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Abstract : Peirce-evanescent baric identities are polynomial identities verified by baric algebras such that their Peirce polynomials are the null polynomial. In this paper procedures for constructing such homogeneous and non homogeneous identities are given. For this we define an algebraic system structure on the free commutative nonassociative algebra generated by a set $\mathcal{T}$ which provides for classes of baric algebras satisfying a given set of identities similar properties to those of the varieties of algebras. Rooted binary trees with labeled leaves are used to explain the Peirce polynomials. It is shown that the mutation algebras satisfy all Peirce-evanescent identities, it results from this that any part of the field K can be the Peirce spectrum of a K-algebra satisfying a Peirce-evanescent identity. We end by giving methods to obtain generators of homogeneous and non-homogeneous Peirce-evanescent identities that are applied in several univariate and multivariate cases.

Key words : Baric algebras, polynomial identities, algebraic systems, variety of algebraic systems, T-ideal, labeled rooted binary trees, altitude of a polynomial, Peirce polynomial, mutation algebras.

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1. Introduction

Non-associative algebras are not necessarily associative algebras in which the associative identity $(xy)\,z - x\,(yz) = 0$ is replaced by one or more polynomial identities [1]. These polynomial identities can be univariate or multivariate, with constant or variable coefficients. In cases where these algebras admit an idempotent, the Peirce decomposition, a fundamental tool for their study, is obtained from the Peirce polynomial which is an annihilator polynomial of the left multiplication operator $L_e : x \mapsto ex$, where $e \neq 0$ is an idempotent.

However, there are algebras defined by polynomial identities for which the Peirce polynomial is the null polynomial. Let us illustrate this situation by an example, let A be a commutative K-algebra on a field K of characteristic $\neq 2$ verifying the identity

$$x^2x^2 - \alpha\omega(x)\,x^3 - (1-\alpha)\,\omega(x)^2\,x^2 = 0 \tag{1.1}$$

where $\omega : A \to K$ is a nonzero algebra morphism. It is assumed that there is in A an idempotent element $e \neq 0$, from $e^2 = e$ we deduce that $\omega(e)^2 = \omega(e)$ from where $\omega(x) \in \{0, 1\}$, we can assert that $\omega(e) \neq 0$ otherwise by setting $x = e$ in the identity 1.1 we would have $e = 0$. First linearization of (1.1) gives:

$$4x^2\,(xy) - \alpha\left[\omega(y)\,x^3 + \omega(x)\left(x^2y + 2x\,(xy)\right)\right] - 2\,(1-\alpha)\left[\omega(x)\,\omega(y)\,x^2 + \omega(x)^2\,xy\right] = 0$$

[1] «Without associativity, rings and algebras are not in general well enough behaved to have much of a structure theory. For this reason, the nonassociative algebraists normally studies the class of rings which satisfy some particular identity or set of identities.» M. Osborn [15].

taking $y \in \ker \omega$ this relation becomes

$$4x^2 (xy) - \alpha\omega (x) \left(x^2 y + 2x (xy)\right) - 2 (1-\alpha) \omega (x)^2 xy = 0$$

specializing this identity for $x = e$ we get

$$(2-\alpha) (2e (ey) - ey) = 0.$$

Therefore, if $\alpha \neq 2$ the Peirce polynomial is $P(X) = 2X^2 - X$, the spectrum of L_e is $\left\{0, \frac{1}{2}\right\}$ and we have $A = Ke \oplus A(0) \oplus A\left(\frac{1}{2}\right)$ where $A(\lambda) = \ker (L_e - \lambda id)$. The second linearization of (1.1) gives

$$4 (ey) (ez) + (2-\alpha) e (yz) - \alpha ((ey) z + y (ez)) - (1-\alpha) yz = 0$$

with $y, z \in \ker \omega$ which establishes that $A\left(\frac{1}{2}\right)^2 \subset A(0)$, $A(0) A\left(\frac{1}{2}\right) \subset A\left(\frac{1}{2}\right)$ and $A(0)^2 = \{0\}$ if $\alpha \neq 0, 1$ or $A(0)^2 \subset A\left(\frac{1}{2}\right)$ if $\alpha = 0$ and $A(0)^2 \subset A(0)$ if $\alpha = 1$.

On the other hand if $\alpha = 2$, the identity (1.1) is therefor written

$$x^2 x^2 - 2\omega (x) x^3 + \omega (x)^2 x^2 = 0 \tag{1.2}$$

Algebras which verify the identity (1.2) are called backcrossing algebras because of their genetic interpretation (cf. [12]), they appeared for the first time in [11] and subsequently in several other articles (see references in [13]). The linearization and specialization for $x = e$ of 1.2 results in a null Peirce polynomial and therefore does not provide any information on the spectrum of L_e.

In [16], V. Tkachev called degenerate those identities whose Peirce polynomials are null, in this work we prefer to name them evanescent[2].

This paper is organized as follows. In section 2 we provide the commutative groupoid $\mathfrak{M}(\mathcal{T})$ generated by a set at most countable $\mathcal{T}$ with an algebraic system structure, this induces on the commutative free nonassociative algebra generated by $\mathcal{T}$ an algebraic structure which allows to obtain for the classes of baric algebras satisfying a given set of identities, similar properties to those of the varieties of algebras. In section 3 we define the linearizations of the identities defined from the elements of the free algebra obtained in section 2 and the Peirce polynomials of these identities, we show how to calculate the Peirce polynomials by using rooted binary trees with labeled leaves. We define the notions of evanescent polynomials and identities, we show that the mutation algebras verify all evanescent identities and we draw consequences for the Peirce spectrum. We end in section 4 by exposing methods to obtain the generators of the homogeneous and non homogeneous evanescent polynomials, we apply these methods in several cases, we thus obtain a little more than 250 evanescent identities.

2. Varieties for baric algebras.

Throughout this paper, K is a commutative field of characteristic $\neq 2$ and K-algebras are assumed commutative.

[2]Evanescent comes from the present participle *evanescens* of the Latin verb *evanescere* meaning "disappear, vanish". Indeed it is observed during the calculation that the terms of Peirce polynomials disappear gradually.

Let $\mathcal{T} = \{t_n; n \geq 1\}$ be a countable set of symbols, we denote by $\mathfrak{M}(\mathcal{T})$ the commutative groupoid generated by $\mathcal{T}$ equipped with the binary operation, denoted $\cdot$, and defined for all $t_i, t_j \in \mathcal{T}$ and $u, v \in \mathfrak{M}(\mathcal{T})$ by:

$$t_i \cdot t_j = t_i t_j, \qquad t_i \cdot u = t_i(u), \qquad v \cdot t_i = (v)\, t_i, \qquad u \cdot v = (u)(v).$$

The elements of $\mathfrak{M}(\mathcal{T})$ are called nonassociative words (or monomials).

For a monomial $w \in \mathfrak{M}(\mathcal{T})$, the degree of w in $t_i \in \mathscr{T}$, denoted by $|w|_{t_i}$ or $|w|_i$, is the number of occurrences of t_i in the monomial w, the degree of w denoted $|w|$ is the length of the monomial w in other words $|w| = \sum_{i\geq 1} |w|_i$ and the type w is $[|w|_1, \ldots, |w|_n, \ldots]$. Let $\mathfrak{M}(\mathcal{T})_d$ be the set of monomials of degree d and $\mathfrak{M}(\mathcal{T})_{[n_1,\ldots,n_m,\ldots]}$ the set of monomials of type $[n_1, \ldots, n_m, \ldots]$, we get:

$$\mathfrak{M}(\mathcal{T}) = \coprod_{d\geq 1} \mathfrak{M}(\mathcal{T})_d, \tag{2.1}$$

and

$$\mathfrak{M}(\mathcal{T})_d = \coprod_{n_1+\cdots+n_m+\cdots=d} \mathfrak{M}(\mathcal{T})_{[n_1,\ldots,n_m,\ldots]}. \tag{2.2}$$

We also have the following result which will often be used later.

Proposition 1. [Proposition 2 in [18]] *Every nonassociative word w with $|w| \geq 2$ has a unique representation in the form of a product of two nonassociative words of lesser length.*

For what follows, we provide the commutative groupoid $(\mathfrak{M}(\mathcal{T}), \cdot)$ with an algebraic system structure ([2], chap 1) by defining on $\mathfrak{M}(\mathcal{T})$ a non commutative multiplication law noted $\star$ (this law is used in [1] to define generalized baric algebras) such that for any $u, v, u', v', w \in \mathfrak{M}(\mathcal{T})$ we have:

$$(u \star v) \star w = u \star (v \star w) = (u \cdot v) \star w; \tag{2.3}$$

$$(u \star v) \cdot w = u \star (v \cdot w) = v \cdot (u \star w); \tag{2.4}$$

$$(u \star v) \cdot (u' \star v') = (u \cdot u') \star (v \cdot v'); \tag{2.5}$$

$$(u \star v) \star (u' \star v') = (u \cdot v') \star (u' \star v'). \tag{2.6}$$

We note $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$ this algebraic system and to reduce the notation we will write $\mathfrak{M}(\mathcal{T})$ for $(\mathfrak{M}(\mathcal{T}), \cdot)$ when there is no risk of confusion. For all $u, v \in (\mathfrak{M}(\mathcal{T}), \cdot, \star)$ and $i \geq 1$, we inductively define the degrees of the elements of $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$ by $|u \cdot v|_i = |u \star v|_i = |u|_i + |v|_i$.

We have the analogous statement of Proposition 1 for the elements of $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$.

Proposition 2. *Every element w of $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$ admits an unique decomposition in the form of $w = w_1 w_2$ or $w = w_1 \star w_2$ with $w_1, w_2 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star)$ such that $|w_1|, |w_2| < |w|$.*

Proof. This is a consequence of Proposition 1. Indeed, we can consider that every element of $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$ is obtained from an element of $\mathfrak{M}(\mathcal{T})$ by replacing some $\cdot$ operations with $\star$ operations. □

Proposition 3. *We have $(\mathfrak{M}(\mathcal{T}), \cdot, \star) = \mathfrak{M}(\mathcal{T}) \cup (\mathfrak{M}(\mathcal{T}) \star \mathfrak{M}(\mathcal{T}))$.*

Proof. Let us show that for every monomial $w \in (\mathfrak{M}(\mathcal{T}), \cdot, \star)$ such that $w \notin \mathfrak{M}(\mathcal{T})$ with $|w| \geq 2$, there are $u, v \in \mathfrak{M}(\mathcal{T})$ such that $w = u \star v$. By recurrence on the degree $|w|$. If $|w| = 2$, there are $t_i, t_j \in \mathcal{T}$ such that $w = t_i \star t_j$. If $|w| \geq 3$, it is assumed that the property is true for any monomial of length $< |w|$. Let $w \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$ a monomial of degree n, according to proposition 2, w is written in a unique way in the form of $w = w_1 w_2$ or $w = w_1 \star w_2$ with w_1 or w_2 in $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$.

In the case of $w = w_1 w_2$ there are three possible situations:

a) $w_1 \in \mathfrak{M}(\mathcal{T})$ and $w_2 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$, by hypothesis there are $u_2, v_2 \in \mathfrak{M}(\mathcal{T})$ with $w_2 = u_2 \star v_2$, then according to the relation (2.4) we get $w = w_1 (u_2 \star v_2) = u_2 \star (w_1 v_2)$ where $w_1 v_2 \in \mathfrak{M}(\mathcal{T})$;

b) $w_1 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$ and $w_2 \in \mathfrak{M}(\mathcal{T})$, with $w_1 w_2 = w_2 w_1$, we refer to the previous case a);

c) $w_1, w_2 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$, by hypothesis there are $u_1, u_2, v_1, v_2 \in \mathfrak{M}(\mathcal{T})$ such that $w_1 = u_1 \star v_1$ and $w_2 = u_2 \star v_2$, then according to the relation (2.5) we get $w = (u_1 \star v_1)(u_2 \star v_2) = (u_1 u_2) \star (v_1 v_2)$ where $u_1 u_2, v_1 v_2 \in \mathfrak{M}(\mathcal{T})$.

In the case of $w = w_1 \star w_2$ we have four possible situations:

a) $w_1, w_2 \in \mathfrak{M}(\mathcal{T})$, the result is immediate;

b) $w_1 \in \mathfrak{M}(\mathcal{T})$ and $w_2 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$, we have $w_2 = u_2 \star v_2$ with $u_2, v_2 \in \mathfrak{M}(\mathcal{T})$, using relation (2.3) we find $w = w_1 \star w_2 = w_1 \star (u_2 \star v_2) = (w_1 u_2) \star v_2$;

c) $w_1 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$ and $w_2 \in \mathfrak{M}(\mathcal{T})$, we have $w_1 = u_1 \star v_1$ where $u_1, v_1 \in \mathfrak{M}(\mathcal{T})$ then it follows immediately from (2.3) that $w = (u_1 \star v_1) \star w_2 = (u_1 v_1) \star w_2$;

d) $w_1, w_2 \in (\mathfrak{M}(\mathcal{T}), \cdot, \star) \setminus \mathfrak{M}(\mathcal{T})$, it is assumed that $w_1 = u_1 \star v_1$ and $w_2 = u_2 \star v_2$ with $u_1, u_2, v_1, v_2 \in \mathfrak{M}(\mathcal{T})$, then by successively applying the relations (2.6) and (2.4) we get $w = (u_1 \star v_1) \star (u_2 \star v_2) = (u_1 v_1) \star (u_2 \star v_2) = ((u_1 v_1) \star u_2) \star v_2 = ((u_1 v_1) u_2) \star v_2$. □

Let $K(\mathcal{T})$ be the commutative and nonassociative free K-algebra generated by $\mathfrak{M}(\mathcal{T})$ (see [18]). The elements of $K\langle\mathcal{T}\rangle$ are nonassociative polynomials, they are of the form $f = \sum_{k \geq 1} \alpha_k w_k$ with $w_k \in \mathfrak{M}(\mathcal{T})$, $\alpha_k \in K$, then the degree of f, noted $|f|$, is $|f| = \max\{|w_k|; \alpha_k \neq 0\}$ and for all $t_i \in \mathcal{T}$, the degree of f in t_i is defined by $|f|_i = \max\{|w_k|_i; \alpha_k \neq 0\}$. We say that f is homogeneous if for all $t_i \in \mathscr{T}$ and $k \geq 1$ we have $|w_k|_i = |f|_i$ in other words, if all the monomials in f are of the same type.

We denote $(K(\mathcal{T}), \cdot, \star)$ (resp. $K(\mathcal{T})^\star$) the free algebra generated by $(\mathfrak{M}(\mathcal{T}), \cdot, \star)$ (resp. $(\mathfrak{M}(\mathcal{T}), \cdot) \star (\mathfrak{M}(\mathcal{T}), \cdot)$). It follows from the previous proposition that we have:

$$(K(\mathcal{T}), \cdot, \star) = K(\mathcal{T}) \oplus K(\mathcal{T})^\star$$

and so the elements of $(K(\mathcal{T}), \cdot, \star)$ have the form:

$$\sum_i \alpha_i w_i + \sum_j \beta_j u_j \star v_j, \quad (\alpha_i, \beta_j \in K; w_i, u_j, v_j \in \mathfrak{M}(\mathcal{T})).$$

We apply the algebra system $(K(\mathcal{T}), \cdot, \star)$ to identities verified by baric algebras.

A K-algebra A is baric if there is a nonzero algebra homomorphism $\omega : A \to K$ called a weight function of A, we note this (A, ω), the image $\omega(x)$ of an element x of A is called the weight of x, we note $H_{(A,\omega)}$ or more simply H_ω, the affine hyperplane $\{x \in A : \omega(x) = 1\}$ (cf. [3], [17]). For the baric algebras we have the analog of the substitution homomorphism of elements of an algebra for elements of $\mathscr{T}$ ([14], prop. 1.1).

Proposition 4. *Let (A,ω) be a baric K-algebra and let $\varrho : \mathcal{T} \to A$ be a map. Then there exists a unique algebra homomorphism $\widehat{\varrho}$ from $(K(\mathcal{T}),\cdot,\star)$ to (A,ω) such that*

$$\widehat{\varrho}(t_i) = \varrho(t_i), \quad \widehat{\varrho}(t_i t_j) = \varrho(t_i)\,\varrho(t_j), \quad \widehat{\varrho}(t_i \star t_j) = \omega(\varrho(t_i))\,\varrho(t_j).$$

Proof. According to the hypothesis the map $\widehat{\varrho}$ is defined for all monomials of degree 2 of $(\mathfrak{M}(\mathcal{T}),\cdot,\star)$. It is assumed that it is defined for all monomials of $(\mathfrak{M}(\mathcal{T}),\cdot,\star)$ of degree $< n$, let $w \in (\mathfrak{M}(\mathcal{T}),\cdot,\star)$ be a monomial of degree n, we have $w = w_1 w_2$ or $w = w_1 \star w_2$ with $w_1, w_2 \in \mathfrak{M}(\mathcal{T})$ of degree $< n$, by hypothesis $\widehat{\varrho}(w_1)$ and $\widehat{\varrho}(w_2)$ are defined and we set $\widehat{\varrho}(w) = \widehat{\varrho}(w_1)\,\widehat{\varrho}(w_2)$ if $w = w_1 w_2$ and $\widehat{\varrho}(w) = \omega(\widehat{\varrho}(w_1))\,\widehat{\varrho}(w_2)$ if $w = w_1 \star w_2$, with this and by uniqueness of the decomposition of w, the map $\widehat{\varrho}$ is well defined on $(\mathfrak{M}(\mathcal{T}),\cdot,\star)$, it is extended by linearity on $(K(\mathcal{T}),\cdot,\star)$ by setting: $\widehat{\varrho}\left(\sum_{k\geq 1}\alpha_k w_k\right) = \sum_{k\geq 1}\alpha_k \widehat{\varrho}(w_k)$. □

Definition 5. Given f an element of $(K(\mathcal{T}),\cdot,\star)$ such as $f \neq 0$. We say that a baric K-algebra (A,ω) verify the identity f if we have:

$$\widehat{\varrho}(f) = 0, \tag{2.7}$$

for all substitution map $\varrho : \mathcal{T} \to A$.

Remark 6. More generally, the algebraic system $(K(\mathcal{T}),\cdot,\star)$ allows to define the notion of weighted identity introduced in [16]. Let A be a commutative K-algebra and let $\varrho : \mathcal{T} \to A$ be a substitution map. A map $\phi : A \to K$ is said to be polynomial if for any element a and b of A the map $t \mapsto \phi(a+tb)$ is polynomial. Let $\{\phi_w; w \in \mathfrak{M}(\mathcal{T})\}$ be a given family of polynomial maps, there exists a unique algebra homomorphism $\widehat{\varrho}$ from $(K(\mathcal{T}),\cdot,\star)$ to A such that $\widehat{\varrho}(u\cdot v) = \widehat{\varrho}(u)\,\widehat{\varrho}(v)$ and $\widehat{\varrho}(u\star v) = \phi_u(\widehat{\varrho}(u))\,\widehat{\varrho}(v)$ for all $u, v \in \mathfrak{M}(\mathcal{T})$, so we say that the algebra A verify a weighted identity $f \in (K(\mathcal{T}),\cdot,\star)$ if $\widehat{\varrho}(f) = 0$.

Under certain conditions, to show that a baric algebra (A,ω) verify an identity f it is enough to show that A verify f for elements of weight 1.

Proposition 7. *Let be $f \in (K(\mathcal{T}),\cdot,\star)$. If the field K verify* $\mathrm{card}K^* > \max\{|f|_i; i \geq 1\}$, *then a baric K-algebra (A,ω) verify the identity f if and only if we have:*

$$\widehat{\varrho}(f) = 0,$$

for all substitution map $\varrho : \mathcal{T} \to H_\omega$.

Proof. The necessary condition is immediate. Let us show that the condition is sufficient. Let be $f \in K(\mathcal{T})$ with $f = \sum_{r\geq 1}\alpha_r w_r + \sum_{s\geq 1}\beta_s u_s \star v_s$ where $\alpha_r, \beta_s \in K$ and $w_r, u_s, v_s \in \mathfrak{M}(\mathcal{T})$ verifying $\widehat{\varrho}(f) = 0$ for all map $\varrho : \mathcal{T} \to H_\omega$.

For $1 \leq i \leq n$ we set

$$R_i = \{r; |w_r|_i = |f|_i\} \text{ and } S_i = \{s; |u_s \star v_s|_i = |f|_i\}.$$

For all $a_1 \in A$, $\omega(a_1) \neq 0$ we have $\omega(a_1)^{-1} a_1 \in H_\omega$, let be $(x_n)_{n\geq 2}$ where $x_n \in H_\omega$, taking $\varrho(t_1) = \omega(a_1)^{-1} a_1$ and $\varrho(t_i) = x_i$ for $i \geq 2$, the condition $\widehat{\varrho}(f) = 0$ is written:

$$\sum_{r\in R_1}\alpha_r w_r(a_1, x_2, \ldots) + \sum_{r\notin R_1}\alpha_r \omega(a_1)^{|f|_1 - |w_r|_1} w_r(a_1, x_2, \ldots) +$$

$$\sum_{s\in S_1}\beta_s v_s\left(a_1,x_2,\ldots\right)+\sum_{s\notin S_1}\beta_s\omega\left(a_1\right)^{|f|_1-|u_s\star v_s|_1}v_s\left(a_1,x_2,\ldots\right)=0. \tag{2.8}$$

We note $f_1=\sum_{r\in R_1}\alpha_r w_r+\sum_{s\in S_1}\beta_s v_s$, we get $|f_1|_1=|f|_1$ and (2.8) is written:

$$\begin{aligned}f_1\left(a_1,x_2,\ldots\right)+\sum_{r\notin R_1}\alpha_r\omega\left(a_1\right)^{|f|_1-|w_r|_1}w_r\left(a_1,x_2,\ldots\right)+\\ \sum_{s\notin S_1}\beta_s\omega\left(a_1\right)^{|f|_1-|u_s\star v_s|_1}v_s\left(a_1,x_2,\ldots\right)=0,\end{aligned} \tag{2.9}$$

in other words, we got $\widehat{\varrho}\left(f_1+\sum_{r\notin R_1}\alpha_r w_r+\sum_{s\notin S_1}\beta_s u_s\star v_s\right)=0$ where $\widehat{\varrho}(f)=0$ for $\varrho\left(t_1\right)=a_1$ and $\varrho\left(t_i\right)=x_i$ $(2\leq i)$.

Next, for all $(x_n)_{n\geq 1}$ where $x_n\in H_\omega$, all $z\in\ker\omega$ and $\lambda\in K$, as $x_1+\lambda z\in H_\omega$ taking $\varrho\left(t_1\right)=x_1+\lambda z$ and $\varrho\left(t_i\right)=x_i$ for $i\geq 2$, according to (2.9) the identity $\widehat{\varrho}(f)=0$ is written:

$$\lambda^{|f|_1}f_1\left(z,x_2,\ldots\right)+\sum_{k=0}^{|f|_1-1}\lambda^k g_{1,k}\left(z,x_1,\ldots\right)=0$$

where $g_{1,k}\in K\left(\mathcal{T}\right)$ with $|g_{1,k}|_1=k$. By hypothesis we have $\mathrm{card}K>|f|_1$, then replacing λ with pairwise distinct elements $\lambda_0,\ldots,\lambda_{|f|_1}$ of K we get a homogeneous linear system with $|f|_1+1$ equations for unknowns $f_1,g_{1,0},\ldots,g_{1,|f|_1-1}$ whose determinant is non-zero, it follows that $f_1\left(z,x_2,\ldots\right)=0$, this according to (2.8) or (2.9) is the same as $\widehat{\varrho}(f)=0$ for $\varrho\left(t_1\right)=z$ and $\varrho\left(t_i\right)=x_i$ $(2\leq i)$. We have therefore established that $\widehat{\varrho}(f)=0$ for $\varrho\left(t_1\right)=a_1$ and $\varrho\left(t_i\right)=x_i$ $(2\leq i)$ for any $a_1\in A$ and $x_2,\ldots,x_n\in H_\omega$.

Taking $\varrho\left(t_1\right)=a_1$, $\varrho\left(t_2\right)=\omega\left(a_2\right)^{-1}a_2$ and $\varrho\left(t_i\right)=x_i$ $(3\leq i)$ where $a_1\in A$, $a_2\in A$, $\omega\left(a_2\right)\neq 0$, $x_n\in H_\omega$ for $n\geq 3$, the condition $\widehat{\varrho}(f)=0$ leads to

$$\begin{aligned}\sum_{r\in R_2}\alpha_r\omega\left(a_1\right)^{|f|_1-|w_r|_1}w_r\left(\mathbf{x}\right)+\sum_{r\notin R_2}\alpha_r\omega\left(a_1\right)^{|f|_1-|w_r|_1}\omega\left(a_2\right)^{|f|_2-|w_r|_2}w_r\left(\mathbf{x}\right)+\\ \sum_{s\in S_2}\beta_s\omega\left(a_1\right)^{|f|_1-|u_s\star v_s|_1}v_s\left(\mathbf{x}\right)+\sum_{s\notin S_2}\beta_s\omega\left(a_1\right)^{|f|_1-|u_s\star v_s|_1}\omega\left(a_2\right)^{|f|_2-|u_s\star v_s|_2}v_s\left(\mathbf{x}\right)=0\end{aligned} \tag{2.10}$$

where we put $\mathbf{x}=(a_1,a_2,x_3,\ldots,x_n,\ldots)$.

