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Strategic Successive Refinement Coding for Bayesian Persuasion with Two Decoders

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Abstract—We study the multi-user Bayesian persuasion game between one encoder and two decoders, where the first decoder is better informed than the second decoder. We consider two perfect links, one to the first decoder only, and the other to both decoders. We consider that the encoder and both decoders are endowed with distinct and arbitrary distortion functions. We investigate the strategic source coding problem in which the encoder commits to an encoding while the decoders select the sequences of symbols that minimize their respective distortion functions. We characterize the optimal encoder distortion value by considering successive refinement coding with respect to a specific probability distribution which involves two auxiliary random variables, and captures the incentives constraints of both decoders.

A full version of this paper is accessible at: <https://arxiv.org/pdf/2105.06201.pdf>

I. INTRODUCTION

The optimization of distinct and arbitrary distortion functions resulting from the communication between several autonomous devices with non-aligned objectives is under study. This problem was originally formulated in the game theory literature and referred to as the sender-receiver game, where the amount of information transmitted is generally unrestricted. In the seminal paper [1], Crawford and Sobel investigate the Nash equilibrium solution of the cheap talk game, by considering the encoder and the decoder have distinct objectives and choose their coding strategies simultaneously. In [2], Kamenica and formulate the Stackelberg version of the strategic communication game, in which the encoder is the Stackelberg leader and the decoder is the Stackelberg follower. This setting, referred to as the Bayesian persuasion game, is the one under study in this paper by considering two decoders.

This problem is an attractive multi-disciplinary subject of study. The Nash equilibrium solution is investigated for multi-dimensional sources and quadratic distortion functions in [3], [4], whereas the Stackelberg solution is studied in [5]. The computational aspects of the persuasion game are considered in [6]. The strategic communication problem with a noisy channel is investigated in [7], [8], [9], [10], and four different

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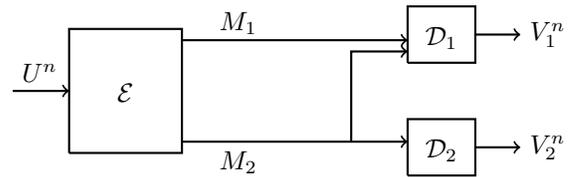


Fig. 1: Successive Refinement Source Coding Setup.

scenarios of strategic communication are studied in [11]. The case where the decoder privately observes a signal correlated to the state, also referred to as the Wyner-Ziv setting [12], is studied in [13], [14] and [15]. Vora and Kulkarni investigate the achievable rates for the strategic communication problem in [16], [17] where the decoder is the Stackelberg leader.

In this paper, we study a multi-receiver Bayesian persuasion game in which the observation of the first decoder contains the observation of the second decoder, as in Fig. 1. More specifically, we assume that the encoder \mathcal{E} selects and announces beforehand the compression scheme to be implemented. Upon receipt of the indexes, the decoders \mathcal{D}_1 and \mathcal{D}_2 update their Bayesian beliefs over the source sequence and select the output sequence that minimizes their respective distortion functions. We characterize the optimal encoder distortion value obtained via the successive refinement coding with respect to the distribution that involves two auxiliary random variables, and that satisfies both decoders incentives constraints.

A. Preliminaries

1) *Notations*: Let \mathcal{E} denote the encoder and \mathcal{D}_i denote the decoder $i \in \{1, 2\}$. Notations U^n and V_i^n denote the sequences of random variables of source information $u^n = (u_1, \dots, u_n) \in \mathcal{U}^n$, and decoder \mathcal{D}_i actions $v_i^n \in \mathcal{V}_i^n$ respectively for $i \in \{1, 2\}$. Calligraphic fonts \mathcal{U} and \mathcal{V}_i denote the alphabets and lowercase letters u and v_i denote the realizations. For a discrete random variable X , we denote by $\Delta(\mathcal{X})$ the probability simplex, i.e. the set of probability distributions over \mathcal{X} , and by $\mathcal{P}_X(x)$ the probability mass function $\mathbb{P}\{X = x\}$. Notation $X \dashv\dashv Y \dashv\dashv Z$ stands for the Markov chain $\mathcal{P}_{Z|XY} = \mathcal{P}_{Z|Y}$. The information source U follows the independent and identically distributed (i.i.d) probability distribution $\mathcal{P}_U \in \Delta(\mathcal{U})$.

II. SYSTEM MODEL

In this section, we aim at formulating the coding problem.

Definition 1. Let $R_1, R_2 \in \mathbb{R}_+^2 = [0, +\infty[^2$, and $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The encoding σ and decoding τ_i strategies of the encoder \mathcal{E} and decoders \mathcal{D}_i for $i \in \{1, 2\}$ respectively, are given by

$$\sigma : U^n \longrightarrow \Delta(\{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\}), \quad (1)$$

$$\tau_1 : \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \longrightarrow \Delta(\mathcal{V}_1^n), \quad (2)$$

$$\tau_2 : \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \longrightarrow \Delta(\mathcal{V}_2^n). \quad (3)$$

We denote by $\mathcal{S}(n, R_1, R_2)$ the set of coding triplets (σ, τ_1, τ_2) .

