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Coexistence of chaotic and elliptic behaviors among analytic, symplectic diffeomorphisms of any surface

Pierre Berger*

May 13, 2021

Abstract

We show the coexistence of chaotic behaviors (positive metric entropy) and elliptic behaviors (integrable elliptic islands) among analytic, symplectic diffeomorphisms in many isotopy classes of any closed surface. In particular this solves a problem introduced by F. Przytycki (1982).

Theorem A (Main result). *For every analytic, symplectic and closed surface (S, Ω) , there is a symplectic, analytic map $f \in \text{Diff}_\Omega^\omega(S)$ such that:*

1. *f has positive metric entropy,*
2. *f displays elliptic islands.*

A symplectic form Ω on an oriented surface is a nowhere-vanishing volume form. This defines a smooth measure Leb on S . A mapping f of (S, Ω) is *symplectic* if it leaves the volume form Ω invariant. This is equivalent to say that it is orientation preserving and leaves Leb invariant. Then for Leb a.e. point $x \in S$ the limit $\Lambda(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_x f^n\|$ exists. The *metric entropy* of f is the mean of Λ . Hence a dynamics has *positive entropy* if it is exponentially sensitive to the initial conditions with positive probability. An *elliptic island* is a domain bounded by a smooth, invariant curve on which the dynamics acts as an irrational rotation. There are many numerical experiments mentioning the coexistence these two phenomena for symplectic, analytic mappings, however so far no example was proved.

Remark 0.1. In the proof of Theorem A, we will show moreover that S without the support of Λ is integrable: the dynamics is equal to the time one of a Hamiltonian flow.

1 Introduction

1.1 History of the problem

This problem enjoys a long history. The first examples of mappings with positive entropy on any surface were discovered by Katok [?]. These examples are isotopic to the identity. Then Katok

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and Gerbert [?] obtained mappings with positive entropy on any surface in the isotopy class of any pseudo-Anosov map. Both constructions were smooth but not analytic. In [?], Gerbert constructed real analytic symplectic pseudo-Anosov maps on any surface, which display positive metric entropy but not the coexistence with an elliptic island. In [?], Przytycki built an example of conservative diffeomorphism of the torus with coexistence of an invariant region with positive entropy and an elliptic island. His construction was infinitely smooth and not analytic. He addressed the problem of whether his construction could be generalized in the analytic class [?, Rk1, P461]. The issue of this problem was recalled as unclear by Liverani in [?, Rk 2.4 P3] where a bifurcation of Przytycki’s example was studied. Note that Theorem A solves in particular Przytycki’s problem.

In [?], Gorodetski proved that typical examples of analytic symplectic surface maps are such that Λ is positive on a set of maximal Hausdorff dimension ($= 2$) and this coexists with elliptic islands. However this leaves open a strong version of the positive entropy conjecture which asserts that “a typical symplectic dynamics has positive metric entropy” (Λ is positive on a set of positive Lebesgue measure). A weaker version of the positive entropy conjecture proposed by Herman [?] asserts the existence of symplectic mappings C^∞ -close to the identity on the disk with positive metric entropy; it implies the density of surface maps with positive metric entropy among those with an elliptic cycle. In [?], the Herman’s positive entropy conjecture was proved with Turaev. Our proof used a quotient similar to the examples of Katok and Przytycki. During Katok’s memorial conference 2019, in a conversation with Gorodetski and Kleptsyn, I claimed that the construction of [?] should be useful to prove the following analytic counterpart of Herman’s positive entropy conjecture [?] and even the next analytic counterpart of our main result with Turaev.

Conjecture 1.1. *There exists an analytic and symplectic perturbation of the identity of the disk with positive metric entropy.*

Conjecture 1.2. *For every analytic and closed symplectic surface (S, Ω) , for every analytic and symplectic $f \in \text{Diff}_\Omega^\omega(S)$ which displays an elliptic periodic point, there are analytic and conservative perturbations of f with positive metric entropy.*

For the analogous strategy* of [?], a first step toward the proof of Conjectures 1.1 and 1.2 is to prove the analytic counterpart of Przytycki’s example.

Following Gorodetski this step was not on reach in a short time, and I bit with him the existence of such an example in a short time †. Corollaries B and C solve this step:

Corollary B. *There exists an analytic and symplectic diffeomorphism f of the closed disk displaying a stochastic island bounded by four heteroclinic bi-links which is robust relative link preservation.*

Let us explain the meaning of the above statement. We recall that a *stochastic island* is a domain \mathcal{I} on which Λ is positive Leb-a.e. A *bi-link* C is a smooth circle equal to the union of two heteroclinic links $C = W^u(P) \cup \{Q\} = W^s(Q) \cup \{P\}$ between saddle fixed points P and Q , see

*For an introduction to the proof of [?], one could look at Arnaud’s Bourbaki Seminar [?].

†More precisely the bit was that someone would prove within five years the existence of an analytic symplectomorphism of the torus, isotopic to the identity, with positive metric entropy and displaying an elliptic island.

Fig. 4 Page 10. Given a perturbation of the dynamics, the *bi-link persists* if the union of the stable and unstable manifolds of the fixed points continue to form a smooth circle. The island is *robust relative link preservation* if for every C^2 -perturbation such that each of the bi-links persist, the domain bounded by the continuation of these bi-links is still a stochastic island.