Then with $\varrho\left(t_1\right)=a_1$, $\varrho\left(t_2\right)=x_2+\lambda_j z$ $(0\leq j\leq|f|_2)$ and $\varrho\left(t_i\right)=x_i$ where $a_1\in A$, $x_i\in H_\omega$ $(i\geq 3)$, $z\in\ker\omega$ and $\lambda_0,\ldots,\lambda_{|f|_2}\in K$ pairwise distinct, we get

$$\sum_{r\in R_2}\alpha_i\omega\left(a_1\right)^{|f|_1-|w_r|_1}w_r\left(a_1,z,x_3,\ldots\right)+\sum_{s\in S_2}\beta_s\omega\left(a_1\right)^{|f|_1-|u_s\star v_s|_1}v_s\left(a_1,z,x_3,\ldots\right)=0. \tag{2.11}$$

From (2.10) and (2.11) we deduce that $\widehat{\varrho}(f)=0$ for $\varrho\left(t_1\right)=a_1$, $\varrho\left(t_2\right)=a_2$ and $\varrho\left(t_i\right)=x_i$ $(3\leq i)$, for all $a_1,a_2\in A$, $x_i\in H_\omega$.

Continuing thus one obtains by induction (2.7). □

The introduction of the multiplicative law $\star$ in $\mathfrak{M}\left(\mathcal{T}\right)$ allows for identities verified by baric algebras to obtain a known result for the varieties of algebras (cf. [18] Theorem 3).

Proposition 8. *Let $f \in (K(\mathcal{T}), \cdot, \star)$ be an identity verified by a baric K-algebra (A, ω). If the field K verify the condition* $\mathrm{card}K^* > \max\{|f|_i ; i \geq 1\}$ *then each homogeneous component of f is an identity verified by A.*

Proof. Let $f \in (K(\mathcal{T}), \cdot, \star)$ be an identity verified by (A, ω). For all $d \geq 0$ we denote by $f_{1,d}$ the sum in f of monomial terms of degree d in t_1, so we have $f = \sum_{d=0}^{|f|_1} f_{1,d}$. To any map $\varrho : \mathcal{T} \to H_\omega$ is associated the map $\varrho_1 : \mathcal{T} \to A$ such that $\varrho_1(t_j) = \varrho(t_j)$ if $j \neq 1$ and $\varrho_1(t_1) = \lambda\varrho(t_1)$ where $\lambda \in K$, $\lambda \neq 0$. Then from $\widehat{\varrho}_1(f) = 0$ it results $\sum_{d=0}^{|f|_1} \lambda^d \widehat{\varrho}(f_{1,d}) = 0$ for all $\lambda \in K$, especially taking λ as pairwise distinct non-zero values $\lambda_0, \ldots \lambda_{|f|_1}$ we get a system of $|f|_1 + 1$ linear equations of unknowns $\widehat{\varrho}(f_{1,d})$ which has a non-zero Vandermonde determinant, therefore we have $\widehat{\varrho}(f_{1,d}) = 0$ for all $d \geq 0$, in other words the polynomials $f_{1,0}, \ldots, f_{1,|f|_1}$ are identities verified by A.

Applying the same procedure for indeterminate t_2 to the polynomials $f_{1,0}, \ldots, f_{1,|f|_1}$ we obtain homogeneous polynomials in t_1 and t_2 which are identities of A. And thus, by continuing for all variables $t_3, \ldots, t_n, \ldots$ the result is established. □

Remark 9. Given the importance of the results obtained in the propositions 7 and 8, it will now be assumed that the field K satisfies the condition stated in these propositions.

Let $(K(\mathcal{T}), \cdot, \star)_{[n_1,\ldots,n_m,\ldots]}$ be the linear subspace of homogeneous polynomials of type $[n_1, \ldots, n_m, \ldots]$, it follows from the proposition 8 that

$$(K(\mathcal{T}), \cdot, \star) = \bigoplus_{(n_1,\ldots,n_m,\ldots)} (K(\mathcal{T}), \cdot, \star)_{[n_1,\ldots,n_m,\ldots]}.$$

From the proposition 8 we immediately deduce the form of the identities verified by baric algebras.

Corollary 10. *The identities verified by a baric algebra (A, ω) are of the form:*

$$\sum_{k=1}^{m} \alpha_k \omega(a_1)^{|f|_1 - |w_k|_1} \ldots \omega(a_n)^{|f|_n - |w_k|_n} w_k(a_1, \ldots, a_n) = 0, \quad \forall a_1, \ldots, a_n \in A;$$

where $\alpha_k \in K$, $\alpha_k \neq 0$ and $w_k \in \mathfrak{M}(\mathcal{T})$ for all $1 \leq k \leq m$.

Proof. Let $f \in (K(\mathcal{T}), \cdot, \star)$ be an identity verified by (A, ω). The set $I = \{i; |f|_i \neq 0\}$ is finite, indeed for $i > |f|$ we have $|f|_i = 0$. Let n be the cardinal number of the set I, we re-index the elements of $\mathcal{T}$ so that $I = \{1, \ldots, n\}$. Then we notice that for each monomial in f of type $u \star v$ with $u, v \in \mathfrak{M}(\mathcal{T})$ such that $|u \star v|_i = |f|_i$, if $\varrho(t_k) = a_k$ $(1 \leq k \leq n)$ we have $\widehat{\varrho}(u \star v) = \omega(\widehat{\varrho}(u))\, \widehat{\varrho}(v)$ with $\omega(\widehat{\varrho}(u)) = \omega(a_1)^{|u|_1} \cdots \omega(a_n)^{|u|_n}$, however we have $|u \star v|_i = |u|_i + |v|_i$ from where $|u|_i = |f|_i - |v|_i$. □

The point of view of the identities considered in the algebraic system $(K(\mathcal{T}), \cdot, \star)$ makes it possible to obtain for the class of baric algebras similar results to those varieties of algebras, what the point of view restricted to the algebra $K(\mathcal{T})$ does not allow to do, for example the proposition 8 is not true in $K(\mathcal{T})$. Nevertheless, the use of the $K(\mathcal{T})$ algebra is very useful for writing more conveniently and manipulating identities. Indeed, in $(K(\mathcal{T}), \cdot, \star)$ the writing of an identity verified by an algebra (A, ω) is not unique, for example, the polynomials $(t_1 t_2)(t_1 t_2) - (t_1 t_2)\, t_1 \star t_2$ and $(t_1 t_2)(t_1 t_2) - (t_1 t_1)\, t_2 \star t_2$

correspond in an algebra (A,ω) to the identity $(xy)^2-\omega(x)^2\omega(y)y=0$, $(x,y\in A)$, which is written $(t_1t_2)(t_1t_2)-t_2.$ in $K(\mathcal{T})$.

Definition 11. Given $f\in K(\mathcal{T})$, $f=\sum_{k\geq 1}\alpha_k w_k$, we call homogenization of f, a polynomial $f^\star\in(K(\mathcal{T}),\cdot,\star)$ defined by:

$$f^\star=\sum_{k\geq 1}\alpha_k\widetilde{w}_k\star w_k,$$

where for all $k,i\geq 1$ we have $\widetilde{w}_k\in\mathfrak{M}(\mathcal{T})$ with $|\widetilde{w}_k|_i=|f|_i-|w_k|_i$.

Proposition 12. *Let (A,ω) be a K-algebra and let $f\in K(\mathcal{T})$ be a polynomial, the following statements are equivalent*

i) A verifies all homogenization $f^\star$ of f,

ii) A verifies one homogenization $f^\star$ of f,

iii) we have $\widehat{\varrho}(f)=0$ for any map $\varrho:\mathcal{T}\to H_\omega$.

Proof. The implication $i)\Rightarrow ii)$ is immediate.

For the rest we notice that for any map $\varrho:\mathcal{T}\to H_\omega$ and all $w\in\mathfrak{M}(\mathcal{T})$ we have $\widehat{\varrho}(w)=1$, from this we deduce that $\widehat{\varrho}(f^\star)=\widehat{\varrho}(f)$ for all $f\in K(\mathcal{T})$ and all $\varrho:\mathcal{T}\to H_\omega$.

$ii)\Rightarrow iii)$ Therefore if $f^\star$ is an identity of A, we have $\widehat{\varrho}(f^\star)=0$ and so $\widehat{\varrho}(f)=\widehat{\varrho}(f^\star)=0$ for any $\varrho:\mathcal{T}\to H_\omega$. .

$iii)\Rightarrow i)$ Reciprocally if we have $\widehat{\varrho}(f)=0$ for any map $\varrho:\mathcal{T}\to H_\omega$ then for any homogenization $f^\star$ of f we have $\widehat{\varrho}(f^\star)=\widehat{\varrho}(f)=0$ this implies from the proposition 7 that the algebra A verifies the identity $f^\star$. □

This result naturally leads to the following definition.

Definition 13. Let (A,ω) be a K-algebra and let $f\in K(\mathcal{T})$, we say that f is an identity verified by A if the algebra A satisfies any homogenization of f.

Baric algebras do not necessarily verify an identity, but in some cases the existence of an identity is assured.

Proposition 14. *If (A,ω) is a finite-dimensional algebra then A verifies an identity.*

Proof. Let $d+1$ be the dimension of (A,ω). The result is true for $d=0$ because in this case $A\simeq Ke$ with $e^2=e$ and $\omega(e)=1$. Suppose $d\geq 1$, let $(e_1,\ldots,e_d)$ be a basis of $\ker\omega$. For $z\in\ker\omega$ let $L_z:x\mapsto zx$, the map L_z is an endomorphism of $\ker\omega$ and the set $L=\{L_z;z\in\ker\omega\}$ is a subspace of $\mathrm{End}(\ker\omega)$ generated by $\{L_{e_1},\ldots,L_{e_d}\}$ therefore ([18], lemma 5, p. 103) L verifies the identity

$$P(t_1,\ldots,t_d)=\sum_{s\in S_d}(-1)^{\mathrm{sgn}\sigma}t_{\sigma(1)}\cdots t_{\sigma(d)}.$$

From $P(L_{z_1},\ldots,L_{z_d})y=0$ for any $z_1,\ldots,z_d\in\ker\omega$ and $y\in A$ we deduce that for all $a_1,\ldots,a_d\in H_\omega$ we have $P\left(L_{a_1^2-a_1},\ldots,L_{a_d^2-a_d}\right)y=0$ for any $y\in A$, in other words A verifies the identity:

$$\sum_{s\in S_d}(-1)^{\mathrm{sgn}\sigma}\left(t_{\sigma(1)}^2-t_{\sigma(1)}\right)\cdots\left(t_{\sigma(d)}^2-t_{\sigma(d)}\right)t_{d+1},$$

which finishes the proof. □

The following example shows that in general this result is not true in infinite dimension.

Example 15. Let $\mathcal{T}=\{t_n; n\geq 1\}$ and let $\mathfrak{M}(\mathcal{T})_n$ be the set of monomials of degree n. We have $\mathfrak{M}(\mathcal{T})=\bigcup_{n\geq 1}\mathfrak{M}(\mathcal{T})_n$ and $\mathfrak{M}(\mathcal{T})_n\subset\bigcup_{p+q=n}\mathfrak{M}(\mathcal{T})_p\times\mathfrak{M}(\mathcal{T})_q$ thus the set $\mathfrak{M}(\mathcal{T})$ is countable. Let $\varphi:\mathfrak{M}(\mathcal{T})\to\mathbb{N}$ be a bijective enumeration. We consider the algebra A with basis $\left(e_{\varphi(w)}\right)_{w\in\mathfrak{M}(\mathcal{T})}$ such that $e_{\varphi(u)}e_{\varphi(v)}=e_{\varphi(v)}e_{\varphi(u)}=e_{\varphi(uv)}$ for any $u,v\in\mathfrak{M}(\mathcal{T})$. The algebra A is commutative, non associative because $\left(e_{\varphi(t_1)}e_{\varphi(t_2)}\right)e_{\varphi(t_3)}=e_{\varphi((t_1t_2)t_3)}$ and $e_{\varphi(t_1)}\left(e_{\varphi(t_2)}e_{\varphi(t_3)}\right)=e_{\varphi(t_1(t_2t_3))}$ with $(t_1t_2)\,t_3\neq t_1\,(t_2t_3)$, it is weighted by $\omega\left(e_{\varphi(w)}\right)=1$. Suppose A verifies an identity $f\in K(\mathcal{T})$ where $f=\sum_{k\geq 1}\alpha_k w_k$, then for $\varrho:\mathcal{T}\to A$ such that $\varrho(t_i)=e_{\varphi(t_i)}$ we get $\widehat{\varrho}(f)=\sum_{k\geq 1}\alpha_k e_{\varphi(w_k)}$ therefore $\widehat{\varrho}(f)\neq 0$, contradiction. Thus the algebra (A,ω) does not verify any identity.

Let (A,ω) be a baric K-algebra, we denote $Id(A)$ (resp. $(Id(A),\cdot,\star)$) the subset of $K(\mathcal{T})$ (resp. $(K(\mathcal{T}),\cdot,\star)$) whose elements are identities vérified by A. According to the proposition 12 the set $Id(A)$ is not empty if and only if it is the same with $(Id(A),\cdot,\star)$. It is clear that $Id(A)$ (resp. $(Id(A),\cdot,\star)$) is a K-algebra and a two-sided ideal of $K(\mathcal{T})$ (resp. $(K(\mathcal{T}),\cdot,\star)$). We have seen in the proposition 8 that the elements of $(Id(A),\cdot,\star)$ are homogeneous. On the other hand, according to the definition 13, the ideal $Id(A)$ can contain both homogeneous and non-homogeneous polynomials, but contrary to what we showed for the lideal $(Id(A),\cdot,\star)$, homogeneous components of a non-homogeneous element of $Id(A)$ are not always identities of A. If $\mathscr{H}(\mathcal{T})$ and $\overline{\mathscr{H}}(\mathcal{T})$ denote respectively the set of homogeneous and non-homogeneous polynomials, the partition $K(\mathcal{T})=\mathscr{H}(\mathcal{T})\sqcup\overline{\mathscr{H}}(\mathcal{T})$ induces the partition of $Id(A)$ into two subsets: $Id(A)=\mathscr{H}(A)\sqcup\overline{\mathscr{H}}(A)$ where $\mathscr{H}(A)=Id(A)\cap\mathscr{H}(\mathcal{T})$ and $\overline{\mathscr{H}}(A)=Id(A)\cap\overline{\mathscr{H}}(\mathcal{T})$. In what follows we study the properties of $Id(A)$ and $(Id(A),\cdot,\star)$.

Proposition 16. *Let (A,ω) be a K-algebra.*

a) For any $f\in Id(A)$ and any $f\in(Id(A),\cdot,\star)$ we have: $f(\mathbb{1})=0$, where $\mathbb{1}=(1,\ldots,1,\ldots)$.

b) If $Id(A)\neq\emptyset$ then we have $\overline{\mathscr{H}}(A)\neq\emptyset$.

c) Given $f\in\overline{\mathcal{H}}(A)$ then for all $i\geq 1$ such that $|f|_i\neq 0$ there are $g,h\in K(\mathcal{T})$ verifying the conditions : $f=g-h$, $|g|_i=|f|_i$, $|g|_i>|h|_i$ and $g(\mathbb{1})=h(\mathbb{1})$.

Proof. a) Let $f\in Id(A)$, $f=\sum_{k\geq 1}\alpha_k w_k$ where $\alpha_k\in K$ and $w_k\in\mathfrak{M}(\mathcal{T})$, from $w_k(\mathbb{1})=1$ it follows $f(\mathbb{1})=\sum_{k\geq 1}\alpha_k w_k(\mathbb{1})=\sum_{k\geq 1}\alpha_k$. But according to the proposition 7 we have $\sum_{k\geq 1}\alpha_k\widehat{\varrho}(w_k)=0$ for any $\varrho:\mathcal{T}\to H_\omega$, or $\omega(\widehat{\varrho}(w_k))=1$ and applying the weight function ω to the relation $\widehat{\varrho}(f)=0$ we get $\sum_{k\geq 1}\alpha_k=0$.

Let $f\in(Id(A),\cdot,\star)$, we have $f=\sum_{k\geq 1}\alpha_k w_k+\sum_{l\geq 1}\beta_l u_l\star v_l$ where $\alpha_k,\beta_l\in K$ and $w_k,u_l,v_l\in\mathfrak{M}(\mathcal{T})$, then $f(\mathbb{1})=\sum_{k\geq 1}\alpha_k+\sum_{l\geq 1}\beta_l$. For any map $\varrho:\mathcal{T}\to H_\omega$ we have $\omega(\widehat{\varrho}(w_k))=1$ and $\omega(\widehat{\varrho}(u_l\star v_l))=\omega(\widehat{\varrho}(u_l))\,\omega(\widehat{\varrho}(v_l))=1$, then by applying the weight function ω to the relation $\widehat{\varrho}(f)=0$ we get $\sum_{k\geq 1}\alpha_k+\sum_{l\geq 1}\beta_l=0$.

b) The result is immediate if $\mathcal{H}(A)=\emptyset$. If $\mathcal{H}(A)\neq\emptyset$, let $f\in\mathcal{H}(A)$ and $i\geq 1$ such that $|f|_i\neq 0$, we have $f\left(t_1,\ldots,t_i^2\ldots,t_n,\ldots\right)\in Id(A)$ with $\left|f\left(t_1,\ldots,t_i^2\ldots,t_n,\ldots\right)\right|_1>|f(t_1,\ldots,t_i\ldots,t_n,\ldots)|$, thus $f\left(t_1,\ldots,t_i^2\ldots,t_n,\ldots\right)-f(t_1,\ldots,t_i\ldots,t_n,\ldots)\in\overline{\mathcal{H}}(A)$.

c) Given $f\in\overline{\mathcal{H}}(A)$, $f=\sum_{k\geq 1}\alpha_k w_k$ where $\alpha_k\in K$, $w_k\in\mathfrak{M}(\mathcal{T})$. Let $i\geq 1$ such that $|f|_i\neq 0$ we note $I_i=\{k;|w_k|_i=|f|_i\}$ then $f=\sum_{k\in I_i}\alpha_k w_k+\sum_{k\notin I_i}\alpha_k w_k$ so taking

$g=\sum_{k\in I_i}\alpha_k w_k$ and $h=g-f$ we have $|g|_i=|f|_i$, $|g|_i>|h|_i$ and from $f(\mathbb{1})=0$ it comes $g(\mathbb{1})-h(\mathbb{1})=0$. □

Another difference between ideals $(Id(A),\cdot,\star)$ and $Id(A)$ concerns the property of T-ideal. A T-ideal of $K(\mathcal{T})$ is a two-sided ideal of $K(\mathcal{T})$ which is stable by substituting indeterminate t_i by any element of $K(\mathcal{T})$ or, which is equivalent, stable by any endomorphism of $K(\mathcal{T})$.

Proposition 17. *Let A be a baric algebra such that $Id(A)\neq\{0\}$, then*

a) The ideal $(Id(A),\cdot,\star)$ is a T-ideal of $(K(\mathcal{T}),\cdot,\star)$.

b) If card$K>2\mu$ where $\mu=\min(\{|f|_i;f\in Id(A),i\geq 1\}\setminus\{0\})$, the ideal $Id(A)$ is not a T-ideal of $K(\mathcal{T})$.

Proof. a) Let $f\in(Id(A),\cdot,\star)$, to any family $(f_n)_{n\geq 1}$ elements of $(K(\mathcal{T}),\cdot,\star)$ and $(a_n)_{n\geq 1}$ elements of A we associate the map $\varrho:\mathcal{T}\to A$, $\varrho(t_i)=f_i(a_1,\ldots,a_n,\ldots)$ then from $\widehat{\varrho}(f)=0$ we deduce $f(f_1,\ldots,f_n,\ldots)\in(Id(A),\cdot,\star)$.

b) Suppose, by absurd, that $Id(A)$ is a T-ideal. Let $f\in\overline{\mathcal{H}}(A)$, $f=\sum_{k\geq 1}\alpha_k w_k$ and $i\geq 1$ such that $|f|_i=\mu$. We note $I=\{k;|w_k|_i=|f|_i\}$, so we have $|w_k|_i<|f|_i$ if $k\notin I$ and $f=\sum_{k\in I}\alpha_k w_k+\sum_{k\notin I}\alpha_k w_k$. From card$K>2\mu$ we deduce that it exists $\alpha\in K$ such that $\sum_{k\in I}\alpha_k\alpha^{2\mu}+\sum_{k\notin I}\alpha_k\alpha^{2|w_k|_i}\neq 0$, then by taking a family $(x_n)_{n\geq 1}$ elements of H_ω and $\varrho:\mathcal{T}\to A$ verifying $\varrho(t_k)=x_k$ if $k\neq i$ and $\varrho(t_i)=\alpha x_i$ we have by hypothesis $\widehat{\varrho}(f)=0$, but $\omega\circ\widehat{\varrho}(f)=\sum_{k\in I}\alpha_k\alpha^{2\mu}+\sum_{k\notin I}\alpha_k\alpha^{2|w_k|_i}$ we have a contradiction, it has been shown that $f\left(t_1,\ldots,\alpha t_i^2,\ldots,t_n,\ldots\right)\notin Id(A)$. □

The ideal $Id(A)$ verifies a weakened notion of T-ideal.

Remark 18. Note $\Delta K(\mathcal{T})$ the set of all polynomials $h\in K(\mathcal{T})$ such that $h(\mathbb{1})=1$. An ideal I of $K(\mathcal{T})$ is a stochastic T-idéal (cf. [1], p. 388) if I is invariant by replacing the symbols t_i by any element h of $\Delta K(\mathcal{T})$. Let (A,ω) be a K-algebra, as for all $(x_n)_{n\geq 1}$ such that $x_n\in H_\omega$ and all $h\in\Delta K(\mathcal{T})$ we have $\omega\big(h(x_1,\ldots,x_n,\ldots)\big)=h(\mathbb{1})=1$, it follows that $\widehat{\varrho}h(t_1,\ldots,t_n,\ldots)\in H_\omega$ therefore if $f\in Id(A)$ according to the definition 13 and the proposition 12, for any family $(h_n)_{n\geq 1}$ elements of $\Delta K(\mathcal{T})$ we have $\widehat{\varrho}f(h_1,\ldots,h_n,\ldots)=0$ in other words $f(h_1,\ldots,h_n,\ldots)\in Id(A)$ and $Id(A)$ is a stochastic T-ideal.

3. Peirce polynomials - Peirce-evanescent identities.

In the rest of this paper, the symbol t is a letter that does not belong to the set $\mathcal{T}$.

Let $f\in(K(\mathcal{T}),\cdot,\star)$, for any $i\geq 1$, we have $f(t_1,\ldots,t_i+t,\ldots)\in(K(\mathcal{T}\cup\{t\}),\cdot,\star)$ and the development of the polynomial $f(t_1,\ldots,t_i+t,\ldots)$ can be written in the form

$$f(t_1,\ldots,t_i+t,\ldots)=\sum_{0\leq k\leq|f|_i}\mathscr{L}_{i,k}(f)(t,t_1,\ldots,t_i,\ldots)$$

where for all $0\leq k\leq|f|_i$ we have $\mathscr{L}_{i,k}(f)\in(K(\mathcal{T}\cup\{t\}),\cdot,\star)$ and $|\mathscr{L}_{i,k}(f)|_t=k$.

The polynomial $\mathscr{L}_{i,k}(f)$ is called the linearization of f of order k in t_i. In particular, we have $\mathscr{L}_{i,0}(f)=f$.

Proposition 19. *Let (A,ω) be a K-algebra verifying an identity $f\in(Id(A),\cdot,\star)$, then for any $i\geq 1$, the linearizations of f in t_i are identities vérified by A.*

Proof. Let $t \notin \mathcal{T}$ and $f \in (Id(A), \cdot, \star)$. For all $i \geq 1$, all $(a_k)_{k\geq 1}$, a elements of A and $\lambda \in K^*$, we consider the maps $\varrho, \varrho_\lambda : \mathcal{T} \cup \{t\} \to A$ such that $\varrho(t_i) = \varrho_\lambda(t_i) = a_i$, $\varrho(t) = a$ and $\varrho_\lambda(t) = \lambda a$, by noticing that $\widehat{\varrho_\lambda}(\mathscr{L}_{i,k}(f)) = \lambda^k \widehat{\varrho}(\mathscr{L}_{i,k}(f))$, from $\widehat{\varrho_\lambda}(f(t_1, \ldots, t_i + t, \ldots)) = 0$ we deduce $\sum_{0\leq k\leq |f|_i} \lambda^k \widehat{\varrho}(\mathscr{L}_{i,k}(f)) = 0$. So considering the hypothesis made about the field K (cf. remark 9), by giving λ a non-zero values $\lambda_0, \ldots \lambda_{|f|_i}$ pairwise distinct, we obtain a linear system consisting of $|f|_i + 1$ equations of unknowns $\widehat{\varrho}(\mathscr{L}_{i,k}(f))$ whose determinant is not zero, therefore we have $\widehat{\varrho}(\mathscr{L}_{i,k}(f)) = 0$, in other words polynomials $\mathscr{L}_{i,0}(f), \ldots, \mathscr{L}_{i,|f|_i}(f)$ are identities verified by A. □

For any $i \geq 1$ and $h \in (K(\mathcal{T}), \cdot, \star)$, we introduce analogs of derivation operators ([8], [10], [18]) $\Delta_{i,h} : (K(\mathcal{T}), \cdot, \star) \to (K(\mathcal{T}), \cdot, \star)$ which are linear maps defined by:

$$\Delta_{i,h}(t_j) = \begin{cases} 0 & \text{if } t_j \neq t_i, \\ h & \text{if } t_j = t_i, \end{cases}$$

$$\Delta_{i,h}(u \cdot v) = \Delta_{i,h}(u) \cdot v + u \cdot \Delta_{i,h}(v), \tag{3.1}$$

$$\Delta_{i,h}(u \star v) = \Delta_{i,h}(u) \star v + u \star \Delta_{i,h}(v), \quad (u, v \in \mathfrak{M}(\mathcal{T})). \tag{3.2}$$

Proposition 20. *Given $f \in (K(\mathcal{T}), \cdot, \star)$, $f = \sum_{k\geq 1} \alpha_k w_k$ where $w_k \in (\mathfrak{M}(\mathcal{T}), \cdot, \star)$, the linearization of f of order d in t_i is obtained by*

$$\mathscr{L}_{i,d}(f) = \sum_{k\geq 1} \alpha_k \Delta^d_{i,t}(w_k).$$

Proof. It is clear that $\mathscr{L}_{i,d}(f) = \sum_{k\geq 1} \alpha_k \mathscr{L}_{i,d}(w_k)$, therefore it is sufficient to show that $\mathscr{L}_{i,d}(w_k) = \Delta^d_{i,t}(w_k)$, which we will do by recurrence on the degree of the monomial w_k. At degree 1 the result follows from the definition of the map $\Delta_{i,t}$. Suppose the result true for all monomials of degree $\leq n$. Let w be a monomial of degree $n+1$, according to the proposition 2 we have $w = u \cdot v$ or $w = u \star v$.