The coding strategies $(\sigma, \tau_1, \tau_2) \in \mathcal{S}(n, R_1, R_2)$ are stochastic and induce a joint probability distribution $\mathcal{P}^{\sigma, \tau_1, \tau_2} \in \Delta(U^n \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times V_1^n \times V_2^n)$ defined by

$$\mathcal{P}^{\sigma, \tau_1, \tau_2} = \left(\prod_{t=1}^n \mathcal{P}_{U_t} \right) \mathcal{P}_{M_1 M_2 | U^n}^\sigma \mathcal{P}_{V_1^n | M_1 M_2}^{\tau_1} \mathcal{P}_{V_2^n | M_2}^{\tau_2}. \quad (4)$$

Definition 2. The single-letter distortion functions of the encoder d_e and both decoders d_i for $i \in \{1, 2\}$ are defined by

$$d_e : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathbb{R}, \quad (5)$$

$$d_1 : \mathcal{U} \times \mathcal{V}_1 \longrightarrow \mathbb{R}, \quad (6)$$

$$d_2 : \mathcal{U} \times \mathcal{V}_2 \longrightarrow \mathbb{R}. \quad (7)$$

Definition 3. The long-run distortion functions are defined by

$$d_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[\frac{1}{n} \sum_{t=1}^n d_e(U_t, V_{1,t}, V_{2,t}) \right]$$

$$= \sum_{u^n, v_1^n, v_2^n} \mathcal{P}_{U^n V_1^n V_2^n}^{\sigma, \tau_1, \tau_2}(u^n, v_1^n, v_2^n) \cdot \left[\frac{1}{n} \sum_{t=1}^n d_e(u_t, v_{1,t}, v_{2,t}) \right],$$

$$d_1^n(\sigma, \tau_1) = \mathbb{E}_{\sigma, \tau_1} \left[\frac{1}{n} \sum_{t=1}^n d_1(U_t, V_{1,t}) \right],$$

$$d_2^n(\sigma, \tau_2) = \sum_{u^n, v_2^n} \mathcal{P}_{U^n V_2^n}^{\sigma, \tau_2}(u^n, v_2^n) \cdot \left[\frac{1}{n} \sum_{t=1}^n d_2(u_t, v_{2,t}) \right].$$

In the above equations, $\mathcal{P}_{U^n V_1^n V_2^n}^{\sigma, \tau_1, \tau_2}$, $\mathcal{P}_{U^n V_1^n}^{\sigma, \tau_1}$ and $\mathcal{P}_{U^n V_2^n}^{\sigma, \tau_2}$ denote the marginal distributions of $\mathcal{P}^{\sigma, \tau_1, \tau_2}$ over the n -sequences (U^n, V_1^n, V_2^n) , (U^n, V_1^n) , and (U^n, V_2^n) respectively. We consider the strategic communication game in which the encoder select σ that minimizes $d_e^n(\sigma, \tau_1, \tau_2)$ and the decoders choose τ_i , $i \in \{1, 2\}$ in order to minimize the long-run distortion functions $d_i^n(\sigma, \tau_i)$, $i \in \{1, 2\}$.

Definition 4. For any encoding strategy σ , the set of decoder i 's best reply strategies for $i \in \{1, 2\}$ is defined as follows

$$BR_i(\sigma) = \{\tau_i, d_i^n(\sigma, \tau_i) \leq d_i^n(\sigma, \tilde{\tau}_i), \forall \tilde{\tau}_i\}. \quad (8)$$

If several pairs of best-reply strategy $(\tau_1, \tau_2) \in BR_1(\sigma) \times BR_2(\sigma)$ are available, we assume that the decoders choose the worst pair from the encoder perspective. For $(R_1, R_2) \in \mathbb{R}_+^2$

and $n \in \mathbb{N}^*$, the encoder has to solve the following coding problem.

$$D_e^n(R_1, R_2) = \inf_{\sigma} \max_{\substack{\tau_1 \in BR_{d_1}(\sigma), \\ \tau_2 \in BR_{d_2}(\sigma)}} d_e^n(\sigma, \tau_1, \tau_2). \quad (9)$$

We now discuss now the operational significance $D_e^n(R_1, R_2)$.

- The encoder \mathcal{E} chooses, announces the encoding σ .
- The sequence U^n is drawn i.i.d with distribution \mathcal{P}_U .
- The messages (M_1, M_2) are encoded according to $\mathcal{P}_{M_1 M_2 | U^n}^\sigma$.
- By knowing σ , the decoder \mathcal{D}_1 observes (M_1, M_2) and draws a sequence pair V_1^n according to the strategy $\tau_1 \in BR_{d_1}(\sigma)$. Similarly, the decoder \mathcal{D}_2 observes M_2 and draws a sequence V_2^n according to the strategy $\tau_2 \in BR_{d_2}(\sigma)$.
- The distortion values are given by $d_e^n(\sigma, \tau_1, \tau_2)$, $d_1^n(\sigma, \tau_1)$, $d_2^n(\sigma, \tau_2)$.