We can wonder also what are the isotopy classes of analytic, symplectic surface mappings which display coexistence phenomena. Our techniques enable (at least) to obtain the following:

Corollary C. *For every analytic and closed symplectic surface (S, Ω) , for any isotopy class C , if*

- *S is the 2-torus for the isotopy class C of the identity,*
- *or S is a surface of genus ≥ 0 and C is the isotopy class of a pseudo-Anosov map of S ,*

then there is a symplectic, analytic map $f \in \text{Diff}_\Omega^\omega(S)$ of isotopy class C , such that f has positive metric entropy and displays elliptic islands.

A natural problem would be to realize any isotopy class of surface diffeomorphisms by an analytic and symplectic dynamics displaying coexistence of positive metric entropy and elliptic islands. It seems that the techniques of this work together with the Nielsen-Thurston's classification of symplectic dynamics on surface should lead to a solution of this problem. Another approach would be to prove Conjecture 1.2 which would imply immediately a solution to the latter problem.

The proof of Theorem A is here completely self contained.

I am grateful to A. Gorodetski and V. Kleptsyn for their encouragements. I am thankful to R. Krikorian and P. Le Calvez for nice conversations. I thank S. Biebler for his careful reading.

1.2 Idea and structure of the proof

All the proofs [?, ?, ?, ?, ?] used bump functions to localize the surgery of the dynamics in a subset of the manifold. We recall that there is no analytic bump function. To deal with the analytic case, Gebert [?] showed that the pseudo-Anosov examples of [?] persist in a finite codimensional submanifold which must intersect the (infinite-dimensional) submanifold of analytic maps. However the examples of [?, ?, ?], displaying the sought coexistence, persist actually along an infinite codimensional submanifold: one have to keep intact heteroclinic links, and I do not see how to do this if the unperturbed map displaying the coexistence is not already analytic... Instead we propose another approach:

We construct an analytic and symplectic extension of the surface punctured by several points, so that the extended surface remains diffeomorphic to the unpunctured surface, and the analytic continuation of the dynamics on the extended surface displays elliptic islands.

We will start with an analytic, conservative dynamics with positive entropy and then we will perform blow-up, quotient, blow-down and connected sums, so that the analytic continuation of the dynamics is well defined after these operations and displays the sought coexistence properties.

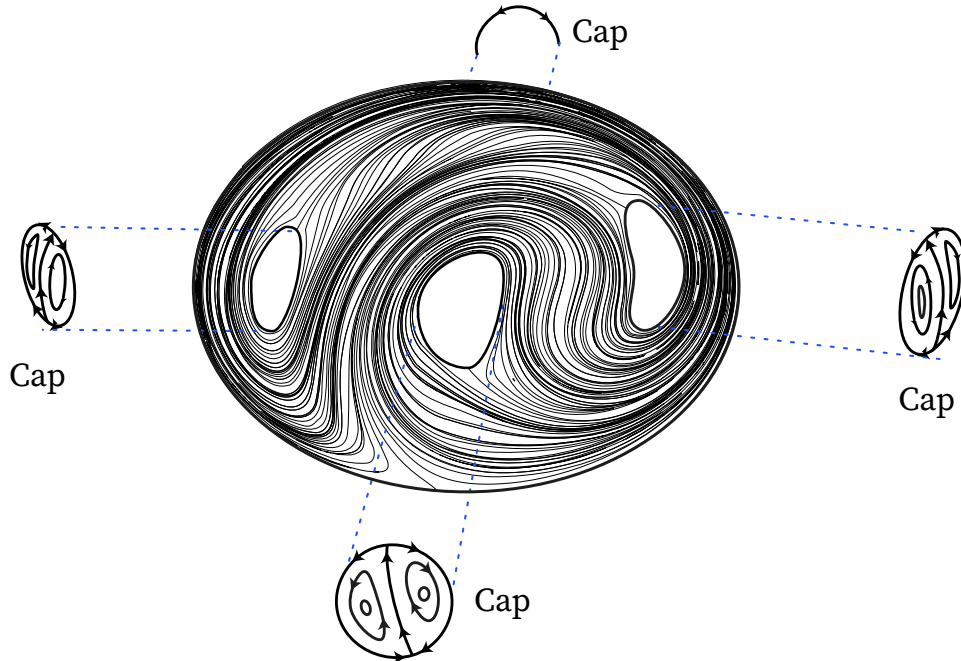


Figure 1: Analytic and conservative dynamics on a sphere displaying coexistence of a stochastic region with elliptic islands.

For the proof of Theorem A, we will start in §2.1 with a linear Anosov map on the 2-torus, then we blow-up four of its fixed point *à la* Przytycki to define an analytic symplectic diffeomorphism of the 2-torus \mathbb{T}^* without four disks, then we quotient it *à la* Katok to define an analytic symplectic diffeomorphism of the 2-sphere \mathbb{S}^* without four disks in §2.2. These steps were already performed in [?] and are depicted in Fig. 2. Then we propose a new construction.

First we regard the continuation of this dynamics on an analytic extension $\hat{\mathbb{S}}^*$ of \mathbb{S}^* in §2.3. Each component of $\hat{\mathbb{S}}^* \setminus \mathbb{S}^*$ is a *collar*. This collar is diffeomorphic to an annulus and equal to a halve neighborhood of two heteroclinic links. In §2.4 we glue two pieces of this annulus to obtain a collar which is a disk without two holes bounded by circle rotations, see Fig. 3. In §2.5 we blow-down them to obtain a collar which is a disk containing two elliptic islands (the dynamics is actually integrable on the whole disk). Such are called *cap's dynamics* on the disk. This forms a *cap* to recap any hole of the sphere \mathbb{S}^* with four holes.