For the $w = u \cdot v$ case we have $\mathscr{L}_{i,d}(w) = \sum_{p+q=d} \mathscr{L}_{i,p}(u) \mathscr{L}_{i,q}(v)$ and with the induction hypothesis we get $\mathscr{L}_{i,d}(w) = \sum_{p+q=d} \Delta^p_{i,t}(u) \Delta^q_{i,t}(v)$, from the relation (3.1) we deduce inductively that for any integer $d \geq 1$ we have $\Delta^d_{i,t}(u \cdot v) = \sum_{p+q=d} \Delta^p_{i,h}(u) \Delta^q_{i,h}(v)$ therefore we got $\mathscr{L}_{i,d}(w) = \Delta^d_{i,t}(u \cdot v) = \Delta^d_{i,t}(w)$.

In the case of $w = u \star v$ we have $\mathscr{L}_{i,d}(w) = \sum_{p+q=d} \mathscr{L}_{i,p}(u) \star \mathscr{L}_{i,q}(v)$ and with the induction hypothesis we deduce that $\mathscr{L}_{i,d}(w) = \sum_{p+q=d} \Delta^p_{i,t}(u) \star \Delta^q_{i,t}(v)$, but with the relation (3.2) we deduce inductively that $\Delta^d_{i,t}(u \star v) = \sum_{p+q=d} \Delta^p_{i,t}(u) \star \Delta^q_{i,t}(v)$ and therefore $\mathscr{L}_{i,d}(u \star v) = \Delta^d_{i,t}(u \star v)$. □

Proposition 21. *Let (A, ω) be a K-algebra and let $f \in (K(\mathcal{T}), \cdot, \star)$ be a homogeneous polynomial such that $f = \sum_{p\geq 1} \alpha_p w_p + \sum_{q\geq 1} \beta_q u_q \star v_q$ where $\alpha_k, \beta_l \in K$ and $w_p, u_q, v_q \in \mathfrak{M}(\mathcal{T})$ with $|w_p|_i = |u_q \star v_q|_i = |f|_i$ for any $i \geq 1$. Then for $t \notin \mathcal{T}$, for any integer $i \geq 1$ and any map $\varrho : \mathcal{T} \cup \{t\} \to A$ we have:*

$$\widehat{\varrho}\mathscr{L}_{i,1}(f) = \sum_{p\geq 1} \alpha_p \widehat{\varrho}\Delta_{i,t}(w_p) + \sum_{q\geq 1} \beta_q \widehat{\varrho}\Delta_{i,t}(u_q \star v_q), \tag{3.3}$$

where for all $q \geq 1$ we have:

$$\widehat{\varrho}\Delta_{i,t}\left(u_q \star v_q\right) = \Big(\prod_{r\geq 1, r\neq i} \omega\left(\varrho\left(t_r\right)\right)^{|u_q|_r}\Big)\omega\left(\varrho\left(t_i\right)\right)^{|u_q|_i-1} \times \Big[|u_q|_i\, \omega\left(\varrho\left(t\right)\right)\widehat{\varrho}\left(v_q\right) + \omega\left(\varrho\left(t_i\right)\right)\widehat{\varrho}\Delta_{i,t}\left(v_q\right)\Big]. \tag{3.4}$$

Proof. According to the proposition 20 and by linearity of $\widehat{\varrho}$ we have:

$$\widehat{\rho}\mathscr{L}_{i,1}\left(f\right) = \sum_{p\geq 1}\alpha_p\widehat{\varrho}\Delta_{i,t}\left(w_p\right) + \sum_{q\geq 1}\beta_q\widehat{\varrho}\Delta_{i,t}\left(u_q \star v_q\right).$$

Then by definition of the maps $\widehat{\varrho}$ and $\Delta_{i,t}$ we have for any integer $q \geq 1$:

$$\widehat{\varrho}\Delta_{i,t}\left(u_q \star v_q\right) = \omega\left(\widehat{\varrho}\Delta_{i,t}\left(u_q\right)\right)\widehat{\varrho}\left(v_q\right) + \omega\left(\widehat{\varrho}\left(u_q\right)\right)\widehat{\varrho}\Delta_{i,t}\left(v_q\right).$$

The maps ω and $\widehat{\varrho}$ being linear, we have

$$\omega\left(\widehat{\varrho}\left(u_q\right)\right) = \prod_{r\geq 1}\omega\left(\varrho\left(t_r\right)\right)^{|u_q|_r}.$$

Let us show that for any monomial u we have

$$\omega\left(\widehat{\varrho}\Delta_{i,t}\left(u\right)\right) = |u|_i\Big(\prod_{r\geq 1, r\neq i}\omega\left(\varrho\left(t_r\right)\right)^{|u|_r}\Big)\omega\left(\varrho\left(t_i\right)\right)^{|u|_i-1}\omega\left(\varrho\left(t\right)\right).$$

This result is true for any monomial u of degree 1, because $\Delta_{i,t}\left(t_j\right) = t$ if $j = i$ and 0 if not, thus $\omega\left(\widehat{\varrho}\Delta_{i,t}\left(t_j\right)\right) = |t_j|_i\,\omega\left(\varrho\left(t\right)\right)$. Assuming this property verified for any monomial of degree $\leq n$, let u be a of degree $n+1$, we have $u = u_1u_2$ with $|u_1|, |u_2| \leq n$,

$$\begin{aligned}
\omega\left(\widehat{\varrho}\Delta_{i,t}\left(u\right)\right) &= \omega\left(\widehat{\varrho}\Delta_{i,t}\left(u_1\right)\right)\omega\left(\widehat{\varrho}\left(u_2\right)\right) + \omega\left(\widehat{\varrho}\left(u_1\right)\right)\omega\left(\widehat{\varrho}\Delta_{i,t}\left(u_2\right)\right)\\
&= |u_1|_i\Big(\prod_{r\neq i}\omega\left(\varrho\left(t_r\right)\right)^{|u_1|_r}\Big)\omega\left(\varrho\left(t_i\right)\right)^{|u_1|_i-1}\omega\left(\varrho\left(t\right)\right) \times \prod_{r\geq 1,}\omega\left(\varrho\left(t_r\right)\right)^{|u_2|_r} +\\
&\qquad \prod_{r\geq 1}\omega\left(\varrho\left(t_r\right)\right)^{|u_1|_r} \times |u_2|_i\Big(\prod_{r\neq i}\omega\left(\varrho\left(t_r\right)\right)^{|u_2|_r}\Big)\omega\left(\varrho\left(t_i\right)\right)^{|u_2|_i-1}\omega\left(\varrho\left(t\right)\right)\\
&= \Big(\prod_{r\neq i}\omega\left(\varrho\left(t_r\right)\right)^{|u_1|_r+|u_2|_r}\Big)\left(|u_1|_i + |u_2|_i\right)\omega\left(\varrho\left(t_i\right)\right)^{|u_1|_i+|u_2|_i-1}\omega\left(\varrho\left(t\right)\right)
\end{aligned}$$

and the result comes from $|u_1|_i + |u_2|_i = |u|_i$. □

An element e of a K-algebra A is an idempotent if $e \neq 0$ and $e^2 = e$. If the algebra A has a weight function ω, from $e^2 = e$ we deduce that $\omega\left(e\right)\left(\omega\left(e\right) - 1\right) = 0$ thus $\omega\left(e\right) = 0$ or 1.

Let (A, ω) be a K-algebra admitting an idempotent e such that $\omega\left(e\right) = 1$, taking in (3.4) the maps $\varrho_{(e,y)}$ defined for any $y \in \ker\omega$ by

$$\begin{aligned}
\varrho_{(e,y)} &: \mathcal{T}\cup\{t\} \to A,\\
\varrho_{(e,y)}\left(t_i\right) &= e \ \ (i \geq 1),\\
\varrho_{(e,y)}\left(t\right) &= y.
\end{aligned}$$

we get $\widehat{\varrho}_{(e,y)}\left(u_q \star v_q\right) = \widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(v_q\right)$ and with this (3.3) becomes:

$$\widehat{\varrho}_{(e,y)}\mathscr{L}_{i,1}\left(f\right) = \sum_{p\geq 1}\alpha_p\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(w_p\right) + \sum_{q\geq 1}\beta_q\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(v_q\right). \tag{3.5}$$

We can deduce

Corollary 22. *Let (A,ω) be a K-algebra admitting an idempotent e of weight 1 and let $f\in K\langle\mathcal{T}\rangle$. For any homogenization f^* of f we have:*

$$\widehat{\varrho}_{(e,y)}\mathscr{L}_{i,1}\left(f^*\right)=\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(f\right),\quad(\forall y\in\ker\omega).$$

Proof. Let i be an integer, we write $f\in K\langle\mathcal{T}\rangle$ in the form $f=\sum_{p\geq1}\alpha_p w_p+\sum_{q\geq1}\beta_q v_q$ with $w_p,v_q\in\mathfrak{M}\langle\mathcal{T}\rangle$ such that $|w_p|_i=|f|_i$ for all $p\geq1$ and $|v_q|_i<|f|_i$ for all $q\geq1$. Let $f^*=\sum_{p\geq1}\alpha_p w_p+\sum_{q\geq1}\beta_q u_q\star v_q$ be a homogenization of f, where for any $q\geq1$ such that $\beta_q\neq0$ we have $u_q\in\mathfrak{M}\langle\mathcal{T}\rangle$ with $|u_q|_i=|f|_i-|v_q|_i$. From (3.5) we get

$$\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(f\right)=\sum_{p\geq1}\alpha_p\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(w_p\right)+\sum_{q\geq1}\beta_q\widehat{\varrho}_{(e,y)}\Delta_{i,t}\left(v_q\right)=\widehat{\varrho}_{(e,y)}\mathscr{L}_{i,1}\left(f^*\right),$$

which completes the proof. □

In the following we note $K\langle t\rangle$ the subalgebra of $K\langle\{t\}\rangle$ generated by the set $\{t^n;n\geq1\}$ where for any integer $n\geq1$ we have: $t^{n+1}=tt^n=t^nt$ with $t^1=t$. For each $i\geq1$ we define the linear map ∂_i by

$$\begin{aligned}&\partial_i:K\langle\mathcal{T}\rangle\to K\langle t\rangle\\&\partial_i\left(t_j\right)=\begin{cases}1&\text{if }j=i\\0&\text{if }j\neq i,\end{cases}\\&\partial_i\left(uv\right)=t\left(\partial_i\left(u\right)+\partial_i\left(v\right)\right),\quad(\forall u,v\in\mathfrak{M}\langle\mathcal{T}\rangle).\end{aligned}\tag{3.6}$$

To simplify the notation we will write $\partial_i f$ instead of $\partial_i\left(f\right)$ for $f\in K\langle\mathcal{T}\rangle$.

Example 23. Let $w=\left(\left(\left(t_1t_2\right)t_2\right)t_3^2\right)\left(\left(t_1^2t_3\right)t_1\right)$ we have:

$$\begin{aligned}\partial_1 w&=t\left(\partial_1\left(\left(\left(t_1t_2\right)t_2\right)t_3^2\right)+\partial_1\left(\left(t_1^2t_3\right)t_1\right)\right)=t\left(t^3+t\left(\partial_1\left(t_1^2t_3\right)+1\right)\right)\\&=t\left(t^3+t^2\partial_1\left(t_1^2\right)+t\right)=3t^4+t^2.\\\partial_2 w&=t\left(t\partial_2\left(\left(t_1t_2\right)t_2\right)\right)=t\left(t^2\left(\partial_2\left(t_1t_2\right)+1\right)\right)=t^4+t^3.\\\partial_3 w&=t\left(t\left(\partial_3\left(\left(t_1t_2\right)t_2\right)+\partial_3\left(t_3^2\right)\right)+t\partial_3\left(t_1^2t_3\right)\right)=t^2\left(2t+t\right)=3t^3.\end{aligned}$$

A more convenient way to calculate polynomials $\partial_i f$ uses the representation of the elements of $\mathfrak{M}\langle\mathcal{T}\rangle$ by rooted binary trees with labeled leaves. The rooted binary trees formalism comes back to Etherington's papers [5, 7] for the study of the combinatorial structure of non-associative monomials, applications of this formalism were originally developed by Tkachev [16] to the Peirce decompositions of algebra identities.

A tree is a graph $T=\left(T^0,T^1\right)$ non oriented, connected, without cycle, where $T^0\neq\emptyset$ (resp. T^1) is the set of vertices (resp. edges).

A tree T is rooted if a vertex, noted $\rho_{_T}$ and called the root, is distinguished.

Two vertices $s_1,s_2\in T^0$ are incident if s_1 and s_2 are the vertices of the same edge. The valence of a vertex s is the number of incident vertices to s. A tree T is binary if the valence of $\rho_{_T}$ is 0 or 2 and if the valence of $s\in T^0$, $s\neq\rho_{_T}$ is equal to 1 or 3.

Univalent vertices of a rooted binary tree T are called leaves, we note $L\left(T\right)$ the set of leaves of T.

A rooted binary tree T is said $\mathcal{T}$-labeled if there is a map $\Lambda:L\left(T\right)\to\mathscr{T}$, we write (T,Λ) such a tree.

Two rooted binary trees T_1 and T_2 are isomorphic if there is a graph isomorphism $\varphi : T_1 \to T_2$ such that $\varphi\left(\rho_{T_1}\right) = \rho_{T_2}$. The rooted trees isomorphism $\varphi : T_1 \to T_2$ remove the leaves: $\varphi\left(L\left(T_1\right)\right) = L\left(T_2\right)$.

Two rooted binary trees $\mathscr{T}$-labeled (T_1, Λ_1) and (T_2, Λ_2) are isomorphic if there is a rooted tree isomorphism $\varphi : T_1 \to T_2$ such that $\Lambda_2 \circ \varphi_{|L(T_1)} = \Lambda_1$.

We denote $\mathscr{T}_{\mathcal{T}}$ the set of isomorphism classes of rooted binary trees $\mathcal{T}$-labeled, we provide $\mathscr{T}_{\mathcal{T}}$ with the grafting law: let $T_1, T_2 \in \mathscr{T}_{\mathcal{T}}$, we associate with (T_1, T_2) the tree $T_1 \cdot T_2$ such that the graph of $T_1 \cdot T_2$ without its root $\rho_{T_1 \cdot T_2}$ and two vertices adjacent to $\rho_{T_1 \cdot T_2}$ has two connected components T_1 and T_2. With the grafting law $\mathscr{T}_{\mathcal{T}}$ is a magma isomorphic to non-commutative magma $\mathrm{Mag}\left(\mathcal{T}\right)$, with this isomorphism $\Psi : \mathrm{Mag}\left(\mathcal{T}\right) \to \mathscr{T}_{\mathcal{T}}$, the degree of $w \in \mathrm{Mag}\left(\mathcal{T}\right)$ on $t_i \in \mathcal{T}$ is equal to the number of leaves labeled t_i in the tree $\Psi\left(w\right)$.

With these notions on rooted and labeled binary trees, we have the following result which provides a convenient and quick way to calculate polynomials $\partial_i w$.

Let (T, Λ) be a rooted binary tree $\mathcal{T}$-labeled, the altitude of a vertex $s \in T^0$, noted $\hbar\left(s\right)$, is the minimum number of edges connecting s to the root ρ_T (cf. [6]).

Proposition 24. *For any $w \in \mathfrak{M}\left(\mathcal{T}\right)$ and any $i \geq 1$, we have:*

$$\partial_i w\left(t\right) = \sum_{s \in \Lambda_w^{-1}(t_i)} t^{\hbar(s)}, \tag{3.7}$$

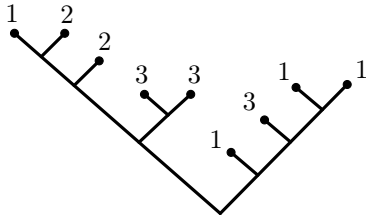
where $\Lambda_w^{-1}\left(t_i\right)$ is the set of leaves labeled t_i in the tree $\Psi\left(w\right)$, in other words, $\Lambda_w^{-1}\left(t_i\right) = \left\{s \in L\left(\Psi\left(w\right)\right); \Lambda\left(s\right) = t_i\right\}$.

Proof. Let us show the result by induction on the degree of w. The result is true if w has degree 1, indeed if $w = t_i$ on the tree $\Psi\left(w\right)$ the leaf labeled t_i is at altitude 0 because it is confused with the root, so $\partial_i w = 1 = t^0$, if $w = t_j$ with $j \neq i$ therefore we get $\Lambda^{-1}\left(t_i\right) = \emptyset$ and by convention the sum is zero. Suppose the result (3.7) true for any monomials of degree n, let w of degree $n+1$, we have $w = uv$ with $u, v \in \mathfrak{M}\left(\mathcal{T}\right)$ of degree at least 1. According to (3.6) we have

$$\partial_i w = t\left(\partial_i u + \partial_i v\right) = \sum_{s \in \Lambda_u^{-1}(t_i)} t^{\hbar(s)+1} + \sum_{s \in \Lambda_v^{-1}(t_i)} t^{\hbar(s)+1} = \sum_{s \in \Lambda_u^{-1}(t_i) \cup \Lambda_v^{-1}(t_i)} t^{\hbar(s)+1},$$

where $\Lambda_u^{-1}\left(t_i\right)$ and $\Lambda_v^{-1}\left(t_i\right)$ denote respectively the set of leaves labeled t_i in the trees $\Psi\left(u\right)$ and $\Psi\left(v\right)$. But the tree $\Psi\left(w\right)$ being the result of the grafting of trees $\Psi\left(u\right)$ and $\Psi\left(v\right)$, it follows that the set of leaves of the tree $\Psi\left(w\right)$ labeled t_i is the union of sets $\Lambda_u^{-1}\left(t_i\right)$ and $\Lambda_v^{-1}\left(t_i\right)$, and by definition of grafting the altitudes of the leaves of $\Lambda_u^{-1}\left(t_i\right)$ and $\Lambda_v^{-1}\left(t_i\right)$ in the tree $\Psi\left(w\right)$ are increased by one unit in relation to their values in thre trees $\Psi\left(u\right)$ and $\Psi\left(v\right)$, we deduce from this that $\sum_{s \in \Lambda_u^{-1}(t_i) \cup \Lambda_v^{-1}(t_i)} t^{\hbar(s)+1} = \sum_{s \in \Lambda_w^{-1}(t_i)} t^{\hbar(s)}$. □

Example 25. To illustrate this result, we use the example 23 with the monomial $w = \left(\left(\left(t_1 t_2\right) t_2\right) t_3^2\right)\left(\left(t_1^2 t_3\right) t_1\right)$. The labeled rooted binary tree associated with w is given below (we only mentioned the indices of the labels).



This tree has 4 leaves labeled t_1 including three of altitude 4 and one of altitude 2, so according to (3.7) we have $\partial_1 w = 3t^4 + t^2$. It has 2 leaves with the label t_2, one of altitude 4, the other of altitude 3 so $\partial_2 w = t^4 + t^3$. Finally the label t_3 is carried by three leaves all located at altitude 3 therefore $\partial_3 w = 3t^3$.

Corollary 26. *For any $w \in \mathfrak{M}(\mathcal{T})$ and any $i \geq 1$, concerning $\partial_i w$ we have:*

a) The coefficients of the polynomial $\partial_i w$ are natural numbers.

b) The polynomial degree of $\partial_i w$ is equal to the maximum altitude of the leaves labeled t_i in the tree $\Psi(w)$, in other words,

$$\deg(\partial_i w) = \max\{\hbar(s); s \in L(\Psi(w)), \Lambda(s) = t_i\}.$$

c) $|\partial_i w| \leq |w| - 1$.

d) $\partial_i w(1) = |w|_i$.

e) If $w = uv$ with $u, v \in \mathfrak{M}(\mathcal{T})$ such that $|u|_i, |v|_i \geq 1$, the polynomial valuation of $\partial_i w$ is:

$$val(\partial_i w) = \begin{cases} \min\{val(\partial_i u), val(\partial_i v)\} + 1 & si\ |u|_i |v|_i \neq 0, \\ \max\{val(\partial_i u), val(\partial_i v)\} + 1 & si\ |u|_i |v|_i = 0. \end{cases}$$

Proof. *a*) and *b*) are immediate consequences of (3.7).

c) Inductively on the degree of w. If $|w| = 1$ the result is immediate because we have $\partial_i w = 0, 1$. If the result is true for any monomial of degree $\leq n$, let $w \in \mathfrak{M}(\mathcal{T})$ of degree $n+1$, there are $u, v \in \mathfrak{M}(\mathcal{T})$ such that $w = uv$, we have $\partial_i w = t(\partial_i u + \partial_i v)$, in view of *a*) we have $|\partial_i w| = \max\{|\partial_i u|, |\partial_i v|\} + 1$, we deduce from this with the induction hypothesis that $|\partial_i w| \leq \max\{|u|, |v|\}$, as $u \neq w$ and $v \neq w$ we have $|u| < |w|$ and $|v| < |w|$ therefore $|\partial_i w| < |w|$.

d) According to (3.7) we have $\partial_i w(1) = \operatorname{card}\left(\Lambda_w^{-1}(t_i)\right)$ and by the magma isomorphism $\Psi : \operatorname{Mag}(\mathcal{T}) \to \mathscr{T}_{\mathcal{T}}$, the number of leaves of the tree $\Psi(w)$ labeled t_i is equal to the degree of w in t_i.

e) This is an immediate consequence of (3.7) and the grafting law of rooted binary trees. □

Let $e \in A$, we denote L_e the K-algebra endomorphism on A defined by $L_e : x \mapsto ex$.

Proposition 27. *Let (A, ω) be a K-algebra admitting an idempotent $e \in H_\omega$ and let $f \in K(\mathcal{T})$. For any integer $i \geq 1$ we have*

$$(\partial_i f)(L_e)(y) = \widehat{\varrho}_{(e,y)} \Delta_{i,t}(f) \quad (\forall y \in \ker \omega).$$

Moreover, if $f \in K(\mathcal{T})$ is an identity verified by A we have: $(\partial_i f)(L_e)(y) = 0$ for any $y \in \ker \omega$.

Proof. By linearity of maps $\partial_i f$ and $\widehat{\varrho}_{(e,y)}\Delta_{i,t}$ it suffices to show that for any $w \in \mathfrak{M}(\mathcal{T})$ we have $\widehat{\varrho}_{(e,y)}\Delta_{i,t}(w) = (\partial_i w)(L_e)(y)$. Let us show this by induction on the degree of w. If the degree of w is 1 we have $\widehat{\varrho}_{(e,y)}\Delta_{i,t}(t_j) = 0$, $(\partial_i t_j)(L_e)(y) = 0$ if $j \neq i$, and $\widehat{\varrho}_{(e,y)}\Delta_{i,t}(t_i) = y = (\partial_i t_i)(L_e)(y)$. Suppose the result true for all monomials of degrees $\leq n$, let $w \in \mathfrak{M}(\mathcal{T})$ of degree $n+1$, the monomial ω writes $w = uv$ with $u, v \in \mathfrak{M}(\mathcal{T})$ of degrees $\leq n$, and according to (3.1) and the inductive hypothesis we have

$$\widehat{\varrho}_{(e,y)}\Delta_{i,t}(w) = \widehat{\varrho}_{(e,y)}(\Delta_{i,t}(u)\, v) + \widehat{\varrho}_{(e,y)}(u\Delta_{i,t}(v)) = L_e\partial_i(u)(L_e)\, y + L_e\partial_i(v)(L_e)\, y,$$

because $\widehat{\varrho}_{(e,y)}(u) = \widehat{\varrho}_{(e,y)}(v) = e$. Finally we have

$$L_e\partial_i(u)(L_e)\, y + L_e\partial_i(v)(L_e)\, y = L_e(\partial_i u + \partial_i v)(L_e)(y) = \partial_i(uv)(L_e)(y).$$

If f is an identity verified by A, according to the proposition 19 we have $\widehat{\varrho}_{(e,y)}\Delta_{i,t}(f) = 0$ for any $y \in \ker\omega$. □

It follows from this proposition that for all$f \in Id(A)$, polynomials $\partial_i f$ are annihilator of the operator L_e whatever idempotent e of weight 1 of A.

Definition 28. Let $f \in K(\mathcal{T})$, for $i \geq 1$, the polynomial $\partial_i f$ is called the Peirce polynomial in t_i of f.

Polynomial f is said Peirce-evanescent if $f \neq 0$ and if all its Peirce polynomials $\partial_i f$, $(i \geq 1)$ are the null polynomial.