III. CHARACTERIZATION

Definition 5. We consider two auxiliary random variables $W_1 \in \mathcal{W}_1$ and $W_2 \in \mathcal{W}_2$ with $|\mathcal{W}_i| = |\mathcal{V}_i|$, for $i \in \{1, 2\}$. For $(R_1, R_2) \in \mathbb{R}_+^2$, we define

$$\mathcal{Q}_0(R_1, R_2) = \left\{ \mathcal{Q}_{W_1 W_2 | U}, \quad R_2 \geq I(U; W_2), \right. \\ \left. R_1 + R_2 \geq I(U; W_1, W_2) \right\}. \quad (10)$$

For every distribution $\mathcal{Q}_{W_1 W_2 | U} \in \Delta(\mathcal{W}_1 \times \mathcal{W}_2)^{|U|}$, we define

$$\mathcal{Q}_1(\mathcal{Q}_{W_1 W_2 | U}) = \arg \min_{\substack{\mathcal{Q}_{V_1 | W_1 W_2} \\ \mathcal{Q}_{V_1 | W_1 W_2}}} \mathbb{E}_{\mathcal{Q}_{W_1 W_2 | U}} \left[d_1(U, V_1) \right], \quad (11)$$

$$\mathcal{Q}_2(\mathcal{Q}_{W_2 | U}) = \arg \min_{\substack{\mathcal{Q}_{V_2 | W_2} \\ \mathcal{Q}_{V_2 | W_2}}} \mathbb{E}_{\mathcal{Q}_{W_2 | U}} \left[d_2(U, V_2) \right]. \quad (12)$$

Note that $\mathcal{Q}_1(\mathcal{Q}_{W_1 W_2 | U}) \in \Delta(\mathcal{V}_1)^{|\mathcal{W}_1 \times \mathcal{W}_2|}$ and $\mathcal{Q}_2(\mathcal{Q}_{W_2 | U}) \in \Delta(\mathcal{V}_2)^{|\mathcal{W}_2|}$. The encoder's optimal distortion is defined by

$$D_e^*(R_1, R_2) \\ = \inf_{\substack{\mathcal{Q}_{W_1 W_2 | U} \\ \in \mathcal{Q}_0(R_1, R_2)}} \max_{\substack{\mathcal{Q}_{V_1 | W_1 W_2} \in \mathcal{Q}_1(\mathcal{Q}_{W_1 W_2 | U}) \\ \mathcal{Q}_{V_2 | W_2} \in \mathcal{Q}_2(\mathcal{Q}_{W_2 | U})}} \mathbb{E} \left[d_e(U, V_1, V_2) \right], \quad (13)$$

where the expectation in (13) is evaluated with respect to $\mathcal{P}_U \mathcal{Q}_{W_1 W_2 | U} \mathcal{Q}_{V_1 | W_1 W_2} \mathcal{Q}_{V_2 | W_2}$.

Remark 1. The random variables U, W_1, W_2, V_1, V_2 satisfy

$$(U, V_2) \text{ --- } (W_1, W_2) \text{ --- } V_1, \quad (U, W_1, V_1) \text{ --- } W_2 \text{ --- } V_2.$$

Definition 6. Given $\mathcal{Q}_{W_1 W_2 | U}$, we denote by $\mathcal{Q}_{U | W_1 W_2} \in \Delta(\mathcal{U})^{|\mathcal{W}_1 \times \mathcal{W}_2|}$ and $\mathcal{Q}_{U | W_2} \in \Delta(\mathcal{U})^{|\mathcal{W}_2|}$ the posterior beliefs of decoders \mathcal{D}_1 and \mathcal{D}_2 that are defined by

$$\mathcal{Q}(u | w_1, w_2) = \frac{\mathcal{P}(u) \mathcal{Q}(w_1, w_2 | u)}{\sum_{u'} \mathcal{P}(u') \mathcal{Q}(w_1, w_2 | u')}, \quad \forall u, w_1, w_2, \quad (14)$$

$$\mathcal{Q}(u | w_2) = \frac{\mathcal{P}(u) \sum_{w_1} \mathcal{Q}(w_1, w_2 | u)}{\sum_{u'} \mathcal{P}(u') \mathcal{Q}(w_2 | u')}, \quad \forall u, w_2. \quad (15)$$

Moreover, for $(w_1, w_2) \in \mathcal{W}_1 \times \mathcal{W}_2$, we introduce the notations $\mathcal{Q}_U^{w_1 w_2} = \mathcal{Q}_{U|W_1 W_2}(\cdot | w_1, w_2) \in \Delta(\mathcal{U})$ and $\mathcal{Q}_U^{w_2} = \mathcal{Q}_{U|W_2}(\cdot | w_2) \in \Delta(\mathcal{U})$.