This allows in §3 to prove the main theorem and the corollaries of its proof. In §3.1, we start by proving Theorem A when the surface is a sphere; the construction is depicted by Fig. 1. Following the number of recaped holes, coexistence phenomena are obtained on a disk (which contains the stochastic island of Corollary B), a cylinder or a pair of pants. The boundary of these can be glued together to form any closed symplectic surface, and so obtain Theorem A. A careful study enables to obtain an analytic, symplectic diffeomorphism of the torus isotopic to the identity, as wondered by Gorodetski and part of Corollary C.

In §3.2, we prove the remaining part of Corollary C regarding surface mappings isotopic to a pseudo-Anosov map. We will start with the example of analytic pseudo-Anosov map of [?], which can represent any isotopy class of orientation preserving pseudo-Anosov maps (see also [?]). Then the punctured surface will be extended following basically the same steps as in §2.3. The only difference is that the normal form [?] at the saddle points is more general and that we will be working on a lifting of the previous construction. Caps will be replaced by a certain *generalized cap* given by Proposition 3.2 and Lemma 3.4. The proof of the lemma follows the same lines as §2.4-2.5. The Proposition enables to recap the surface with holes given by blowing up any periodic saddle orbit. The Lemma enables to bound any cycle of heteroclinic links by a disk on which the dynamics is analytic and integrable.

2 Caps for spheres with four holes

2.1 A non-uniformly hyperbolic map on the torus without four disks

This step is depicted in Fig. 2[left-center].

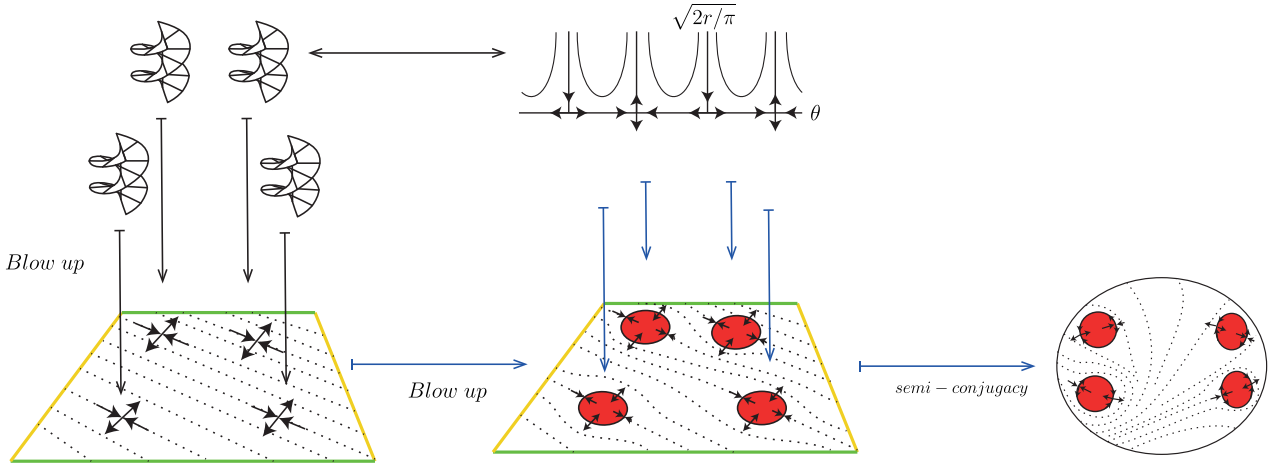


Figure 2: Surgery on an Anosov map

We start with the Anosov map $A(x, y) = (13 \cdot x + 8 \cdot y, 8 \cdot x + 5 \cdot y)$ which acts on the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ endowed with the symplectic form $\Omega = dx \wedge dy$. Let $R \in O_2(\mathbb{R})$ and $\lambda > 0$ be such that $A = R \times \text{diag}(\exp(\lambda), \exp(-\lambda)) \times R^{-1}$. The set $\mathcal{P} := \{0, (1/2, 0), (0, 1/2), (1/2, 1/2)\}$ is formed by four fixed points of the Anosov map A . We perform a symplectic and analytic blow-up at each $P \in \mathcal{P}$. Let $\epsilon > 0$ be small. The $\sqrt{\frac{2}{\pi}}\epsilon$ -neighborhood $P + \mathbb{D}(\sqrt{\frac{2}{\pi}}\epsilon)$ of $P \in \mathbb{R}^2/\mathbb{Z}^2$ is blown up to an annulus via the map:

$$\pi_P : (\theta, r) \in \mathbb{R}/2\mathbb{Z} \times [0, \epsilon] \mapsto P + R \times \left(\sqrt{\frac{2r}{\pi}} \cos(\pi\theta), \sqrt{\frac{2r}{\pi}} \sin(\pi\theta) \right) \in P + \mathbb{D}\left(\sqrt{\frac{2\epsilon}{\pi}}\right).$$

These blow-ups are symplectic and analytic; they define a new surface \mathbb{T}^* as :

$$\mathbb{T}^* := (\mathbb{T}^2 \setminus \mathcal{P}) \sqcup (\mathcal{P} \times \mathbb{R}/2\mathbb{Z} \times [0, \epsilon]) / \sim$$

with \sim the equivalence relation spanned by:

$$(z_1 \in \mathbb{T}^2 \setminus \mathcal{P} \sim (P, z_2) \in \mathcal{P} \times \mathbb{R}/2\mathbb{Z} \times [0, \epsilon]) \iff z_1 = \pi_P(z_2) .$$

The surface \mathbb{T}^* is analytic and symplectic; it is a torus without four disks. Also A can be lifted to \mathbb{T}^* as the map A^* whose restriction to $\mathbb{T}^2 \setminus \mathcal{P}$ is A and whose restriction to a neighborhood of $\mathcal{P} \times \mathbb{R}/2\mathbb{Z} \times \{0\}$ is the time 1 map of the Hamiltonian:

$$(2.1) \quad H : (\theta, r) \mapsto \frac{\lambda}{\pi} \cdot r \cdot \sin(2\pi\theta) .$$

Indeed nearby each fixed point in \mathcal{P} , in the coordinate induced by R , the map A is the time one of the flow of the Hamiltonian $H_1(x, y) = \lambda \cdot x \cdot y = \lambda \cdot \sqrt{\frac{2r}{\pi}} \cos(\pi\theta) \cdot \sqrt{\frac{2r}{\pi}} \sin(\pi\theta) = H(\theta, r)$.