Polynomial f is a Peirce-evanescent identity (in short, an evanescent identity) if $f(\mathbb{1}) = 0$ and if f is Peirce-evanescent.

A K-algebra (A, ω) admitting an idempotent $e \in H_\omega$ and verifying an identity $f \in K(\mathcal{T})$ is said Peirce-evanescent for f if the polynomial f is an evanescent identity.

Example 29. Let (A, ω) be an algebra verifying the identity

$$f(x, y) = x^2(xy^2) - x(xy^2) - x^2y + xy.$$

If we assume that there exists an idempotent in A, relative to this idempotent we find $\partial_x f(t) = 3t^2 - (t^2 + t) - 2t^2 + t = 0$ and $\partial_y f(t) = 2t^3 - 2t^3 - t + t = 0$, so A is Peirce-evanescent.

We denote by $Ev(\mathcal{T})$ the subset of $K(\mathcal{T})$ whose elements are evanescent polynomials, so for all K-algebra (A, ω) admitting an idempotent e of weight 1, the set $Ev(A) = Ev(\mathcal{T}) \cap Id(A)$ refers to the set of evanescent identities relative to e verified by A.

Proposition 30. *The set $Ev(\mathcal{T})$ is an ideal of $K(\mathcal{T})$.*

Proof. It is immediate that$Ev(\mathcal{T})$ is a linear subspace of $K(\mathcal{T})$. Let us show that for any $f, g \in K(\mathcal{T})$ and for all $i \geq 1$ we have

$$\partial_i(fg) = t(f(\mathbb{1})\,\partial_i g + g(\mathbb{1})\,\partial_i f). \tag{3.8}$$

Let $f = \sum_{p\geq 1}\alpha_p u_p$ and $g = \sum_{q\geq 1}\beta_q v_q$ where $\alpha_p, \beta_q \in K$ and $u_p, v_q \in \mathfrak{M}(\mathcal{T})$, we get

$$\partial_i(fg) = \sum_{p,q\geq 1}\alpha_p\beta_q\partial_i(u_p v_q) = t\Big(\Big(\sum_{p\geq 1}\alpha_p\Big)\partial_i g + \Big(\sum_{q\geq 1}\beta_q\Big)\partial_i u\Big),$$

but $\sum_{p\geq 1}\alpha_p = f(\mathbb{1})$ and $\sum_{q\geq 1}\beta_q = g(\mathbb{1})$.

In particular, if we take $f \in Ev(\mathcal{T})$ and $g \in K(\mathcal{T})$ we get $\partial_i f = 0$ and according to the proposition 16 we have $f(\mathbb{1}) = 0$ from where $\partial_i(fg) = 0$. □

On the other hand, the ideal $Ev(A)$ is neither a T-ideal of $(K(\mathcal{T}),\cdot,\star)$ nor a stochastic T-ideal (cf. remark 18) as we can see in the following example.

Example 31. Starting from the evanescent identity characterizing backcrossing algebras $f(x)=x^2x^2-2x^3+x^2$, we consider the identity $g(x)=f\left(\frac{1}{2}\left(x^2+x\right)\right)$.

We have $g(x)=\frac{1}{16}\left[\left(x^2+x\right)^2\left(x^2+x\right)^2-4\left(x^2+x\right)^3+4\left(x^2+x\right)^2\right]$.

From $\partial_x\left(\left(x^2+x\right)^2\right)=4t(2t+1)$ we deduce:

$$\partial_x\left(\left(x^2+x\right)^2\left(x^2+x\right)^2\right)=2t\partial_x\left(\left(x^2+x\right)^2\right)=8t^2(2t+1)$$

$$\partial_x\left(\left(x^2+x\right)^3\right)=t\Big(\partial_x\left(\left(x^2+x\right)^2\right)+\partial_x\left(x^2+x\right)\Big)=t(2t+1)(4t+1)$$

finally we get $\partial_x g(x)=\frac{1}{16}(2t-(4t+1)+4)4t(2t+1)=\frac{1}{4}t(2t+1)(3-2t)\neq 0$.

Relation (3.8) gives a simple method for constructing evanescent identities.

Proposition 32. *Let (A,ω) be an algebra admitting an idempotent $e\in H_\omega$ and verifying an identity of the type fg where $f,g\in K(\mathcal{T})$. If we have $f(\mathbb{1})=g(\mathbb{1})=0$ then the identity fg is evanescent.*

Given (A,ω) a K-algebra verifying an identity f, the Peirce spectum is the set of roots of Peirce polynomials $\partial_i f$, relative to an idempotent $e\in H_\omega$, these are the eigenvalues of the operator L_e which are involved in the Peirce decomposition of the K-algebra (A,ω) verifying f. It is obvious that if the algebra (A,ω) verifies an evanescent identity, in the absence of Peirce polynomials, the spectrum of the operator L_e is undetermined. In the following we specify this by showing that the spectrum of L_e can be any part of K containing 1, for this we use mutation algebras.

A mutation algebra (A,M,ω) is a K-algebra defined by a vector space A over K, a linear map $M:A\to A$, a linear form $\omega:A\to K$ such that $\omega\neq 0$, $\omega\circ M=\omega$ and by the multiplicative law $xy=\frac{1}{2}(\omega(y)M(x)+\omega(x)M(y))$ where $x,y\in A$. It follows from the definition that $\omega(xy)=\omega(x)\omega(y)$ therefore ω is a weight function.

Example 33. Mutation algebras verify a multitude of identities. The construction of these identities is based on the following property: in a mutation algebra (A,M,ω) we have $(\ker\omega)^2=0$, then taking x,y,x',y' in H_ω such that $x-y\neq 0$ and $x'-y'\neq 0$ we have $(x-y)(x'-y')=0$. With this process we build ad libitum identities verified by all mutation algebras, for example $(t_1-t_2)^2$, $\left(t_1^2-t_2\right)^2$, $\left(t_1^2-t_1\right)\left(t_2^2-t_2\right)$, $(t_1+t_2-t_3-t_4)(t_1-t_2+t_3-t_4)$, $t_1^2t_2^2-(t_1t_2)^2$ et cetera ...

Mutation algebras verify all evanescent identities.

Proposition 34. *Let (A,M,ω) be a mutation algebra, for any $f\in Ev(\mathcal{T})$ the algebra A verifies the identity f.*

Proof. Let $f\in Ev(\mathcal{T})$ be an evanescent identity, $f=\sum_{k\geq 1}\alpha_k w_k$ where $\alpha_k\in K$ and $w_k\in\mathfrak{M}(\mathcal{T})$. Let (A,M,ω) be a mutation algebra, for any family $(x_n)_{n\geq 1}$ of elements of H_ω we define the map $\varrho:\mathcal{T}\to H_\omega$ by $\varrho(t_i)=x_i$, $(i\geq 1)$ and the algebra morphisms $\varphi_i:K\langle t\rangle\to A$ by:

$$\varphi_i(t^n)=\frac{1}{2^n}M^n(x_i).$$

Let us show by induction on the degree that for all $w \in \mathfrak{M}(\mathcal{T})$ we have:

$$\widehat{\varrho}(w) = \sum_{i\geq 1} \varphi_i(\partial_i w).$$

The result is immediate if w is of degree 1. Suppose the result true for all monomials of degrees $\leq n$. Let w be a monomial of degree $n+1$, there are $u, v \in \mathfrak{M}(\mathcal{T})$ of degrees $\leq n$ such that $w = uv$, then $\widehat{\varrho}(w) = \widehat{\varrho}(uv) = \widehat{\varrho}(u)\,\widehat{\varrho}(v)$, using the mutation algebra structure of A we get $\widehat{\varrho}(u)\,\widehat{\varrho}(v) = \frac{1}{2}M(\widehat{\varrho}(u)) + \frac{1}{2}M(\widehat{\varrho}(v))$, with the induction hypothesis this becomes $\frac{1}{2}M(\widehat{\varrho}(u)) + \frac{1}{2}M(\widehat{\varrho}(v)) = \frac{1}{2}M\left(\sum_{i\geq 1}\varphi_i(\partial_i u) + \varphi_i(\partial_i v)\right)$, but we have $\varphi_i(t\partial_i u) = \frac{1}{2}M\varphi_i(\partial_i u)$, therefore $\widehat{\varrho}(u)\,\widehat{\varrho}(v) = \sum_{i\geq 1}\varphi_i(t(\partial_i u + \partial_i v)) = \sum_{i\geq 1}\varphi_i(\partial_i uv)$ from which the result. We deduce that

$$\widehat{\varrho}(f) = \sum_{k\geq 1}\alpha_k\widehat{\varrho}(w_k) = \sum_{k\geq 1}\alpha_k\sum_{i\geq 1}\varphi_i(\partial_i w_k) = \sum_{i\geq 1}\varphi_i\Big(\sum_{k\geq 1}\alpha_k\partial_i w_k\Big) = \sum_{i\geq 1}\varphi_i(\partial_i f),$$

and since f is evanescent we have $\partial_i f = 0$ for all $i \geq 1$ therefore $\widehat{\varrho}(f) = 0$, and according to the proposition 7 we have shown that algebra (A, M, ω) verifies the identity f. □

Proposition 35. *For any subset P of K such that $1 \in P$, there is a mutation algebra (A, M, ω) having an idempotent e whose spectrum of the operator L_e is P.*

Proof. Consider the K-space A with basis $(e_n)_{n\in\mathbb{N}}$, equipped with the mutation algebra structure defined by $M : A \to A$, $M(e_0) = e_0$, $M(e_i) = 2e_{i+1}$ for all $i \geq 1$ and with $\omega : A \to K$ such that $\omega(e_0) = 1$, $\omega(e_i) = 0$ for any $i \geq 1$, then we get $e_0^2 = e_0$ and $e_0 e_i = e_{i+1}$ therefore the element e_0 is an idempotent and the spectrum of L_{e_0} is $P = \{1\}$.

Let I be a non empty set and let $P = \{1\} \cup \{\lambda_i; i \in I\}$ be a subset of K. We consider the vector space A over K with basis $\{e\} \cup \{e_i; i \in I\}$ equipped with the mutation algebra structure by the maps $M : A \to A$, $M(e) = e$, $M(e_i) = 2\lambda_i e_i$ and $\omega : A \to K$ such that $\omega(e) = 1$, $\omega(e_i) = 0$. This algebra (A, M, ω) admits e for idempotent element and for all $i \in I$ we have $ee_i = \frac{1}{2}M(e_i) = \lambda_i e_i$ therefore the spectrum of L_e is P. □

4. Evanescent identities of type $[n]$, $[n,1]$, $[n,2]$, $[n,1,1]$.

4.1. **Method of obtaining generators for homogeneous and non-homogeneous evanescent polynomials.**

Generators of evanescent identities are searched among non-homogeneous polynomials defined as follows.

Definition 36. A non-homogeneous polynomial $f \in K(t_1, \ldots, t_n)$ is called a train polynomial of degree $(d_1, \ldots, d_n)$ if $f = g - \sum_{i=1}^{r} h_i$, with $g, h_1, \ldots, h_r \in K(t_1, \ldots, t_n)$ satisfying the following conditions:

a) $f(\mathbb{1}) = 0$,
b) polynomial g is homogeneous of type $[d_1, \ldots, d_n]$,
c) for all $1 \leq i \leq r$, polynomial h_i is homogeneous of type $[\delta_i, d_2, \ldots, d_n]$,
d) we have $0 \leq \delta_1 < \ldots < \delta_r < d_1$.

Remark 37. For $n = 1$, if we take polynomials g_i and h_j in the set $\{x^k; k \geq 1\}$, we find the definition of principal train polynomials introduced by Etherington [4] and with g_i and h_j in $\{x^{[k]}; k \geq 1\}$ where $x^{[n+1]} = x^{[n]}x^{[n]}$, $x^{[1]} = x$ we obtain plenary train polynomials studied in [9].

In the cases studied below, the following method is used to obtain the generators of the evanescent polynomials in the form of train polynomial.

For a given n-tuple $(d_1,\dots,d_n)$ and for $w\in\mathfrak{M}(t_1,\dots,t_n)$ of type $[d_1,\dots,d_n]$, we are looking for a polynomial $P_w\in K(t_1,\dots,t_n)$ such that $w-P_w$ is a train polynomial of degree $(d_1,\dots,d_n)$ verifying $\partial_i(w-P_w)=0$ for every $1\leq i\leq n$. For that we choose a set $\mathscr{F}=\{w_{1,k},\dots,w_{m,k};k\geq 0\}$ where

- for all $1\leq j\leq m$ and all $k\geq 0$ we have $w_{j,k}\in\mathfrak{M}(t_1,\dots,t_n)$ and $w_{j,k}$ is of type $[k,d_2,\dots,d_n]$ or $[k+1,d_2,\dots,d_n]$,
- for all $1\leq i\leq n$ there is $1\leq j\leq m$ such that the integer sequence $\left(\left|\partial_i(w_{j,k})\right|\right)_{k\geq 0}$ is strictly increasing and the set of integers $\left\{\left|\partial_i(w_{j,k})\right|;k\geq 0\right\}$ is $\mathbb{N}$ or $\mathbb{N}^*$.

Then for $w\in\mathfrak{M}(t_1,\dots,t_n)$, $w\notin\mathscr{F}$, we put $P_w=\sum_{j=1}^{m}\left(\sum_{k=0}^{\delta_j}\alpha_{j,k}w_{j,k}\right)$ where $\delta_j=\left|\partial_i w\right|$, for each $1\leq i\leq n$ we have $\partial_i(w-P_w)\in K\langle t\rangle$, therefore the search for the polynomial P_w verifying $\partial_i(w-P_w)=0$ $(1\leq i\leq n)$ is equivalent to solving a system of linear equations of unknowns $(\alpha_{j,k})_{\substack{1\leq j\leq m\\0\leq k\leq\delta_j}}$.

With regard to the generators of homogeneous evanescent polynomials of type $[d_1,\dots,d_n]$. We note by N the cardinality of $\mathfrak{M}(t_1,\dots,t_n)_{[d_1,\dots,d_n]}$ and by $(w_k)_{1\leq k\leq N}$ the elements of this set. Let be $f=\sum_{k=1}^{N}\alpha_k w_k$, we search $(\alpha_k)_{1\leq k\leq N}$ such that $\partial_i f=0$ for any $1\leq i\leq n$ and $\sum_{k=1}^{N}\alpha_k=0$, we have $\partial_i f\in K\langle t\rangle$ and according to the corollary 26, $|\partial_i f|\leq\sum_{j=1}^{n}d_j-1$, therefore the conditions $\sum_{k=1}^{N}\alpha_k(\partial_i w_k)=0$ and $\sum_{k=1}^{N}\alpha_k=0$ give a system of at most $k\left(\sum_{j=1}^{n}d_j-1\right)+1$ linear equations of unknowns $(\alpha_k)_{1\leq k\leq N}$.

4.2. **Evanescent train identities of degree** (n) **and evanescent homogeneous identities of type** $[n]$**.**

In this section to simplify the notations we will write $\mathfrak{M}(x)$ instead of $\mathfrak{M}(\{x\})$ and $K(x)$ instead of $K(\{x\})$.

For any $n\geq 1$ we denote $\mathfrak{M}(x)_{[n]}$ the subset of $\mathfrak{M}(x)$ whose elements are monomials of type $[n]$. The numbers $W_{[n]}=\operatorname{card}\mathfrak{M}(x)_{[n]}$ are the Wedderburn-Etherington numbers, they satisty to the following recursive relations depending on the parity of n. Starting from $W_{[1]}=1$, we get:

$$W_{[2p]}=\sum_{i=1}^{p-1}W_{[i]}W_{[2p-i]}+\binom{W_{[p]}+1}{2},\qquad W_{[2p+1]}=\sum_{i=1}^{p}W_{[i]}W_{[2p+1-i]},\quad (p\geq 1).$$

The first values of $W_{[n]}$ are:

n	0	1	2	3	4	5	6	7	8	9	10
$W_{[n]}$	0	1	1	1	2	3	6	11	23	46	98

4.2.1. *Evanescent train identities of degree* (n).

In the following we note $\mathbb{Q}\langle x\rangle$ the vector space over $\mathbb{Q}$ generated by $\{x^n;n\geq 1\}$.

Proposition 38. *There is no train evanescent identity of degrees* (2) *and* (3).

For any $n \geq 4$ *and all* $w \in \mathfrak{M}(x)_{[n]}$ *verifying* $w \neq x^n$, *there is an unique polynomial* $P_w \in \mathbb{Q}\langle x \rangle$ *of degree* $< n$ *such that the polynomial* $w - P_w$ *is an evanescent train identity of degree* (n).

Proof. Let be $f(x) = \alpha x^2 + \beta x$, we have $\partial_x f(t) = 2\alpha t + \beta$ thus $\partial_x f = 0$ if $\alpha = \beta = 0$. Let be $f(x) = \alpha x^3 + \beta x^2 + \gamma x$, we have $\partial_x f(t) = 2\alpha t^2 + (\alpha + 2\beta) t + \gamma$ and consequently $\partial_x f = 0$ only if $f = 0$.

Let be $w \in \mathfrak{M}(x)_{[n]}$ such that $w \neq x^n$, according to the result c) of corollary 26 we have $|\partial_x w| \leq n - 1$ and for any $k \geq 3$ we have

$$\partial_x \left(x^k\right) = 2t^{k-1} + \sum_{i=1}^{k-2} t^i. \tag{4.1}$$

Let be $p = |\partial_x w|$, we have $\partial_x w(t) = \sum_{k=0}^{p} \alpha_k t^k$ and we are looking for $P_w(x) = \sum_{k=1}^{p+1} \beta_k x^k$ such that $\partial_x (w - P_w) = 0$ and $P_w(1) = w(1) = 1$. A simple calculation gives $\partial_x P_w(t) = 2\beta_{p+1} t^p + \sum_{k=1}^{p-1} \left(2\beta_{k+1} + \sum_{i=k+2}^{p+1} \beta_i\right) t^k + 2\beta_1$ and we have $\partial_x w = \partial_x P_w$ if and only if $(\beta_i)_{1 \leq i \leq p+1}$ is solution of the linear system:

$$2\beta_{p+1} = \alpha_p,\ 2\beta_{k+1} + \sum_{i=k+2}^{p+1} \beta_i = \alpha_k, (1 \leq k \leq p-1),\ 2\beta_1 = \alpha_0,\ \sum_{k=1}^{p+1} \beta_k = 1,$$

which is equivalent to the triangular linear system: $2\beta_1 = \alpha_0$, $\beta_k - \sum_{i=1}^{k-1} \beta_i = \alpha_k - 1$ $(2 \leq k \leq p+1)$ as according to corollary 26 we have $\alpha_k \in \mathbb{N}$ for any $0 \leq k \leq p$, the solution $(\beta_i)_{1 \leq i \leq p+1}$ of this system verify $\beta_i \in \mathbb{Q}$ for all $1 \leq i \leq p+1$. □

From this, the following corollary is immediately deduced.

Corollary 39. *For any* $n \geq 4$, *the vector space of evanescent train identities of degree* (n) *has dimension* $W_{[n]} - 1$.

The proof of proposition 38 gives a method based on the resolution of triangular linear systems to obtain evanescent polynomials, unfortunately it is difficult to apply for large values of n, fortunately the following result gives an algorithm easier to implement.

Theorem 40. *For any integers* $p, q \geq 1$ *we put:*

$$E_{p,q}(x) = x^p x^q - x^{p+1} - x^{q+1} + x^2.$$

Let $\mathscr{E}$ *be the ideal generated by the family* $\{E_{p,q}; p, q \geq 1\}$ *and* $\pi : K(x) \to {}^{K(x)}/_{\mathscr{E}}$ *the canonical surjection. So for all* $n \geq 4$ *and* $w \in \mathfrak{M}(x)_{[n]}$, $w \neq x^n$ *we have* $\pi(w) = P_w$ *and for any* $f \in K(x)$ *of degree* ≥ 4, *the polynomial* $f - \pi(f)$ *is an evanescent train identity of degree* (n).

Proof. Using the relation (4.1) we show by a simple calculation that the polynomials $E_{p,q}$ are evanescent. Let us show by induction on the degree $n \geq 4$ that $w - \pi(w)$ is evanescent and that $\pi(w) \in \mathbb{Z}\langle x \rangle$ for any $w \in \mathfrak{M}(x)_{[n]}$, $w \neq x^n$. We have $\pi\left(x^2 x^2\right) = 2x^3 - x^2$ and we know that the polynomial $x^2 x^2 - \left(2x^3 - x^2\right)$ is evanescent. If the result is true for any $u \in \mathfrak{M}(x)_{[k]}$, $u \neq x^k$ where $4 \leq k \leq n$, let be $w \in \mathfrak{M}(x)_{[n+1]}$, $w \neq x^{n+1}$, there is $u \in \mathfrak{M}(x)_{[p]}$ and $v \in \mathfrak{M}(x)_{[q]}$ such that $w = uv$ where $1 \leq p \leq q$

and $p+q=n+1$. If $u=x^p$ or $v=x^q$ we have $\pi(u)=u$ ou $\pi(v)=v$ so we have $\partial_x(w-\pi(w))=\partial_x(uv-\pi(u)\pi(v))=t(\partial_x(u-\pi(u))+\partial_x(v-\pi(v)))=0$. Moreover if $\pi(u),\pi(v)\in\mathbb{Z}\langle x\rangle$ then $\pi(w)=\pi(u)\pi(v)\in\mathbb{Z}\langle x\rangle$.

For all $w\in\mathfrak{M}(x)_{[n]}$ we have $\pi(w)\in\mathbb{Z}\langle x\rangle$ and $w-\pi(w)$ is evanescent, then according to the proposition 4.1, by uniqueness of the polynomial P_w we have $\pi(w)=P_w$. Finally for any $f\in K(x)$, $f=\sum_{k\geq 1}\alpha_k w_k$ we have $f-\pi(f)=\sum_{k\geq 1}\alpha_k(w_k-\pi(w_k))$ thus $f-\pi(f)\in\mathscr{E}$ in other words, the polynomial $f-\pi(f)$ is evanescent. □

Remark 41. This theorem makes it possible to specify a property stated in the proposition 38: for all $w\in\mathfrak{M}(x)_{[n]}$ such that $w\neq x^n$, the polynomial P_w verify $P_w\in\mathbb{Z}\langle x\rangle$.

Theorem 40 gives a very practical and very fast way to obtain evanescent polynomials.

Example 42. Let be $w=\left(\left(x^3x^3\right)x^2\right)\left(\left(x^2x^4\right)x^3\right)$, we have:

$$\begin{aligned}\pi(w)&=\left(x^2\left(2x^4-x^2\right)\right)\left(x^3\left(x^3+x^5-x^2\right)\right)=\left(2x^5-x^2\right)\left(x^6+2x^4-x^3-x^2\right)\\&=x^7+2x^6+2x^5-x^4-2x^3-x^2\end{aligned}$$

we thus obtain the evanescent identity train of degree (17):

$$\left(\left(x^3x^3\right)x^2\right)\left(\left(x^2x^4\right)x^3\right)-x^7-2x^6-2x^5+x^4+2x^3+x.$$

Using this method we obtain the following generators of evanescent train identities:

– of degree (4):

$x^2x^2-2x^3+x^2$.

– of degree (5):

$\left(x^2x^2\right)x-2x^4+x^3$;

$x^3x^2-x^4-x^3+x^2$.

– of degree (6):

$\left(x^2x^2\right)x^2-2x^4+x^2$;

$\left(x^3x^2\right)x-x^5-x^4+x^3$;

$x^3x^3-2x^4+x^2$;

$x^4x^2-x^5-x^3+x^2$.

$\left(\left(x^2x^2\right)x\right)x-2x^5+x^4$;

– of degree (7):

$\left(x^2x^2\right)x^3-3x^4+x^3+x^2$;

$x^5x^2-x^6-x^3+x^2$;

$\left(x^3x^3\right)x-2x^5+x^3$;

$\left(x^4x^2\right)x-x^6-x^4+x^3$;

$x^4x^3-x^5-x^4+x^2$;

$\left(\left(x^3x^2\right)x\right)x-x^6-x^5+x^4$;

$\left(x^3x^2\right)x^2-x^5-x^4+x^2$;

$\left(\left(\left(x^2x^2\right)x\right)x\right)x-2x^6+x^5$.

$\left(\left(x^2x^2\right)x\right)x^2-2x^5+x^4-x^3+x^2$;

– of degree (8):

$x^4x^4-2x^5+x^2$;

$\left(\left(\left(x^2x^2\right)x^2\right)x\right)x-2x^6+x^4$;

$\left(x^3x^3\right)x^2-2x^5+x^2$;

$\left(x^4x^3\right)x-x^6-x^5+x^3$;

$\left(\left(x^2x^2\right)x^2\right)x^2-2x^5+x^2$;
$\left(\left(x^2x^2\right)x\right)x^3-2x^5+x^2$;
$\left(x^3x^2\right)x^3-x^5-2x^4+x^3+x^2$;
$\left(\left(x^2x^2\right)x^3\right)x-3x^5+x^4+x^3$;
$\left(x^2x^2\right)x^4-x^5-2x^4+x^3+x^2$;
$\left(x^2x^2\right)\left(x^2x^2\right)-4x^4+2x^3+x^2$;
$x^5x^3-x^6-x^4+x^2$;
$\left(\left(x^3x^3\right)x\right)x-2x^6+x^4$;
$\left(x^4x^2\right)x^2-x^6-x^4+x^2$;
$\left(\left(x^3x^2\right)x\right)x^2-x^6-x^5+x^4-x^3+x^2$;
$\left(\left(\left(x^2x^2\right)x\right)x\right)x^2-2x^6+x^5-x^3+x^2$;
$\left(\left(\left(x^2x^2\right)x\right)x^2\right)x-2x^6+x^5-x^4+x^3$;
$\left(x^5x^2\right)x-x^7-x^4+x^3$;
$\left(\left(x^3x^2\right)x^2\right)x-x^6-x^5+x^3$;
$\left(\left(\left(x^2x^2\right)x\right)x^2\right)x-2x^6+x^5-x^4+x^3$;
$x^6x^2-x^7-x^3+x^2$;
$\left(\left(x^4x^2\right)x\right)x-x^7-x^5+x^4$;
$\left(\left(\left(x^3x^2\right)x\right)x\right)x-x^7-x^6+x^5$.

