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NADARAYA-WATSON ESTIMATOR FOR I.I.D. PATHS OF DIFFUSION PROCESSES

NICOLAS MARIE[†] AND AMÉLIE ROSIER[◊]

ABSTRACT. This paper deals with a nonparametric Nadaraya-Watson estimator \hat{b} of the drift function computed from independent continuous observations of a diffusion process. Risk bounds on \hat{b} and its discrete-time approximation are established. An extension of the leave-one-out cross validation bandwidth selection method for \hat{b} and some numerical experiments are provided.

MSC2020 subject classifications. 62G05 ; 62M05.

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1. INTRODUCTION

Consider the stochastic differential equation

$$(1) \quad X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s,$$

where $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are two continuous functions and $W = (W_t)_{t \in [0, T]}$ is a Brownian motion.

Since the 1980's, the statistical inference for stochastic differential equations (SDE) has been widely investigated by many authors in the parametric and in the nonparametric frameworks. Classically (see Kutoyants [15]), the estimators of the drift function are computed from one path of the solution to Equation (1) and converge when T goes to infinity. The existence and the uniqueness of the stationary solution to Equation (1) are then required, and obtained thanks to restrictive conditions on b .

Let $\mathcal{I} : (x, w) \mapsto \mathcal{I}(x, w)$ be the Itô map for Equation (1) and, for N copies W^1, \dots, W^N of W , consider

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$X^i = \mathcal{I}(x_0, W^i)$ for every $i \in \{1, \dots, N\}$. The estimation of the drift function b from continuous-time and discrete-time observations of (X^1, \dots, X^N) is a functional data analysis problem already investigated in the parametric framework (see Ditlevsen and De Gaetano [12], Overgaard et al. [17], Picchini, De Gaetano and Ditlevsen [18], Picchini and Ditlevsen [19], Comte, Genon-Catalot and Samson [6], Delattre and Lavielle [10], Delattre, Genon-Catalot and Samson [9], Dion and Genon-Catalot [11], Delattre, Genon-Catalot and Larédo [8], etc.) and more recently in the nonparametric framework (see Comte and Genon-Catalot [4, 5]). In [4, 5], the authors have extended to the diffusion processes framework the projection least squares estimators already well studied in the regression framework (see Cohen et al. [1] and Comte and Genon-Catalot [3]).

Under the appropriate conditions on b and σ recalled at Section 2, the distribution of X_t has a density $p_t(x_0, \cdot)$ for every $t \in (0, T]$, and then one can define

$$f(x) := \frac{1}{T - t_0} \int_{t_0}^T p_t(x_0, x) dt ; x \in \mathbb{R}$$

for any $t_0 > 0$. Clearly, f is a density function:

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{T - t_0} \int_{t_0}^T \int_{-\infty}^{\infty} p_t(x_0, x) dx dt = 1.$$

Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be a kernel and consider $K_h(x) := h^{-1}K(h^{-1}x)$ with $h \in (0, 1]$. In the spirit of Comte and Genon-Catalot [4, 5], our paper deals first with the continuous-time Nadaraya-Watson estimator

$$\widehat{b}_{N,h}(x) := \frac{\widehat{bf}_{N,h}(x)}{\widehat{f}_{N,h}(x)}$$

of the drift function b , where

$$\widehat{f}_{N,h}(x) := \frac{1}{N(T - t_0)} \sum_{i=1}^N \int_{t_0}^T K_h(X_t^i - x) dt$$

is an estimator of f and

$$\widehat{bf}_{N,h}(x) := \frac{1}{N(T - t_0)} \sum_{i=1}^N \int_{t_0}^T K_h(X_t^i - x) dX_t^i$$

is an estimator of bf . From independent copies of X continuously observed on $[0, T]$, $\widehat{b}_{N,h}$ is a natural extension of the Nadaraya-Watson estimator already well-studied in the regression framework (see Comte [2], Chapter 4 or Györfi et al. [13], Chapter 5). The paper also deals with a discrete-time approximate of the previous Nadaraya-Watson estimator:

$$\widehat{b}_{n,N,h}(x) := \frac{\widehat{bf}_{n,N,h}(x)}{\widehat{f}_{n,N,h}(x)},$$

where

$$\widehat{f}_{n,N,h}(x) := \frac{1}{nN} \sum_{i=1}^N \sum_{j=0}^{n-1} K_h(X_{t_j}^i - x)$$

is an estimator of f ,

$$\widehat{bf}_{n,N,h}(x) := \frac{1}{N(T - t_0)} \sum_{i=1}^N \sum_{j=0}^{n-1} K_h(X_{t_j}^i - x)(X_{t_{j+1}}^i - X_{t_j}^i)$$

is an estimator of bf , and (t_0, t_1, \dots, t_n) is the dissection of $[t_0, T]$ such that $t_j = t_0 + (T - t_0)j/n$ for every $j \in \{1, \dots, n\}$.

Section 2 deals with the existence and the regularity of the density $p_t(x_0, \cdot)$ of X_t for every $t \in (0, T]$, and with a Nikol'skii type condition fulfilled by f . Section 3 deals with a risk bound on $\widehat{b}_{N,h}$ and Section 4 with

a risk bound on $\widehat{b}_{n,N,h}$. Finally, Section 5 provides an extension of the leave-one-out cross-validation bandwidth selection method for $\widehat{b}_{N,h}$ and some numerical experiments. The proofs are postponed to Section 6.

Notations and basic definitions:

- For every $A, B \in \mathbb{R}$ such that $A < B$, $C^0([A, B]; \mathbb{R})$ is equipped with the uniform norm $\|\cdot\|_{\infty, A, B}$, and $C^0(\mathbb{R})$ is equipped with the uniform (semi-)norm $\|\cdot\|_{\infty}$.
- For every $p \in \overline{\mathbb{N}}$, $C_b^p(\mathbb{R}) := \cap_{j=0}^p \{\varphi \in C^p(\mathbb{R}) : \varphi^{(j)} \text{ is bounded}\}$.
- For every $p \geq 1$, $\mathbb{L}^p(\mathbb{R}) = \mathbb{L}^p(\mathbb{R}, dx)$ is equipped with its usual norm $\|\cdot\|_p$ such that

$$\|\varphi\|_p := \left(\int_{-\infty}^{\infty} \varphi(x)^p dx \right)^{1/p}; \quad \forall \varphi \in \mathbb{L}^p(\mathbb{R}).$$

- \mathbb{H}^2 is the space of the processes $(Y_t)_{t \in [0, T]}$, adapted to the filtration generated by W , such that

$$\mathbb{E} \left(\int_0^T Y_t^2 dt \right) < \infty.$$

2. PRELIMINARIES: REGULARITY OF THE DENSITY AND ESTIMATES

This section deals with the existence and the regularity of the density $p_t(x_0, \cdot)$ of X_t for every $t \in (0, T]$, with the Kusuoka-Stroock bounds on $(t, x) \mapsto p_t(x_0, x)$ and its derivatives, and then with a Nikol'skii type condition fulfilled by f .

In the sequel, in order to ensure the existence and the uniqueness of the (strong) solution to Equation (1), b and σ fulfill the following regularity assumption.

Assumption 2.1. *The functions b and σ are Lipschitz continuous.*

Now, assume that the solution X to Equation (1) fulfills the following assumption.

Assumption 2.2. *There exists $\beta \in \mathbb{N}^*$ such that, for any $t \in (0, T]$, the distribution of X_t has a β times continuously derivable density $p_t(x_0, \cdot)$. Moreover, for every $x \in \mathbb{R}$,*

$$p_t(x_0, x) \leq \frac{\mathbf{c}_{2.2,1}}{t^{1/2}} \exp \left[-\mathbf{m}_{2.2,1} \frac{(x - x_0)^2}{t} \right]$$

and

$$|\partial_x^\ell p_t(x_0, x)| \leq \frac{\mathbf{c}_{2.2,2}(\ell)}{t^{q_2(\ell)}} \exp \left[-\mathbf{m}_{2.2,2}(\ell) \frac{(x - x_0)^2}{t} \right]; \quad \forall \ell \in \{1, \dots, \beta\},$$

where all the constants are positive and not depending on t and x .

At Section 4, the following assumption on X is also required.

Assumption 2.3. *For any $x \in \mathbb{R}$, the function $t \in (0, T] \mapsto p_t(x_0, x)$ is continuously derivable. Moreover,*

$$|\partial_t p_t(x_0, x)| \leq \frac{\mathbf{c}_{2.3,3}}{t^{q_3}} \exp \left[-\mathbf{m}_{2.3,3} \frac{(x - x_0)^2}{t} \right]; \quad \forall t \in (0, T],$$

where $\mathbf{c}_{2.3,3}$, $\mathbf{m}_{2.3,3}$ and q_3 are three positive constants not depending on t and x .

Let us provide some examples of diffusion processes categories satisfying Assumptions 2.2 and/or 2.3.

Examples:

(1) Assume that the functions b and σ belong to $C_b^\infty(\mathbb{R})$, and that there exists $\alpha > 0$ such that

(2) $|\sigma(x)| > \alpha; \quad \forall x \in \mathbb{R}$.

Then, by Kusuoka and Stroock [14], Corollary 3.25, X fulfills Assumptions 2.2 and 2.3.

- (2) Assume that b is Lipschitz continuous (but not bounded) and that $\sigma \in C_b^1(\mathbb{R})$. Assume also that σ satisfies the non-degeneracy condition (2) and that σ' is Hölder continuous. Then, by Menozzi et al. [16], Theorem 1.2, X fulfills Assumption 2.2 with $\beta = 1$ (but not necessarily Assumption 2.3).

Note that the conditions required to apply Menozzi et al. [16], Theorem 1.2 are fulfilled by the so-called Ornstein-Uhlenbeck process, that is the solution to the Langevin equation:

$$(3) \quad X_t = x_0 - \theta \int_0^t X_s ds + \sigma W_t ; t \in \mathbb{R}_+,$$

where $\theta, \sigma > 0$ and $x_0 \in \mathbb{R}_+$. In this special case, since it is well-known that the solution to Equation (3) is a Gaussian process such that

$$\mathbb{E}(X_t) = x_0 e^{-\theta t} \quad \text{and} \quad \text{var}(X_t) = \frac{\sigma^2}{2\theta} (1 - e^{-2\theta t}) ; \forall t \in [0, T],$$

one can show that X also fulfills Assumption 2.3.