2.2 A non-uniformly hyperbolic map on the sphere without four disks

Note that A is equivariant by the involution $-id$ on \mathbb{T}^2 . The action of the involution on $\mathbb{T}^2 \setminus \mathcal{P}$ is free and $-id$ fixes each point in \mathcal{P} . The involution $-id$ lifts to \mathbb{T}^* as the involution J defined by:

$$J|\mathbb{T}^2 \setminus \mathcal{P} = -id|\mathbb{T}^2 \setminus \mathcal{P} \quad \text{and} \quad J|\mathcal{P} \times \mathbb{R}/2\mathbb{Z} \times [0, \epsilon] : (P, \theta, r) \mapsto (P, \theta + 1, r) .$$

Let \mathbb{S}^* be the space \mathbb{T}^* quotiented by the involution J . This step is depicted in Fig. 2[center-right]. Note that \mathbb{S}^* is an analytic sphere without four disks. A neighborhood of the boundary of these four holes is canonically parametrized by $\mathcal{P} \times \mathbb{R}/2\mathbb{Z} \times [0, \epsilon]/J = \mathcal{P} \times \mathbb{R}/\mathbb{Z} \times [0, \epsilon]$. Also the associated projection $\pi^* : \mathbb{T}^* \rightarrow \mathbb{S}^*$ is a 2-covering. Since the symplectic form Ω is equivariant by $-id$, we can endow \mathbb{S}^* with the push forward of Ω by π^* that we still denote by Ω . We notice that the dynamics A^* descends to an analytic and symplectic dynamics f^* on \mathbb{S}^* . In other words, there is $f^* \in \text{Diff}_\Omega^\omega(\mathbb{S}^*)$ such that:

$$f^* \circ \pi^* = \pi^* \circ A^* .$$

As $A|\mathbb{T}^2 \setminus \mathcal{P}$ is a 2-covering of $f^*|\mathbb{S}^* \setminus \partial\mathbb{S}^*$, the map f^* has positive metric entropy.

Now we shall embed the surface \mathbb{S}^* via a symplectic and analytic map into a sphere so that the dynamics can be extended to one which is analytic, symplectic and displays non-degenerated elliptic points. As depicted in Fig. 3, this will be done first by implementing an explicit formula for the collar lemma, so that the each hole can be identified to the interior of a disk endowed with a dynamics at the neighborhood of the boundary. On the boundary it lies two saddle points; we will perform a surgery to glue a segment of the unstable branch of one to a segment of a stable branch of the other, so that the dynamics is extended to the disk without two disks, and finally we will blow-down each of these latter two disks to create two non-degenerated elliptic points.

2.3 Explicit collar lemma: the holes are surrounded by heteroclinic links

Let us now precise the dynamics of f^* at the boundary of \mathbb{S}^* . By Eq. (2.1), nearby each component, the dynamics f^* is equal to the time 1 of the flow of the following Hamiltonian:

$$(2.2) \quad H : (\theta, r) \in \mathbb{R}/\mathbb{Z} \times [0, \epsilon] \mapsto \frac{\lambda}{\pi} \cdot r \cdot \sin(2\pi\theta) .$$

Observe that $\Omega = d\theta \wedge dr$ and H extend canonically to a neighborhood V of $\mathbb{R}/\mathbb{Z} \times \{0\}$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Let $W \subset V$ be a neighborhood of $\mathbb{R}/\mathbb{Z} \times \{0\}$ in V such that the time $t = 1$ of the Hamiltonian flow $\phi_H^t : W \rightarrow V$ of H is well defined. For $\pm \in \{-, +\}$, let $V^\pm := V \cap \mathbb{R}/\mathbb{Z} \times \mathbb{R}_\pm$ and $W^\pm := W \cap \mathbb{R}/\mathbb{Z} \times \mathbb{R}_\pm$. On the boundary $\partial W^- = \partial W^+ = \mathbb{R}/\mathbb{Z} \times \{0\}$, the map ϕ_H^1 displays two saddle fixed points $Q = (0, 0)$ and $Q' := (1/2, 0)$, so that $W^s(Q) \setminus \{Q\} = W^u(Q') \setminus \{Q'\} = \partial W \setminus \{Q, Q'\}$. In particular ∂W^\pm is a bi-link.

Note that $\bigsqcup_{\mathcal{P}} W^+$ and $\bigsqcup_{\mathcal{P}} V^+$ are neighborhoods of $\partial \mathbb{S}^*$ in \mathbb{S}^* . So we can extend the surface \mathbb{S}^* by gluing canonically $\mathcal{P} \times V$ of V at $\mathcal{P} \times V^+ \subset \mathbb{S}^*$. This defines an open surface $\hat{\mathbb{S}}^*$ which contains \mathbb{S}^* and such that $\hat{\mathbb{S}}^* \setminus \text{int} \mathbb{S}^*$ is equal to $\bigsqcup_{\mathcal{P}} V^-$. On a neighborhood of \mathbb{S}^* in $\hat{\mathbb{S}}^*$, the map f^* extends analytically to a map denoted by \hat{f}^* and whose restriction to $\mathcal{P} \times W^-$ is ϕ_H^1 .