4.2.2. *Evanescent homogeneous identities of type* $[n]$.

Proposition 43. *There is no evanescent homogeneous identity of type* $[n]$ *for all* $n \leq 5$. *For* $n \geq 6$, *the vector space of evanescent homogeneous identities of type* $[n]$ *is generated by at least* $W_{[n]} - n + 2$ *evanescent homogeneous polynomials.*

Proof. The result is immediate for types [2] and [3] because according to proposition 38 there are no evanescent polynomials. For the type [4] there is only a unique evanescent polynomial that is not homogeneous. Let be $n \geq 5$, to simplify the notation we put $N = W_{[n]}$, we note $w_1, \ldots, w_N$ the elements of $\mathfrak{M}(x)_{[n]}$. Say that there exists an evanescent homogeneous polynomial of type $[n]$ is equivalent to saying that there exists non-zero elements $\alpha_1, \ldots, \alpha_N$ in K such that $\sum_{k=1}^{N} \alpha_k w_k = 0$, $\sum_{k=1}^{N} \alpha_k = 0$ and $\sum_{k=1}^{N} \alpha_k \partial_x(w_k) = 0$. However, with result c) of corollary 26 we have $|\partial_x w| \leq n-1$ for any $w \in \mathfrak{M}_{[n]}(x)$, thus for all $1 \leq k \leq N$ we have $\partial_x(w_k) = \sum_{i=1}^{n-1} \lambda_{k,i} t^i$ with $\lambda_{k,i} = 0$ for $i > |\partial_x w_k|$ and $\sum_{k=1}^{N} \alpha_k \partial_x(w_k) = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N} \lambda_{k,i}\alpha_k\right) t^i$, which amounts to solving the $n-1$ linear equations $S : \sum_{k=1}^{N} \lambda_{k,i}\alpha_k = 0, (1 \leq i \leq n-1)$ of unknowns $\alpha_1, \ldots, \alpha_N$. Considering the fact $\sum_{k=1}^{N} \alpha_k = 0$, the rank of the linear system (S) is $\leq n-2$ and its solutions form a vector space of dimension $\geq N-(n-2)$. If $n = 5$, we have $\partial_x\left(x^5\right) = 2t^4 + t^3 + t^2 + t$, $\partial_x\left(\left(x^2x^2\right)x\right) = 4t^3 + t$, $\partial_x\left(x^3x^2\right) = 2t^3 + 3t^2$, the system (S) is of rank 3 and it has $(0,0,0)$ for unique solution. □

Using the method described in the proof we obtain the generators of evanescent homogeneous identities:

– of type [6]
$x^3x^3 + \left(\left(x^2x^2\right)x\right)x - x^4x^2 - \left(x^3x^2\right)x$;
$\left(x^2x^2\right)x^2 + \left(\left(x^2x^2\right)x\right)x - x^4x^2 - \left(x^3x^2\right)x$.

– of type [7]
$x^4x^3 - \left(x^3x^2\right)x^2$;
$\left(x^2x^2\right)x^3 + \left(\left(x^2x^2\right)x\right)x^2 - 2x^4x^3$;
$x^4x^3 + \left(x^4x^2\right)x - \left(\left(x^3x^2\right)x\right)x - \left(x^2x^2\right)x^3$;

$x^5x^2+\left(x^2x^2\right)x^3+\left(\left(x^3x^2\right)x\right)x-\left(\left(\left(x^2x^2\right)x\right)x\right)x-2x^4x^3;$
$x^4x^3+\left(x^3x^3\right)t+\left(\left(\left(x^2x^2\right)x\right)x\right)x-2\left(\left(x^3x^2\right)x\right)x-\left(x^2x^2\right)x^3;$
$x^4x^3+\left(\left(x^2x^2\right)x^2\right)t+\left(\left(\left(x^2x^2\right)x\right)x\right)x-2\left(\left(x^3x^2\right)x\right)x-\left(x^2x^2\right)x^3.$

– of type [8]
$x^5x^3-\left(x^4x^2\right)x^2;$
$\left(x^2x^2\right)x^4-\left(x^3x^2\right)x^3;$
$\left(\left(x^3x^2\right)x^2\right)x-\left(x^4x^3\right)x;$
$\left(x^4x^2\right)x^2+\left(\left(x^4x^2\right)x\right)x-\left(x^4x^3\right)x-x^6x^2;$
$\left(x^3x^2\right)x^3+\left(\left(\left(x^2x^2\right)x\right)x\right)x^2-2\left(x^4x^2\right)x^2;$
$\left(\left(x^2x^2\right)x^3\right)x+\left(\left(\left(x^2x^2\right)x\right)x^2\right)x-2\left(x^4x^3\right)x;$
$x^4x^4+\left(x^4x^3\right)x-\left(\left(x^2x^2\right)x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(x^3x^2\right)x^3+\left(\left(x^3x^3\right)x\right)x-\left(x^4x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(x^3x^3\right)x^2+\left(x^4x^3\right)x-\left(\left(x^2x^2\right)x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(x^4x^3\right)x+\left(\left(x^2x^2\right)x\right)x^3-\left(\left(x^2x^2\right)x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(x^3x^2\right)x^3+\left(\left(\left(x^2x^2\right)x^2\right)x\right)x-\left(x^4x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(\left(x^2x^2\right)x^2\right)x^2+\left(x^4x^3\right)x-\left(\left(x^2x^2\right)x^3\right)x-\left(x^4x^2\right)x^2;$
$\left(x^4x^2\right)x^2+\left(\left(x^2x^2\right)x^3\right)x+\left(\left(\left(x^3x^2\right)x\right)x\right)x-2\left(x^4x^3\right)x-x^6x^2;$
$\left(x^3x^2\right)x^3+\left(x^4x^3\right)x+\left(\left(x^3x^2\right)x\right)x^2-\left(\left(x^2x^2\right)x^3\right)x-2\left(x^4x^2\right)x^2;$
$\left(x^2x^2\right)\left(x^2x^2\right)+\left(\left(x^2x^2\right)x^3\right)x+\left(x^4x^2\right)x^2-\left(x^4x^3\right)x-2\left(x^3x^2\right)x^3;$
$\left(x^5x^2\right)x+\left(\left(x^2x^2\right)x^3\right)x+2\left(x^4x^2\right)x^2-2\left(x^4x^3\right)x-x^6x^2-\left(x^3x^2\right)x^3;$
$\left(\left(x^2x^2\right)x^3\right)x+\left(\left(\left(\left(x^2x^2\right)x\right)x\right)x\right)x+3\left(x^4x^2\right)x^2-2\left(x^4x^3\right)x-2x^6x^2-\left(x^3x^2\right)x^3.$

4.3. **Evanescent train identities of degree** $(n,1)$ **and evanescent homogeneous identities of type** $[n,1]$**.**

Let $W_{[n,1]}$ be the cardinality of $\mathfrak{M}\left(x,y\right)_{[n,1]}$ the set of monomials of type $[n,1]$, because we can write any $w\in\mathfrak{M}\left(x,y\right)_{[n,1]}$ in the form $w=w_1w_2$ with $w_1\in\mathfrak{M}\left(x,y\right)_{[n-i]}$ and $w_2\in\mathfrak{M}\left(x,y\right)_{[i,1]}$ where $0\leq i\leq n-1$ we immediately deduce that

$$W_{[n,1]}=\sum_{i=0}^{n-1}W_{[n-i]}W_{[i,1]},\quad W_{[0,1]}=1.$$

And the first values of $W_{[n,1]}$ are:

n	0	1	2	3	4	5	6	7	8	9	10
$W_{[n,1]}$	1	1	2	4	9	20	46	106	248	582	1376

4.3.1. *Evanescent train identities of degree* $(n,1)$.

For any integer $r \geq 1$, we define $x^{\{r\}}y = x\left(x^{\{r-1\}}y\right)$ where $x^{\{0\}}y = y$, we put $\mathscr{F} = \left\{x^{n+1}y, x^{\{n\}}y; n \geq 0\right\}$ and $\mathbb{Q}\langle\mathscr{F}\rangle$ denote the vector space over $\mathbb{Q}$ generated by the set $\mathscr{F}$.

Lemma 44. *For any integer $p \geq 2$ and $r \geq 0$ we have:*

$$\partial_x\left(x^p y\right) = 2t^p + \sum_{i=2}^{p-1} t^i, \qquad \partial_y\left(x^p y\right) = t; \tag{4.2}$$

$$\partial_x\left(x^{\{r\}}y\right) = \sum_{i=1}^{r} t^i, \qquad \partial_y\left(x^{\{r\}}y\right) = t^r. \tag{4.3}$$

Proof. We have $\partial_x\left(x^p y\right) = t\left(\partial_x\left(x^p\right) + \partial_x\left(y\right)\right) = t\partial_x\left(x^p\right)$ and using the relation (4.1) we deduce the result, we also have $\partial_y\left(x^p y\right) = t\left(\partial_y\left(x^p\right) + \partial_y\left(y\right)\right) = t$. For $r \geq 1$ we get $\partial_x\left(x^{\{r\}}y\right) = t\left(1 + \partial_x\left(x^{\{r-1\}}y\right)\right)$ and $\partial_y\left(x^{\{r\}}y\right) = t\partial_y\left(x^{\{r-1\}}y\right)$, we deduce the results by recursivity. □

Proposition 45. *There is no evanescent train identity of degree* $(2,1)$.

For any $n \geq 3$ and any $w \in \mathfrak{M}\left(x,y\right)_{[n,1]}$ verifying $w \neq x^n y$ and $w \neq x^{\{n\}}y$, there is an unique polynomial $P_w \in \mathbb{Q}\langle\mathscr{F}\rangle$ with $\left|P_w\right|_x < n$ such that the polynomial $w - P_w$ is an evanescent train identity of degree $(n,1)$.

Proof. We have $\partial_x\left(x^2y\right) = 2t^2$, $\partial_y\left(x^2y\right) = t$, $\partial_x\left(x\left(xy\right)\right) = t^2 + t$ and $\partial_y\left(x\left(xy\right)\right) = t^2$, but $\partial_x\left(xy\right) = \partial_y\left(xy\right) = t$, $\partial_x\left(y\right) = 0$ and $\partial_y\left(y\right) = 1$, with this we easily show that we can not find $(\alpha,\beta,\gamma,\delta) \neq (0,0,0,0)$ such as the polynomial $f = \alpha x^2 y + \beta x\left(xy\right) + \gamma xy + \delta y$ verify $\partial_x f = \partial_y f = 0$.

Given $w \in \mathfrak{M}\left(x,y\right)_{[n,1]}$ such that $w \neq x^n y$ and $w \neq x^{\{n\}}y$ with $n \geq 3$. Let be $p = \left|\partial_x w\right|$ and $q = \left|\partial_y w\right|$, with result $c)$ of corollary 26 it comes $p,q \leq n$. From result $d)$ of corollary 26 we deduce that $\partial_x w\left(0\right) = \partial_y w\left(0\right) = 0$ and thus the valuations of $\partial_x w$ and $\partial_y w$ are greater than 1, we put $\partial_x w = \sum_{k=1}^{n} \alpha_k t^k$ and $\partial_y w = \sum_{k=1}^{n} \beta_k t^k$ with $\alpha_k = 0$ if $k > p$ and $\beta_k = 0$ as soon as $k > q$. We are locking for $P_w = \sum_{k=1}^{n} \lambda_k x^k y + \sum_{k=1}^{n} \mu_k x^{\{k\}}y$ verifying $\partial_x P_w = \partial_x w$, $\partial_y P_w = \partial_y w$ and $P_w\left(1,1\right) = 1$. Using lemma 44 we get

$$\partial_x P_w = \left(2\lambda_n + \mu_n\right)t^n + \sum_{i=1}^{n-1}\Big(2\lambda_i + \sum_{k=i+1}^{n} \lambda_k + \sum_{k=i}^{n} \mu_k\Big)t^i$$

$$\partial_y P_w = \sum_{i=2}^{n} \mu_i t^i + \Big(\sum_{k=1}^{n} \lambda_k + \mu_1\Big)t.$$

From the equation $\partial_y P_w = \partial_y w$ it results $\mu_i = \beta_i$ for $2 \leq i \leq n$ and $\sum_{k=1}^{n} \lambda_k + \mu_1 = \beta_1$. From the equation $\partial_x P_w = \partial_x w$ we deduce $2\lambda_n + \mu_n = \alpha_n$ and $2\lambda_i + \sum_{k=i+1}^{n} \lambda_k + \sum_{k=i}^{n} \mu_k = \alpha_i$ for $1 \leq i \leq n-1$, so we have $\lambda_n = \frac{1}{2}\left(\alpha_n - \beta_n\right)$ and for any $2 \leq i \leq n-1$ we find $\lambda_i = \frac{1}{2}\left(\alpha_i - \beta_i\right) + \sum_{k=i+1}^{n} \frac{1}{2^{k-i+1}}\left(\alpha_k + \beta_k\right)$, finally by writing $\alpha_1 = 2\lambda_1 + \sum_{k=2}^{n} \lambda_k + \sum_{k=1}^{n} \mu_k$ in the form $\alpha_1 = \lambda_1 + (\sum_{k=1}^{n} \lambda_k + \mu_1) + \sum_{k=2}^{n} \beta_k$, we get $\lambda_1 = \alpha_1 - \sum_{k=1}^{n} \beta_k = \alpha_1 - 1$, all this makes it possible to determine $\mu_1 = \beta_1 - \sum_{k=1}^{n} \lambda_k$. And we can verify that $P_w\left(1,1\right) = \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i + \sum_{i=2}^{n} \beta_i + (\beta_1 - \sum_{k=1}^{n} \lambda_k) = \sum_{i=1}^{n} \beta_i = \partial_y w\left(1\right) = 1$ according to the result $d)$ of corollary 26. It has been shown that the system

of equations $\partial_x P_w = \partial_x w$, $\partial_y P_w = \partial_y w$, $P_w(1,1) = 1$ admits an unique solution P_w, moreover according to corollary 26 we have $\alpha_k, \beta_k \in \mathbb{N}$ and the above makes it possible to affirm that $\lambda_k, \mu_k \in \mathbb{Q}$ for any $1 \leq k \leq n$ and therefore $P_w \in \mathbb{Q}\langle \mathscr{F} \rangle$. □

Corollary 46. *For any $n \geq 3$, the vector space of evanescent train polynomials lof degree $(n,1)$ has dimension $W_{[n,1]} - 2$.*

Proof. According to the previous proposition the space of the evanescent polynomials of type $[n,1]$ is generated by polynomials $w - P_w$ for any $w \in \mathfrak{M}(x,y)_{[n,1]}$ such that $w \notin \mathscr{F}$. □

Theorem 47. *For any integers $p, q \geq 2$ and $r \geq 0$ we put:*

$$\begin{aligned} E_{p,q}(x) &= x^p x^q - x^{p+1} - x^{q+1} + x^2, \\ F_{p,q}(x,y) &= x^p(x^q y) - x(xy) - x^p y - x^{q+1} y + x^2 y + xy, \\ F_{p,\{r\}}(x,y) &= x^p\left(x^{\{r\}} y\right) - x^p y - x^{\{r+1\}} y + xy, \\ F_{\{r\},p}(x,y) &= x^{\{r\}}(x^p y) - x^{\{r+1\}} y - x^{p+r} y + x^{r+1} y. \end{aligned}$$

Let $\mathscr{I}$ be the ideal generated by the family of polynomials $\left(E_{p,q}, F_{p,q}, F_{p,\{r\}}, F_{\{r\},p}\right)_{\substack{p,q\geq 2 \\ r\geq 0}}$ and $\pi : K(x) \to {}^{K(x)}/_{\mathscr{I}}$ the canonical surjection. Then for any $n \geq 3$ and any monomial $w \in \mathfrak{M}(x,y)_{[n,1]}$ such that $w \notin \mathscr{F}$ we have $\pi(w) = P_w$ and for any $f \in \bigoplus_{n\geq 3} K(x,y)_{[n,1]}$, the polynomial $f - \pi(f)$ is an evanescent train identity of degree $(n,1)$.

Proof. We saw for theorem 40 that polynomials $E_{p,q}$ are evanescent, let us show that the same is true for polynomials $F_{p,q}$, $F_{p,\{r\}}$ and $F_{\{r\},q}$.

For $p, q \geq 2$, we have $\partial_x\left(x^p\left(x^q y\right)\right) = t\left(\partial_x\left(x^p\right) + \partial_x\left(x^q y\right)\right) = t\partial_x\left(x^p\right) + t^2\partial_x\left(x^q\right)$ and $\partial_y\left(x^p\left(x^q y\right)\right) = t\partial_y\left(x^q y\right)$, with relation (4.1) and lemma 44 we get:

$$\partial_x\left(x^p\left(x^q y\right)\right) = 2t^p + 2t^{q+1} + \sum_{i=2}^{p-1} t^i + \sum_{i=3}^{q} t^i, \qquad \partial_y\left(x^p\left(x^q y\right)\right) = t^2. \tag{4.4}$$

For any $p \geq 2$ and $r \geq 0$ we have $\partial_x\left(x^p\left(x^{\{r\}} y\right)\right) = t\left(\partial_x\left(x^p\right) + \partial_x\left(x^{\{r\}} y\right)\right)$ and $\partial_y\left(x^p\left(x^{\{r\}} y\right)\right) = t\partial_y\left(x^{\{r\}} y\right)$, from relation (4.1) and lemma 44 we deduce

$$\partial_x\left(x^p\left(x^{\{r\}} y\right)\right) = 2t^p + \sum_{i=2}^{p-1} t^i + \sum_{i=2}^{r+1} t^i, \qquad \partial_y\left(x^p\left(x^{\{r\}} y\right)\right) = t^{r+1}. \tag{4.5}$$

For any $r \geq 1$ we have $\partial_x\left(x^{\{r\}}\left(x^p y\right)\right) = t\left(1 + \partial_x\left(x^{\{r-1\}}\left(x^p y\right)\right)\right)$ therefore we deduce that $\partial_x\left(x^{\{r\}}\left(x^p y\right)\right) = \sum_{i=1}^{r} t^i + t^r\partial_x\left(x^p y\right)$ and from $\partial_y\left(x^{\{r\}}\left(x^p y\right)\right) = t\partial_y\left(x^{\{r-1\}}\left(x^p y\right)\right)$ it results $\partial_y\left(x^{\{r\}}\left(x^p y\right)\right) = t^r\partial_y\left(x^p y\right)$, and with lemma 44 we have

$$\partial_x\left(x^{\{r\}}\left(x^p y\right)\right) = 2t^{r+p} + \sum_{i=r+1}^{r+p-1} t^i + \sum_{i=1}^{r} t^i, \qquad \partial_y\left(x^{\{r\}}\left(x^p y\right)\right) = t^{r+1}. \tag{4.6}$$

A simple calculation using the relations obtained above and those of the lemma 44 shows that polynomials $F_{p,q}$, $F_{p,\{r\}}$ and $F_{\{r\},q}$ are evanescent.

Let us show that for any $w \in \mathfrak{M}(x,y)_{[n,1]}$ such that $w \notin \mathscr{F}$, polynomial $w - \pi(w)$ is evanescent. According to the relation (4.4), it is immediate for any $w \in \mathfrak{M}(x,y)_{[3,1]}$ such that $w \neq x^3y$ and $w \neq x^{\{3\}}y$. Assume the property true for all monomials of $\mathfrak{M}(x,y)_{[p,1]} \setminus \left\{x^py, x^{\{p\}}y\right\}$ where $3 \leq p \leq n$, let be $w \in \mathfrak{M}(x,y)_{[n+1,1]}$ verifying $w \neq x^{n+1}y$ and $w \neq x^{\{n+1\}}y$. We have two cases:

– there are $u \in \mathfrak{M}(x,y)_{[p]}$ and $v \in \mathfrak{M}(x,y)_{[q,1]}$ such that $w = uv$ with $u \neq x$, $1 \leq p,q$ and $p+q = n+1$, then we have $\partial_x(w - \pi(w)) = \partial_x(uv - \pi(u)\pi(v)) = t(\partial_x(u - \pi(u)) + \partial_x(v - \pi(v))) = 0$ and likewise $\partial_y(w - \pi(w)) = 0$;

– there are $u \in \mathfrak{M}(x,y)_{[p]}$ and $v \in \mathfrak{M}(x,y)_{[q]}$ such that $w = (uv)y$ with $uv \neq x^{n+1}$, $1 \leq p,q$ and $p+q = n+1$, so $\partial_x(w - \pi(w)) = \partial_x(((uv) - \pi(u)\pi(v))y)$, it results $\partial_x(w - \pi(w)) = t^2(\partial_x(u - \pi(u)) + \partial_x(v - \pi(v))) = 0$, in a similar way we have $\partial_y(w - \pi(w)) = 0$.

It is easy to show by induction that for all $w \in \mathfrak{M}(x,y)_{[n,1]}$ such that $w \neq x^ny$ and $w \neq x^{\{n\}}y$ we have $|\pi(w)|_x < n$ and $\pi(w) \in \mathbb{Z}\langle\mathscr{F}\rangle$, then by uniqueness of the polynomial P_w such that $w - P_w$ is evanescent, we get $\pi(w) = P_w$. □

This theorem gives an efficient algorithm for constructing evanescent train identities train of degree $(n,1)$, as illustrated by an example.

Example 48. Let be $w = x^5\left(x\left(x\left(x^4\left(\left(x^2x^3\right)y\right)\right)\right)\right)$, in the algebra ${}^{K(x)}/_{\mathscr{F}}$ we find modulo $E_{p,q}$: $\left(x^2x^3\right)y = x^4y + x^3y - x^2y$, next modulo $F_{p,q}$ we have: $x^4\left(\left(x^2x^3\right)y\right) = x(xy) + x^5y + 2x^4y - x^3y - x^2y - xy$. Modulo $F_{\{r\},p}$ we get $x^{\{3\}}\left(x^4\left(\left(x^2x^3\right)y\right)\right) = x^{\{5\}}y + x^8y + 2x^7y - x^6y - x^5y - x^4y$, at last modulo $F_{p,\{r\}}$ and $F_{p,q}$ we get $\pi(w) = x^{\{6\}}y + x^9y + 2x^8y - x^7y - x^6y - xy$, we can therefore say that the polynomial of type $[17,1]$:

$$x^5\left(x\left(x\left(x^4\left(\left(x^2x^3\right)y\right)\right)\right)\right) - x(x(x(x(x(xy))))) - x^9y - 2x^8y + x^7y + x^6y + xy$$

is an evanescent identity.

By applying this algorithm we easily obtain the generators of evanescent train identities

– of degree $(3,1)$:

$x^2(xy) - x(xy) - x^2y + xy$;
$x\left(x^2y\right) - x(xy) - x^3y + x^2y$.

– of degree $(4,1)$:

$\left(x^2x^2\right)y - 2x^3y + x^2y$;
$x^2(x(xy)) - x(x(xy)) - x^2y + xy$;
$x^2\left(x^2y\right) - x(xy) - x^3y + xy$;
$x\left(x^2(xy)\right) - x(x(xy)) - x^3y + x^2y$;
$x^3(xy) - x(xy) - x^3y + xy$;
$x\left(x\left(x^2y\right)\right) - x(x(xy)) - x^4y + x^3y$.
$x\left(x^3y\right) - x(xy) - x^4y + x^2y$;

– of degree $(5,1)$:

$x^2\left(x^2(xy)\right) - x(x(xy)) - x^3y + xy$;
$x^3\left(x^2y\right) - x(xy) - 2x^3y + x^2y + xy$;
$x\left(x\left(x^3y\right)\right) - x(x(xy)) - x^5y + x^3y$;
$x\left(x^2\left(x^2y\right)\right) - x(x(xy)) - x^4y + x^2y$;

$x^3(x(xy)) - x(x(xy)) - x^3y + xy$;
$x\left(x^3(xy)\right) - x(x(xy)) - x^4y + x^2y$;
$x^2\left(x^3y\right) - x(xy) - x^4y + xy$;
$\left(x^3x^2\right)y - x^4y - x^3y + x^2y$;
$x^4(xy) - x(xy) - x^4y + xy$;
$x\left(x^4y\right) - x(xy) - x^5y + x^2y$;
$\left(\left(x^2x^2\right)x\right)y - 2x^4y + x^3y$;
$\left(x^2x^2\right)(xy) - x(xy) - 2x^3y + x^2y + xy$;
$x\left(\left(x^2x^2\right)y\right) - x(xy) - 2x^4y + x^3y + x^2y$;
$x^2(x(x(xy))) - x(x(x(xy))) - x^2y + xy$;
$x\left(x^2(x(xy))\right) - x(x(x(xy))) - x^3y + x^2y$;
$x\left(x\left(x^2(xy)\right)\right) - x(x(x(xy))) - x^4y + x^3y$;
$x\left(x\left(x\left(x^2y\right)\right)\right) - x(x(x(xy))) - x^5y + x^4y$;
$x^2\left(x\left(x^2y\right)\right) - x(x(xy)) - x^4y + x^3y - x^2y + xy$.