Remark 2.4. Under Assumptions 2.1 and 2.2, for any $p \geq 1$ and any continuous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ having polynomial growth, $t \in [0, T] \mapsto \mathbb{E}(|\varphi(X_t)|^p)$ is bounded. Indeed, for any $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}(|\varphi(X_t)|^p) &\leq \mathbf{c}_1 (1 + \mathbb{E}(|X_t|^{pq})) = \mathbf{c}_1 \int_{-\infty}^{\infty} (1 + |x|^{pq}) p_t(x_0, x) dx \\ &\leq \mathbf{c}_1 \mathbf{c}_{2.2,1} \int_{-\infty}^{\infty} (1 + |t^{1/2}x + x_0|^{pq}) e^{-\mathbf{m}_{2.2,1}x^2} dx \leq \mathbf{c}_2 (1 \vee T^{pq/2}) \end{aligned}$$

where

$$\mathbf{c}_2 = \mathbf{c}_1 \mathbf{c}_{2.2,1} \int_{-\infty}^{\infty} [1 + (|x| + |x_0|)^{pq}] e^{-\mathbf{m}_{2.2,1}x^2} dx$$

and the constants $\mathbf{c}_1, q > 0$ only depend on φ . Moreover,

$$\begin{aligned} N_{p,f}(\varphi) &:= \int_{-\infty}^{\infty} |\varphi(x)|^p f(x) dx \\ &= \frac{1}{T - t_0} \int_{t_0}^T \mathbb{E}(|\varphi(X_t)|^p) dt \leq \mathbf{c}_2 (1 \vee T^{pq/2}). \end{aligned}$$

Then, $\varphi \in \mathbb{L}^p(\mathbb{R}, f(x)dx)$ and $N_{p,f}(\varphi)$ is bounded by a constant which doesn't depend on t_0 . In particular, the remark applies to b and σ with $q = 1$ by Assumption 2.1.

Finally, let us show that f fulfills a Nikol'skii type condition.

Corollary 2.5. Under Assumptions 2.1 and 2.2, there exists $\mathbf{c}_{2.5} > 0$, not depending on t_0 , such that for every $\ell \in \{0, \dots, \beta - 1\}$ and $\theta \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} [f^{(\ell)}(x + \theta) - f^{(\ell)}(x)]^2 dx \leq \frac{\mathbf{c}_{2.5}}{t_0^{2q_2(\ell+1)}} (\theta^2 + |\theta|^3).$$

Remark 2.6. Assumption 2.2, Assumption 2.3 and Corollary 2.5 are crucial in the sequel, but t_0 has to be chosen carefully to get reasonable risk bounds on the estimators $\widehat{b}_{N,h}$ and $\widehat{b}_{n,N,h}$. Indeed, the behavior of the Kusuoka-Stroock bounds on $(t, x) \mapsto p_t(x_0, x)$ and its derivatives is singular at point $(0, x_0)$. This is due to the fact that the distribution of X at time 0 is a Dirac measure while that it has a smooth density with respect to Lebesgue's measure for every $t \in (0, T]$. In the sequel, each risk bound derived from Assumption 2.2, Assumption 2.3 and Corollary 2.5 only depends on t_0 through a multiplicative constant of order $1/\min\{t_0^\alpha, T - t_0\}$ ($\alpha > 0$). So, to take $t_0 \in [1, T - 1]$ when $T > 1$ gives constants of same order than in the classic regression framework.

3. RISK BOUND ON THE CONTINUOUS-TIME NADARAYA-WATSON ESTIMATOR

This section deals with risk bounds on $\widehat{f}_{N,h}$, $\widehat{bf}_{N,h}$, and then on the Nadaraya-Watson estimator $\widehat{b}_{N,h}$.

In the sequel, the kernel K fulfills the following usual assumption.

Assumption 3.1. *The kernel K is symmetric and belongs to $\mathbb{L}^2(\mathbb{R})$. Moreover,*

$$\int_{-\infty}^{\infty} |z^{\beta+1}K(z)|dz < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} z^\ell K(z)dz = 0 ; \forall \ell \in \{1, \dots, \beta\}.$$

The following proposition provides a risk bound on $\widehat{f}_{N,h}$.

Proposition 3.2. *Under Assumptions 2.1, 2.2 and 3.1,*

$$\mathbb{E}(\|\widehat{f}_{N,h} - f\|_2^2) \leq \mathfrak{c}_{3.2}(t_0)h^{2\beta} + \frac{\|K\|_2^2}{Nh}$$

with

$$\mathfrak{c}_{3.2}(t_0) = \frac{\mathfrak{c}_{2.5}}{(\beta-1)!2^{2q_2(\beta)}} \left(\int_{-\infty}^{\infty} |z|^\beta (1 + |z|^{1/2})K(z)dz \right)^2.$$

Note that thanks to Proposition 3.2, the bias-variance tradeoff is reached by (the risk bound on) $\widehat{f}_{N,h}$ when h is of order $N^{-1/(2\beta+1)}$. Moreover, by Remark 2.6, to take $t_0 \geq 1$ when $T > 1$ gives

$$\mathbb{E}(\|\widehat{f}_{N,h} - f\|_2^2) \leq \mathfrak{c}_{3.2}h^{2\beta} + \frac{\|K\|_2^2}{Nh} \quad \text{with} \quad \mathfrak{c}_{3.2} = \frac{\mathfrak{c}_{2.5}}{(\beta-1)!^2} \left(\int_{-\infty}^{\infty} |z|^\beta (1 + |z|^{1/2})K(z)dz \right)^2.$$

In the sequel, $\mathbb{L}^2(\mathbb{R}, f(x)dx)$ is equipped with the f -weighted norm $\|\cdot\|_f$ defined by

$$\|\varphi\|_f := \left(\int_{-\infty}^{\infty} \varphi(x)^2 f(x)dx \right)^{1/2}$$

for every $\varphi \in \mathbb{L}^2(\mathbb{R}, f(x)dx)$. Let us recall that by Remark 2.4, for every $\varphi \in \mathbb{L}^2(\mathbb{R}, f(x)dx)$, $\|\varphi\|_f$ is bounded by a constant which doesn't depend on t_0 .

The following proposition provides a risk bound on $\widehat{bf}_{N,h}$.

Proposition 3.3. *Under Assumptions 2.1, 2.2 and 3.1,*

$$\mathbb{E}(\|\widehat{bf}_{N,h} - bf\|_2^2) \leq \|(bf)_h - bf\|_2^2 + \frac{\mathfrak{c}_{3.3}(t_0)}{Nh}$$

with $(bf)_h := K_h * (bf)$ and

$$\mathfrak{c}_{3.3}(t_0) = 2\|K\|_2^2 \left(\|b\|_f^2 + \frac{1}{T-t_0} \|\sigma\|_f^2 \right).$$

Assume that b is β times continuously derivable and that there exists $\varphi \in \mathbb{L}^1(\mathbb{R}, |z|^{\beta-1}K(z)dz)$ such that, for every $\theta \in \mathbb{R}$ and $h \in (0, 1]$,

$$(4) \quad \int_{-\infty}^{\infty} [(bf)^{(\beta-1)}(x+h\theta) - (bf)^{(\beta-1)}(x)]^2 dx \leq \varphi(\theta)h^2.$$

Then, $\|(bf)_h - bf\|_2^2$ is of order $h^{2\beta}$, and by Proposition 3.3, the bias-variance tradeoff is also reached by $\widehat{bf}_{N,h}$ when h is of order $N^{-1/(2\beta+1)}$. Moreover, by Remark 2.6, to take $t_0 \leq T-1$ when $T > 1$ gives

$$\mathbb{E}(\|\widehat{bf}_{N,h} - bf\|_2^2) \leq \|(bf)_h - bf\|_2^2 + \frac{\mathfrak{c}_{3.3}}{Nh} \quad \text{with} \quad \mathfrak{c}_{3.3} = 2\|K\|_2^2(\|b\|_f^2 + \|\sigma\|_f^2).$$

Finally, Propositions 3.2 and 3.3 allow to provide a risk bound on the Nadaraya-Watson estimator $\widehat{b}_{N,h}$.

Proposition 3.4. *Consider the 2 bandwidths Nadaraya-Watson (2bNW) estimator*

$$\widehat{b}_{N,h,\eta}(x) := \frac{\widehat{bf}_{N,h}(x)}{\widehat{f}_{N,\eta}(x)} \quad \text{with } h, \eta > 0,$$

and assume that $f(x) > m > 0$ for every $x \in [A, B]$ ($A, B \in \mathbb{R}$ such that $A < B$). Under Assumptions 2.1, 2.2 and 3.1,

$$\mathbb{E}(\|\widehat{b}_{N,h,\eta} - b\|_{f,A,B}^2) \leq \frac{\mathbf{c}_{3.4}(A, B)}{m^2} \left[\|(bf)_h - bf\|_2^2 + \frac{\mathbf{c}_{3.3}(t_0)}{Nh} + 2\|b\|_f^2 \left(\mathbf{c}_{3.2}(t_0)\eta^2 + \frac{\|K\|_2^2}{N\eta} \right) \right]$$

with $\mathbf{c}_{3.4}(A, B) := 8(\|f\|_\infty \vee \|b^2 f\|_{\infty, A, B})$ and $\|\varphi\|_{f, A, B} := \|\varphi \mathbf{1}_{[A, B]}\|_f$ for every $\varphi \in \mathbb{L}^2(\mathbb{R}, f(x)dx)$.

Proposition 3.4 says that the risk of $\widehat{b}_{N,h,\eta}$ can be controlled by the sum of those of $\widehat{bf}_{N,h}$ and $\widehat{f}_{N,\eta}$ up to a multiplicative constant. However, the limitation of this result is that m is unknown in general and must be replaced by an estimator as well. Most of the time, as stated in Comte [2], Chapter 4, the minimum of an estimator of f is taken to choose m in practice.

Note also that there is no big deal with the fact that $\mathbf{c}_{3.4}(A, B)$ depends on $\|b^2 f\|_{\infty, A, B}$ because thanks to Assumption 2.2, f has a sub-Gaussian tail, so there is a large class of functions b such that $b^2 f$ is bounded on \mathbb{R} .

Finally, if $h = \eta$ and bf satisfies Condition (4), then the risk bound on $\widehat{b}_{N,h,h} = \widehat{b}_{N,h}$ is of order $h^{2\beta} + 1/(Nh)$, and the bias-variance tradeoff is again reached when h is of order $N^{-1/(2\beta+1)}$.

4. RISK BOUND ON THE DISCRETE-TIME APPROXIMATE NADARAYA-WATSON ESTIMATOR

This section deals with risk bounds on $\widehat{f}_{n,N,h}$, $\widehat{bf}_{n,N,h}$ and then on the approximate Nadaraya-Watson estimator $\widehat{b}_{n,N,h}$.

In the sequel, in addition to Assumption 3.1, K fulfills the following one.

Assumption 4.1. *The kernel K is two times continuously derivable on \mathbb{R} and $K', K'' \in \mathbb{L}^2(\mathbb{R})$.*

Compactly supported kernels belonging to $C^2(\mathbb{R})$, or Gaussian kernels, fulfill Assumption 3.1.