2.4 From holes surrounded by heteroclinic links to holes surrounded by rotations

The idea is to shape W^- as in Fig. 3 [left] to perform the surgery depicted in Fig. 3 [left-center].

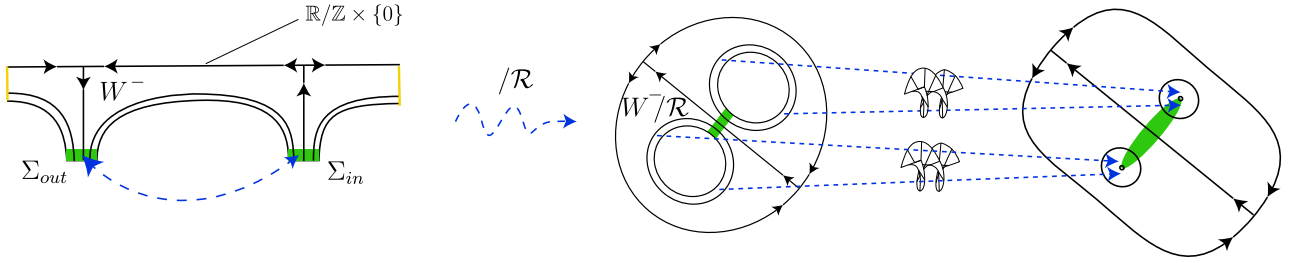


Figure 3: Making an integrable cap by gluing the green rectangles together and then blowing down.

Recall that H extends to $V \supset W$. For $\eta > 0$ small, we can shrink W to have W^- of the form:

$$W^- := \{(\theta, r) \in \mathbb{R}/\mathbb{Z} \times [-\eta, 0] : |H(\theta, r)| \leq \eta^3\}.$$

By Eq. (2.2), the boundary of W^- is formed by three curves; the union of two of them is:

$$\Sigma := \{\theta \in \mathbb{R}/\mathbb{Z} : |H(\theta, -\eta)| \leq \eta^2\} \times \{-\eta\}$$

Let Σ_{out} be the component of Σ which intersects $W_\eta^u(Q) = \{0\} \times [-\eta, 0]$. Let Σ_{in} be the component of Σ which intersects $W_\eta^s(Q') = \{\frac{1}{2}\} \times [-\eta, 0]$. To perform the surgery depicted in Fig. 3 [left-center], we define:

$$\phi : \bigcup_{t \in [-1, 0]} \phi_H^t(\Sigma_{out}) \rightarrow \bigcup_{t \in [0, 1]} \phi_H^t(\Sigma_{in}),$$

such that for $(\theta, r) \in \Sigma_{out}$, $\phi(\theta, r) = \phi_H^1(\frac{1}{2} - \theta, r)$ and for $t \in [0, 1]$, $\phi \circ \phi_H^{t-1}(r, \theta) = \phi_H^t(\frac{1}{2} - \theta, r)$.

Note that the map ϕ is analytic and symplectic with range $\bigcup_{t \in [0, 1]} \phi_H^t(\Sigma_{in})$. Moreover it respects H . Let W^-/\mathcal{R} be equal to W^- quotiented by the equivalence relation \mathcal{R} spanned by:

$$\forall (r, \theta) \in \bigcup_{t \in [-1, 0]} \phi_H^t(\Sigma_{out}), \text{ and } (r', \theta') \in \bigcup_{t \in [0, 1]} \phi_H^t(\Sigma_{in}), \quad (r, \theta)\mathcal{R}(r', \theta') \quad \text{iff } (r, \theta) = \phi(r', \theta').$$

As ϕ is an analytic diffeomorphism which sends the intersection of its domain with $\partial W^- \setminus \Sigma$ into $\partial W^- \setminus \Sigma$, the space W^-/\mathcal{R} is an analytic surface with boundary. Moreover, we notice that W^-/\mathcal{R} is equal to the closed disk $\bar{\mathbb{D}}$ without two disks \mathbb{D}_+ and \mathbb{D}_- , as depicted in Fig. 3 [center]:

$$W^-/\mathcal{R} = \bar{\mathbb{D}} \setminus (\mathbb{D}_+ \sqcup \mathbb{D}_-)$$

This surgery respects the symplectic form Ω and the Hamiltonian H , which remains analytic on W^-/\mathcal{R} . Hence they push-forward to a symplectic form and a Hamiltonian still denoted by Ω and H . Note also that the analytic continuation of f on W^-/\mathcal{R} is still equal to the time one map of the Hamiltonian H .

2.5 Blowing down holes surrounded by rotations

Now we would like to blow-down the circles ($\partial\mathbb{D}_- \sqcup \partial\mathbb{D}_+$) to fixed points P_- and P_+ . These blow-downs will construct a pair of disks depicted in Fig. 3 [Center-Right]. In order to do so, we first observe that on $\partial\mathbb{D}_+ \sqcup \partial\mathbb{D}_-$, the function H is constant (equal to resp. η^3 and $-\eta^3$) and its symplectic gradient does not vanish. So we can apply the classical action-angle coordinate change:

Lemma 2.1. *Let V_0 be a neighborhood V_0 of $C = \mathbb{R}/\mathbb{Z} \times \{0\}$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^+$ and $H : V_0 \rightarrow \mathbb{R}$ be an analytic Hamiltonian constant on C and whose differential does not vanish on C . Then there are $\delta > 0$, a neighborhood $V'_0 \subset V_0$ of C and an analytic and symplectic map $\psi : V'_0 \rightarrow \mathbb{R}/\mathbb{Z} \times [0, \delta]$ so that $H \circ \psi^{-1}$ sends the orbit of each point of V'_0 to a horizontal circle $\mathbb{R}/\mathbb{Z} \times \{\rho\}$.*