Theorem 1. Let $(R_1, R_2) \in \mathbb{R}_+^2$, we have

$$\begin{aligned} \forall \varepsilon > 0, \exists \hat{n} \in \mathbb{N}, \forall n \geq \hat{n}, D_e^n(R_1, R_2) &\leq D_e^*(R_1, R_2) + \varepsilon, \\ \forall n \in \mathbb{N}, D_e^n(R_1, R_2) &\geq D_e^*(R_1, R_2). \end{aligned}$$

Together with Fekete's Lemma for the sub-additive sequence $(nD_e^n(R_1, R_2))_{n \in \mathbb{N}^*}$, this result describes two features of the asymptotic behavior of the encoder's long run distortion function $D_e^*(R_1, R_2)$:

- 1) It converges to the optimal distortion $D_e^*(R_1, R_2)$,
- 2) It does not go below $D_e^*(R_1, R_2)$.

In other words,

$$\lim_{n \rightarrow \infty} D_e^n(R_1, R_2) = \inf_{n \in \mathbb{N}^*} D_e^n(R_1, R_2) = D_e^*(R_1, R_2). \quad (16)$$

IV. CONVERSE PROOF

Let $(R_1, R_2) \in \mathbb{R}_+^2$ and $n \in \mathbb{N}^*$. We consider $(\sigma, \tau_1, \tau_2) \in \mathcal{S}(R_1, R_2)$ and a random variable T uniformly distributed over $\{1, 2, \dots, n\}$ and independent of $(U^n, M_1, M_2, V_1^n, V_2^n)$. We introduce the auxiliary random variables $W_1 = (M_1, T)$, $W_2 = (M_2, T)$, $(U, V_1, V_2) = (U_T, V_{1,T}, V_{2,T})$, distributed according to $\mathcal{P}_{UW_1W_2V_1V_2}^{\sigma\tau_1\tau_2}$ defined for all $(u, w_1, w_2, v_1, v_2) = (u_t, m_1, m_2, t, v_{1,t}, v_{2,t})$ by

$$\begin{aligned} &\mathcal{P}_{UW_1W_2V_1V_2}^{\sigma\tau_1\tau_2}(u, w_1, w_2, v_1, v_2) \\ &= \mathcal{P}_{UM_1M_2TV_1TV_2}^{\sigma\tau_1\tau_2}(u_t, m_1, m_2, t, v_{1,t}, v_{2,t}) \\ &= \frac{1}{n} \sum_{\substack{u^{t-1} \\ u^{t+1}}} \sum_{\substack{v_1^{t-1}, v_1^{t+1} \\ v_2^{t-1}, v_2^{t+1}}} \left(\prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_1M_2|U^n}^\sigma(m_1, m_2 | u^n) \\ &\quad \times \mathcal{P}_{V_1^n|M_1M_2}^{\tau_1}(v_1^n | m_1, m_2) \mathcal{P}_{V_2^n|M_2}^{\tau_2}(v_2^n | m_2). \end{aligned}$$

Lemma 1. The distribution $\mathcal{P}_{UW_1W_2V_1V_2}^{\sigma\tau_1\tau_2}$ has marginal on $\Delta(\mathcal{U})$ given by \mathcal{P}_U and satisfies the Markov chain properties

$$(U, V_2) \text{---} (W_1, W_2) \text{---} V_1, \quad (U, W_1, V_1) \text{---} W_2 \text{---} V_2.$$

Proof. [Lemma 1] The i.i.d. property of the source ensures that the marginal distribution is \mathcal{P}_U . By the definition of the decoding functions τ_1 and τ_2 we have

$$\begin{aligned} (U_T, V_{2,T}) &\text{---} (M_1, M_2, T) \text{---} V_{1,T}, \\ (U_T, M_1, V_{1,T}) &\text{---} (M_2, T) \text{---} V_{2,T}. \end{aligned}$$

□ Therefore $\mathcal{P}_{UW_1W_2V_1V_2}^{\sigma\tau_1\tau_2} = \mathcal{P}_U \mathcal{P}_{W_1W_2|U}^\sigma \mathcal{P}_{V_1|W_1W_2}^{\tau_1} \mathcal{P}_{V_2|W_2}^{\tau_2}$.

Lemma 2. For all σ , the distribution $\mathcal{P}_{W_1W_2|U}^\sigma \in \mathbb{Q}_0$.

Proof. [Lemma 2] We consider an encoding strategy σ , then

$$nR_2 \geq H(M_2) \geq I(M_2; U^n) \quad (17)$$

$$= \sum_{t=1}^n I(U_t; M_2 | U^{t-1}) \quad (18)$$

$$= nI(U_T; M_2 | U^{T-1}, T) \quad (19)$$

$$= nI(U_T; M_2, U^{T-1}, T) \quad (20)$$

$$\geq nI(U_T; M_2, T) \quad (21)$$

$$= nI(U; W_2). \quad (22)$$

In fact, (19) follows from the introduction of the uniform random variable $T \in \{1, \dots, n\}$, (20) comes from the i.i.d. property of the source and (22) follows from the identification of the auxiliary random variables (U, W_2) . Similarly,

$$n(R_1 + R_2) \geq H(M_1, M_2) \geq I(U^n; M_1, M_2) \quad (23)$$

$$= \sum_{t=1}^n I(U_t; M_1, M_2 | U^{t-1}) \quad (24)$$

$$= nI(U_T; M_1, M_2 | U^{T-1}, T) \quad (25)$$

$$\geq nI(U_T; M_1, M_2, T) \quad (26)$$

$$= nI(U; W_1, W_2). \quad (27)$$

□

Lemma 3. For all (σ, τ_1, τ_2) and $i \in \{1, 2\}$, we have

$$d_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}[d_e(U, V_1, V_2)], \quad (28)$$

