hence φ^m acts trivially on $F_1(Y)$ for some m (in fact, $m \leq 3$). Recall that there are exactly three lines in Y passing through a general point $y \in Y$. This implies that φ^m acts trivially on Y , a contradiction. \square

Now we use Mukai's realization of $X = X_{22} \subseteq \mathbf{P}^{13}$ as $VSP(C, 6)$ where C is a plane quartic [Muk89]. Take a general quartic $C \subseteq \mathbf{P}^2$ and let $X = VSP(C, 6)$. Then the curve $F_1(X)$ is also a smooth plane quartic F_C which is *covariant* of C [Sch01, Theorem 6.1]. The curve $F_1(X) = F_C$ has a natural $(3, 3)$ -correspondence of intersecting lines which defines an even theta characteristic Θ on F_C . There is a map $C \mapsto (F_C, \Theta)$ of the corresponding moduli spaces which is called *Scorza map*. It is birational [DK93, Theorem 7.8]. In particular, this implies that the curve F_C is general in the moduli space of plane quartics. Since the plane quartic F_C is general, we have $\text{Aut}(F_C) = \{1\}$. Hence $\text{Aut}(X) = \{1\}$ for $X = VSP(C, 6)$ by Claim A.1.1. \square

Remark A.2. Note that in contrast with the cases $g \leq 10$ the automorphism group of a prime Fano threefold of genus $g = 12$ can be infinite. We refer to [Pro90], [KPS18],[KP18] for description of infinite automorphisms.

REFERENCES

- [DK93] Igor Dolgachev and Vassil Kanev, *Polar covariants of plane cubics and quartics*, Adv. Math. **98** (1993), 216–301.
- [IP99] V.A. Iskovskikh and Yu. Prokhorov, Fano varieties, Algebraic geometry V, 1–247, Encyclopaedia Math. Sci. **47**, Springer 1999.
- [KP18] Kuznetsov A., Prokhorov Y., *Prime Fano threefolds of genus 12 with a G_m -action*, Épijournal de Géométrie Algébrique **2** (2018).
- [KPS18] Kuznetsov A., Prokhorov Y., Shramov C., *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*, Jpn. J. Math. **13** (2018), 109–185.
- [Muk89] Shigeru Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Nat. Acad. Sci. U.S.A. **86** (1989), no. 9, 3000–3002.
- [Pro90] Yu. Prokhorov, *Automorphism groups of Fano 3-folds*, Russian Math. Surveys **45** (1990), 222–223.
- [Sch01] Frank-Olaf Schreyer, *Geometry and algebra of prime Fano 3-folds of genus 12*, Compositio Math. **127** (2001), 297–319.

STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES & LABORATORY OF ALGEBRAIC GEOMETRY, NRU HSE & DEPARTMENT OF ALGEBRA, MOSCOW STATE UNIVERSITY, MOSCOW, RUSSIA

Email address: prokhoro@mi-ras.ru