4.3.2. *Evanescent homogeneous identities of type* $[n,1]$.

Proposition 49. *There is no evanescent homogeneous identity of type* $[n,1]$ *for* $n \leq 3$. *For any* $n \geq 4$, *the vector space of evanescent homogeneous identities of type* $[n,1]$ *is generated by at least* $W_{[n,1]} - 2(n-1)$ *evanescent homogeneous identities.*

Proof. Let be $\mathfrak{M}(x,y)_{[n,1]} = \{w_k; 1 \leq k \leq N\}$ where $N = W_{[n,1]}$ and $f = \sum_{i=1}^{N} \alpha_i w_i$, we search $(\alpha_i)_{1\leq i\leq N} \in K^N$ such that $\partial_x f = \partial_y f = 0$. with $\partial_x f = \sum_{i=1}^{N} \alpha_i \partial_x w_i$ and $\partial_y f = \sum_{i=1}^{N} \alpha_i \partial_y w_i$. For any $1 \leq i \leq N$ we have $\partial_x w_i, \partial_y w_i \in K[t]$ with $|\partial_x w_i| \leq n$ and $|\partial_y w_i| \leq n$ therefore equations $\partial_x f = 0$ and $\partial_y f = 0$ give two linear systems of n equations with N unknowns.

So if $n = 2$, with $f(x,y) = \alpha x^2 y + \beta x(xy)$, from $\partial_y(f) = \alpha t + \beta t^2$ we deduce $\alpha = \beta = 0$, so there is no evanescent homogeneous identity of type $[2,2]$.

If $n = 3$, taking $f(x,y) = \alpha x^3 y + \beta x^2(xy) + \gamma x\left(x^2y\right) + \delta x(x(xy))$, we get $\partial_y(f) = \alpha t + \beta t^2 + \gamma t^2 + \delta t^3$, from $\partial_y f = 0$ we deduce $\alpha = \delta = 0$ and $\beta + \gamma = 0$, with this $\partial_x f = 0$ gives $\partial_x\left(\beta x^2(xy) + \gamma x\left(x^2y\right)\right) = \beta\left(3t^2\right) + \gamma\left(2t^3 + t\right) = 0$ therefore $\beta = \gamma = 0$, there is no evanescent homogeneous identity of type $[3,2]$.

Suppose $n \geq 4$, from $f(1,1) = 0$ we deduce $\sum_{i=1}^{N} \alpha_i = 0$, therefore the rank of the linear system $\partial_x f = 0$ is $\leq n-1$, the same is true for $\partial_y f = 0$, so the rank of the linear system $\partial_x f = \partial_y f = 0$ is $\leq 2(n-1)$ therefore the dimension of the solution space is $\geq W_{[n,1]} - 2(n-1)$. □

The method used in the proof allows to give the generators of evanescent homogeneous identities:

– of type $[4,1]$
$\left(x^2x^2\right)y - x^3(xy)$;
$x^4y + x\left(x^2(xy)\right) - \left(x^2x^2\right)y - x\left(x\left(x^2y\right)\right)$;
$x\left(x^3y\right) + x^2(x(xy)) - x^3(xy) - x\left(x\left(x^2y\right)\right)$.

– of type $[5,1]$
$x\left(x^2\left(x^2y\right)\right) - x\left(x^3(xy)\right)$;
$\left(x^3x^2\right)y + x\left(x\left(x^3y\right)\right) - x^5y - x\left(x^3(xy)\right)$;
$\left(\left(x^2x^2\right)x\right)y + x\left(x^4y\right) - x^5y - x\left(\left(x^2x^2\right)y\right)$;

$x^4(xy)+x\left(x\left(x^3y\right)\right)-x^2\left(x\left(x^2y\right)\right)-x\left(x^4y\right);$
$x^2\left(x^3y\right)+x\left(x\left(x^3y\right)\right)-x^2\left(x\left(x^2y\right)\right)-x\left(x^4y\right);$
$x^2\left(x\left(x^2y\right)\right)+x\left(x^2\left(x\left(xy\right)\right)\right)-x\left(x^3\left(xy\right)\right)-x^2\left(x\left(x\left(xy\right)\right)\right);$
$x^2\left(x\left(x^2y\right)\right)+x\left(x\left(x\left(x^2y\right)\right)\right)-x^2\left(x\left(x\left(xy\right)\right)\right)-x\left(x\left(x^3y\right)\right);$
$x^3\left(x^2y\right)+x\left(\left(x^2x^2\right)y\right)+2x\left(x\left(x^3y\right)\right)-x^2\left(x\left(x^2y\right)\right)-2x\left(x^4y\right)-x\left(x^3\left(xy\right)\right);$
$x^3\left(x\left(xy\right)\right)+x\left(\left(x^2x^2\right)y\right)+x\left(x\left(x^3y\right)\right)-x^2\left(x\left(x^2y\right)\right)-x\left(x^4y\right)-x\left(x^3\left(xy\right)\right);$
$x\left(\left(x^2x^2\right)y\right)+x\left(x\left(x^3y\right)\right)+x^2\left(x^2(xy)-x\left(x^4y\right)-x\left(x^3\left(xy\right)\right)-x^2\left(x\left(x^2y\right)\right);$
$\left(x^2x^2\right)(xy)+x\left(\left(x^2x^2\right)y\right)+2x\left(x\left(x^3y\right)\right)-x^2\left(x\left(x^2y\right)\right)-2x\left(x^4y\right)-x\left(x^3\left(xy\right)\right);$
$x^2\left(x\left(x^2y\right)\right)+x\left(x^4y\right)+x\left(x\left(x^2\left(xy\right)\right)\right)-x^2\left(x\left(x\left(xy\right)\right)\right)-x\left(\left(x^2x^2\right)y\right)-x\left(x\left(x^3y\right)\right).$

4.4. **Evanescent train identities of degree $(n,2)$ and evanescent homogeneous identities of type $[n,2]$.**

Let $W_{[n,2]}$ be the cardinality of $\mathfrak{M}(x,y)_{[n,2]}$ the set of monomials of type $[n,2]$. For $w\in\mathfrak{M}(x,y)_{[n,2]}$ there are two ways to decompose w as a product of two monomials $w=w_1w_2$. Either by taking $w_1\in\mathfrak{M}(x,y)_{[n-i]}$ and $w_2\in\mathfrak{M}(x,y)_{[i,2]}$ for $0\leq i\leq n-1$ and in this case we have $\sum_{i=0}^{n-1}W_{[n-i]}W_{[i,2]}$ possible writings. Either with $w_1\in\mathfrak{M}(x,y)_{[i,1]}$ and $w_2\in\mathfrak{M}(x,y)_{[j,1]}$ for $0\leq i\leq j\leq n$ such that $i+j=n$, hence $2i\leq n$ and according to the parity of n we have two cases. If n is odd, $n=2p+1$, for all $0\leq i\leq p$ words w_1 and w_2 have different degrees in x so there are $\sum_{i=0}^{p}W_{[2p+1-i,1]}W_{[i,1]}$ possible decompositions for w in product of two monomials. If n is even, $n=2p$, for all $0\leq i<p$ monomials w_1 and w_2 have different degrees in x so there are $\sum_{i=0}^{p-1}W_{[2p-i,1]}W_{[i,1]}$ decompositions for w, and for $i=p$ monomials w_1 and w_2 are in $\mathfrak{M}(x,y)_{[p,1]}$ hence we have $\binom{W_{[p,1]}+1}{2}$ decompositions of w.

In summary, we obtained:

$W_{[0,2]}=1,$

$$W_{[n,2]}=\sum_{i=0}^{n-1}W_{[2p+1-i]}W_{[i,2]}+\sum_{i=0}^{\lfloor n/2\rfloor}W_{[n-i,1]}W_{[i,1]}+\begin{cases}0 & \text{if } n \text{ odd}\\ \binom{W_{[n/2,1]}}{2} & \text{if } n \text{ even}\end{cases},(n\geq 1).$$

First values of $W_{[n,2]}$ are

n	0	1	2	3	4	5	6	7	8	9	10
$W_{[n,2]}$	1	2	6	15	41	106	280	726	1891	4886	12622

4.4.1. *Evanescent train identities of degree $(n,2)$.*

For any $f\in K(x,y)$ and any integer $r\geq 1$, we define $x^{\{r\}}f=x\left(x^{\{r-1\}}f\right)$ where $x^{\{0\}}f=f$.

Lemma 50. *For any integer $r\geq 0$ we have:*

$$\partial_x\left(\left(x^{\{r\}}y\right)y\right)=\sum_{i=2}^{r+1}t^i,\qquad \partial_y\left(\left(x^{\{r\}}y\right)y\right)=t^{r+1}+t; \tag{4.7}$$

$$\partial_x\big(x^{\{r\}}y^2\big) \;=\; \sum_{i=1}^{r} t^i, \qquad\qquad \partial_y\big(x^{\{r\}}y^2\big) \;=\; 2t^{r+1}. \tag{4.8}$$

Proof. Starting from $\partial_x\big(\big(x^{\{r\}}y\big)y\big) = t\partial_x\big(x^{\{r\}}y\big) = t\left(1+\partial_x\big(x^{\{r-1\}}y\big)\right)$ and $\partial_y\left(\big(x^{\{r\}}y\big)y\right) = t\left(\partial_y\big(x^{\{r\}}y\big)+1\right) = t^2\partial_y\big(x^{\{r-1\}}y\big)+t$, and from $\partial_x\big(x^{\{r\}}y^2\big) = t\left(1+\partial_x\big(x^{\{r-1\}}y^2\big)\right)$ and $\partial_y\big(x^{\{r\}}y^2\big) = t\partial_y\big(x^{\{r-1\}}y^2\big)$ the stated results are obtained recursively. □

We note $\mathscr{F} = \left\{\big(x^{\{r\}}y\big)\,y, x^{\{r\}}y^2; r\geq 0\right\}$ and $\mathbb{Q}\left\langle\mathscr{F}\right\rangle$ the vector space over $\mathbb{Q}$ generated by the set $\mathscr{F}$.

Proposition 51. *There is no evanescent train identity of type* $(1,2)$.

For any $n\geq 2$ *and any* $w\in\mathfrak{M}\left(x,y\right)_{[n,2]}$ *where* $w\neq\big(x^{\{n\}}y\big)\,y$ *and* $w\neq x^{\{n\}}y^2$, *there is an unique polynomial* $P_w\in\mathbb{Q}\left\langle\mathscr{F}\right\rangle$ *with* $\left|P_w\right|_x\leq n$ *such that polynomial* $w-P_w$ *is an evanescent train identity of degree* $(n,2)$.

Proof. We have seen that there is no train evanescent identity of degree $(2,1)$ so by permutation of x and y there is no train evanescent identity of degree $(1,2)$.

Let be $n\geq 2$. For any $w\in\mathfrak{M}\left(x,y\right)_{[n,2]}$ we have $\left|\partial_x w\right|\leq n+1$ with, in particular, $\left|\partial_x\left(x^{\{n-k\}}\left(\big(x^{\{k\}}y\big)\,y\right)\right)\right| = n+1$ for every $1\leq k<n$.

Let be $w\in\mathfrak{M}\left(x,y\right)_{[n,2]}$ verifying $w\notin\mathscr{F}$, we look for $P_w = \sum_{r=1}^{n}\alpha_r\big(x^{\{r\}}y\big)\,y + \sum_{s=0}^{n}\beta_s x^{\{s\}}y^2$ such that $f = w-P_w$ verify $\partial_x f = \partial_y f = 0$ and $f\left(1,1\right)=0$. Let be $\partial_x w = \sum_{i=1}^{n+1}\lambda_i t^i$ and $\partial_y w = \sum_{i=1}^{n+1}\mu_i t^i$, we have:

$$\partial_x f = \left(\lambda_{n+1}-\alpha_n\right)t^{n+1} + \sum_{i=2}^{n}\Big(\lambda_i - \Big(\sum_{r=i-1}^{n}\alpha_r + \sum_{s=i}^{n}\beta_s\Big)\Big)t^i + \Big(\lambda_1 - \sum_{s=1}^{n}\beta_s\Big)t,$$

$$\partial_y f = \sum_{i=2}^{n+1}\Big(\mu_i - \Big(\alpha_{i-1}+2\beta_{i-1}\Big)\Big)\,t^i + \Big(\mu_1 - \Big(\sum_{r=1}^{n}\alpha_r + 2\beta_0\Big)\Big)t.$$

And the solution of the linear system $\partial_x f = \partial_y f = 0$ and $\sum_{r=1}^{n}\alpha_r + \sum_{s=0}^{n}\beta_s = 1$ is

$$\alpha_i = \lambda_{i+1} - \sum_{k=i+2}^{n+1}\frac{1}{2^{k-i-1}}\left(\mu_k+\lambda_k\right), \qquad\qquad (1\leq i\leq n)$$

$$\beta_i = \frac{1}{2}\left(\mu_{i+1}-\lambda_{i+1}\right) + \sum_{k=i+2}^{n+1}\frac{1}{2^{k-i}}\left(\mu_k+\lambda_k\right), \qquad\qquad (1\leq i\leq n)$$

$$\beta_0 = 1 - \sum_{k=2}^{n+1}\frac{1}{2^{k-1}}\left(\mu_k+\lambda_k\right).$$

However, according to the result *a)* of corollary 26 we have $\lambda_i,\mu_i\in\mathbb{N}$ for any $1\leq i\leq n$, therefore we have $\alpha_i,\beta_i\in\mathbb{Q}$, which completes the proof. □

We can deduce from this that

Corollary 52. *For any* $n\geq 2$, *the vector space of evanescent train polynomials of degree* $(n,2)$ *has dimension* $W_{[n,2]}-2$.

The following result provides a procedure for quickly constructing evanescent identities from elements taken from $\mathfrak{M}\left(x,y\right)_{[n,2]}$.

Theorem 53. *For all integers $p,q\geq 2$ and $r\geq 0$ we put:*

$$\begin{aligned}
E_{p,q}^{[n]}(x) &= x^p x^q - x^{p+1} - x^{q+1} + x^2;\\
E_{p,q}^{[n,1]}(x,y) &= x^p\left(x^q y\right) - x\left(xy\right) - x^p y - x^{q+1}y + x^2 y + xy;\\
E_{p,\{r\}}^{[n,1]}(x,y) &= x^p\big(x^{\{r\}}y\big) - x^p y - x^{\{r+1\}}y + xy;\\
E_{\{r\},p}^{[n,1]}(x,y) &= x^{\{r\}}\left(x^p y\right) - x^{\{r+1\}}y - x^{p+r}y + x^{r+1}y;\\
E_{p,\{r\}}^{[n,2]}(x,y) &= x^p\big(\big(x^{\{r\}}y\big)y\big) - 2\big(x^{\{p-1\}}y\big)y - \big(x^{\{r+1\}}y\big)y + x^{\{p-1\}}y^2\\
&\quad + \left(xy\right)y - xy^2 + y^2;\\
E_{\{r\},\{s\}}^{[n,2]}(x,y) &= x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big) - \big(x^{\{r+s\}}y\big)y + \big(x^{\{r\}}y\big)y - x^{\{r\}}y^2;\\
F_{p,\{r\}}^{[n,2]}(x,y) &= x^p\big(x^{\{r\}}y^2\big) - 2\big(x^{\{p-1\}}y\big)y + x^{\{p-1\}}y^2 - x^{\{r+1\}}y^2 + y^2;\\
G_{p,q}^{[n,2]}(x,y) &= \left(x^p y\right)\left(x^q y\right) - 2\big(x^{\{p\}}y\big)y - 2\big(x^{\{q\}}y\big)y + x^{\{p\}}y^2 + x^{\{q\}}y^2\\
&\quad + 2\left(xy\right)y - 2xy^2 + y^2;\\
G_{p,\{r\}}^{[n,2]}(x,y) &= \left(x^p y\right)\left(x^{\{r\}}y\right) - 2\big(x^{\{p\}}y\big)y - \big(x^{\{r\}}y\big)y + x^{\{p\}}y^2\\
&\quad + \left(xy\right)y - xy^2 + y^2;\\
G_{\{r\},\{s\}}^{[n,2]}(x,y) &= \big(x^{\{r\}}y\big)\big(x^{\{s\}}y\big) - \big(x^{\{r\}}y\big)y - \big(x^{\{s\}}y\big)y + y^2.
\end{aligned}$$

Let $\mathscr{G}$ be the ideal generated by the family of polynomials

$$\left(E_{p,q}^{[n]}, E_{p,q}^{[n,1]}, E_{p,\{r\}}^{[n,1]}, E_{\{r\},p}^{[n,1]}, E_{p,\{r\}}^{[n,2]}, E_{p,\{r\}}^{[n,2]}, E_{\{r\},\{s\}}^{[n,2]}, F_{p,\{r\}}^{[n,2]}, G_{p,q}^{[n,2]}, G_{p,\{r\}}^{[n,2]}, G_{\{r\},\{s\}}^{[n,2]}\right)_{\substack{p,q\geq 2\\ r,s\geq 0}}$$

and $\pi : K\left(x\right)\to {}^{K(x)}\!/_{\mathscr{G}}$ the canonical surjection.

So for any integer $n\geq 2$ and any monomial $w\in\mathfrak{M}\left(x,y\right)_{[n,2]}$ such that $w\neq\big(x^{\{n\}}y\big)\,y$ and $w\neq x^{\{n\}}y^2$ we have $\pi\left(w\right)=P_w$ and for any $f\in\bigoplus_{n\geq 2}K\left(x,y\right)_{[n,2]}$, the polynomial $f-\pi\left(f\right)$ is an evanescent train identity of degree $(n,2)$.

Proof. We have shown in theorems 40 and 47 that polynomials $E_{p,q}^{[n]}$, $E_{p,q}^{[n,1]}$, $E_{p,\{r\}}^{[n,1]}$, $E_{\{r\},p}^{[n,1]}$ and $E_{p,\{r\}}^{[n,2]}$ are evanescent, let us show that this is also the case for the other polynomials of the statement.

For $p\geq 2$ and $r\geq 0$ we have $\partial_x\left(x^p\big(\big(x^{\{r\}}y\big)y\big)\right)=t\left(\partial_x\left(x^p\right)+\partial_x\big(x^{\{r\}}y\big)\right)$ and $\partial_y\left(x^p\big(\big(x^{\{r\}}y\big)y\big)\right)=t\partial_y\left(\big(x^{\{r\}}y\big)y\right)$, with relations (4.2), (4.3) and (4.7) we get:

$$\partial_x\left(x^p\big(\big(x^{\{r\}}y\big)y\big)\right)=2t^p+\sum_{i=2}^{p-1}t^i+\sum_{i=3}^{r+2}t^i,\qquad \partial_y\left(x^p\big(\big(x^{\{r\}}y\big)y\big)\right)=\sum_{i=2}^{r+2}t^i.$$

For any $r,s\geq 0$ we have $\partial_x\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=t\left(1+\partial_x\left(x^{\{r-1\}}\big(\big(x^{\{s\}}y\big)y\big)\right)\right)$ we deduce that $\partial_x\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=\sum_{i=1}^{r}t^i+t^r\partial_x\left(\big(x^{\{s\}}y\big)y\right)$ and $\partial_y\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=t\partial_y\left(x^{\{r-1\}}\big(\big(x^{\{s\}}y\big)y\big)\right)$ hence $\partial_y\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=t^r\partial_y\left(\big(x^{\{s\}}y\big)y\right)$, using relations (4.7) we have:

$$\partial_x\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=\sum_{i=1}^{r+s+1}t^i-t^{r+1},\qquad \partial_y\left(x^{\{r\}}\big(\big(x^{\{s\}}y\big)y\big)\right)=t^{r+s+1}+t^{r+1}.$$

For any $p \geq 2$ and $r \geq 0$ we have $\partial_x\left(x^p\left(x^{\{r\}}y^2\right)\right) = t\left(\partial_x\left(x^p\right) + \partial_x\left(x^{\{r\}}y^2\right)\right)$ and $\partial_y\left(x^p\left(x^{\{r\}}y^2\right)\right) = t\partial_y\left(x^{\{r\}}y^2\right)$ with relations (4.8) we deduce:

$$\partial_x\left(x^p\left(\left(x^{\{r\}}y\right)y\right)\right) = \sum_{i=1}^{r+s+1} t^i - t^{r+1}, \qquad \partial_y\left(x^p\left(\left(x^{\{r\}}y\right)y\right)\right) = t^{r+s+1} + t^{r+1}.$$

For $p, q \geq 2$ we have $\partial_x\left(\left(x^py\right)\left(x^qy\right)\right) = t\left(\partial_x\left(x^py\right) + \partial_x\left(x^qy\right)\right) = t^2\left(\partial_x\left(x^p\right) + \partial_x\left(x^q\right)\right)$ and $\partial_y\left(\left(x^py\right)\left(x^qy\right)\right) = t\left(\partial_y\left(x^py\right) + \partial_y\left(x^qy\right)\right) = 2t^2\partial_y\left(y\right)$ from this and relation (4.1) we deduce

$$\partial_x\left(\left(x^py\right)\left(x^qy\right)\right) = 2t^{p+1} + 2t^{q+1} + \sum_{i=3}^{p} t^i + \sum_{i=3}^{q} t^i, \qquad \partial_y\left(\left(x^py\right)\left(x^qy\right)\right) = 2t^2.$$

Let be $p \geq 2$ and $r \geq 0$, we have $\partial_x\left(\left(x^py\right)\left(x^{\{r\}}y\right)\right) = t\left(\partial_x\left(x^py\right) + \partial_x\left(x^{\{r\}}y\right)\right)$ and $\partial_y\left(\left(x^py\right)\left(x^{\{r\}}y\right)\right) = t\left(\partial_y\left(x^py\right) + \partial_y\left(x^{\{r\}}y\right)\right)$, using relations (4.2) and (4.3) we get

$$\partial_x\left(\left(x^py\right)\left(x^{\{r\}}y\right)\right) = 2t^{p+1} + \sum_{i=3}^{p} t^i + \sum_{i=2}^{r+1} t^i, \qquad \partial_y\left(\left(x^py\right)\left(x^{\{r\}}y\right)\right) = t^{r+1} + t^2.$$

For any integers $r, s \geq 0$, from $\partial_x\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}y\right)\right) = t\left(\partial_x\left(x^{\{r\}}y\right) + \partial_x\left(x^{\{r\}}y\right)\right)$ and $\partial_y\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}y\right)\right) = t\left(\partial_y\left(x^{\{r\}}y\right) + \partial_y\left(x^{\{r\}}y\right)\right)$ and from relation (4.5) we deduce

$$\partial_x\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}y\right)\right) = \sum_{i=2}^{r+1} t^i + \sum_{i=2}^{s+1} t^i, \qquad \partial_y\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}y\right)\right) = t^{r+1} + t^{s+1}.$$

Using these results and the relations of lemma 50 we establish that polynomials $E^{[n,2]}_{p,\{r\}}$, $E^{[n,2]}_{\{r\},\{s\}}$, $F^{[n,2]}_{p,\{r\}}$, $G^{[n,2]}_{p,q}$, $G^{[n,2]}_{p,\{r\}}$ and $G^{[n,2]}_{\{r\},\{s\}}$ are evanescent.

Let be $w \in \mathfrak{M}\left(x,y\right)_{[n,2]}$ such that $w \neq \left(x^{\{n\}}y\right)y$ and $w \neq x^{\{n\}}y^2$. Let us show by induction on the degree n in x of w, that polynomial $w - \pi\left(w\right)$ is evanescent. This is true for $n = 2$, as can be seen on generators of evanescent train identity of degree $(2,2)$ given below. Suppose the result true for any $v \in \mathfrak{M}\left(x,y\right)_{[p,2]}$ and any $2 \leq p < n$. There are $u, v \in \mathfrak{M}\left(x,y\right)$ such that $w = uv$ with $u \in \mathfrak{M}\left(x,y\right)_{[n-k,1]}$, $v \in \mathfrak{M}\left(x,y\right)_{[k,1]}$ or $u \in \mathfrak{M}\left(x,y\right)_{[n-k]}$ and $v \in \mathfrak{M}\left(x,y\right)_{[k,2]}$ with $1 < k < n$. We have $\partial_x\left(w - \pi\left(w\right)\right) = \partial_x\left(uv - \pi\left(u\right)\pi\left(v\right)\right) = t\left(\partial_x\left(u - \pi\left(u\right)\right) + \partial_x\left(v - \pi\left(v\right)\right)\right)$, in the same way we have $\partial_y\left(w - \pi\left(w\right)\right) = t\partial_y\left(u - \pi\left(u\right)\right) + t\partial_y\left(v - \pi\left(v\right)\right)$.

Therefore, if $u \in \mathfrak{M}\left(x,y\right)_{[n-k,1]}$, $v \in \mathfrak{M}\left(x,y\right)_{[k,1]}$, it results from theorem 47 that the polynomial $w - \pi\left(w\right)$ is evanescent.

In the case of $u \in \mathfrak{M}\left(x,y\right)_{[n-k]}$ and $v \in \mathfrak{M}\left(x,y\right)_{[k,2]}$, with theorem 40 we have that $u - \pi\left(u\right)$ is evanescent and by induction hypothesis it is the same of the polynomial $v - \pi\left(v\right)$.