$$d_i^n(\sigma, \tau_i) = \mathbb{E}[d_i(U, V_i)], \quad (29)$$

evaluated with respect to $\mathcal{P}_U \mathcal{P}_{W_1W_2|U}^\sigma \mathcal{P}_{V_1|W_1W_2}^{\tau_1} \mathcal{P}_{V_2|W_2}^{\tau_2}$. Moreover for all σ , we have

$$\begin{aligned} \mathbb{Q}_1(\mathcal{P}_{W_1W_2|U}^\sigma) &= \left\{ \mathcal{Q}_{V_1|W_1W_2}, \right. \\ &\quad \left. \exists \tau_1 \in BR_1(\sigma), \mathcal{Q}_{V_1|W_1W_2} = \mathcal{P}_{V_1|W_1W_2}^{\tau_1} \right\}, \quad (30) \end{aligned}$$

$$\begin{aligned} \mathbb{Q}_2(\mathcal{P}_{W_2|U}^\sigma) &= \left\{ \mathcal{Q}_{V_2|W_2}, \right. \\ &\quad \left. \exists \tau_2 \in BR_2(\sigma), \mathcal{Q}_{V_2|W_2} = \mathcal{P}_{V_2|W_2}^{\tau_2} \right\}. \quad (31) \end{aligned}$$

Proof. [Lemma 3] By Definition 3 we have

$$\begin{aligned} d_e^n(\sigma, \tau_1, \tau_2) &= \sum_{\substack{u^n, m_1, m_2, \\ v_1^n, v_2^n}} \left(\prod_{t=1}^n \mathcal{P}_U(u_t) \right) \\ &\quad \times \mathcal{P}_{M_1M_2|U^n}^\sigma(m_1, m_2 | u^n) \mathcal{P}_{V_1^n|M_1M_2}^{\tau_1}(v_1^n | m_1, m_2) \\ &\quad \times \mathcal{P}_{V_2^n|M_2}^{\tau_2}(v_2^n | m_2) \cdot \left[\frac{1}{n} \sum_{t=1}^n d_e(u_t, v_{1,t}, v_{2,t}) \right] \quad (32) \end{aligned}$$

$$\begin{aligned} &= \sum_{t=1}^n \sum_{\substack{u_t, m_1, m_2, \\ v_{1,t}, v_{2,t}}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(u_t, m_1, m_2, t, v_{1,t}, v_{2,t}) \\ &\quad \times d_e(u_t, v_{1,t}, v_{2,t}) = \mathbb{E}[d_e(U, V_1, V_2)]. \quad (33) \end{aligned}$$

Given $\mathcal{Q}_{V_1|W_1W_2} \in \mathbb{Q}_1(\mathcal{P}_{W_1W_2|U}^\sigma)$, we consider τ_1 such that

$$\mathcal{P}_{V_1^n|M_1M_2}^{\tau_1}(v_1^n | m_1, m_2) = \prod_{t=1}^n \mathcal{Q}_{V_1|W_1W_2}(v_{1,t} | m_1, m_2, t).$$

Therefore

$$d_1^n(\sigma, \tau_1) = \mathbb{E}_{\substack{\mathcal{P}_{W_1 W_2 | U}^\sigma \\ \mathcal{Q}_{V_1 | W_1 W_2}}} [d_1(U, V_1)] \quad (34)$$

$$= \min_{\mathcal{P}_{V_1 | W_1 W_2}} \mathbb{E}_{\substack{\mathcal{P}_{W_1 W_2 | U}^\sigma \\ \mathcal{P}_{V_1 | W_1 W_2}}} [d_1(U, V_1)] \quad (35)$$

$$\leq \min_{\tilde{\tau}_1} \mathbb{E}_{\substack{\mathcal{P}_{W_1 W_2 | U}^\sigma \\ \mathcal{P}_{V_1 | W_1 W_2}^{\tilde{\tau}_1}}} [d_1(U, V_1)] = \min_{\tilde{\tau}_1} d_1^n(\sigma, \tilde{\tau}_1), \quad (36)$$

hence $\tau_1 \in BR_1(\sigma)$. The other inclusion is direct and the same arguments imply (31). \square