Proposition 4.2. *Under Assumptions 2.1, 2.2, 2.3, 3.1 and 4.1, if*

$$\frac{1}{nh^2} \leq 1,$$

then there exists a constant $\mathbf{c}_{4.2} > 0$, not depending on h, N, n and t_0 , such that

$$\mathbb{E}(\|\widehat{f}_{n,N,h} - f\|_2^2) \leq \frac{\mathbf{c}_{4.2}}{\min\{t_0^{2q_2(\beta)}, t_0^{2q_3}\}} \left(h^{2\beta} + \frac{1}{Nh} + \frac{1}{n^2} \right) + \frac{1}{Nnh^3}.$$

Assume that $\beta = 1$ (extreme case) and h is of order $N^{-1/3}$. As mentioned at Section 3, under this condition, the bias-variance tradeoff is reached by $\widehat{f}_{N,h}$. Then, the approximation error of $\widehat{f}_{n,N,h}$ is of order $1/n$, which is the order of the variance of the Brownian motion increments along the dissection (t_0, t_1, \dots, t_n) of $[t_0, T]$. For this reason, the risk bound established in Proposition 4.2 is satisfactory. Moreover, by Remark 2.6, to take $t_0 \geq 1$ when $T > 1$ gives

$$\mathbb{E}(\|\widehat{f}_{n,N,h} - f\|_2^2) \leq \mathbf{c}_{4.2} \left(h^{2\beta} + \frac{1}{Nh} + \frac{1}{n^2} \right) + \frac{1}{Nnh^3}.$$

Proposition 4.3. *Consider $\varepsilon \in (0, 1)$. Under Assumptions 2.1, 2.2, 2.3, 3.1 and 4.1, if*

$$\frac{1}{nh^{2-\varepsilon}} \leq 1,$$

the kernel K belongs to $\mathbb{L}^4(\mathbb{R})$ and $z \mapsto zK'(z)$ belongs to $\mathbb{L}^2(\mathbb{R})$, then there exist a constant $\mathbf{c}_{4.3} > 0$, not depending on ε, h, N, n and t_0 , and a constant $\mathbf{c}_{4.3}(\varepsilon) > 0$, depending on ε but not on h, N, n and t_0 ,

such that

$$\mathbb{E}(\|\widehat{bf}_{n,N,h} - bf\|_2^2) \leq \frac{\mathbf{c}_{4.3}}{\min\{t_0^{1/2}, t_0^{2q_3}, T - t_0\}} \left(\|(bf)_h - bf\|_2^2 + \frac{1}{Nh} + \frac{1}{n} \right) + \frac{\mathbf{c}_{4.3}(\varepsilon)}{\min\{1, t_0^{(1-\varepsilon)/2}\}} \cdot \frac{1}{Nnh^{3+\varepsilon}}.$$

Remark 4.4. Note that if b and σ are bounded, Proposition 4.3 can be improved. Precisely, with $\varepsilon = 0$ and without the additional conditions $K \in \mathbb{L}^4(\mathbb{R})$ and $z \mapsto zK'(z)$ belongs to $\mathbb{L}^2(\mathbb{R})$, the risk bound on $\widehat{bf}_{n,N,h}$ is of same order than in Proposition 4.2 (see Remark 6.3 for details).

Assume that $\beta = 1$ (extreme case), bf fulfills Condition (4) and h is of order $N^{-1/3}$. Then, for $\varepsilon > 0$ as close as possible to 0, the approximation error of $\widehat{bf}_{n,N,h}$ is of order $N^{\varepsilon/3}/n$. If in addition b and σ are bounded, thanks to Remark 4.4, with $\varepsilon = 0$ and without the additional conditions $K \in \mathbb{L}^4(\mathbb{R})$ and that $z \mapsto zK'(z)$ belongs to $\mathbb{L}^2(\mathbb{R})$, then the approximation error of $\widehat{bf}_{n,N,h}$ is of order $1/n$ as the error of $\widehat{f}_{n,N,h}$. Moreover, by Remark 2.6, to take $t_0 \in [1, T - 1]$ when $T > 1$ gives

$$\mathbb{E}(\|\widehat{bf}_{n,N,h} - bf\|_2^2) \leq \mathbf{c}_{4.3} \left(\|(bf)_h - bf\|_2^2 + \frac{1}{Nh} + \frac{1}{n} \right) + \frac{\mathbf{c}_{4.3}(\varepsilon)}{Nnh^{3+\varepsilon}}.$$

Finally, Propositions 4.2 and 4.3 allow to provide a risk bound on the approximate Nadaraya-Watson estimator $\widehat{b}_{n,N,h}$.

Proposition 4.5. Consider $\varepsilon > 0$, and assume that $f(x) > m > 0$ for every $x \in [A, B]$ ($A, B \in \mathbb{R}$ such that $A < B$). Under the assumptions of Proposition 4.3, there exist a constant $\mathbf{c}_{4.5} > 0$, not depending on $\varepsilon, A, B, h, N, n$ and t_0 , and a constant $\mathbf{c}_{4.5}(\varepsilon) > 0$, depending on ε but not on A, B, h, N, n and t_0 , such that

$$\mathbb{E}(\|\widehat{b}_{n,N,h} - b\|_{f,A,B}^2) \leq \frac{\mathbf{c}_{3.4}(A, B)}{m^2 \min\{1, t_0^{(1-\varepsilon)/2}, t_0^{1/2}, t_0^{2q_2(\beta)}, t_0^{2q_3}, T - t_0\}} \times \left[\mathbf{c}_{4.5} \left(\|(bf)_h - bf\|_2^2 + h^{2\beta} + \frac{1}{Nh} + \frac{1}{n} \right) + \frac{\mathbf{c}_{4.5}(\varepsilon)}{Nnh^{3+\varepsilon}} \right].$$

The proof and the meaning of Proposition 4.5 given Propositions 4.2 and 4.3 are exactly the same than for Proposition 3.4 given Propositions 3.2 and 3.3.

5. BANDWIDTH SELECTION AND NUMERICAL EXPERIMENTS

In the nonparametric regression framework, it has been established in Comte and Marie [7] that the leave-one-out cross-validation (looCV) bandwidth selection method for Nadaraya-Watson's estimator is numerically more satisfactory than two alternative procedures based on Goldenshluger-Lepski's method and on the PCO method. For this reason, this section deals with an extension of the looCV method to the Nadaraya-Watson estimator studied in this paper, even if it seems difficult to prove a theoretical risk bound on the associated adaptive estimator. Numerical experiments are provided at Subsection 5.2.

5.1. An extension of the leave-one-out cross-validation method. First of all, note that

$$\widehat{b}_{N,h}(x) = \sum_{i=1}^N \int_{t_0}^T \omega_t^i(x) dX_t^i$$

with

$$\omega_t^i(x) := \frac{K_h(X_t^i - x)}{\sum_{j=1}^N \int_{t_0}^T K_h(X_t^j - x) dt} ; \forall (t, i) \in [t_0, T] \times \{1, \dots, N\},$$

and that

$$\sum_{i=1}^N \int_{t_0}^T \omega_t^i(x) dt = 1.$$

This nice (weighted) representation of $\widehat{b}_{N,h}(x)$ allows us to consider the following extension of the well-known looCV criterion in our framework:

$$\text{CV}(h) := \sum_{i=1}^N \left(\int_{t_0}^T \widehat{b}_{N,h}^{-i}(X_t^i)^2 dt - 2 \int_{t_0}^T \widehat{b}_{N,h}^{-i}(X_t^i) dX_t^i \right)$$

with

$$\widehat{b}_{N,h}^{-i}(x) := \sum_{j \in \{1, \dots, N\} \setminus \{i\}} \int_{t_0}^T \omega_t^j(x) dX_t^j ; \forall i \in \{1, \dots, N\}.$$

Let us explain heuristically this extension of the looCV criterion. By assuming that $dX_t = Y_t dt$, Equation (1) can be expressed as a regression-like model:

$$Y_t = b(X_t) + \sigma(X_t) \dot{W}_t.$$

Then, clearly, a natural extension of the looCV criterion is

$$\begin{aligned} \text{CV}^*(h) &:= \sum_{i=1}^N \int_{t_0}^T (Y_t^i - \widehat{b}_{N,h}^{-i}(X_t^i))^2 dt \\ &= \text{CV}(h) + \sum_{i=1}^N \int_{t_0}^T (Y_t^i)^2 dt. \end{aligned}$$

Of course $\text{CV}^*(h)$ is not satisfactory because the last term of its previous decomposition doesn't exist, but since this term doesn't depend on h , to minimize $\text{CV}^*(\cdot)$ is equivalent to minimize $\text{CV}(\cdot)$ which only involves quantities existing without the condition $dX_t = Y_t dt$.

Finally, for X^1, \dots, X^N observed at times t_0, \dots, t_n such that $t_j = t_0 + j(T - t_0)/n$ for every $j \in \{0, \dots, N\}$, thanks to Section 4, it is reasonable to minimize the following discrete-time criterion in numerical experiments:

$$\text{CV}_n(h) := \sum_{i=1}^N \left[\frac{T - t_0}{n} \sum_{j=0}^{n-1} \widehat{b}_{n,N,h}^{-i}(X_{t_j}^i)^2 - 2 \sum_{j=0}^{n-1} \widehat{b}_{n,N,h}^{-i}(X_{t_j}^i)(X_{t_{j+1}}^i - X_{t_j}^i) \right].$$

5.2. Numerical experiments. Some numerical experiments on our estimation method are presented in this subsection. The discrete-time approximate Nadaraya-Watson estimator is computed on 4 datasets generated by SDEs with various types of vector fields. In each case, the bandwidth of the NW estimator is selected via the looCV method introduced at Subsection 5.1. On the one hand, we consider two models with the same linear drift function but with an additive noise for the first one and a multiplicative noise for the second one:

1. The so-called Langevin equation, that is

$$X_t = x_0 - \int_0^t X_s ds + 0.1 \cdot W_t.$$

2. The hyperbolic diffusion process, that is

$$X_t = x_0 - \int_0^t X_s ds + 0.1 \int_0^t \sqrt{1 + X_s^2} dW_s.$$

On the other hand, we consider two models having the same non-linear drift function, involving $\sin(\cdot)$, but here again with an additive noise for the first one and a multiplicative noise for the second one:

3. The third model is defined by

$$X_t = x_0 - \int_0^t (X_s + \sin(4X_s)) ds + 0.1 \cdot W_t,$$

4. and the fourth model is defined by

$$X_t = x_0 - \int_0^t (X_s + \sin(4X_s))ds + 0.1 \int_0^t (2 + \cos(X_s))dW_s.$$

The models and the estimator are implemented by taking $N = 50$, $n = 50$, $T = 5$, $x_0 = 2$, $t_0 = 1$ and K the Gaussian kernel $z \mapsto (2\pi)^{-1/2}e^{-z^2/2}$. For Models 1 and 2, the estimator of the drift function is computed for the bandwidths set

$$\mathcal{H}_1 := \{0.02k ; k = 1, \dots, 10\},$$

and for Models 3 and 4, it is computed for the bandwidths set

$$\mathcal{H}_2 := \{0.01k ; k = 1, \dots, 10\}.$$

Each set of bandwidths has been chosen after testing different values of h , to see with which ones the estimation performs better. To choose smaller values for the second set of simulations allows to check that the looCV method does not systematically select the smallest bandwidth.