Proof of Lemma 2.1. For $\delta' > 0$ small enough, the section $\Pi := \{0\} \times [0, \delta']$ is transverse to the orbits of the Hamiltonian H . Every point z in Π is periodic of a period $T(z)$. The union of the orbits of points in $z \in \Pi$ is equal to a neighborhood $V'_0 \subset V_0$ of C . Let us define the following flow box:

$$\psi_0 : V'_0 \rightarrow \mathbb{R} \times \Pi / \sim \quad \text{with } (t, z) \sim (t + kT(z), z), \quad k \in \mathbb{Z}.$$

In these coordinates the flow ϕ_H^t of H is the translation by $(t, 0)$. We shall reshape the range of ψ_0 so that it equals $\mathbb{R}/\mathbb{Z} \times [0, \delta]$ with $\delta = \text{Leb}(\mathbb{R} \times \Pi / \sim)$. To this end we consider a primitive \hat{T} of $T|\Pi$ which vanishes at $0 \in \Pi \cap C$:

$$\hat{T} : (0, r) \in \Pi = \{0\} \times [0, \delta'] \mapsto \int_0^r T(0, x) dx.$$

Note that $\delta := \hat{T}(\delta')$. Let $\zeta : \mathbb{R} \times \Pi / \sim \rightarrow \mathbb{R}/\mathbb{Z} \times [0, \delta]$ be defined by:

$$\zeta(z, t) = (-t/T(z), \hat{T}(z)).$$

We observe that ζ is analytic and symplectic. Thus $\psi = \zeta \circ \psi_0$ is an analytic and symplectic map from V'_0 onto $\mathbb{R}/2\mathbb{Z} \times [0, \hat{T}(\delta)]$ which sends each orbit to a horizontal circle $\mathbb{R}/\mathbb{Z} \times \{\rho\}$. \square

Thus for every $\pm \in \{-, +\}$, there exist a neighborhood V_\pm of $\partial\mathbb{D}_\pm$ endowed with analytic and symplectic coordinates $\psi_\pm : V_\pm \rightarrow \mathbb{R}/\mathbb{Z} \times [0, \delta]$ such that $H \circ \psi_\pm^{-1}(\theta, r) = h_\pm(r)$ for an analytic maps h_\pm . So we can perform a blow-down of the circle $\partial\mathbb{D}_\pm$ to a fixed point P_\pm . This blow-down sends V_\pm to a disk of radius $\sqrt{\delta/\pi}$ and the dynamics on this disk is the time one map given by the Hamiltonian $P_\pm + (x, y) \mapsto h_\pm(x^2 + y^2)$, which is indeed integrable and displays an elliptic island.

3 Application of the construction

3.1 Proof of Theorem A and Corollary B

Steps §2.1 and §2.2 constructed an analytic diffeomorphism f^* on the sphere \mathbb{S}^* without four holes with positive metric entropy. At step §2.3, we saw that this dynamics could be extended at the neighborhood $\equiv \mathbb{R}/\mathbb{Z} \times (-\epsilon, \epsilon)$ of boundary $\equiv \mathbb{R}/\mathbb{Z} \times \{0\}$ of each of these holes by the time one map of the Hamiltonian:

$$H(\theta, r) = \frac{\lambda}{\pi} \cdot r \cdot \sin(2\pi\theta).$$

Steps §2.4 and §2.5 constructed a cap for any of the holes of \mathbb{S}^* . This is an analytic dynamics \hat{f} on the disks which displays two elliptic fixed points and which is integrable: it is the time one map of an Hamiltonian. Moreover this Hamiltonian coincide at the neighborhood boundary the disk $\approx \mathbb{R}/\mathbb{Z} \times \{0\}$ with the Hamiltonian H . As a matter of fact, we can fillup any hole of \mathbb{S}^* endowed with f^* , by a a disk endowed with \hat{f} .

Let us perform surgeries with these two objects in order to deduce Theorem A and Corollary B.

Proof of Theorem A. Case where S is the sphere. In the above construction, we recap each of the four holes of \mathbb{S}^* with a disk, and we endow \mathbb{S}^* with f^* and each four disks with the cap's dynamics \hat{f} . We obtained an analytic, symplectic map of the sphere with positive metric entropy and displaying four elliptic islands.

Case where S is the torus. In the above construction, we fill up two holes of \mathbb{S}^* with two disks, and we endow \mathbb{S}^* with f^* and each disk with the cap's dynamics \hat{f} . This defines a symplectic dynamics $f^{\mathbb{A}}$ of the annulus \mathbb{A} with positive entropy and four elliptic points (two in each cap). A neighborhood of $\partial\mathbb{A}$ in \mathbb{A} is diffeomorphic to $\bigsqcup_{\pm \in \{+, -\}} \mathbb{R}/\mathbb{Z} \times [\pm 1, \pm(1 - \epsilon)]$ and in these coordinates the dynamics $f^{\mathbb{A}}$ is the time one of the flow of $H(\theta, r \pm 1)$ with:

$$H(\theta, r \pm 1) = \frac{\lambda}{\pi} \cdot r \cdot \sin(2\pi\theta).$$

We glue both boundaries of \mathbb{A} by: $(\theta, r + 1) \sim (\theta, r - 1)$ for r small. This defines an analytic and symplectic map on the torus with a priori non-trivial isotopy class.