For any strategy σ , we have

$$\begin{aligned} & \max_{\tau_1, \tau_2} d_e^n(\sigma, \tau_1, \tau_2) \\ &= \max_{\tau_1, \tau_2} \mathbb{E}_{\substack{\mathcal{P}_{W_1 W_2 | U}^\sigma \\ \mathcal{P}_{V_1 | W_1 W_2}^{\tau_1} \mathcal{P}_{V_2 | W_2}^{\tau_2}}} [d_e(U, V_1, V_2)] \quad (37) \end{aligned}$$

$$= \max_{\substack{\mathcal{Q}_{V_1 | W_1 W_2} \in \mathcal{Q}_1(\mathcal{P}_{W_1 W_2 | U}^\sigma) \\ \mathcal{Q}_{V_2 | W_2} \in \mathcal{Q}_2(\mathcal{P}_{W_2 | U}^\sigma)}} \mathbb{E}_{\substack{\mathcal{P}_{W_1 W_2 | U}^\sigma \\ \mathcal{Q}_{V_1 | W_1 W_2} \mathcal{Q}_{V_2 | W_2}}} [d_e(U, V_1, V_2)] \quad (38)$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_1 W_2 | U} \in \mathcal{Q}_0(R_1, R_2) \\ \mathcal{Q}_{V_1 | W_1 W_2} \in \mathcal{Q}_1(\mathcal{Q}_{W_1 W_2 | U}) \\ \mathcal{Q}_{V_2 | W_2} \in \mathcal{Q}_2(\mathcal{Q}_{W_2 | U})}} \mathbb{E} [d_e(U, V_1, V_2)] \quad (39)$$

$$= D_e^*(R_1, R_2). \quad (40)$$

Equations (37) and (38) comes from Lemma 3, whereas (39) comes from Lemma 2. This concludes the converse proof of Theorem 1.

V. ACHIEVABILITY PROOF

A. Alternative Formulation

Definition 7. We denote by $V_1^*(q_1)$ and $V_2^*(q_2)$, the sets of optimal outputs of decoders \mathcal{D}_1 and \mathcal{D}_2 for any distributions $q_1 \in \Delta(U)$ and $q_2 \in \Delta(U)$.

$$V_1^*(q_1) = \arg \min_{v_1 \in V_1} \sum_u q_1(u) d_1(u, v_1), \quad (41)$$

$$V_2^*(q_2) = \arg \min_{v_2 \in V_2} \sum_u q_2(u) d_2(u, v_2). \quad (42)$$

Definition 8. Given a strategy $\mathcal{Q}_{W_1 W_2 | U}$ we denote by $\tilde{A}(\mathcal{Q}_{W_1 W_2 | U}, w_1, w_2) \subset \mathcal{V}_1 \times \mathcal{V}_2$ the set of action pairs (v_1, v_2) that are optimal for the decoders and worst for the encoder. This set is defined by

$$\begin{aligned} \tilde{A}(\mathcal{Q}_{W_1 W_2 | U}, w_1, w_2) = \\ \arg \max_{(v_1, v_2) \in V_1^*(\mathcal{Q}_{W_1 W_2 | U}^{w_1}) \times V_2^*(\mathcal{Q}_{W_2 | U}^{w_2})} \left\{ \sum_u \mathcal{Q}_U^{w_1, w_2}(u) d_e(u, v_1, v_2) \right\}. \quad (43) \end{aligned}$$

Definition 9. Given $(R_1, R_2) \in \mathbb{R}_+^2$, we define

$$\begin{aligned} \tilde{\mathcal{Q}}_0(R_1, R_2) = \left\{ \mathcal{Q}_{W_1 W_2 | U} \text{ s.t. } R_2 > I(U; W_2), \right. \\ \left. R_1 + R_2 > I(U; W_1, W_2), \right. \\ \left. \max_{w_1, w_2} |\tilde{A}(\mathcal{Q}_{W_1 W_2 | U}, w_1, w_2)| = 1 \right\}. \quad (44) \end{aligned}$$

Definition 10. Consider the following program:

$$\begin{aligned} \tilde{D}_e(R_1, R_2) = \\ \inf_{\mathcal{Q}_{W_1 W_2 | U} \in \tilde{\mathcal{Q}}_0(R_1, R_2)} \max_{\substack{\mathcal{Q}_{V_1 | W_1 W_2} \in \mathcal{Q}_1(\mathcal{Q}_{W_1 W_2 | U}) \\ \mathcal{Q}_{V_2 | W_2} \in \mathcal{Q}_2(\mathcal{Q}_{W_2 | U})}} \mathbb{E} [d_e(U, V_1, V_2)]. \quad (45) \end{aligned}$$

where the expectation in (45) is evaluated with respect to $\mathcal{P}_U \mathcal{Q}_{W_1 W_2 | U} \mathcal{Q}_{V_1 | W_1 W_2} \mathcal{Q}_{V_2 | W_2}$.

Lemma 4. For $(R_1, R_2) \in \mathbb{R}_+^2$, we have

$$D_e^*(R_1, R_2) = \tilde{D}_e(R_1, R_2) \quad (46)$$

The proof of lemma 4 relies on showing that $\tilde{\mathcal{Q}}_0(R_1, R_2)$ is dense in $\mathcal{Q}_0(R_1, R_2)$. It is provided in the full version of the paper.

B. Special Cases

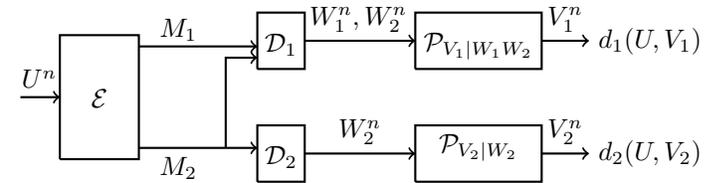


Fig. 2: Achievability of Successive Refinement Source Coding Setup.