On each figure, the true drift function (in red) and the looCV adaptive NW estimator (in blue) are plotted on the left-hand side, and the beam of proposals is plotted in green on the right-hand side.

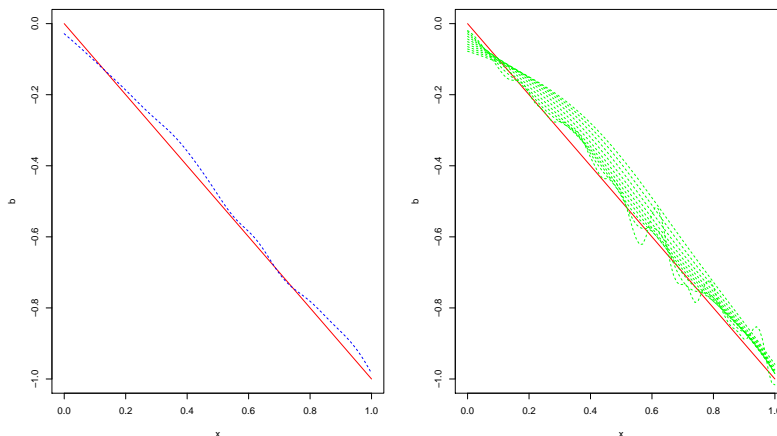


FIGURE 1. LooCV NW estimation for Model 1 (Langevin equation), $\hat{h} = 0.04$.

On Figures 1 and 2, one can see that the drift function is well estimated by the looCV adaptive NW estimator, with a mean MSE (computed over the 10 proposals) equal to 0.00055 for the Langevin equation and to 0.00094 for the hyperbolic diffusion process. As presumed, the multiplicative noise in Model 2 slightly complicates the estimation. Note that when the bandwidth is too small, the estimation degrades, but the looCV method selects a higher value of h which performs better on the estimation. This means that, as expected, the looCV method selects a reasonable approximation of the bandwidth for which the NW estimator reaches the bias-variance tradeoff.

On Figures 3 and 4, one can see that the drift function of Models 3 and 4 is still well estimated by our method. However, note that there is a significant degradation of the mean MSE, which is equal to 0.00384 for Model 3 and to 0.01185 for Model 4. This is related to the *complexity* of the drift function to estimate. Once again, to consider a multiplicative noise in Model 4 degrades the estimation quality with respect to Model 3. As for Models 1 and 2, note that the looCV does not systematically select the smallest bandwidth.

Finally, for each model, Table 1 gathers the mean of the MSEs of 10 looCV NW estimations of the drift function. The values of the MSEs are relatively small but goes higher when the drift function becomes complicated. As stated below, the MSEs are also impacted when the dataset is generated by a SDE with multiplicative noise as the hyperbolic diffusion process (Model 2) or Model 4. Note that the standard

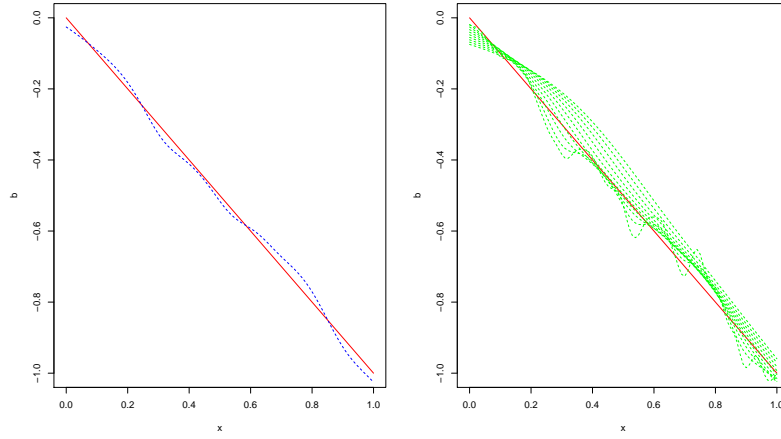


FIGURE 2. LooCV NW estimation for Model 2 (hyperbolic diffusion process), $\hat{h} = 0.06$.

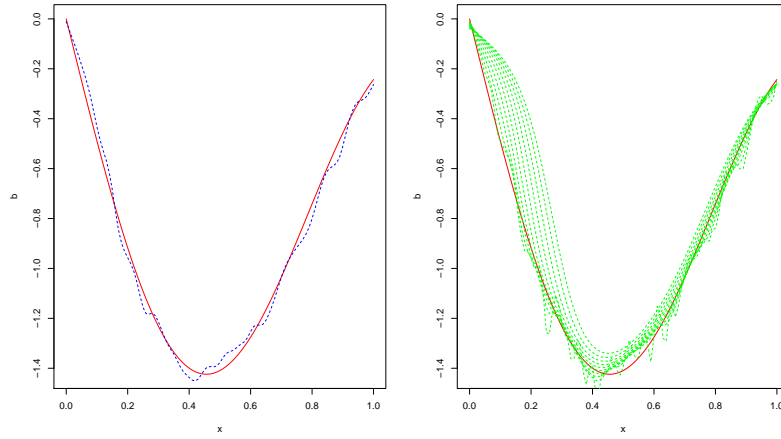
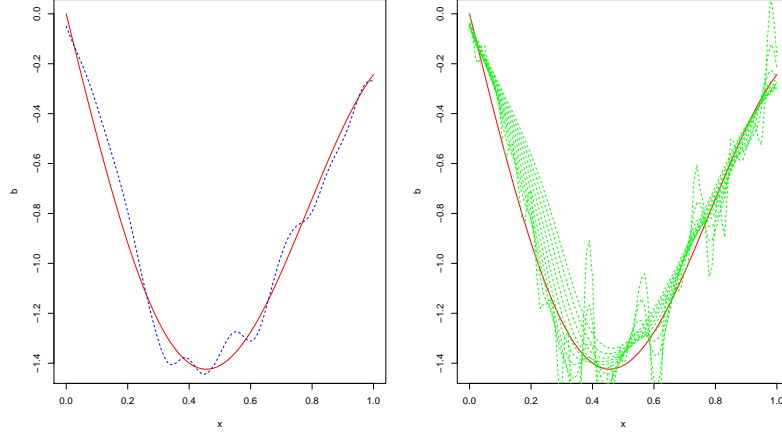


FIGURE 3. LooCV NW estimation for Model 3, $\hat{h} = 0.02$.

deviation remains quite small for all simulations, which means that the MSE is not biased by a few too high or too low values.

	Model 1	Model 2	Model 3	Model 4
$100 \times \text{MSE}$	0.0878	0.1004	0.2633	0.9632
$100 \times \text{StD}$	0.0463	0.0548	0.0819	0.4568

TABLE 1. Mean MSE and StD for 10 LooCV NW estimations.


 FIGURE 4. LooCV NW estimation for Model 4, $\hat{h} = 0.04$.

6. PROOFS

6.1. Proof of Corollary 2.5. Consider $\ell \in \{0, \dots, \beta - 1\}$ and $\theta \in \mathbb{R}_+$. Thanks to the bound on $(t, x) \mapsto \partial_x^{\ell+1} p_t(x_0, x)$ given in Assumption 2.2,

$$\begin{aligned}
 \|f^{(\ell)}(\cdot + \theta) - f^{(\ell)}\|_2^2 &= \int_{-\infty}^{\infty} [f^{(\ell)}(x + x_0 + \theta) - f^{(\ell)}(x + x_0)]^2 dx \\
 &\leq \frac{1}{T - t_0} \int_{t_0}^T \int_{-\infty}^{\infty} (\partial_2^\ell p_t(x_0, x + x_0 + \theta) - \partial_2^\ell p_t(x_0, x + x_0))^2 dx dt \\
 &\leq \frac{\theta^2}{T - t_0} \int_{t_0}^T \int_{-\infty}^{\infty} \sup_{z \in [x, x + \theta]} |\partial_2^{\ell+1} p_t(x_0, z + x_0)|^2 dx dt \\
 &\leq \mathbf{c}_{2,2,2}(\ell + 1)^2 \frac{\theta^2}{T - t_0} \int_{t_0}^T \frac{1}{t^{2q_2(\ell+1)}} \int_{-\infty}^{\infty} \sup_{z \in [x, x + \theta]} \exp\left(-2\mathbf{m}_{2,2,2}(\ell + 1) \frac{z^2}{t}\right) dx dt \\
 &= \mathbf{c}_{2,2,2}(\ell + 1)^2 \frac{\theta^2}{T - t_0} \int_{t_0}^T \frac{1}{t^{2q_2(\ell+1)}} \times \\
 &\quad \left[\int_{-\infty}^{-\theta} \exp\left(-2\mathbf{m}_{2,2,2}(\ell + 1) \frac{(x + \theta)^2}{t}\right) dx + \theta + \int_0^{\infty} \exp\left(-2\mathbf{m}_{2,2,2}(\ell + 1) \frac{x^2}{t}\right) dx \right] dt \\
 &\leq \frac{1}{t_0^{2q_2(\ell+1)}} \left[\mathbf{c}_1 \theta^2 + \theta^3 \max_{k \in \{0, \dots, \beta - 1\}} \mathbf{c}_{2,2,2}(k + 1)^2 \right]
 \end{aligned}$$

with

$$\mathbf{c}_1 = 2 \max_{k \in \{0, \dots, \beta - 1\}} \left\{ \mathbf{c}_{2,2,2}(k + 1)^2 \int_0^{\infty} \exp\left(-2\mathbf{m}_{2,2,2}(k + 1) \frac{x^2}{T}\right) dx \right\},$$

and the same way,

$$\begin{aligned}
\|f^{(\ell)}(\cdot - \theta) - f^{(\ell)}\|_2^2 &\leq \frac{\theta^2}{T - t_0} \int_{t_0}^T \int_{-\infty}^{\infty} \sup_{z \in [x - \theta, x]} |\partial_2^{\ell+1} p_t(x_0, z + x_0)|^2 dx dt \\
&\leq \mathfrak{c}_{2.2,2}(\ell + 1)^2 \frac{\theta^2}{T - t_0} \int_{t_0}^T \frac{1}{t^{2q_2(\ell+1)}} \times \\
&\quad \left[\int_{-\infty}^0 \exp\left(-2\mathfrak{m}_{2.2,2}(\ell + 1) \frac{x^2}{t}\right) dx + \theta + \int_{\theta}^{\infty} \exp\left(-2\mathfrak{m}_{2.2,2}(\ell + 1) \frac{(x - \theta)^2}{t}\right) dx \right] dt \\
&\leq \frac{1}{t_0^{2q_2(\ell+1)}} \left[\mathfrak{c}_1 \theta^2 + \theta^3 \max_{k \in \{0, \dots, \beta-1\}} \mathfrak{c}_{2.2,2}(k + 1)^2 \right].
\end{aligned}$$

This concludes the proof.