Case where S is a surface of higher genus. In the above construction, we recap only one holes of \mathbb{S}^* with a disk to form a pair of pants \mathbb{P} : a disk with two holes. We endow \mathbb{S}^* with f^* and the disk with the cap dynamics \hat{f} . This defines a symplectic dynamics $f^{\mathbb{P}}$ on \mathbb{P} . We recall that every closed, oriented surface S of genus ≥ 2 displays a pants decomposition. We glue canonically (as above) the pants at their boundaries to obtain the sought dynamics. \square

Proof of Corollary C for S equal to the torus and f isotopic to the identity. We constructed above a symplectic and analytic maps $f^{\mathbb{A}}$ on the annulus \mathbb{A} such that at the neighborhood $\bigsqcup_{\pm \in \{+, -\}} \mathbb{R}/\mathbb{Z} \times [\pm 1, \pm(1 - \epsilon)]$ of the boundary $\partial\mathbb{A}$ the dynamics $f^{\mathbb{A}}$ is the times one of the flow of $H(\theta, r \pm 1)$ with:

$$H(\theta, r \pm 1) = \frac{\lambda}{\pi} \cdot r \cdot \sin(2\pi\theta).$$

We saw that if we glue both boundaries of \mathbb{A} by $(\theta, r + 1) \sim (\theta, r - 1)$ for r small, the dynamics is *a priori* in a non-trivial isotopy class. To vanish this isotopy class, the idea is to glue f^* with its inverse

f^{*-1} . At the boundary of \mathbb{A} , the map f^{*-1} is the time one of the flow of $-H(\theta, r \pm 1) = H(-\theta, r \pm 1)$. So we glue two copies \mathbb{A}_1 and \mathbb{A}_2 of \mathbb{A} along their respective boundaries by $(\theta_1, r_1 + 1) \sim (-\theta_2, r_2 - 1)$ and $(\theta_1, r_1 - 1) \sim (-\theta_2, r_2 + 1)$ for r_1, r_2 small and any θ_1, θ_2 . The space $\hat{\mathbb{A}}$ obtained is a 2-covering of a Klein bottle, which is a torus. The dynamics induced by f^* and f^{*-1} on $\hat{\mathbb{A}}$ is analytic, symplectic and isotopic to the identity. Moreover the dynamics on the torus displays the coexistence phenomenon. \square

Proof of Corollary B. We first start with the above analytic diffeomorphism f^* on the sphere \mathbb{S}^* with four holes. We recap three holes with caps and we endow each of them with the dynamics \hat{f} . This defines an analytic and symplectic map $f^{\mathbb{D}}$ on the disk \mathbb{D} . Note that the disk is not endowed with its standard symplectic form, but using [?], we can analytically conjugate it to one which leaves invariant the standard symplectic form on \mathbb{D} . The image \mathcal{I} of \mathbb{S}^* in \mathbb{D} is depicted Fig. 4. Therein the Lyapunov exponent function Λ is Leb a.e. equal to a positive constant. In the sense of [?] (inspired

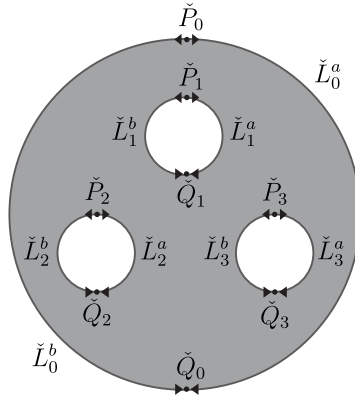


Figure 4: Stochastic island \mathcal{I} in grey.

from [?, ?, ?]), the set \mathcal{I} is called a *stochastic island*. This means that \mathcal{I} is a disk with three holes; and that the boundary of \mathcal{I} is formed by four pairs of heteroclinic bi-links $\{(\check{L}_i^a, \check{L}_i^b) : 0 \leq i \leq 3\}$. Each $\check{L}_i^a \cup \check{L}_i^b$ is a smooth circle included in the stable and unstable manifolds of hyperbolic fixed points \check{P}_i and \check{Q}_i respectively:

$$\check{L}_i^a \cup \check{L}_i^b \subset W^u(\check{P}_i; f^{\mathbb{D}}) \cup W^s(\check{Q}_i; f^{\mathbb{D}}).$$

For every f which is C^1 -close to $f^{\mathbb{D}}$, for every $0 \leq i \leq 3$, the *hyperbolic continuations* P_i and Q_i of \check{P}_i and \check{Q}_i are uniquely defined hyperbolic fixed points for f . If $\{W^u(P_i; f) \cup W^s(Q_i; f) : 0 \leq i \leq 3\}$ form four heteroclinic bi-links $\{L_i^a \cup L_i^b : 0 \leq i \leq 3\}$ close to $\{\check{L}_i^a \cup \check{L}_i^b : 0 \leq i \leq 3\}$, then we say that the *bi-links are persistent for the perturbation* f .