1) $R_1 = R_2 = 0$: The auxiliary random variables (W_1, W_2) are independent of U . Moreover, message sets are singletons, and the only possible encoding strategy σ_0 is given by $\sigma_0 : \mathcal{U}^n \rightarrow \{1\} \times \{1\}$. The codebook consists of two sequences $W_2^n(1)$ and $W_1^n(1, 1)$ only. Therefore, the following result holds:

Lemma 5. $D_e^*(0, 0) = D_e^n(0, 0) \quad \forall n \in \mathbb{N}^*$.

2) $R_1 > 0$ & $R_2 = 0$: Random variables W_2 and U are independent for $R_1 > 0$ and $R_2 = 0$, i.e. $\mathcal{Q}_{W_1 W_2 | U} = \mathcal{Q}_{W_2} \mathcal{Q}_{W_1 | W_2 U}$. This means that decoder \mathcal{D}_2 will repeatedly chose the action $v_{2,0} \in V^*(\mathcal{P}_U)$ that corresponds to its prior belief \mathcal{P}_U and maximizes the encoder's distortion. The persuasion game is thus reduced to the point-to-point problem with one decoder \mathcal{D}_1 as in [11].

3) $R_1 = 0$ & $R_2 > 0$: The auxiliary random variable W_1 is independent of U . Hence, the encoder transmits the same index to both decoders. Therefore, both decoders will have the same posterior belief $\mathcal{Q}_U^{w_2} \in \Delta(U)$, $\forall w_2 \in \mathcal{W}_2$. The output sequences V_1^n and V_2^n are drawn i.i.d. according to the optimal $\mathcal{Q}_{V_1 | W_2} \in \mathcal{Q}_1(\mathcal{Q}_{W_2 | U})$ and $\mathcal{Q}_{V_2 | W_2} \in \mathcal{Q}_2(\mathcal{Q}_{W_2 | U})$ respectively. This case is also reduced to the point-to-point problem solved in [10], by considering a unique decoder that selects the optimal pair of outputs (v_1, v_2) .

Definition 11. A family of pairs $(\lambda_{w_2}, \mathcal{Q}_U^{w_2})_{w_2 \in \mathcal{W}_2} \in ([0, 1] \times \Delta(\mathcal{U}))^{|\mathcal{W}_2|}$ is a splitting of \mathcal{P}_U if

$$\sum_{w_2 \in \mathcal{W}_2} \lambda_{w_2} = 1, \quad (47)$$

$$\sum_{w_2 \in \mathcal{W}_2} \lambda_{w_2} \mathcal{Q}_U^{w_2} = \mathcal{P}_U. \quad (48)$$

In that case, the optimal distortion can be reformulated in terms of a convexification of its expected distortion as follows:

$$D_e^*(0, R_2) = \inf_{(\lambda_{w_2}, \mathcal{Q}_U^{w_2})_{w_2 \in \mathcal{W}_2}} \sum_{w_2 \in \mathcal{W}_2} \lambda_{w_2} \Psi_e(\mathcal{Q}_U^{w_2}). \quad (49)$$

where $\Psi_e(q) = \max_{\substack{(v_1, v_2) \in \\ V_1^*(q) \times V_2^*(q)}} \mathbb{E}_q \left[d_e(U, v_1, v_2) \right]$.

4) $(R_1, R_2) \in]0, +\infty[^2$: Fix a conditional probability distribution $\mathcal{Q}_{W_1, W_2|U}$. There exists $\eta > 0$ such that

$$R_2 = I(U; W_2) + \eta, \quad (50)$$

$$R_1 = I(U; W_1|W_2) + \eta. \quad (51)$$

Codebook generation: Randomly and independently generate $2^{\lfloor nR_2 \rfloor}$ sequences $w_2^n(m_2)$ for $m_2 \in [1 : 2^{\lfloor nR_2 \rfloor}]$, according to the i.i.d distribution $\mathcal{P}_{W_2^n} = \prod_{t=1}^n \mathcal{P}_{W_2}(w_{2t})$. For each $(m_1, m_2) \in [1 : 2^{\lfloor nR_1 \rfloor}] \times [1 : 2^{\lfloor nR_2 \rfloor}]$ generate a sequence $w_1^n(m_1, m_2)$ randomly and conditionally independently according to the i.i.d conditional distribution $\mathcal{P}_{W_1^n|M_1 W_2^n} = \prod_{t=1}^n \mathcal{P}_{W_1|M_1 W_2}(w_{1t}|m_1, w_{2t}(m_2))$.

Coding algorithm: Encoder \mathcal{E} observes u^n and looks in the codebook for a pair (m_1, m_2) such that $(u^n, w_1^n(m_1, m_2), w_2^n(m_2)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{P}_{W_1 W_2|U})$, i.e. the sequences are jointly typical with tolerance parameter $\delta > 0$. If such a jointly typical tuple doesn't exist, the source encoder sets (m_1, m_2) to $(1, 1)$. Then, it sends m_2 to decoder \mathcal{D}_2 , and (m_1, m_2) to decoder \mathcal{D}_1 .