6.2. Proof of Proposition 3.2. First of all, the bias of $\widehat{f}_{N,h}(x)$ is denoted by $\mathfrak{b}(x)$ and its variance by $\mathfrak{v}(x)$. Moreover, let us recall the bias-variance decomposition of the \mathbb{L}^2 -risk of $\widehat{f}_{N,h}$:

$$\mathbb{E}(\|\widehat{f}_{N,h} - f\|_2^2) = \int_{-\infty}^{\infty} \mathfrak{b}(x)^2 dx + \int_{-\infty}^{\infty} \mathfrak{v}(x) dx.$$

On the one hand, let us find a suitable bound on the integrated variance of $\widehat{f}_{N,h}$. Since X^1, \dots, X^N are i.i.d. copies of X , and thanks to Jensen's inequality,

$$\begin{aligned}
\mathfrak{v}(x) &= \text{var} \left(\frac{1}{N(T - t_0)} \sum_{i=1}^N \int_{t_0}^T K_h(X_t^i - x) dt \right) \\
&= \frac{1}{N(T - t_0)^2} \text{var} \left(\int_{t_0}^T K_h(X_t - x) dt \right) \leq \frac{1}{N} \mathbb{E} \left[\left(\int_{t_0}^T K_h(X_t - x) \frac{dt}{T - t_0} \right)^2 \right] \\
&\leq \frac{1}{N(T - t_0)} \int_{t_0}^T \mathbb{E}(K_h(X_t - x)^2) ds = \frac{1}{N} \int_{-\infty}^{\infty} K_h(z - x)^2 f(z) dz.
\end{aligned}$$

Thus, since K is symmetric,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathfrak{v}(x) dx &\leq \frac{1}{N} \int_{-\infty}^{\infty} f(z) \int_{-\infty}^{\infty} K_h(z - x)^2 dx dz \\
&= \frac{1}{Nh} \left(\int_{-\infty}^{\infty} f(z) dz \right) \left(\int_{-\infty}^{\infty} K(x)^2 dx \right) = \frac{\|K\|_2^2}{Nh}.
\end{aligned}$$

On the other hand, let us find a suitable bound on the integrated squared-bias of $\widehat{f}_{N,h}(x)$. Since X^1, \dots, X^N are i.i.d. copies of X ,

$$\begin{aligned}
\mathfrak{b}(x) &= \frac{1}{T - t_0} \int_{t_0}^T \mathbb{E}(K_h(X_t - x)) dt - f(x) \\
&= \frac{1}{h} \int_{-\infty}^{\infty} K \left(\frac{z - x}{h} \right) f(z) dz - f(x) \\
&= \int_{-\infty}^{\infty} K(z) (f(hz + x) - f(x)) dz.
\end{aligned}$$

First, assume that $\beta = 1$. By Assumption 3.1, the generalized Minkowski inequality and Corollary 2.5,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathfrak{b}(x)^2 dx &\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K(z) (f(hz + x) - f(x)) dz \right)^2 dx \\
&\leq \left[\int_{-\infty}^{\infty} K(z) \left(\int_{-\infty}^{\infty} (f(hz + x) - f(x))^2 dx \right)^{1/2} dz \right]^2 \leq \mathfrak{c}_1(t_0) h^2
\end{aligned}$$

with

$$\mathbf{c}_1(t_0) = \frac{\mathbf{c}_{2.5}}{t_0^{2q_2(1)}} \left(\int_{-\infty}^{\infty} |z|(1+|z|^{1/2})K(z)dz \right)^2.$$

Now, assume that $\beta \geq 2$. By the Taylor formula with integral remainder, for every $z \in \mathbb{R}$,

$$f(hz+x) - f(x) = \mathbf{1}_{\beta \geq 3} \sum_{\ell=1}^{\beta-2} \frac{(hz)^\ell}{\ell!} f^{(\ell)}(x) + \frac{(hz)^{\beta-1}}{(\beta-1)!} \int_0^1 (1-\tau)^{\beta-2} f^{(\beta-1)}(\tau hz+x) d\tau.$$

Then, by Assumption 3.1, the generalized Minkowski inequality and Corollary 2.5,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{b}(x)^2 dx &\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K(z)(f(hz+x) - f(x)) dz \right)^2 dx \\ &= \frac{h^{2(\beta-1)}}{|(\beta-1)!|^2} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} z^{\beta-1} K(z) \int_0^1 (1-\tau)^{\beta-2} [f^{(\beta-1)}(\tau hz+x) - f^{(\beta-1)}(x)] d\tau dz \right)^2 dx \\ &\leq \frac{h^{2(\beta-1)}}{|(\beta-1)!|^2} \times \\ &\quad \left[\int_{-\infty}^{\infty} |z|^{\beta-1} K(z) \int_0^1 (1-\tau)^{\beta-2} \left(\int_{-\infty}^{\infty} [f^{(\beta-1)}(\tau hz+x) - f^{(\beta-1)}(x)]^2 dx \right)^{1/2} d\tau dz \right]^2 \\ &\leq \frac{\mathbf{c}_{2.5} h^{2\beta}}{|(\beta-1)!|^2 t_0^{2q_2(\beta)}} \left(\int_{-\infty}^{\infty} |z|^{\beta-1} K(z) \int_0^1 (1-\tau)^{\beta-2} [\tau|z| + (\tau|z|)^{3/2}] d\tau dz \right)^2 \leq \mathbf{c}_2(t_0) h^{2\beta} \end{aligned}$$

with

$$\mathbf{c}_2(t_0) = \frac{\mathbf{c}_{2.5}}{|(\beta-1)!|^2 t_0^{2q_2(\beta)}} \left(\int_{-\infty}^{\infty} |z|^\beta (1+|z|^{1/2})K(z)dz \right)^2.$$

This concludes the proof.

6.3. Proof of Proposition 3.3. First of all,

$$\mathbb{E}(\|\widehat{bf}_{N,h} - bf\|_2^2) = \int_{-\infty}^{\infty} \mathbf{b}(x)^2 dx + \int_{-\infty}^{\infty} \mathbf{v}(x) dx$$

where $\mathbf{b}(x)$ (resp. $\mathbf{v}(x)$) is the bias (resp. the variance) term of $\widehat{bf}_{N,h}(x)$ for any $x \in \mathbb{R}$. On the one hand, let us find a suitable bound on the integrated variance of $\widehat{bf}_{N,h}$. Since X^1, \dots, X^N are i.i.d. copies of X ,

$$\begin{aligned} \mathbf{v}(x) &= \text{var} \left(\frac{1}{N(T-t_0)} \sum_{i=1}^N \int_{t_0}^T K_h(X_t^i - x) dX_t^i \right) \leq \frac{1}{N(T-t_0)^2} \mathbb{E} \left[\left(\int_{t_0}^T K_h(X_t - x) dX_t \right)^2 \right] \\ &\leq \frac{2}{N} \mathbb{E} \left[\left(\int_{t_0}^T K_h(X_t - x) b(X_t) \frac{dt}{T-t_0} \right)^2 + \frac{1}{(T-t_0)^2} \left(\int_{t_0}^T K_h(X_t - x) \sigma(X_t) dW_t \right)^2 \right]. \end{aligned}$$

In the right-hand side of the previous inequality, Jensen's inequality on the first term and the isometry property for Itô's integral on the second one give

$$\begin{aligned} \mathbf{v}(x) &\leq \frac{2}{N(T-t_0)} \int_{t_0}^T \mathbb{E}[K_h(X_t - x)^2 b(X_t)^2] dt + \frac{2}{N(T-t_0)^2} \int_{t_0}^T \mathbb{E}[K_h(X_t - x)^2 \sigma(X_t)^2] dt \\ &= \frac{2}{N} \int_{-\infty}^{\infty} K_h(z-x)^2 b(z)^2 f(z) dz + \frac{2}{N(T-t_0)} \int_{-\infty}^{\infty} K_h(z-x)^2 \sigma(z)^2 f(z) dz. \end{aligned}$$

Moreover, K is symmetric and $K \in \mathbb{L}^2(\mathbb{R}, dx)$ by Assumption 3.1, and $b, \sigma \in \mathbb{L}^2(\mathbb{R}, f(x)dx)$ by Remark 2.4. Then,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{v}(x)dx &\leq \frac{2}{N} \int_{\mathbb{R}^2} K_h(z-x)^2 b(z)^2 f(z) dz dx + \frac{2}{N(T-t_0)} \int_{\mathbb{R}^2} K_h(z-x)^2 \sigma(z)^2 f(z) dz dx \\ &= \frac{2}{Nh} \int_{-\infty}^{\infty} b(z)^2 f(z) \int_{-\infty}^{\infty} K(x)^2 dx dz + \frac{2}{N(T-t_0)h} \int_{-\infty}^{\infty} \sigma(z)^2 f(z) \int_{-\infty}^{\infty} K(x)^2 dx dz \\ &\leq \frac{2\|K\|_2^2}{Nh} \left(\int_{-\infty}^{\infty} b(z)^2 f(z) dz + \frac{1}{T-t_0} \int_{-\infty}^{\infty} \sigma(z)^2 f(z) dz \right). \end{aligned}$$

On the other hand, let us find a suitable bound on the integrated squared-bias of $\widehat{b}_{N,h}^{\widehat{f}}(x)$. Again, since X^1, \dots, X^N are i.i.d. copies of X , and since Itô's integral restricted to \mathbb{H}^2 is a martingale-valued map,

$$\begin{aligned} \mathbf{b}(x) &= \mathbb{E} \left[\frac{1}{N(T-t_0)} \sum_{i=1}^N \int_{t_0}^T K_h(X_t^i - x) dX_t^i \right] - b(x)f(x) \\ &= \frac{1}{T-t_0} \mathbb{E} \left(\int_{t_0}^T K_h(X_t - x) dX_t \right) - b(x)f(x) \\ &= \frac{1}{T-t_0} \left[\mathbb{E} \left(\int_{t_0}^T K_h(X_t - x) b(X_t) dt \right) + \mathbb{E} \left(\int_{t_0}^T K_h(X_t - x) \sigma(X_t) dW_t \right) \right] - b(x)f(x) \\ &= \frac{1}{T-t_0} \int_{t_0}^T \mathbb{E}(K_h(X_t - x) b(X_t)) dt - b(x)f(x) = \int_{-\infty}^{\infty} K_h(z-x) b(z) f(z) dz - b(x)f(x). \end{aligned}$$

Then,

$$\mathbf{b}(x)^2 = ((bf)_h - bf)(x)^2 \quad \text{with} \quad (bf)_h = K_h * (bf).$$

Therefore, since f is bounded and b belongs to $\mathbb{L}^2(\mathbb{R}, f(x)dx)$ by Remark 2.4,

$$\int_{-\infty}^{\infty} \mathbf{b}(x)^2 dx = \|bf - (bf)_h\|_2^2.$$

This concludes the proof.