It is clear that for all $w \in \mathfrak{M}\left(x,y\right)_{[n,2]}$ we have $\pi\left(w\right) \in \mathbb{Z}\left\langle\mathscr{F}\right\rangle$ then by uniqueness of polynomial P_w we have $\pi\left(w\right) = P_w$. □

Applying this theorem we obtain the generators of evanescent train identities:

– of degree $(2,2)$

$x^2y^2 - 2\left(xy\right)y + y^2$;

$(xy)^2 - 2(xy)y + y^2;$
$x((xy)y) - (x(xy))y + (xy)y - xy^2;$
$(x^2y)y - 2(x(xy))y + (xy)y + x(xy^2) - xy^2.$

– of degree $(3,2)$
$x^2((xy)y) - (x(xy))y - (xy)y + y^2;$
$x(xy)^2 - 2(x(xy))y + 2(xy)y - xy^2;$
$x(x^2y^2) - 2(x(xy))y + 2(xy)y - xy^2;$
$(x(xy))(xy) - (x(xy))y - (xy)y + y^2;$
$x^3y^2 - 2(x(xy))y + x(xy^2) - xy^2 + y^2;$
$x^2(xy^2) - 2(xy)y - x(xy^2) + xy^2 + y^2;$
$x((x(xy))y) - x(x(xy))y + (xy)y - xy^2;$
$(xy)(x^2y) - 2(x(xy))y + x(xy^2) - xy^2 + y^2;$
$(x^2(xy))y - 3(x(xy))y + 2(xy)y + x(xy^2) - xy^2;$
$x(x((xy)y)) - (x(x(xy)))y + (x(xy))y - x(xy^2);$
$(x^3y)y - 2(x(x(xy)))y + (xy)y + x(x(xy^2)) - xy^2;$
$(x(x^2y))y - 2(x(x(xy)))y + (x(xy))y + x(x(xy^2)) - x(xy^2);$
$x((x^2y)y) - 2x(x(xy))y + (x(xy))y + (xy)y + x(x(xy^2)) - x(xy^2) - xy^2.$

– of degree $(4,2)$
$x^2(xy)^2 - 2(x(xy))y + y^2;$
$x^3(xy^2) - 2(x(xy))y + y^2;$
$(x(xy))^2 - 2(x(xy))y + y^2;$
$x^2(x^2y^2) - 2(x(xy))y + y^2;$
$x^2((x(xy)y) - (x(x(xy)))y - (xy)y + y^2;$
$x^4y^2 - 2(x(x(xy)))y + x(x(xy^2)) - xy^2 + y^2;$
$x^2(x(xy^2)) - 2(xy)y - x(x(xy^2)) + xy^2 + y^2;$
$(x^3y)(xy) - 2(x(x(xy)))y + x(x(xy^2)) - xy^2 + y^2;$
$x^3((xy)y) - 3(x(xy))y + (xy)y + x(xy^2) - xy^2 + y^2;$
$(x^2y)^2 - 4(x(xy))y + 2(xy)y + 2x(xy^2) - 2xy^2 + y^2;$
$(x^2x^2)y^2 - 4(x(xy))y + 2x(xy) + 2x(xy^2) - 2xy^2 + y^2;$
$(x^2y)(x(xy)) - 3(x(xy))y + (xy)y + x(xy^2) - xy^2 + y^2;$
$(x^2(xy))(xy) - 3(x(xy))y + (xy)y + x(xy^2) - xy^2 + y^2;$

$\left(x^4y\right)y-2\left(x\left(x\left(x\left(xy\right)\right)\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)+\left(xy\right)y-xy^2;$
$\left(x\left(x\left(xy\right)\right)\right)\left(xy\right)-\left(x\left(x\left(xy\right)\right)\right)y-\left(xy\right)y+y^2;$
$x\left(y\left(x\left(x\left(xy\right)\right)\right)\right)-\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(xy\right)y-xy^2;$
$x\left(x\left(xy\right)^2\right)-2\left(x\left(x\left(xy\right)\right)\right)y+2\left(x\left(xy\right)\right)y-x\left(xy^2\right);$
$x\left(x\left(x^2y^2\right)\right)-2\left(x\left(x\left(xy\right)\right)\right)y+2\left(x\left(xy\right)\right)y-x\left(xy^2\right);$
$x\left(x\left(y\left(x\left(xy\right)\right)\right)\right)-\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(x\left(xy\right)\right)y-x\left(xy^2\right);$
$x\left(x^2\left(\left(xy\right)y\right)\right)-\left(x\left(x\left(xy\right)\right)\right)y-\left(x\left(xy\right)\right)y+2\left(xy\right)y-xy^2;$
$x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)-\left(x\left(x\left(xy\right)\right)\right)y-\left(x\left(xy\right)\right)y+2\left(xy\right)y-xy^2;$
$x\left(x\left(x\left(\left(xy\right)y\right)\right)\right)-\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(x\left(x\left(xy\right)\right)\right)y-x\left(x\left(xy^2\right)\right);$
$x\left(x^3y^2\right)-2\left(x\left(x\left(xy\right)\right)\right)y+2\left(xy\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right)-xy^2;$
$x\left(x^2\left(xy^2\right)\right)-2\left(x\left(xy\right)\right)y+2\left(xy\right)y-x\left(x\left(xy^2\right)\right)+x\left(xy^2\right)-xy^2;$
$\left(x\left(x^2\left(xy\right)\right)\right)y-3\left(x\left(x\left(xy\right)\right)\right)y+2\left(x\left(xy\right)\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right);$
$\left(x^3\left(xy\right)\right)y-2\left(x\left(x\left(xy\right)\right)\right)y-\left(x\left(xy\right)\right)y+2\left(xy\right)y+x\left(x\left(xy^2\right)\right)-xy^2;$
$\left(x^2\left(x\left(xy\right)\right)\right)y-\left(x\left(x\left(xy\right)\right)\right)y-2\left(x\left(xy\right)\right)y+2\left(xy\right)y+x\left(xy^2\right)-xy^2;$
$\left(\left(x^2x^2\right)y\right)y-2\left(x\left(x\left(xy\right)\right)\right)y-\left(x\left(xy\right)\right)y+2\left(xy\right)y+x\left(x\left(xy^2\right)\right)-xy^2;$
$\left(x\left(x^3y\right)\right)y-2\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)+\left(x\left(xy\right)\right)y-x\left(xy^2\right);$
$x\left(\left(xy\right)\left(x^2y\right)\right)-2\left(x\left(x\left(xy\right)\right)\right)y+2\left(xy\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right)-xy^2;$
$\left(\left(x^2x^2\right)y\right)y-2\left(x\left(x\left(xy\right)\right)\right)y-\left(x\left(xy\right)\right)y+2\left(xy\right)y+x\left(x\left(xy^2\right)\right)-xy^2;$
$x^2\left(x\left(\left(xy\right)y\right)\right)-\left(x\left(x\left(xy\right)\right)\right)y+\left(x\left(xy\right)\right)y-2\left(xy\right)y-x\left(xy^2\right)+xy^2+y^2;$
$x^2\left(\left(x^2y\right)y\right)-2\left(x\left(x\left(xy\right)\right)\right)y+\left(x\left(xy\right)\right)y-\left(xy\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right)+y^2;$
$\left(xy\right)\left(x\left(x^2y\right)\right)-2\left(x\left(x\left(xy\right)\right)\right)y+\left(x\left(xy\right)\right)y-\left(xy\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right)+y^2;$
$\left(x\left(x\left(x^2y\right)\right)\right)y-2\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(x\left(x\left(xy\right)\right)\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)-x\left(x\left(xy^2\right)\right);$
$x\left(\left(x^2\left(xy\right)\right)y\right)-3\left(x\left(x\left(xy\right)\right)\right)y+2\left(x\left(xy\right)\right)y+\left(xy\right)y+x\left(x\left(xy^2\right)\right)-x\left(xy^2\right)-xy^2;$
$x\left(\left(x^3y\right)y\right)-2\left(x\left(x\left(x\left(xy\right)\right)\right)y+\left(x\left(xy\right)\right)y+\left(xy\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)-x\left(xy^2\right)-xy^2;$
$x\left(y\left(x\left(x^2y\right)\right)\right)-2\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(x\left(x\left(xy\right)\right)\right)y+\left(xy\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)$
$\quad-x\left(x\left(xy^2\right)\right)-xy^2;$
$x\left(x\left(\left(x^2y\right)y\right)\right)-2\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y+\left(x\left(x\left(xy\right)\right)\right)y+\left(x\left(xy\right)\right)y+x\left(x\left(x\left(xy^2\right)\right)\right)$
$\quad-x\left(x\left(xy^2\right)\right)-x\left(xy^2\right).$

4.4.2. *Evanescent homogeneous identities of type* $[n,2]$.

Proposition 54. *For any* $n\geq 2$*, the vector space of evanescent homogeneous identities of type* $[n,2]$ *is generated by at least* $W_{[n,2]}-2n$ *evanescent homogeneous identities.*

Proof. Let be $n\geq 2$, we note $N=W_{[n,2]}$. Let be $\mathfrak{M}\left(x,y\right)_{[n,2]}=\left\{w_k;1\leq k\leq N\right\}$ and $f=\sum_{i=1}^{N}\alpha_i w_i$, we look for $\left(\alpha_i\right)_{1\leq i\leq N}\in K^N$ such that $f\left(1,1\right)=1$ and $\partial_x f=$

$\partial_y f = 0$ where $\partial_x f = \sum_{i=1}^{N} \alpha_i \partial_x w_i$ and $\partial_y f = \sum_{i=1}^{N} \alpha_i \partial_y w_i$. For all $1 \leq i \leq N$ we have $\partial_x w_i, \partial_y w_i \in K[t]$ with $|\partial_x w_i| \leq n+1$ and $|\partial_y w_i| \leq n+1$, moreover, there is $1 \leq i, j \leq N$ such that $|\partial_x w_i| = n+1$ and $|\partial_y w_j| = n+1$, therefore the solutions $(\alpha_i)_{1 \leq i \leq N}$ of equations $\partial_x f = 0$ and $\partial_y f = 0$ are solutions of two linear systems with $n+1$ equations and N unknowns. From $f(1,1) = 1$ we deduce that $\sum_{i=1}^{N} \alpha_i = 0$, therefore the rank of the linear system $\partial_x f = 0$ is $\leq n$, the same is true for the system $\partial_y f = 0$, thus the rank of linear system $\partial_x f = \partial_y f = 0$ is $\leq 2n$ therefore the space of the solutions is of dimension $\geq W_{[n,2]} - 2n$. □

With the method used in the proof we obtain the following generators of the evanescent homogeneous polynomials:

– of type $[2,2]$

$x^2y^2 - (xy)^2$;

$\left(x^2y\right) y + x\left(xy^2\right) - x\left((xy)\, y\right) - \left(x\,(xy)\right) y$;

– of type $[3,2]$

$x^3y^2 - (xy)\left(x^2y\right)$;

$x\left(x^2y^2\right) - x\,(xy)^2$;

$x^2\left((xy)\, y\right) - (xy)\,(x\,(xy))$;

$x^3y^2 + x^2\left(xy^2\right) - 2\,(xy)\,(x\,(xy))$;

$x^3y^2 - \left(x^3y\right) y + \left(x\left(x^2y\right)\right) y - (xy)\,(x\,(xy))$;

$\left(x^3y\right) y - \left(x^2\,(xy)\right) y + x\left(x^2y^2\right) - x\left(\left(x^2y\right) y\right)$;

$\left(x^3y\right) y + x\left(x\left(xy^2\right)\right) - x\left((x\,(xy))\, y\right) - \left(x\,(x\,(xy))\right) y$;

$\left(x^2\,(xy)\right) y - (x\,(x\,(xy)))\, y - x\left(x^2y^2\right) + x\left(x\left((xy)\, y\right)\right)$;

$x^3y^2 + x\left(x^2y^2\right) - \left(x^2\,(xy)\right) y - (x\,(xy))\,(xy) - x\left(y\,(x\,(xy))\right) + \left(x\,(x\,(xy))\right) y$;

– of type $[4,2]$

$(x\,(xy))^2 - x^2\left(x^2y^2\right)$;

$x^3\left(xy^2\right) - x^2\left(x^2y^2\right)$;

$x^2\,(xy)^2 - x^2\left(x^2y^2\right)$;

$x\left(x^3y^2\right) - x\left((xy)\left(x^2y\right)\right)$;

$\left(x^3\,(xy)\right) y - \left(x^2\left(x^2y\right)\right) y$;

$x\left(x\left(x^2y^2\right)\right) - x\left(x\,(xy)^2\right)$;

$\left(x^2y\right)(x\,(xy)) - (xy)\left(x^2\,(xy)\right)$;

$x^2\left((x\,(xy))\, y\right) - (xy)\,(x\,(x\,(xy)))$;

$x\left(x^2\left((xy)\, y\right)\right) - x\left((xy)\,(x\,(xy))\right)$;

$$x\left(x^3y^2\right)-2x\left((xy)\left(x\left(xy\right)\right)\right)+x\left(x^2\left(xy^2\right)\right);$$
$$x^2\left(x^2y^2\right)-x\left(x^3y^2\right)-x^2\left(x\left(xy^2\right)\right)+x\left(x\left(x^2y^2\right)\right);$$
$$x\left(x^3y^2\right)-x^2\left(x^2y^2\right)-x\left(\left(x^2\left(xy\right)\right)y\right)+x^2\left(\left(x\left(xy\right)\right)y\right);$$
$$\left(x^4y\right)y+x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)-\left(x^2\left(x\left(xy\right)\right)\right)y-x\left(\left(x^3y\right)y\right);$$
$$\left(x\left(x^3y\right)\right)y+x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)-x\left(x^3y^2\right)-\left(x\left(x\left(x^2y\right)\right)\right)y;$$
$$x\left(x^3y^2\right)+x\left(\left(x\left(x^2y\right)\right)y\right)-x\left(\left(x^3y\right)y\right)-x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x\left(x^3y^2\right)+x^2\left(x\left(xy^2\right)\right)-x^2\left(x\left(\left(xy\right)y\right)\right)-x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x^2\left(x^2y^2\right)-x^2\left(\left(x^2y\right)y\right)-x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)+x\left(\left(x^2\left(xy\right)\right)y\right);$$
$$\left(\left(x^2x^2\right)y\right)y-\left(x^3\left(xy\right)\right)y-x\left(y\left(x^2\left(xy\right)\right)\right)+x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x^2\left(x^2y^2\right)-\left(xy\right)\left(x\left(x^2y\right)\right)-x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)+x\left(y\left(x^2\left(xy\right)\right)\right);$$
$$\left(x\left(x^3y\right)\right)y-x\left(\left(x^3y\right)y\right)+x\left(\left(x\left(x\left(xy\right)\right)\right)y\right)-\left(x\left(x\left(x\left(xy\right)\right)\right)\right)y;$$
$$x^4y^2+2x\left(x^3y^2\right)-2x^2\left(x^2y^2\right)-2x\left(\left(x^2\left(xy\right)\right)y\right)+x^2\left(x\left(xy^2\right)\right);$$
$$x\left(\left(x^3y\right)y\right)+x\left(x\left(x\left(xy^2\right)\right)\right)-x\left(x\left(y\left(x\left(xy\right)\right)\right)\right)-x\left(y\left(x\left(x\left(xy\right)\right)\right)\right);$$
$$\left(x^3y\right)\left(xy\right)+2x\left(x^3y^2\right)+x^2\left(x\left(xy^2\right)\right)-2x^2\left(x^2y^2\right)-2x\left(\left(x^2\left(xy\right)\right)y\right);$$
$$x\left(x^3y^2\right)+\left(x^2\left(x\left(xy\right)\right)\right)y+x\left(\left(x^2\left(xy\right)\right)y\right)-\left(\left(x^2x^2\right)y\right)y-2x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x\left(x^3y^2\right)+x^2\left(x\left(xy^2\right)\right)+x\left(\left(x^3y\right)y\right)-x^2\left(x^2y^2\right)-x\left(x\left(\left(x^2y\right)y\right)\right)-x\left(\left(x^2\left(xy\right)\right)y\right);$$
$$x^2\left(x^2y^2\right)+\left(x\left(x^2\left(xy\right)\right)\right)y+x\left(\left(x^2\left(xy\right)\right)y\right)-x\left(x^3y^2\right)-x^2\left(x\left(xy^2\right)\right)-\left(\left(x^2x^2\right)y\right)y;$$
$$2x\left(x^3y^2\right)+x^3\left(\left(xy\right)y\right)+x^2\left(x\left(xy^2\right)\right)-2x^2\left(x^2y^2\right)-x\left(\left(x^2\left(xy\right)\right)y\right)-x\left(\left(xy\right)\left(x(xy)\right)\right);$$
$$\left(x^2y\right)^2+4x\left(x^3y^2\right)+2x^2\left(x\left(xy^2\right)\right)-3x^2\left(x^2y^2\right)-2x\left(\left(x^2\left(xy\right)\right)y\right)-2x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x^2\left(x^2y^2\right)+x\left(\left(x^2\left(xy\right)\right)y\right)+x\left(x\left(x\left(\left(xy\right)y\right)\right)\right)-x\left(x^3y^2\right)-x^2\left(x\left(xy^2\right)\right)-x\left(\left(x\left(x\left(xy\right)\right)\right)y\right);$$
$$\left(x^2x^2\right)y^2+4x\left(x^3y^2\right)+2x^2\left(x\left(xy^2\right)\right)-3x^2\left(x^2y^2\right)-2x\left(\left(x^2\left(xy\right)\right)y\right)-2x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$2x\left(x^3y^2\right)+x^2\left(x\left(xy^2\right)\right)+\left(x^2y\right)\left(x\left(xy\right)\right)-2x^2\left(x^2y^2\right)-x\left(\left(x^2\left(xy\right)\right)y\right)-x\left(\left(xy\right)\left(x\left(xy\right)\right)\right);$$
$$x^2\left(x^2y^2\right)+\left(x\left(x^3y\right)\right)y+2x\left(\left(x^2\left(xy\right)\right)y\right)-x\left(x^3y^2\right)-x^2\left(x\left(xy^2\right)\right)-\left(\left(x^2x^2\right)y\right)y$$
$$-x\left(\left(x^3y\right)y\right);$$
$$x^2\left(x^2y^2\right)+x\left(\left(x^2\left(xy\right)\right)y\right)+x\left(\left(xy\right)\left(x\left(xy\right)\right)\right)+x\left(x\left(y\left(x\left(xy\right)\right)\right)\right)-2x\left(x^3y^2\right)$$
$$-x^2\left(x\left(xy^2\right)\right)-x\left(y\left(x\left(x\left(xy\right)\right)\right)\right);$$

4.5. **Evanescent train identities of degree** $(n,1,1)$ **and evanescent homogeneous identities of type** $[n,1,1]$**.**

We can write all $w\in\mathfrak{M}\left(x,y,z\right)_{[n,1,1]}$ in the form $w=w_1w_2$ where $(w_1,w_2)\in\mathfrak{M}\left(x\right)_{[n,-i]}\times\mathfrak{M}\left(x,y,z\right)_{[i,1,1]}$ for $0\leq i\leq n-1$, or $(w_1,w_2)\in\mathfrak{M}\left(x,y\right)_{[n-i,1]}\times\mathfrak{M}\left(x,y,z\right)_{[i,0,1]}$ with $0\leq i\leq n$, as $\sharp\mathfrak{M}\left(x,y,z\right)_{[i,0,1]}=\sharp\mathfrak{M}\left(x,z\right)_{[i,1]}$ we have

$$W_{[n,1,1]}=\sum_{i=0}^{n-1}W_{[n-i]}W_{[i,1,1]}+\sum_{i=0}^{n}W_{[n-i,1]}W_{[i,1]},\quad W_{[0,1,1]}=1.$$

The first values of $W_{[n,1,1]}$ are

n	0	1	2	3	4	5	6	7	8	9	10
$W_{[n,1,1]}$	1	3	9	25	69	186	497	1314	3453	9019	23454

4.5.1. *Evanescent train identities of degree* $(n,1,1)$.

For all $f \in K(x,y,z)$ and all integer $r \geq 1$, we define $x^{\{r\}}f = x\left(x^{\{r-1\}}f\right)$ where $x^{\{0\}}f = f$.

Lemma 55. *For any integer* $r \geq 0$ *we have:*

w	$\partial_x(w)$	$\partial_y(w)$	$\partial_z(w)$
$x^{\{r\}}((xy)z)$	$\sum_{i=1}^{r} t^i + t^{r+2}$,	t^{r+2},	t^{r+1};
$x^{\{r\}}((xz)y)$	$\sum_{i=1}^{r} t^i + t^{r+2}$,	t^{r+1},	t^{r+2};
$x^{\{r\}}(yz)$	$\sum_{i=1}^{r} t^i$,	t^{r+1},	t^{r+1}.

Proof. Indeed, we have $\partial_x\left(x^{\{r\}}((xy)z)\right) = t\left(1+\partial_x\left(x^{\{r-1\}}((xy)z)\right)\right)$, from this we deduce recursively that $\partial_x\left(x^{\{r\}}((xy)z)\right) = \sum_{i=1}^{r} t^i + t^r\partial_x((xy)z)$ with $\partial_x((xy)z) = t^2$. Next, $\partial_y\left(x^{\{r\}}((xy)z)\right) = t\partial_y\left(x^{\{r-1\}}((xy)z)\right)$ and $\partial_z\left(x^{\{r\}}((xy)z)\right) = t\partial_z\left(x^{\{r-1\}}((xy)z)\right)$ from which we deduce that $\partial_y\left(x^{\{r\}}((xy)z)\right) = t^r\partial_y\left((xy)z\right)$ and $\partial_z\left(x^{\{r\}}((xy)z)\right) = t^r\partial_z\left((xy)z\right)$ with $\partial_y\left((xy)z\right) = t^2$ and $\partial_z\left((xy)z\right) = t$. From witch, by exchanging the roles of y and z, we deduce the results concerning monomials $x^{\{r\}}((xz)y)$. From $\partial_x\left(x^{\{r\}}(yz)\right) = t\left(1+\partial_x\left(x^{\{r-1\}}(yz)\right)\right)$ it results $\partial_x\left(x^{\{r\}}(yz)\right) = \sum_{i=1}^{r} t^i$, next with $\partial_y\left(x^{\{r\}}(yz)\right) = t\partial_y\left(x^{\{r-1\}}(yz)\right)$ we get $\partial_y\left(x^{\{r\}}(yz)\right) = t^r\partial_y(yz) = t^{r+1}$, we deduce from this after exchanging y and z that $\partial_z\left(x^{\{r\}}(yz)\right) = t^{r+1}$. □

We consider the set $\mathscr{F} = \left\{x^{\{r\}}((xy)z), x^{\{r\}}((xz)y), x^{\{r\}}(yz); r \geq 0\right\}$ and we note $\mathbb{Q}\langle\mathscr{F}\rangle$ the vector space over $\mathbb{Q}$ generated by $\mathscr{F}$.

Proposition 56. *There is no evanescent train identity of degree* $(1,1,1)$.

For any $n \geq 2$ *and any* $w \in \mathfrak{M}(x,y,z)_{[n,1,1]}$ *such that* $w \notin \mathscr{F}$, *there is an unique polynomial* $P_w \in \mathbb{Q}\langle\mathscr{F}\rangle$ *with* $|P_w|_x \leq n$ *such that the polynomial* $w - P_w$ *is an evanescent train identity of degree* $(1,1,1)$.

Proof. Let be $f = \lambda_1 x(yz) + \lambda_2(xy)z + \lambda_3(xz)y + \mu_1 xy + \mu_2 xz + \mu_3 yz$, we have $\partial_x f = (\lambda_2+\lambda_3)t^2 + (\lambda_1+\mu_1+\mu_2)t$, $\partial_y f = (\lambda_1+\lambda_2)t^2 + (\lambda_3+\mu_1+\mu_3)t$ and $\partial_z f = (\lambda_1+\lambda_3)t^2 + (\lambda_2+\mu_2+\mu_3)t$, it is easy to deduce that $\partial_x f = \partial_y f = \partial_z f = 0$ if and only if we have $f = 0$, therefore there is no evanescent train polynomial of degree $(1,1,1)$.

We take $n \geq 2$, let be $w \in \mathfrak{M}(x,y,z)_{[n,1,1]}$ such that $w \notin \mathscr{F}$, according to result c) of corollary 26, polynomials $\partial_x w$, $\partial_y w$ and $\partial_z w$ are of degrees in x less than $n+1$, let be

$\partial_x w = \sum_{k=1}^{n+1} \alpha_k t^k$, $\partial_y w = \sum_{k=1}^{n+1} \beta_k t^k$ and $\partial_z w = \sum_{k=1}^{n+1} \gamma_k t^k$. We put

$$P_w = \sum_{k=0}^{n-1} \lambda_k x^{\{k\}} ((xy)\, z) + \sum_{k=0}^{n-1} \mu_k x^{\{k\}} ((xz)\, y) + \sum_{k=0}^{n} \nu_k x^{\{k\}} (yz)$$

by applying the relations of lemma 55 we get

$$\begin{aligned}
\partial_x P_w =& \left(\lambda_{n-1} + \mu_{n-1}\right) t^{n+1} + \left(\lambda_{n-2} + \mu_{n-2} + \nu_n\right) t^n \\
&+ \sum_{i=2}^{n-1} \left(\sum_{k=i}^{n-1} \left(\lambda_k + \mu_k + \nu_k\right) + \left(\lambda_{i-2} + \mu_{i-2}\right) + \nu_n \right) t^i + \left(\sum_{k=1}^{n} \left(\lambda_k + \mu_k + \nu_k\right) + \nu_n \right) t, \\
\partial_y P_w =& \left(\lambda_{n-1} + \nu_n\right) t^{n+1} + \sum_{k=2}^{n} \left(\lambda_{k-2} + \mu_{k-1} + \nu_{k-1}\right) t^k + \left(\mu_0 + \nu_0\right) t, \\
\partial_z P_w =& \left(\mu_{n-1} + \nu_n\right) t^{n+1} + \sum_{k=2}^{n} \left(\lambda_{k-1} + \mu_{k-2} + \nu_{k-1}\right) t^k + \left(\lambda_0 + \nu_0\right) t.
\end{aligned}$$

So we have $\partial_x w = \partial_x P_w$ if and only if

$$\begin{cases}
\lambda_{n-1} + \mu_{n-1} & = \alpha_{n+1} \\
\lambda_{n-2} + \mu_{n-2} + \nu_n & = \alpha_n \\
\left(\lambda_{k-2} + \mu_{k-2}\right) + \sum_{i=k}^{n-1} \left(\lambda_i + \mu_i + \nu_i\right) + \nu_n & = \alpha_k, \quad (2 \le k \le n-1) \\
\sum_{i=1}^{n-1} \left(\lambda_i + \mu_i + \nu_i\right) + \nu_n & = \alpha_1
\end{cases}$$

and we have $\partial_y w = \partial_y P_w$, $\partial_z w = \partial_z P_w$ if and only if we have respectively

$$\begin{cases}
\lambda_{n-1} + \nu_n & = \beta_{n+1} \\
\lambda_{k-2} + \mu_{k-1} + \nu_{k-1} & = \beta_k, \quad (2 \le k \le n), \\
\mu_0 + \nu_0 & = \beta_1,
\end{cases}$$

and

$$\begin{cases}
\mu_{n-1} + \nu_n & = \gamma_{n+1} \\
\lambda_{k-1} + \mu_{k-2} + \nu_{k-1} & = \gamma_k, \quad (2 \le k \le n) \\
\lambda_0 + \nu_0 & = \gamma_1.
\end{cases}$$

According to result d) of corollary 26, we can note that in these two systems we have $\sum_{k=0}^{n-1} \left(\lambda_k + \mu_k\right) + \sum_{k=0}^{n} \nu_k = \sum_{k=1}^{n+1} \beta_k = \partial_y P_w\,(1) = 1$, therefore $P_w\,(1,1,1) = 1$.

