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A Harnack type inequality for an elliptic equation with singular source.

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Abstract

We derive a Harnack type inequality for an equation with logarithmic and singular weight having interior singularity.

Keywords: singular weight, logarithmic singularity, interior singularity, ideal point, a priori estimate, maximum principle, sup, inf, sup = f(inf).

MSC: 35J60, 35B44, 35B45, 35B50

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open bounded domain Ω of \mathbb{R}^2 .

We consider the following equation:

$$(P) \begin{cases} -\Delta u = -\log \frac{|x|}{2d} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

Here:

$$0 < a \leq V \leq b < +\infty, u \in L_{loc}^\infty(\Omega),$$

and,

$$d = \text{diam}(\Omega), 0 \in \Omega$$

This equation is defined in the sense of the distributions. Equations of the previous type were studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1–20], where one can find some existence and compactness results.

Among other results, we can see in [12] the following important Theorem

Theorem A (Brezis-Merle [12]). *If (u_i) is a sequence of solutions of problem (P) with (V_i) satisfying $0 < a \leq V_i \leq b < +\infty$ and without the term $-\log \frac{|x|}{2d}$, then, for any compact subset K of Ω , it holds:*

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$$\sup_K u_i \leq c,$$

with c depending on a, b, K, Ω

One can find in [12] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} , namely, we have:

Theorem B (Brezis-Merle [12]). For $(u_i)_i$ and $(V_i)_i$ two sequences of functions relative to the problem (P) without the term $-\log \frac{|x|}{2d}$ and with,

$$0 \leq V_i \leq b < +\infty \text{ and } \int_{\Omega} e^{u_i} dy \leq C,$$

then for all compact set K of Ω it holds;

$$\sup_K u_i \leq c,$$

with c depending on b, C, K and Ω .

If we assume V with more regularity, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [20], that, if $(u_i)_i$ is a sequence of functions solutions of the previous equation without assumption on the boundary with V_i satisfying $0 < a \leq V_i \leq b < +\infty$, then we have a sup + inf inequality.

The point $0 \in \Omega$ is a conical singularity and an interior singularity and called a divisor and an ideal point. See for example the papers of Hulin and Troyanov, [16, 17, 22].

Here, we have:

Theorem 1.1 For sequences $(u_i)_i$ and $(V_i)_i$ of the Problem (P), for all compact subset K of Ω we have:

$$\|u_i\|_{L^\infty(K)} \leq c(a, b, K, \Omega).$$

Consider a positive number $M > 0$, if we assume $u_i \geq -M$, we can extend the previous result to any function u bounded from below by $-M$. If we consider the function $v_i = u_i + M$, then v_i satisfies all the condition of the previous theorem.

Here we have, if we replace the condition $u_i \geq 0$ by $u_i \geq -M$:

Corollary 1.2 For sequences $(u_i)_i$ and $(V_i)_i$ of the first equation of (P), for all compact subset K of Ω we have:

$$\|u_i\|_{L^\infty(K)} \leq c(a, b, M, K, \Omega).$$

1) We have an example of the previous problem with $u_i \geq 0$ if we add the Dirichlet condition and use the maximum principle (in this case we assume $u \in W_0^{1,1}(\Omega)$ and $-\log \frac{|x|}{2d} e^u \in L^1(\Omega)$ or $u \in W_0^{1,2}(\Omega) \Rightarrow e^u \in L^k(\Omega), \forall k \geq 1$ as a solution of a variational problem). From the counterexample constructed in the paper of Brezis and Merle, see [12], one can have an example of solutions with Dirichlet condition (but blowing-up solutions on the boundary).

2) We can replace the assumption $u \in L_{loc}^\infty(\Omega)$ by:

$$u \in L_{loc}^1(\Omega) \text{ and } \left(-\log \frac{|x|}{2d}\right)e^u \in L_{loc}^1(\Omega),$$

this imply that $u \in L_{loc}^\infty(\Omega)$.

Indeed, by solving a Dirichlet Problem and use Theorem 1 of the paper of Brezis and Merle, see [12], and Weyl theorem one can have $e^{|u|} \in L_{loc}^k(\Omega)$, $\forall k \geq 1$, the resut follow by the elliptic estimates of Agmon.

A consequence of the previous corollary we have:

Corollary 1.3 *For a solution u with V of the first equation of (P), for all compact subset K of Ω we have:*

$$\sup_K u \leq c(a, b, \inf_\Omega u, K, \Omega).$$

It is an estimate of the maximum on each compact subset of Ω of the solutions by mean of the infimum on Ω and a, b, K and Ω . (Also we have an a priori estimate).