6.4. Proof of Proposition 3.4. First of all,

$$\widehat{b}_{N,h,\eta} - b = \left[\frac{\widehat{b}f_{N,h} - bf}{\widehat{f}_{N,\eta}} + \left(\frac{1}{\widehat{f}_{N,\eta}} - \frac{1}{f} \right) bf \right] \mathbf{1}_{\widehat{f}_{N,\eta}(\cdot) > m/2} - b \mathbf{1}_{\widehat{f}_{N,\eta}(\cdot) \leq m/2}.$$

Then,

$$\|\widehat{b}_{N,h,\eta} - b\|_{f,A,B}^2 \leq 2 \left\| \left[\frac{\widehat{b}f_{N,h} - bf}{\widehat{f}_{N,\eta}} + \left(\frac{1}{\widehat{f}_{N,\eta}} - \frac{1}{f} \right) bf \right] \mathbf{1}_{\widehat{f}_{N,\eta}(\cdot) > m/2} \right\|_{f,A,B}^2 + 2\|b \mathbf{1}_{\widehat{f}_{N,\eta}(\cdot) \leq m/2}\|_{f,A,B}^2.$$

Moreover, for any $x \in [A, B]$, since $f(x) > m$, for every $\omega \in \{\widehat{f}_{N,\eta}(\cdot) \leq m/2\}$,

$$|f(x) - f_{N,\eta}(x, \omega)| \geq f(x) - f_{N,\eta}(x, \omega) > m - \frac{m}{2} = \frac{m}{2}.$$

Thus,

$$\begin{aligned}
 \|\widehat{b}_{N,h,\eta} - b\|_{f,A,B}^2 &\leq \frac{8}{m^2} \|\widehat{b}f_{N,h} - bf\|_f^2 + \frac{8}{m^2} \|(f - \widehat{f}_{N,\eta})b\|_{f,A,B}^2 + 2\|b\mathbf{1}_{|f(\cdot) - \widehat{f}_{N,\eta}(\cdot)| > m/2}\|_{f,A,B}^2 \\
 &\leq \frac{8}{m^2} \int_{-\infty}^{\infty} (\widehat{b}f_{N,h} - bf)(x)^2 f(x) dx \\
 &\quad + \frac{8}{m^2} \int_A^B (f(x) - \widehat{f}_{N,\eta}(x))^2 b(x)^2 f(x) dx \\
 &\quad + 2 \int_A^B b(x)^2 f(x) \mathbf{1}_{|f(x) - \widehat{f}_{N,\eta}(x)| > m/2} dx.
 \end{aligned}$$

Since f is bounded (see Corollary 2.5) and $b^2 f$ is bounded on $[A, B]$ because b and f are continuous functions, this leads to

$$\begin{aligned}
 \|\widehat{b}_{N,h,\eta} - b\|_{f,A,B}^2 &\leq \frac{8\|f\|_{\infty}}{m^2} \|\widehat{b}f_{N,h} - bf\|_f^2 \\
 &\quad + \frac{8\|b^2 f\|_{\infty,A,B}}{m^2} \|\widehat{f}_{N,\eta} - f\|_2^2 + 2\|b^2 f\|_{\infty,A,B} \int_{-\infty}^{\infty} \mathbf{1}_{|f(x) - \widehat{f}_{N,\eta}(x)| > m/2} dx.
 \end{aligned}$$

Therefore, thanks to Markov's inequality,

$$\begin{aligned}
 \mathbb{E}(\|\widehat{b}_{N,h,\eta} - b\|_{f,A,B}^2) &\leq \frac{8\|f\|_{\infty}}{m^2} \mathbb{E}(\|\widehat{b}f_{N,h} - bf\|_f^2) \\
 &\quad + \frac{8\|b^2 f\|_{\infty,A,B}}{m^2} \mathbb{E}(\|\widehat{f}_{N,\eta} - f\|_2^2) + \frac{8\|b^2 f\|_{\infty,A,B}}{m^2} \int_{-\infty}^{\infty} \mathbb{E}(|f(x) - \widehat{f}_{N,\eta}(x)|^2) dx \\
 &\leq \frac{8(\|f\|_{\infty} \vee \|b^2 f\|_{\infty,A,B})}{m^2} [\mathbb{E}(\|\widehat{b}f_{N,h} - bf\|_f^2) + 2\mathbb{E}(\|\widehat{f}_{N,\eta} - f\|_2^2)].
 \end{aligned}$$

Propositions 3.3 and 3.2 allow to conclude.

6.5. Proof of Proposition 4.2. First of all, note that

$$\begin{aligned}
 \mathbb{E}(\|\widehat{f}_{n,N,h} - f\|_2^2) &\leq 2\mathbb{E}(\|\widehat{f}_{N,h} - f\|_2^2) + 2\mathbb{E}(\|\widehat{f}_{N,h} - \widehat{f}_{n,N,h}\|_2^2) \\
 &\leq 2 \left[\mathbf{c}_{3.2}(t_0)h^{2\beta} + \frac{1}{Nh} + \int_{-\infty}^{\infty} \mathbb{E}(\widehat{f}_{N,h}(x) - \widehat{f}_{n,N,h}(x))^2 dx \right. \\
 &\quad \left. + \int_{-\infty}^{\infty} \text{var}(\widehat{f}_{N,h}(x) - \widehat{f}_{n,N,h}(x)) dx \right]
 \end{aligned}$$

by Proposition 3.2, and note also that

$$\widehat{f}_{N,h}(x) - \widehat{f}_{n,N,h}(x) = \frac{1}{N(T-t_0)} \sum_{i=1}^N \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (K_{h,x}(X_t^i) - K_{h,x}(X_{t_j}^i)) dt$$

with $K_{h,x}(\cdot) := K_h(\cdot - x)$. On the one hand, for every $s, u \in [t_0, T]$ such that $s \leq u$, by Itô's formula, Jensen's inequality, the isometry property for Itô's integral and Remark 2.4,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_u) - K_{h,x}(X_s))^2] dx &= \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\int_s^u K'_{h,x}(X_t) dX_t + \frac{1}{2} \int_s^u K''_{h,x}(X_t) d\langle X \rangle_t \right)^2 \right] dx \\
&\leq \mathbf{c}_1 \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\int_s^u K'_{h,x}(X_t) b(X_t) dt \right)^2 \right. \\
&\quad \left. + \mathbb{E} \left[\left(\int_s^u K''_{h,x}(X_t) \sigma(X_t)^2 dt \right)^2 \right] \right. \\
&\quad \left. + \mathbb{E} \left[\left(\int_s^u K'_{h,x}(X_t) \sigma(X_t) dW_t \right)^2 \right] \right] dx \\
&\leq \mathbf{c}_1 \left[(u-s) \int_s^u \mathbb{E} \left(b(X_t)^2 \int_{-\infty}^{\infty} K'_{h,x}(X_t)^2 dx \right) dt \right. \\
&\quad \left. + (u-s) \int_s^u \mathbb{E} \left(\sigma(X_t)^4 \int_{-\infty}^{\infty} K''_{h,x}(X_t)^2 dx \right) dt \right. \\
&\quad \left. + \int_s^u \mathbb{E} \left(\sigma(X_t)^2 \int_{-\infty}^{\infty} K'_{h,x}(X_t)^2 dx \right) dt \right] \\
&\leq \mathbf{c}_2 \left[\frac{(u-s)^2}{h^3} + \frac{(u-s)^2}{h^5} + \frac{u-s}{h^3} \right]
\end{aligned}$$

where \mathbf{c}_1 and \mathbf{c}_2 are two positive constants not depending on s, u, h, N, n and t_0 . Then,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \text{var}(\widehat{f}_{n,N,h}(x) - \widehat{f}_{N,h}(x)) dx \\
&= \frac{1}{N(T-t_0)^2} \int_{-\infty}^{\infty} \text{var} \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (K_{h,x}(X_t^1) - K_{h,x}(X_{t_j}^1)) dt \right] dx \\
&\leq \frac{1}{N(T-t_0)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t^i) - K_{h,x}(X_{t_j}^i))^2] dx dt \leq \frac{\mathbf{c}_3}{Nnh^3}
\end{aligned}$$

where the constant $\mathbf{c}_3 > 0$ is not depending on h, N, n and t_0 . On the other hand, by Assumption 2.3,

$$\begin{aligned}
|\mathbb{E}(\widehat{f}_{N,h}(x) - \widehat{f}_{n,N,h}(x))| &\leq \frac{1}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\mathbb{E}(K_{h,x}(X_t)) - \mathbb{E}(K_{h,x}(X_{t_j}))| dt \\
&\leq \frac{1}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} K_h(z-x) |p_t(x_0, z) - p_{t_j}(x_0, z)| dz dt \\
&\leq \frac{1}{T-t_0} \sum_{j=0}^{n-1} \left[\int_{t_j}^{t_{j+1}} (t-t_j) dt \right] \\
&\quad \times \left[\int_{-\infty}^{\infty} K_h(z-x) \sup_{u \in [t_0, T]} |\partial_u p_u(x_0, z)| dz \right] \\
&\leq \mathbf{c}_{2.3,3} \frac{T-t_0}{nt_0^{q_3}} \int_{-\infty}^{\infty} K(z) \exp \left[-\mathbf{m}_{2.3,3} \frac{(hz+x-x_0)^2}{T} \right] dz.
\end{aligned}$$

Then, by Jensen's inequality,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{E}(\widehat{f}_{N,h}(x) - \widehat{f}_{n,N,h}(x))^2 dx &\leq \frac{\mathfrak{c}_5}{n^2 t_0^{2q_3}} \int_{-\infty}^{\infty} K(z) \int_{-\infty}^{\infty} \exp\left[-2\mathfrak{m}_{2,3,3} \frac{(hz + x - x_0)^2}{T}\right] dx dz \\ &= \frac{\mathfrak{c}_6}{n^2 t_0^{2q_3}} \end{aligned}$$

where

$$\mathfrak{c}_6 = \mathfrak{c}_5 \int_{-\infty}^{\infty} \exp\left[-2\mathfrak{m}_{2,3,3} \frac{(x - x_0)^2}{T}\right] dx$$

and the constant $\mathfrak{c}_5 > 0$ is not depending on h, N, n and t_0 . This concludes the proof.