The solution of these systems of linear equations are:

$$\begin{aligned}
\lambda_k &= \sum_{i=k+2}^{n+1} \frac{1}{2^{i-k-1}} \left(\alpha_i + \beta_i - \left(2^{i-k-1} - 1\right) \gamma_i\right), \quad (0 \le k \le n-1) \\
\mu_k &= \sum_{i=k+2}^{n+1} \frac{1}{2^{i-k-1}} \left(\alpha_i - \left(2^{i-k-1} - 1\right) \beta_i + \gamma_i\right), \quad (0 \le k \le n-1) \\
\nu_k &= \frac{1}{2} \left(-\alpha_{k+1} + \beta_{k+1} + \gamma_{k+1}\right) \\
&\quad + \sum_{i=k+2}^{n+1} \frac{1}{2^{i-k}} \left(-3\alpha_i + \left(2^{i-k} - 3\right) \beta_i + \left(2^{i-k} - 3\right) \gamma_i\right), \quad (1 \le k \le n-1)
\end{aligned}$$

$$\nu_0 = 1 - \sum_{i=2}^{n+1} \frac{1}{2^{i-1}} \left(\alpha_i + \beta_i + \gamma_i\right),$$
$$\nu_n = \frac{1}{2} \left(-\alpha_{n+1} + \beta_{n+1} + \gamma_{n+1}\right).$$

□

We can deduce the following.

Corollary 57. *For $n \geq 2$, the vector space of evanescent train identities of degree $(n,1,1)$ has dimension $W_{[n,1,1]} - 3$.*

As before we can give a procedure for quickly constructing evanescent identities from elements taken in $\mathfrak{M}(x,y,z)_{[n,1,1]}$.

Theorem 58. *For all integers $p, q \geq 2$, $r \geq 0$ and for any $t, t_1, t_2 \in \{y, z\}$, $t_1 \neq t_2$, we put*

$$E_{p,q}^{[n]}(x) = x^p x^q - x^{p+1} - x^{q+1} + x^2;$$
$$E_{p,q}^{[n,1]}(x,t) = x^p \left(x^q t\right) - x\left(xt\right) - x^p t - x^{q+1} t + x^2 t + xt;$$
$$E_{p,\{r\}}^{[n,1]}(x,t) = x^p \left(x^{\{r\}} t\right) - x^p t - x^{\{r+1\}} t + xt;$$
$$E_{\{r\},p}^{[n,1]}(x,t) = x^{\{r\}} \left(x^p t\right) - x^{\{r+1\}} t - x^{p+r} t + x^{r+1} t;$$
$$E_{p,q}^{[n,1,1]}(x,y,z) = \left(x^p y\right)\left(x^q z\right) + x^{\{p\}}\left(yz\right) + x^{\{q\}}\left(yz\right) - \sum_{i=0}^{p-1} x^{\{i\}} \Big(\left(xy\right) z + \left(xz\right) y - 2yz \Big)$$
$$- \sum_{i=1}^{q-1} x^{\{i\}} \Big(\left(xy\right) z + \left(xz\right) y - 2yz \Big) - 4x\left(yz\right) - yz;$$
$$E_{\{r\},\{s\}}^{[n,1,1]}(x,y,z) = \left(x^{\{r\}} y\right)\left(x^{\{s\}} z\right) - \sum_{i=0}^{r-1} x^{\{i\}} \Big(\left(xy\right) z - yz \Big) - \sum_{i=0}^{s-1} x^{\{i\}} \Big(\left(xy\right) z - yz \Big) - yz;$$
$$E_{p}^{[n,1,1]}(x,t_1,t_2) = \left(x^p t_1\right) t_2 - \sum_{i=1}^{p-1} x^{\{i\}} \Big(\left(xt_1\right) t_2 + \left(xt_2\right) t_1 - 2t_1 t_2 \Big) + x^{\{p\}}\left(t_1 t_2\right)$$
$$- \left(xt_1\right) t_2 - x\left(t_1 t_2\right);$$
$$E_{p,\{r\}}^{[n,1,1]}(x,t_1,t_2) = \left(x^p t_1\right) \left(x^{\{r+1\}} t_2 \right) - \sum_{i=0}^{p-1} x^{\{i\}} \Big(\left(xt_1\right) t_2 + \left(xt_2\right) t_1 - 2t_1 t_2 \Big)$$
$$- \sum_{i=1}^{r} x^{\{i\}} \Big(\left(xt_2\right) t_1 - t_1 t_2 \Big) + x^{\{p\}}\left(t_1 t_2\right) - x\left(t_1 t_2\right) - t_1 t_2;$$
$$F_{p,\{r\}}^{[n,1,1]}(x,y,z) = x^p \left(x^{\{r\}}\left(yz\right) \right) - \sum_{i=0}^{p-2} x^{\{i\}} \Big(\left(xy\right) z + \left(xz\right) y - 2yz \Big)$$
$$+ x^{\{p-1\}}\left(yz\right) - x^{\{r+1\}}\left(yz\right) - yz;$$

$$G_{p,\{r\}}^{[n,1,1]}(x,t_1,t_2)=x^p\left(x^{\{r\}}\left((xt_1)\,t_2\right)\right)-\sum_{i=0}^{p-2}x^{\{i\}}\Big((xt_1)\,t_2+(xt_2)\,t_1-2t_1t_2\Big)$$
$$-x^{\{r+1\}}\left((xt_1)\,t_2\right)+x^{\{p-1\}}\left(t_1t_2\right)-t_1t_2;$$

Let $\mathscr{H}$ be the ideal generated by the family of polynomials

$$\left(E_{p,q}^{[n]},E_{p,q}^{[n,1]},E_{p,\{r\}}^{[n,1]},E_{\{r\},p}^{[n,1]},E_{p,q}^{[n,1,1]},E_{\{r\},\{s\}}^{[n,1,1]},E_{p}^{[n,1,1]},E_{p,\{r\}}^{[n,1,1]},F_{p,\{r\}}^{[n,1,1]},G_{p,\{r\}}^{[n,1,1]}\right)_{\substack{p,q\geq 2\\ r,s\geq 0}}$$

and $\pi:K(x)\to {}^{K(x)}\!/_{\mathscr{H}}$ the canonical surjection.

So for any $n\geq 2$ and any monomial $w\in\mathfrak{M}_{[n,1,1]}(x,y,z)$ such that $w\notin\mathscr{F}$ we have $\pi(w)=P_w$ and for any $f\in\bigoplus_{n\geq 2}K(x,y,z)_{[n,1,1]}$, the polynomial $f-\pi(f)$ is an evanescent train identity of degree $(n,1,1)$.

Proof. We have already shown to the theorems 40 and 47 that polynomials $E_{p,q}^{[n]}$, $E_{p,q}^{[n,1]}$, $E_{p,\{r\}}^{[n,1]}$ and $E_{\{r\},p}^{[n,1]}$ are evanescent, let us show it for the other polynomials of the statement.

For any $p,q\geq 2$ and any $r,s\geq 0$, we have $\partial_x\left(\left(x^py\right)\left(x^qz\right)\right)=t\left(\partial_x\left(x^py\right)+\partial_x\left(x^qz\right)\right)$, then $\partial_y\left(\left(x^py\right)\left(x^qz\right)\right)=t\partial_y\left(x^py\right)=t^2$, likewise $\partial_z\left(\left(x^py\right)\left(x^qz\right)\right)=t^2$ and with relation (4.2) we get:

$$\partial_x\left(\left(x^py\right)\left(x^qz\right)\right)=2\sum_{i=3}^{p}t^i+2t^{p+1}+2t^{q+1},\quad \partial_y\left(\left(x^py\right)\left(x^qz\right)\right)=t^2,\quad \partial_z\left(\left(x^py\right)\left(x^qz\right)\right)=t^2.$$

We have $\partial_x\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}z\right)\right)=t\left(\partial_x\left(x^{\{r\}}y\right)+\partial_x\left(x^{\{s\}}z\right)\right)$, next $\partial_y\left(\left(x^{\{s\}}y\right)\left(x^{\{s\}}z\right)\right)=t\partial_y\left(x^{\{r\}}y\right)=t^2\partial_y\left(x^{\{r-1\}}y\right)$ and likewise $\partial_z\left(\left(x^{\{s\}}y\right)\left(x^{\{s\}}z\right)\right)=t^2\partial_z\left(x^{\{s-1\}}z\right)$, with relation (4.3) we deduce recursively that

$$\partial_x\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}z\right)\right)=\sum_{i=2}^{r+1}t^i+\sum_{i=2}^{s+1}t^i,$$
$$\partial_y\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}z\right)\right)=t^{r+1},\qquad \partial_z\left(\left(x^{\{r\}}y\right)\left(x^{\{s\}}z\right)\right)=t^{s+1}.$$

From $\partial_x\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=t\left(\partial_x\left(x^py\right)+\partial_x\left(x^{\{r\}}z\right)\right)$, $\partial_y\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=t\partial_y\left(x^py\right)$, $\partial_z\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=t\partial_z\left(x^{\{r\}}z\right)$ and with (4.2) and (4.3) we get

$$\partial_x\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=2t^{p+1}+\sum_{i=3}^{p}t^i+\sum_{i=2}^{r+1}t^i,$$
$$\partial_y\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=t^2,\qquad \partial_z\left(\left(x^py\right)\left(x^{\{r\}}z\right)\right)=t^{r+1}.$$

We have $\partial_x\left(x^p\left(x^{\{r\}}\left(yz\right)\right)\right)=t\left(\partial_x\left(x^p\right)+\partial_x\left(x^{\{r\}}\left(yz\right)\right)\right)=t\partial_x\left(x^p\right)+t^2\partial_x\left(x^{\{r-1\}}\left(yz\right)\right)$, $\partial_y\left(x^p\left(x^{\{r\}}\left(yz\right)\right)\right)=t\partial_y\left(x^{\{r\}}\left(yz\right)\right)=t^2\partial_y\left(x^{\{r-1\}}\left(yz\right)\right)$ and likewise $\partial_z\left(x^p\left(x^{\{r\}}\left(yz\right)\right)\right)=t^2\partial_z\left(x^{\{r-1\}}\left(yz\right)\right)$.

Similarly, we have $\partial_x\left(x^p\left(x^{\{r\}}\left(\left(xy\right)z\right)\right)\right)=t\left(\partial_x\left(x^p\right)+\partial_x\left(x^{\{r\}}\left(\left(xy\right)z\right)\right)\right)=t\partial_x\left(x^p\right)+t^2\partial_x\left(x^{\{r-1\}}\left(\left(xy\right)z\right)\right)$, $\partial_y\left(x^p\left(x^{\{r\}}\left(\left(xy\right)z\right)\right)\right)=t\partial_y\left(x^{\{r\}}\left(\left(xy\right)z\right)\right)=t^2\partial_y\left(x^{\{r-1\}}\left(\left(xy\right)z\right)\right)$ and $\partial_z\left(x^p\left(x^{\{r\}}\left(\left(xy\right)z\right)\right)\right)=t^2\partial_z\left(x^{\{r-1\}}\left(\left(xy\right)z\right)\right)$. Applying relation (4.1) we deduce

recursively that

$$\partial_x\left(x^p\left(x^{\{r\}}(yz)\right)\right)=2t^p+\sum_{i=2}^{p-1}t^i+\sum_{i=2}^{r+1}t^i,$$
$$\partial_y\left(x^p\left(x^{\{r\}}(yz)\right)\right)=t^{r+2},\qquad \partial_z\left(x^p\left(x^{\{r\}}(yz)\right)\right)=t^{r+2}.$$

and

$$\partial_x\left(x^p\left(x^{\{r\}}((xy)z)\right)\right)=2t^p+t^{r+3}+\sum_{i=2}^{p-1}t^i+\sum_{i=2}^{r+1}t^i,$$
$$\partial_y\left(x^p\left(x^{\{r\}}((xy)z)\right)\right)=t^{r+3},\qquad \partial_z\left(x^p\left(x^{\{r\}}((xy)z)\right)\right)=t^{r+2}.$$

With these results and the relations of lemma 55 we show by simple calculations that polynomials $E^{[n,1,1]}_{p,q}$, $E^{[n,1,1]}_{\{r\},\{s\}}$, $E^{[n,1,1]}_{p}$, $E^{[n,1,1]}_{p,\{r\}}$, $F^{[n,1,1]}_{p,\{r\}}$ and $G^{[n,1,1]}_{p,\{r\}}$ are evanescent.

Let be $w\in\mathfrak{M}(x,y,z)_{[n,1,1]}$ such that $w\notin\mathscr{F}$, let us show by induction on the degree n in x of w that the polynomial $w-\pi(w)$ is evanescent. The result is true for $n=2$ as can be verified on the generators of train polynomial of degree $(2,1,1)$ given below. Suppose the result true for all monomials of type $[p,1,1]$ with $2\leq p<n$. There are $u,v\in\mathfrak{M}(x,y,z)$ such that $w=uv$ with $\left|u\right|_x,\left|v\right|_x<n$ so we have either $u\in\mathfrak{M}(x,y)_{[n-p,1]}$, $v\in\mathfrak{M}(x,z)_{[p,1]}$, or else $u\in\mathfrak{M}(x)_{[n-p]}$, $v\in\mathfrak{M}(x,y,z)_{[p,1,1]}$. We have $\partial_x(w-\pi(w))=\partial_x(uv-\pi(u)\pi(v))=t\partial_x(u-\pi(u))+t\partial_x(v-\pi(v))$ and likewise $\partial_y(w-\pi(w))=t\partial_y(u-\pi(u))+t\partial_y(v-\pi(v))$ and $\partial_z(w-\pi(w))=t\partial_z(u-\pi(u))+t\partial_z(v-\pi(v))$.

In the case of $u\in\mathfrak{M}(x,y)_{[n-p,1]}$, $v\in\mathfrak{M}(x,z)_{[p,1]}$, according to theorem 55 polynomials $u-\pi(u)$ and $v-\pi(v)$ are evanescent, so we have $\partial_x(w-\pi(w))=\partial_y(w-\pi(w))=\partial_z(w-\pi(w))=0$.

When $u\in\mathfrak{M}(x)_{[n-p]}$, $v\in\mathfrak{M}(x,y,z)_{[p,1,1]}$, according to theorem 40 and by induction hypothesis, polynomials $u-\pi(u)$ and $v-\pi(v)$ are evanescent.

It is clear that for all $w\in\mathfrak{M}(x,y,z)_{[n,1,1]}$ such that $w\notin\mathscr{F}$ we have $\pi(w)\in\mathbb{Z}[\mathscr{F}]$ and therefore by uniqueness of the polynomial P_w we have $\pi(w)=P_w$. □

Using this theorem we can give the generators of evanescent train identities:

– of degree $(2,1,1)$

$x^2(yz)-(xy)z-(xz)y+yz,$
$(xy)(xz)-(xy)z-(xz)y+yz,$
$(x(xy))z-x((xy)z)-(xy)z+x(yz),$
$(x(xz))y-x((xz)y)-(xz)y+x(yz),$
$\left(x^2y\right)z+x(x(yz))-x((xy)z)-x((xz)y)-(xy)z+x(yz),$
$\left(x^2z\right)y+x(x(yz))-x((xy)z)-x((xz)y)-(xz)y+x(yz).$

– of degree $(3,1,1)$

$x\left(x^2(yz)\right)-x((xy)z)-x((xz)y)+x(yz),$

$$x\left(\left(xy\right)\left(xz\right)\right)-x\left(\left(xy\right)z\right)-x\left(\left(xz\right)y\right)+x\left(yz\right),$$
$$x\left(\left(x\left(xy\right)\right)z\right)-x\left(x\left(\left(xy\right)z\right)\right)-x\left(\left(xy\right)z\right)+x\left(x\left(yz\right)\right),$$
$$x\left(\left(x\left(xz\right)\right)y\right)-x\left(x\left(\left(xz\right)y\right)\right)-x\left(\left(xz\right)y\right)+x\left(x\left(yz\right)\right),$$
$$x^{2}\left(x\left(yz\right)\right)-x\left(x\left(yz\right)\right)-\left(xz\right)y-\left(xy\right)z+x\left(yz\right)+yz,$$
$$x^{2}\left(\left(xy\right)z\right)-x\left(\left(xy\right)z\right)-\left(xz\right)y-\left(xy\right)z+x\left(yz\right)+yz,$$
$$x^{2}\left(y\left(xz\right)\right)-x\left(y\left(xz\right)\right)-\left(xz\right)y-\left(xy\right)z+x\left(yz\right)+yz,$$
$$\left(x\left(xy\right)\right)\left(xz\right)-x\left(\left(xy\right)z\right)+x\left(yz\right)-\left(xy\right)z-y\left(xz\right)+yz,$$
$$\left(x\left(xz\right)\right)\left(xy\right)-x\left(\left(xz\right)y\right)+x\left(yz\right)-\left(xy\right)z-y\left(xz\right)+yz,$$
$$\left(x^{2}\left(xy\right)\right)z+x\left(x\left(yz\right)\right)-2x\left(\left(xy\right)z\right)-x\left(y\left(xz\right)\right)+2x\left(yz\right)-\left(xy\right)z,$$
$$\left(x^{2}\left(xz\right)\right)y+x\left(x\left(yz\right)\right)-2x\left(\left(xz\right)y\right)-x\left(\left(xy\right)z\right)+2x\left(yz\right)-\left(xz\right)y,$$
$$\left(x\left(x\left(xy\right)\right)\right)z-x\left(x\left(\left(xy\right)z\right)\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)-\left(xy\right)z+x\left(yz\right),$$
$$\left(x\left(x\left(xz\right)\right)\right)y-x\left(x\left(y\left(xz\right)\right)\right)+x\left(x\left(yz\right)\right)-x\left(y\left(xz\right)\right)-y\left(xz\right)+x\left(yz\right),$$
$$x^{3}\left(yz\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)-x\left(y\left(xz\right)\right)+x\left(yz\right)-y\left(xz\right)-z\left(xy\right)+yz,$$
$$\left(x^{2}y\right)\left(xz\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)-x\left(y\left(xz\right)\right)+x\left(yz\right)-y\left(xz\right)-\left(xy\right)z+yz,$$
$$\left(x^{2}z\right)\left(xy\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)-x\left(y\left(xz\right)\right)+x\left(yz\right)-y\left(xz\right)-\left(xy\right)z+yz,$$
$$x\left(\left(x^{2}y\right)z\right)+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right),$$
$$x\left(\left(x^{2}z\right)y\right)+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)+x\left(x\left(yz\right)\right)-x\left(y\left(xz\right)\right),$$
$$\left(x\left(x^{2}y\right)\right)z+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)+x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)$$
$$-\left(xy\right)z+x\left(yz\right),$$
$$\left(x\left(x^{2}z\right)\right)y+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)+x\left(x\left(yz\right)\right)-x\left(y\left(xz\right)\right)$$
$$-y\left(xz\right)+x\left(yz\right),$$
$$\left(x^{3}y\right)z+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)+2x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)$$
$$-x\left(y\left(xz\right)\right)-\left(xy\right)z+x\left(yz\right),$$
$$\left(x^{3}z\right)y+x\left(x\left(x\left(yz\right)\right)\right)-x\left(x\left(y\left(xz\right)\right)\right)-x\left(x\left(\left(xy\right)z\right)\right)+2x\left(x\left(yz\right)\right)-x\left(\left(xy\right)z\right)$$
$$-x\left(y\left(xz\right)\right)-\left(xz\right)y+x\left(yz\right).$$

4.5.2. *Evanescent homogeneous identities of type* $[n,1,1]$.

Proposition 59. *For any* $n\geq 2$*, the vector space of evanescent homogeneous identities of type* $[n,1,1]$ *is generated by at least* $W_{[n,1,1]}-3n$ *evanescent homogeneous identities of type* $[n,1,1]$ *.*

Proof. In what follows we put $N=W_{[n,1,1]}$ and $\mathfrak{M}\left(x,y,z\right)_{[n,1,1]}=\{w_k;1\leq k\leq N\}$. Let be $f=\sum_{k=1}^{N}\alpha_k w_k$, we look for $(\alpha_k)_{1\leq k\leq N}$ such that $f\left(1,1,1\right)=0$, $\partial_x f=\partial_y f=\partial_z f=0$. Because for all $w\in\mathfrak{M}_{[n,1,1]}$ we have $\left|\partial_x w\right|,\left|\partial_y w\right|,\left|\partial_z w\right|\leq n+1$ and according to lemma 55 there is a monomial $w\in\mathfrak{M}\left(x,y,z\right)_{[n,1,1]}$ such that $\partial_x w$, $\partial_y w$ or $\partial_z w$ is of degree $n+1$, we deduce that the polynomials $\partial_x f$, $\partial_y f$ and $\partial_z f$ are of degree $n+1$. Therefore relations $\partial_x f=0$, $\partial_y f=0$ and $\partial_z f=0$ are equivalent to three linear systems of $n+1$ equations of unknowns $(\alpha_k)_{1\leq k\leq N}$, condition $f\left(1,1,1\right)=0$ implies that each

of these systems is of rank $\leq n$, it follows that the rank of system of linear equations $\partial_x f = \partial_y f = \partial_z f = 0$ is $\leq 3n$ and therefore the space of solutions is of dimension $\geq W_{[n,1,1]} - 3n$. □

By using the method followed in the above proof, we explain the generators of evanescent homogeneous identities:

– of type $[2,1,1]$

$x^2(yz) - (xy)(xz)$,

$\left(x^2y\right)z - (x(xy))z - x(y(xz)) + x(x(yz))$,

$\left(x^2z\right)y - (x(xz))y - x((xy)z) + x(x(yz))$.

– of type $[3,1,1]$

$x^3(yz) - \left(x^2y\right)(xz)$,

$x^3(yz) - (xy)\left(x^2z\right)$,

$x\left(x^2(yz)\right) - x((xy)(xz))$,

$x^2((xy)z) - (x(xy))(xz)$,

$x^3(yz) - \left(x^3y\right)z + \left(x\left(x^2y\right)\right)z - x^2((xy)z)$,

$\left(x^3y\right)z + x((xy)(xz)) - x\left(\left(x^2z\right)y\right) - \left(x^2(xy)\right)z$,

$\left(x^3y\right)z - x\left(\left(x^2z\right)y\right) - (x(x(xy)))z + x(x(z(xy)))$,

$\left(x^3y\right)z - \left(x\left(x^2y\right)\right)z - x((x(xz))y) + x(x(y(xz)))$,

$\left(x^3y\right)z - (x(x(xy)))z - x((x(xz))y) + x(x(x(yz)))$,

$x\left(\left(x^2y\right)z\right) - \left(x\left(x^2y\right)\right)z - x((x(xy))z) + (x(x(xy)))z$,

$\left(x^3y\right)z - \left(x^3z\right)y + x((x(xy))z) - (x(x(xy)))z + \left(x\left(x^2z\right)\right)y - x\left(yx^2zy\right)$,

$x^3(yz) - \left(x^3y\right)z + (x(x(xy)))z - x((x(xy))z) + x\left(\left(x^2z\right)y\right) - x^2((xz)y)$,

$x^3(yz) - \left(x^3y\right)z + x\left(\left(x^2z\right)y\right) - x((x(xy))z) + (x(x(xy)))z - (xy)(x(xz))$,

$\left(x^3y\right)z - \left(x^3z\right)y + x((x(xy))z) - (x(x(xy)))z + (x(x(xz)))y - x((x(xz))y)$,

$\left(x^3z\right)y - \left(x^2(xz)\right)y + x((xy)(xz)) - x((x(xy))z) + (x(x(xy)))z - \left(x\left(x^2y\right)\right)z$,

$x^3(yz) - 2\left(x^3y\right)z + (x(x(xy)))z - x((x(xy))z) + x\left(\left(x^2z\right)y\right) + \left(x\left(x^2y\right)\right)z - x^2(x(yz))$.

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