We ask the following questions about explicit dependance of $\sup_K u$ in terms of $\inf_\Omega u$ and inequality of type $\sup + \inf$, as in the work of Shafrir [20] and the work of Tarantello, see [21] and Bartolucci-Tarantello, see [9]:

Problems. 1) Consider the Problem (P) without the boundary condition (without Dirichlet condition) and assume that:

$$0 < a \leq V \leq b < +\infty,$$

Does exists constants $C_1 = C_1(a, b, K, \Omega), C_2 = C_2(a, b, K, \Omega)$ such that:

$$\sup_K u + C_1 \inf_\Omega u \leq C_2,$$

for all solution u of (P) ?

2) If we add the condition $\|\nabla V\|_\infty \leq A$, can we have a sharp inequality:

$$\sup_K u + \inf_\Omega u \leq c(a, b, A, K, \Omega)?$$

2 Proof of the Theorem

We have:

$$u_i \in L_{loc}^\infty(\Omega).$$

Thus by the local boundedness elliptic esitmates of Agmon and the Sobolev embedding, see [1, 2], we have:

$$u_i \in W_{loc}^{2,k}(\Omega) \cap C^{1,\epsilon}(\Omega), \quad k > 2.$$

Step 1: For all $x_0 \in \Omega$, there is $r > 0$ such that: $\int_{B(x_0,r)} -\log \frac{|x|}{2d} e^{u_i} dx$ is bounded.

Let us consider $x_0 \in \Omega$ and set ϕ_1 the first eigenfunction of the Laplacian with Dirichlet condition and with corresponding eigenvalue $\lambda_1 > 0$ for the ball

$B(x_0, 2r) \subset \Omega$. We use integration by parts between u_i and ϕ_1 . The Stokes formula gives:

$$\int_{B(x_0, 2r)} -\log \frac{|x|}{2d} V_i e^{u_i} \phi_1 dx = \lambda_1 \int_{B(x_0, 2r)} u_i \phi_1 dx - \int_{\partial B(x_0, 2r)} \partial_\nu \phi_1 u_i dx,$$

We write:

$$u_i \phi_1 = u_i \sqrt{-\log \frac{|x|}{2d}} \phi_1 \times \sqrt{\frac{1}{-\log \frac{|x|}{2d}}} \phi_1,$$

We use Cauchy-Schwarz inequality to have:

$$\int_{B(x_0, 2r)} u_i \phi_1 \leq C \|(-\log \frac{|x|}{2d}) u_i^2 \phi_1\|_{L^1(B(x_0, 2r))}^{1/2},$$

But, $u_i \geq 0$ and $\partial_\nu \phi_1 \geq 0$, thus:

$$\int_{B(x_0, 2r)} -\log \frac{|x|}{2d} e^{u_i} \phi_1 dx \leq C,$$

since $\phi_1 > 0$ the result follows.

Step 2: $u_i^+ = u_i$ is locally bounded in L^1 .

We have, $u_i \geq 0$ and we write:

$$u_i = u_i \left(-\log \frac{|x|}{2d}\right) \times \left(\frac{1}{-\log \frac{|x|}{2d}}\right) \leq C u_i \left(-\log \frac{|x|}{2d}\right) \leq C e^{u_i} \left(-\log \frac{|x|}{2d}\right)$$

By the result of step 1, we obtain:

$$\|u_i\|_{L^1(B(x_0, r_0))} \leq C.$$

Since,

$$\int_{B(y, r)} -\log \frac{|x|}{2d} V_i e^{u_i} dx \leq C,$$

We have a convergence to a nonnegative measure μ :

$$\int_{B(y, r)} -\log \frac{|x|}{2d} V_i e^{u_i} \phi dx \rightarrow \int_{B(y, r)} \phi d\mu, \quad \forall \phi \in C_c(B(y, r)).$$

We set S the following set:

$$S = \{x \in B(y, r), \exists (x_i) \in \Omega, x_i \rightarrow x, u_i(x_i) \rightarrow +\infty\}.$$

We say that x_0 is a regular point of μ if there function $\psi \in C_c(B(y, r))$, $0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of x_0 such that:

$$\int \psi d\mu < 4\pi. \quad (1)$$

We can deduce that a point x_0 is non-regular if and only if $\mu(x_0) \geq 4\pi$.