Here comes the main difference with the successive refinement coding, which is due to the strategic nature of the problem. Instead of declaring $w_1^n(m_1, m_2)$ and $w_2^n(m_2)$ and selecting V_1^n and V_2^n i.i.d. with respect to $\mathcal{Q}_{V_1|W_1 W_2} \in \mathcal{Q}_1(\mathcal{Q}_{W_1 W_2|U})$ and $\mathcal{Q}_{V_2|W_2} \in \mathcal{Q}_2(\mathcal{Q}_{W_2|U})$, the decoders 1 and 2 compute their Bayesian posterior beliefs $\mathcal{P}_{U^n|M_1 M_2}^\sigma$ and $\mathcal{P}_{U^n|M_2}^\sigma$ and select the output sequences V_1^n and V_2^n that minimize their long-run distortion functions.

Error Event: Given a tolerance $\delta > 0$, the error event is given by $\mathcal{E} = \{(U^n, W_2^n(m_2), W_1^n(m_2, m_1)) \notin \mathcal{T}_\delta^n\}$. We have by the union of events bound $\mathcal{P}(\mathcal{E}) \leq \mathcal{P}(\mathcal{E}_1) + \mathcal{P}(\mathcal{E}_2(M_2) \cap \mathcal{E}_1^c)$, where

$$\mathcal{E}_1 = \{(U^n, W_2^n(m_2)) \notin \mathcal{T}_\delta^n \forall m_2\}, \quad (52)$$

$$\mathcal{E}_2(m_2) = \{(U^n, W_2^n(m_2), W_1^n(m_2, m_1)) \notin \mathcal{T}_\delta^n \forall m_1\}. \quad (53)$$

By [18, Lemma 3.3, pp. 62], $\mathcal{P}(\mathcal{E}_1)$ tends to zero as $n \rightarrow \infty$ if

$$R_2 > I(U; W_2) + \eta. \quad (54)$$

$\mathcal{P}(\mathcal{E}_1 \cap \mathcal{E}_2(M_2))$ goes to zero by [18, Lemma 3.3, pp. 62] if

$$R_1 + R_2 > I(U; W_1, W_2) + \eta. \quad (55)$$

Since the expected error probability evaluated with respect to the random codebook is small, we have that for all $\varepsilon_2 > 0$, for all $\eta > 0$, there exists $\bar{\delta} > 0$, for all $\delta \leq \bar{\delta}$, there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$ we have:

$$\mathbb{E}[\mathcal{P}(\mathcal{E}_1)] \leq \varepsilon_2, \quad (56)$$

$$\mathbb{E}[\mathcal{P}(\mathcal{E}_2(m_2))] \leq \varepsilon_2. \quad (57)$$

5) **Control of Beliefs:** We introduce the indicator of error events $E_\delta^1 \in \{0, 1\}$ for decoder \mathcal{D}_1 defined as follows

$$E_\delta^1 = \begin{cases} 1, & \text{if } (u^n, w_1^n, w_2^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_2|U}). \\ 0, & \text{otherwise.} \end{cases} \quad (58)$$

Assuming the distribution $\mathcal{P}_{U|W_1 W_2}$ is fully supported, the beliefs of decoder \mathcal{D}_1 are controlled as follows

$$\mathbb{E} \left[\frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_t^{m_1, m_2} \| \mathcal{P}_{U|W_1 W_2}(\cdot | W_{1t}, W_{2t})) \middle| E_\delta^1 = 0 \right] \quad (59)$$

$$= \sum_{\substack{m_1, m_2, \\ w_1^n, w_2^n}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(m_1, m_2, w_1^n, w_2^n \mid E_\delta^1 = 0) \\ \times \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_2}(u) \log_2 \frac{\mathcal{P}_t^{m_1 m_2}(u)}{\mathcal{P}_{U|W_1 W_2}(u | w_{1t}, w_{2t})} \quad (60)$$

$$= \sum_{\substack{m_1, m_2, \\ w_1^n, w_2^n}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(m_1, m_2, w_1^n, w_2^n \mid E_\delta^1 = 0) \\ \times \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_2}(u) \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_2}(u | w_{1t}, w_{2t})} \\ - \sum_{\substack{m_1, m_2, \\ w_1^n, w_2^n}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(m_1, m_2, w_1^n, w_2^n \mid E_\delta^1 = 0) \times \\ \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_2}(u) \log_2 \frac{1}{\mathcal{P}_t^{m_1 m_2}(u)} \quad (61)$$

$$\leq \frac{1}{n} I(U^n; M_1, M_2 \mid E_\delta^1 = 0) - I(U; W_1, W_2) + \delta \\ + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma, \tau_1, \tau_2}(E_\delta^1 = 1) \quad (62)$$

$$\leq \eta + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma, \tau_1, \tau_2}(E_\delta^1 = 1). \quad (63)$$

Similarly, we can control the beliefs of decoder \mathcal{D}_2 . This completes the proof of achievability. \square

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