6.6. Proof of Proposition 4.3. The proof of Proposition 4.3 relies on the two following technical lemmas.

Lemma 6.1. *Consider a symmetric and continuous function $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that $\bar{\varphi}_1 : z \mapsto z\varphi_1(z)$ belongs to $\mathbb{L}^2(\mathbb{R})$. Consider also $\varphi_2, \psi \in C^0(\mathbb{R})$ having polynomial growth. Under Assumptions 2.1 and 2.2, for every $p > 0$, there exists a constant $\mathfrak{c}_{6.1}(p) > 0$, not depending on φ_1 and t_0 , such that for every $s, t \in [t_0, T]$ satisfying $s < t$,*

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\int_s^t \varphi_1(x - X_u) \varphi_2(X_u) dW_u \right)^2 \psi(X_t)^2 \right] dx \\ \leq \mathfrak{c}_{6.1}(p)(t - s) \left[\|\varphi_1\|_2^2 + \|\bar{\varphi}_1\|_2^2 + \frac{1}{t_0^{1/(2p)}} \left(\int_{-\infty}^{\infty} \varphi_1(z)^{2p} dz \right)^{1/p} \right]. \end{aligned}$$

Lemma 6.2. *Consider $\varphi \in C^0(\mathbb{R})$. Under Assumptions 2.1 and 2.2, for every $s, t \in [t_0, T]$ such that $s < t$,*

$$\int_{-\infty}^{\infty} \mathbb{E}(K_{h,x}(X_s) \varphi(X_s, X_t))^2 dx \leq \frac{\mathfrak{c}_{2.2,1}}{t_0^{1/2}} \mathbb{E}[\varphi(X_s, X_t)^2].$$

The proof of Lemma 6.1 (resp. Lemma 6.2) is postponed to Subsubsection 6.6.1 (resp. Subsubsection 6.6.2).

First of all, note that

$$\begin{aligned} \mathbb{E}(\|\widehat{bf}_{n,N,h} - bf\|_2^2) &\leq 2\mathbb{E}(\|\widehat{bf}_{N,h} - bf\|_2^2) + 2\mathbb{E}(\|\widehat{bf}_{N,h} - \widehat{bf}_{n,N,h}\|_2^2) \\ &\leq 2 \left[\|(bf)_h - bf\|_2^2 + \frac{\mathfrak{c}_{3.3}(t_0)}{Nh} + \int_{-\infty}^{\infty} \mathbb{E}(\widehat{bf}_{N,h}(x) - \widehat{bf}_{n,N,h}(x))^2 dx \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \text{var}(\widehat{bf}_{N,h}(x) - \widehat{bf}_{n,N,h}(x)) dx \right] \\ &=: 2 \left[\|(bf)_h - bf\|_2^2 + \frac{\mathfrak{c}_{3.3}(t_0)}{Nh} + B_{n,N,h} + V_{n,N,h} \right] \end{aligned}$$

by Proposition 3.3, and note also that

$$\widehat{bf}_{N,h}(x) - \widehat{bf}_{n,N,h}(x) = \frac{1}{N(T - t_0)} \sum_{i=1}^N \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (K_{h,x}(X_t^i) - K_{h,x}(X_{t_j}^i)) dX_t^i.$$

The proof is dissected in two steps. The term $V_{n,N,h}$ is controlled in the first step, and then $B_{n,N,h}$ is controlled in the second one.

Step 1. First of all, by Jensen's inequality,

$$\begin{aligned} V_{n,N,h} &= \frac{1}{N(T-t_0)^2} \int_{-\infty}^{\infty} \text{var} \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (K_{h,x}(X_t) - K_{h,x}(X_{t_j})) dX_t \right] dx \\ &\leq \frac{2}{N(T-t_0)^2} \int_{-\infty}^{\infty} \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 b(X_t)^2] dt \right] dx + V_{n,N,h}^{\sigma} \end{aligned}$$

with

$$V_{n,N,h}^{\sigma} := \frac{2}{N(T-t_0)^2} \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (K_{h,x}(X_t) - K_{h,x}(X_{t_j})) \sigma(X_t) dW_t \right)^2 \right] dx.$$

In order to control $V_{n,N,h}$ as in the proof of Proposition 4.2, a preliminary bound on $V_{n,N,h}^{\sigma}$ has to be established via the isometry property of Itô's integral:

$$\begin{aligned} V_{n,N,h}^{\sigma} &= \frac{2}{N(T-t_0)^2} \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\int_{t_0}^T \left(\sum_{j=0}^{n-1} (K_{h,x}(X_t) - K_{h,x}(X_{t_j})) \sigma(X_t) \mathbf{1}_{[t_j, t_{j+1}]}(t) \right) dW_t \right)^2 \right] dx \\ &= \frac{2}{N(T-t_0)^2} \int_{-\infty}^{\infty} \int_{t_0}^T \mathbb{E} \left[\left(\sum_{j=0}^{n-1} (K_{h,x}(X_t) - K_{h,x}(X_{t_j})) \sigma(X_t) \mathbf{1}_{[t_j, t_{j+1}]}(t) \right)^2 \right] dt dx \\ &= \frac{2}{N(T-t_0)^2} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 \sigma(X_t)^2] dt \right) dx. \end{aligned}$$

Then,

$$\begin{aligned} V_{n,N,h} &\leq \frac{2}{N(T-t_0)^2} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 b(X_t)^2] dx dt \\ &\quad + \frac{2}{N(T-t_0)^2} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 \sigma(X_t)^2] dx dt. \end{aligned}$$

For $\varphi = b$ or $\varphi = \sigma$, by Itô's formula,

$$\begin{aligned} &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 \varphi(X_t)^2] dx dt \\ &\leq \mathbf{c}_1 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \left[\mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) b(X_u) du \right)^2 \varphi(X_t)^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_{t_j}^t K''_{h,x}(X_u) \sigma(X_u)^2 du \right)^2 \varphi(X_t)^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) \sigma(X_u) dW_u \right)^2 \varphi(X_t)^2 \right] \right] dx dt \end{aligned}$$

where the constant $\mathbf{c}_1 > 0$ is not depending on φ , h , N , n and t_0 . Moreover, for every $j \in \{0, \dots, n-1\}$ and $t \in [t_j, t_{j+1}]$, by Lemma 6.1 with $p = 1/(1-\varepsilon)$,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) \sigma(X_u) dW_u \right)^2 \varphi(X_t)^2 \right] dx &\leq \mathbf{c}_{6.1}(p)(t-t_j) \int_{-\infty}^{\infty} K'_h(z)^2 dz \\ &\quad + \mathbf{c}_{6.1}(p)(t-t_j) \int_{-\infty}^{\infty} z^2 K'_h(z)^2 dz \\ &\quad + \frac{\mathbf{c}_{6.1}(p)}{t_0^{1/(2p)}}(t-t_j) \left(\int_{-\infty}^{\infty} K'_h(z)^{2p} dz \right)^{1/p} \\ &\leq \mathbf{c}_2(\varepsilon)(t-t_j) \left[1 + \frac{1}{h^3} + \frac{t_0^{-(1-\varepsilon)/2}}{h^{3+\varepsilon}} \right] \end{aligned}$$

where the constant $\mathbf{c}_2(\varepsilon) > 0$ depends on ε , but not on φ , j , t , h , N , n and t_0 . Thus, by Jensen's inequality and Remark 2.4,

$$\begin{aligned} &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 \varphi(X_t)^2] dx dt \\ &\leq \mathbf{c}_3(\varepsilon) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[(t-t_j) \int_{t_j}^t \mathbb{E} \left(\varphi(X_t)^2 b(X_u)^2 \int_{-\infty}^{\infty} K'_{h,x}(X_u)^2 dx \right) du \right. \\ &\quad \left. + (t-t_j) \int_{t_j}^t \mathbb{E} \left(\varphi(X_t)^2 \sigma(X_u)^4 \int_{-\infty}^{\infty} K''_{h,x}(X_u)^2 dx \right) du + (t-t_j) \left[1 + \frac{1}{h^3} + \frac{t_0^{-(1-\varepsilon)/2}}{h^{3+\varepsilon}} \right] \right] dt \\ &\leq \frac{\mathbf{c}_4(\varepsilon)}{\min\{1, t_0^{(1-\varepsilon)/2}\}} (T-t_0)^3 \left(\frac{1}{n^2 h^3} + \frac{1}{n^2 h^5} + \frac{1}{n h^{3+\varepsilon}} \right) \end{aligned}$$

where $\mathbf{c}_3(\varepsilon)$ and $\mathbf{c}_4(\varepsilon)$ are two positive constants depending on ε , but not on φ , h , N , n and t_0 . Therefore,

$$V_{n,N,h} \leq \frac{\mathbf{c}_5(\varepsilon)}{\min\{1, t_0^{(1-\varepsilon)/2}\}} \cdot \frac{1}{N n h^{3+\varepsilon}}$$

where the constant $\mathbf{c}_5(\varepsilon) > 0$ depends on ε , but not on h , N , n and t_0 .

Step 2. First of all, since Itô's integral restricted to \mathbb{H}^2 is a martingale-valued map, since $K_{h,x}$ is a

kernel, by Lemma 6.2, by Assumptions 2.2 and 2.3, and since b is Lipschitz continuous,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \mathbb{E}(\widehat{bf}_{N,h}(x) - \widehat{bf}_{n,N,h}(x))^2 dx \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}((K_{h,x}(X_t) - K_{h,x}(X_{t_j}))b(X_t)) dt \right. \\
&\quad \left. + \frac{1}{T-t_0} \sum_{j=0}^{n-1} \mathbb{E} \left[\int_{t_j}^{t_{j+1}} (K_{h,x}(X_t) - K_{h,x}(X_{t_j}))\sigma(X_t) dW_t \right] \right)^2 dx \\
&\leq \frac{2}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}(K_{h,x}(X_t)b(X_t) - K_{h,x}(X_{t_j})b(X_{t_j}))^2 dx dt \\
&\quad + \frac{2}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}(K_{h,x}(X_{t_j})|b(X_t) - b(X_{t_j})|)^2 dx dt \\
&\leq \frac{2}{T-t_0} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K_{h,x}(z) dx \right) b(z)^2 (p_t(x_0, z) - p_{t_j}(x_0, z))^2 dz dt \\
&\quad + \frac{2c_{2.2,1}}{t_0^{1/2}(T-t_0)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[(b(X_t) - b(X_{t_j}))^2] dt \\
&\leq \frac{4}{T-t_0} \sum_{j=0}^{n-1} \left[\int_{t_j}^{t_{j+1}} (t-t_j)^2 dt \right] \left[\int_{-\infty}^{\infty} b(z)^2 \sup_{u \in [t_0, T]} |\partial_u p_u(x_0, z)|^2 dz \right] \\
&\quad + \frac{2c_{2.2,1}}{t_0^{1/2}(T-t_0)} \|b'\|_{\infty}^2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[(X_t - X_{t_j})^2] dt \\
&\leq c_6 \left(\frac{1}{t_0^{2q_3} n^2} + \frac{1}{t_0^{1/2}(T-t_0)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathbb{E}[(X_t - X_{t_j})^2] dt \right)
\end{aligned}$$