A consequence of this fact is that if x_0 is a regular point then:

$$\exists R_0 > 0 \text{ such that one can bound } (u_i) = (u_i^+) \text{ in } L^\infty(B_{R_0}(x_0)). \quad (2)$$

We deduce (2) from corollary 4 of Brezis-Merle paper, and $-\log \frac{|x|}{2d} \in L^r, \forall 1 \leq r < +\infty$, we have by step 2 the inequality:

$$\|u_i^+\|_1 = \|u_i\|_1 \leq c.$$

We denote by Σ the set of non-regular points.

Step 3: $S = \Sigma$.

We have $S \subset \Sigma$. Let's consider $x_0 \in \Sigma$. Then we have:

$$\forall R > 0, \lim \|u_i^+\|_{L^\infty(B_R(x_0))} = +\infty. \quad (3)$$

Suppose contrary that:

$$\|u_i^+\|_{L^\infty(B_{R_0}(x_0))} \leq C.$$

Then:

$$\|e^{u_{i_k}}\|_{L^\infty(B_{R_0}(x_0))} \leq C, \text{ and}$$

$$\int_{B_R(x_0)} -\log \frac{|x|}{2d} V_{i_k} e^{u_{i_k}} = o(1).$$

For R small enough, which imply (1) for a function ψ and x_0 will be regular, contradiction. Then we have (3). We choose $R_0 > 0$ small such that $B_{R_0}(x_0)$ contain only x_0 as non -regular point. Σ . Let's $x_i \in B_R(x_0)$ such that:

$$u_i^+(x_i) = \max_{B_R(x_0)} u_i^+ \rightarrow +\infty.$$

We have $x_i \rightarrow x_0$. Else, there exists $x_{i_k} \rightarrow \bar{x} \neq x_0$ and $\bar{x} \notin \Sigma$, i.e. \bar{x} is a regular point. It is impossible because we would have (2).

Since the measure is finite, if there are blow-up points, or non-regular points, $S = \Sigma$ is finite.

Step 4: $\Sigma = \{\emptyset\}$.

Now: suppose contrary that there exists a non-regular point x_0 . We choose a radius $R > 0$ such that $B_R(x_0)$ contain only x_0 as non-regular point. Thus outside Σ we have local uniform boundedness of u_i , also in C^1 norm. Also, we have weak *-convergence of V_i to $V \geq 0$ with $V \leq b$.

Let's consider (by a variational method):

$$z_i \in W_0^{1,2}(B_R(x_0)),$$

$$-\Delta z_i = f_i = -\log \frac{|x|}{2d} V_i e^{u_i} \text{ in } B_R(x_0), \text{ et } z_i = 0 \text{ on } \partial B_R(x_0).$$

By a duality theorem:

$$z_i \in W_0^{1,q}(B_R), \|\nabla z_i\|_q \leq C_q.$$

By the maximum principle, $u_i \geq z_i$ in $B_R(x_0)$.

$$\int -\log \frac{|x|}{2d} e^{z_i} \leq \int -\log \frac{|x|}{2d} e^{u_i} \leq C. \quad (4)$$

On the other hand, $z_i \rightarrow z$ a.e. (uniformly on compact sets of $B_R(x_0) - \{x_0\}$) with z solution of :

$$-\Delta z = \mu \text{ in } B_R(x_0), \text{ et } z = 0 \text{ on } \partial B_R(x_0).$$

Also, we have up to a subsequence, $z_i \rightarrow z$ in $W_0^{1,q}(B_R(x_0)), 1 \leq q < 2$ weakly, and thus $z \in W_0^{1,q}(B_R(x_0))$.

Then by Fatou lemma:

$$\int -\log \frac{|x|}{2d} e^z \leq C. \quad (5)$$

As $x_0 \in S$ is not regular point we have $\mu(\{x_0\}) \geq 4\pi$, which imply that, $\mu \geq 4\pi\delta_{x_0}$ and by the maximum principle in $W_0^{1,1}(B_R(x_0))$ (obtained by Kato's inequality)

$$z(x) \geq 2 \log \frac{1}{|x - x_0|} + O(1) \text{ if } x \rightarrow x_0.$$

Because,

$$z_1 \equiv 2 \log \frac{1}{|x - x_0|} + 2 \log R \in W_0^{1,s}(B_R(x_0)), 1 \leq s < 2.$$

Thus,

$$-\log \frac{|x|}{2d} e^z \geq \frac{-C \log \frac{|x|}{2d}}{|x - x_0|^2}, C > 0.$$

Both in the cases $x_0 = 0$ and $x_0 \neq 0$ we have:

$$\int_{B_R(x_0)} -\log \frac{|x|}{2d} e^z = \infty.$$

However, by (5):

$$\int -\log \frac{|x|}{2d} e^z \leq C.$$

which is a contradiction.

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