where the constant $c_6 > 0$ is not depending on h, N, n and t_0 . Moreover, for any $j \in \{0, \dots, n-1\}$ and $t \in [t_j, t_{j+1}]$,

$$X_t - X_{t_j} = \int_{t_j}^t b(X_u) du + \int_{t_j}^t \sigma(X_u) dW_u$$

and then, by Jensen's inequality, the isometry property of Itô's integral and Remark 2.4,

$$\begin{aligned}
\mathbb{E}[(X_t - X_{t_j})^2] &\leq (t-t_j) \int_{t_j}^t \mathbb{E}(b(X_u)^2) du + \int_{t_j}^t \mathbb{E}(\sigma(X_u)^2) du \\
&\leq (t-t_j)^2 \sup_{u \in [t_0, T]} \mathbb{E}(b(X_u)^2) + (t-t_j) \sup_{u \in [t_0, T]} \mathbb{E}(\sigma(X_u)^2) \leq c_7(t-t_j)
\end{aligned}$$

where the constant $c_7 > 0$ is not depending on j, t, h, N, n and t_0 . Therefore,

$$\int_{-\infty}^{\infty} \mathbb{E}(\widehat{bf}_{N,h}(x) - \widehat{bf}_{n,N,h}(x))^2 dx \leq \frac{c_8}{\min\{t_0^{1/2}, t_0^{2q_3}\}} \left(\frac{1}{n^2} + \frac{1}{n} \right)$$

where the constant $c_8 > 0$ is not depending on n, N, h and t_0 .

Remark 6.3. Assume that b and σ are bounded. Then, in Step 1, for $\varphi = b$ or $\varphi = \sigma$,

$$\begin{aligned}
& \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \mathbb{E}[(K_{h,x}(X_t) - K_{h,x}(X_{t_j}))^2 \varphi(X_t)^2] dx dt \\
& \leq \mathbf{c}_1 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \left[\mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) b(X_u) du \right)^2 \varphi(X_t)^2 \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left(\int_{t_j}^t K''_{h,x}(X_u) \sigma(X_u)^2 du \right)^2 \varphi(X_t)^2 \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) \sigma(X_u) dW_u \right)^2 \varphi(X_t)^2 \right] \right] dx dt \\
& \leq \mathbf{c}_1 \|\varphi\|_{\infty}^2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{-\infty}^{\infty} \left[\mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) b(X_u) du \right)^2 \right] \right. \\
& \quad \left. + \mathbb{E} \left[\left(\int_{t_j}^t K''_{h,x}(X_u) \sigma(X_u)^2 du \right)^2 \right] + \mathbb{E} \left[\left(\int_{t_j}^t K'_{h,x}(X_u) \sigma(X_u) dW_u \right)^2 \right] \right] dx dt.
\end{aligned}$$

So, in this special case, the bound on $V_{n,N,h}$ is established by using the exact same arguments than in the proof of Proposition 4.2. In particular, one can take $\varepsilon = 0$, the additional conditions $K \in \mathbb{L}^4(\mathbb{R})$ and $z \mapsto zK(z)$ belongs to $\mathbb{L}^2(\mathbb{R})$ are not required, and the bound on $V_{n,N,h}$ is of order $1/(Nnh^3)$ and doesn't depend on t_0 .

6.6.1. *Proof of Lemma 6.1.* Consider $\varphi(x, z) := \varphi_1(x - z)\varphi_2(z)$ for every $z \in \mathbb{R}$, $q > 0$ such that $1/p + 1/q = 1$, and $s, t \in [0, T]$ such that $s < t$. First of all, by the isometry property of Itô's integral, Burkholder-Davis-Gundy's inequality, Hölder's inequality, Markov's inequality, Remark 2.4, and the generalized Minkowski inequality,

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_s^t \varphi(x, X_u) dW_u \right)^2 \psi(X_t)^2 \right] \\
& \leq x^2 \mathbb{E} \left[\left(\int_s^t \varphi(x, X_u) dW_u \right)^2 \mathbf{1}_{\psi(X_t)^2 \leq x^2} \right] \\
& \quad + \mathbb{E} \left[\left(\int_s^t \varphi(x, X_u) dW_u \right)^{2p} \right]^{1/p} \mathbb{E}(\psi(X_t)^{4q})^{1/(2q)} \mathbb{P}(\psi(X_t)^2 > x^2)^{1/(2q)} \\
& \leq x^2 \int_s^t \mathbb{E}[\varphi(x, X_u)^2] du + \mathbf{c}_1(p) \mathbb{E} \left[\left(\int_s^t \varphi(x, X_u)^2 du \right)^p \right]^{1/p} \mathbb{E}(\psi(X_t)^{4q})^{1/(2q)} \\
& \quad \times \left[\frac{\mathbb{E}(\psi(X_t)^2)^{1/(2q)}}{x^{1/q}} \mathbf{1}_{[-1,1]}(x) + \frac{\mathbb{E}(\psi(X_t)^{4q})^{1/(2q)}}{x^2} \mathbf{1}_{\mathbb{R} \setminus [-1,1]}(x) \right] \\
& \leq x^2 \int_s^t \mathbb{E}[\varphi(x, X_u)^2] du + \mathbf{c}_2(p) \left(\int_s^t \mathbb{E}[\varphi(x, X_u)^{2p}]^{1/p} du \right) \left(\frac{1}{x^{1/q}} \mathbf{1}_{[-1,1]}(x) + \frac{1}{x^2} \mathbf{1}_{\mathbb{R} \setminus [-1,1]}(x) \right)
\end{aligned}$$

where $c_1(p)$ and $c_2(p)$ are two positive constants depending on p , but not on x, s, t, φ and t_0 . On the one hand, since φ_2 has polynomial growth, by Remark 2.4, for every $u \in [s, t]$,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 \mathbb{E}[\varphi(x, X_u)^2] dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 \varphi_1(x-z)^2 \varphi_2(z)^2 p_u(x_0, z) dx dz \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+z)^2 \varphi_1(x)^2 \varphi_2(z)^2 p_u(x_0, z) dx dz \\ &\leq 2 \left(\int_{-\infty}^{\infty} x^2 \varphi_1(x)^2 dx \right) \left(\int_{-\infty}^{\infty} \varphi_2(z)^2 p_u(x_0, z) dz \right) \\ &\quad + 2 \left(\int_{-\infty}^{\infty} \varphi_1(x)^2 dx \right) \left(\int_{-\infty}^{\infty} z^2 \varphi_2(z)^2 p_u(x_0, z) dz \right) \\ &\leq c_3 (\|\bar{\varphi}_1\|_2^2 + \|\varphi_1\|_2^2) \end{aligned}$$

where the constant $c_3 > 0$ is not depending on u, φ_1 and t_0 . Then,

$$\int_{-\infty}^{\infty} x^2 \int_s^t \mathbb{E}[\varphi(x, X_u)^2] du dx \leq c_3 (t-s) (\|\bar{\varphi}_1\|_2^2 + \|\varphi_1\|_2^2).$$

On the other hand, since φ_2 has polynomial growth, by Assumption 2.2, for every $x \in \mathbb{R}$,

$$\begin{aligned} \int_s^t \mathbb{E}[\varphi(x, X_u)^{2p}]^{1/p} du &= \int_s^t \left(\int_{-\infty}^{\infty} \varphi_1(z)^{2p} \varphi_2(z+x)^{2p} p_u(x_0, z+x) dz \right)^{1/p} du \\ &\leq \frac{c_4(p)}{t_0^{1/(2p)}} (t-s) \left(\int_{-\infty}^{\infty} \varphi_1(z)^{2p} dz \right)^{1/p} \end{aligned}$$

where the constant $c_4(p) > 0$ depends on p , but not on x, s, t, φ_1 and t_0 . Then,

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\int_s^t \mathbb{E}[\varphi(x, X_u)^{2p}]^{1/p} du \right) \left(\frac{1}{x^{1/q}} \mathbf{1}_{[-1,1]}(x) + \frac{1}{x^2} \mathbf{1}_{\mathbb{R} \setminus [-1,1]}(x) \right) dx \\ \leq \frac{c_5(p)}{t_0^{1/(2p)}} (t-s) \left(\int_{-\infty}^{\infty} \varphi_1(z)^{2p} dz \right)^{1/p} \end{aligned}$$

with

$$c_5(p) = c_4(p) \left(\int_{-1}^1 \frac{dx}{x^{1/q}} + \int_{\mathbb{R} \setminus [-1,1]} \frac{dx}{x^2} \right) < \infty.$$

6.6.2. Proof of Lemma 6.2. Consider $s, t \in [t_0, T]$ such that $s < t$, and let $p_{s,t}$ (resp. $p_{t|s}$) be the density of (X_s, X_t) (resp. the conditional density of X_t with respect to X_s). Moreover, for the sake of readability, $p_s(x_0, \cdot)$ is denoted by $p_s(\cdot)$ in this proof. Since $K_{h,x}$ is a kernel, and by Assumption 2.2,

$$\begin{aligned} \mathbb{E}(K_{h,x}(X_s) \varphi(X_s, X_t))^2 &= \left[\int_{-\infty}^{\infty} K_{h,x}(y) \left(\int_{-\infty}^{\infty} \varphi(y, z) p_{t|s}(z|y) dz \right) p_s(y) dy \right]^2 \\ &\leq \int_{-\infty}^{\infty} K_{h,x}(y) \left(\int_{-\infty}^{\infty} \varphi(y, z) p_{t|s}(z|y) dz \right)^2 p_s(y)^2 dy \\ &\leq \sup_{s \in [t_0, T]} \left\{ \sup_{y \in \mathbb{R}} p_s(y) \right\} \int_{-\infty}^{\infty} K_{h,x}(y) \left(\int_{-\infty}^{\infty} \varphi(y, z)^2 p_{t|s}(z|y) dz \right) p_s(y) dy \\ &\leq \frac{c_{2.2,1}}{t_0^{1/2}} \int_{-\infty}^{\infty} K_{h,x}(y) \int_{-\infty}^{\infty} \varphi(y, z)^2 p_{s,t}(y, z) dz dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{E}(K_{h,x}(X_s)\varphi(X_s, X_t))^2 dx &\leq \frac{c_{2.2,1}}{t_0^{1/2}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K_{h,x}(y) dy \right) \left(\int_{-\infty}^{\infty} \varphi(y, z)^2 p_{s,t}(y, z) dz \right) dy \\ &= \frac{c_{2.2,1}}{t_0^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y, z)^2 p_{s,t}(y, z) dz dy = \frac{c_{2.2,1}}{t_0^{1/2}} \mathbb{E}[\varphi(X_s, X_t)^2]. \end{aligned}$$

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