Then the next proposition implies Corollary B. \square

Proposition 3.1 ([?, prop. 2.1]). *For every conservative map f which is C^2 -close to $f^{\mathbb{D}}$ and for which the bi-links are persistent, then the continuations of these bi-links bound a stochastic island. In particular, the metric entropy of f is positive.*

3.2 Proof of Corollary C

It remains the case of mappings isotopic to pseudo-Anosov maps. To carry them we will need to generalize the construction of cap to prove:

Proposition 3.2. *Let $U \subset \mathbb{R}^2$ be a neighborhood of 0. Let $f : U \rightarrow \mathbb{R}^2$ be an analytic and symplectic map with 0 as a saddle fixed point with positive eigenvalues. Then there exist $\rho > \rho' > 0$ arbitrarily small, an analytic and symplectic map $\phi : \mathbb{D}(\rho) \setminus \mathbb{D}(\rho') \rightarrow \mathbb{D}(\sqrt{\rho^2 - \rho'^2}) \setminus \{0\}$ such that on:*

$$\hat{U} := (U \setminus \{0\}) \sqcup \mathbb{D}(\rho) / \sim \quad \text{with } U \ni u \sim d \in \mathbb{D}(\rho) \text{ iff } u = \phi(d),$$

the map $f|_{U \setminus \{0\}}$ extends to a symplectic and analytic map \hat{f} on \hat{U} which leaves invariant the disk $\mathbb{D}(\rho')$ and on which its restriction is integrable and displays three elliptic fixed points.

Proof of Corollary C for f isotopic to a pseudo-Anosov map. Let (S, Ω) be a symplectic orientable, closed surface. Then by [?, ?], any orientation preserving pseudo-Anosov isotopy class is represented by an analytic and symplectic map f on S .

Lemma 3.3. *The map f displays a hyperbolic periodic cycle $(P_i)_{i \in \mathbb{Z}_q}$ with positive eigenvalues.*

Proof. As f has positive topological entropy, it displays a horseshoe [?] with at least two rectangles. There are two possibilities: Either one of these rectangles is not rotated by the induced dynamics, and so we get immediately a saddle periodic cycle with positive eigenvalues. Or both rectangle are rotated by a half turn. Then we can compose the induced dynamics by these two rectangles to obtain a hyperbolic periodic cycle with positive eigenvalues. \square

Let U_0 be neighborhood of P_0 and let $U_i := f^i(U_0)$ for every i . We assume U_0 small enough so that $U_0 \cup U_q$ can be identified to a subset of \mathbb{R}^2 . Then each U_i , $1 \leq i \leq q-1$, can also be identified to subset of \mathbb{R}^2 using the diffeomorphism $f^{q-i} : U_i \rightarrow U_q$. In these identifications, $U_1 \equiv \dots \equiv U_{q-1} \equiv U_q$, $f|_{U_i} \rightarrow U_{i+1}$ is the identity for $1 \leq i \leq q-1$ and $f|_{U_0} \equiv f^q|_{U_0}$.

We apply Proposition 3.2 to $f^q|_{U_0}$. This blows up P_0 to a disk, and so U_0 and U_q to \hat{U}_0 and \hat{U}_q . Furthermore we can lift $f : U_0 \rightarrow U_1$ to $\hat{f} : \hat{U}_0 \rightarrow \hat{U}_1$ so that $\hat{f}^q|_{\hat{U}_0}$ displays three elliptic islands. Using the above identifications, we also blow up each $U_1 \equiv \dots \equiv U_{q-1} \equiv U_q$ to $\hat{U}_1 \equiv \dots \equiv \hat{U}_{q-1} \equiv \hat{U}_q$. We lift each $f|_{U_i} \equiv id$ to $\hat{f}|_{\hat{U}_i} \equiv id$ for $1 \leq i \leq q-1$.

All these blow-ups can define a blow up \hat{S} of S along the orbit $(P_i)_{i \in \mathbb{Z}_q}$, and a lifting \hat{f} of f which displays the sought properties. \square

Proof of Proposition 3.2. By [?], there exists analytic and symplectic coordinates of a neighborhood of 0 for which the dynamics is of the following form with λ an analytic function:

$$f(x, y) = (\exp(\lambda(x \cdot y)) \cdot x, \exp(-\lambda(x \cdot y)) \cdot y), \quad \text{with } \lambda(x \cdot y) > 0.$$

Let Λ be an integral of the function λ so that $\Lambda(0) = 0$. Note that f is the time one of the flow of the Hamiltonian $H_1 : (x, y) \mapsto \Lambda(x \cdot y)$. Let us follow the same lines as in Sections 2.3 to 2.5.

The difference is that the function λ here is not constant. Also we will not perform the quotient $\mathbb{R}/2\mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{R}$. Hence basically, we will construct a 2-covering of the previous cap.

First, we perform a symplectic and analytic blow-up of 0. For $\epsilon > 0$ small, let $r_\epsilon := \sqrt{\frac{2}{\pi}\epsilon}$. The r_ϵ -neighborhood $\mathbb{D}(r_\epsilon)$ of 0 is blown up to an annulus via the map:

$$\pi_0 : (\theta, r) \in \mathbb{R}/2\mathbb{Z} \times [0, \epsilon] \mapsto \left(\sqrt{\frac{2r}{\pi}} \cos(\pi\theta), \sqrt{\frac{2r}{\pi}} \sin(\pi\theta) \right) \in \mathbb{D}(r_\epsilon).$$

This blow up defines a symplectic and analytic annulus \mathbb{D}^* as :

$$\mathbb{D}^* := (\mathbb{D}(r_\epsilon) \setminus \{0\}) \sqcup \mathbb{R}/2\mathbb{Z} \times [0, \epsilon]/\pi_0.$$

Note that H_1 lifts to $\mathbb{R}/2\mathbb{Z} \times [0, \epsilon]$ as the mapping:

$$H : (\theta, r) \mapsto \Lambda\left(\frac{r}{\pi} \sin(2\theta)\right)$$

Observe that $\Omega = d\theta \wedge dr$ and H extend canonically to $V := \mathbb{R}/2\mathbb{Z} \times (-\epsilon, \epsilon)$. On $C = \mathbb{R}/2\mathbb{Z} \times \{0\}$, the dynamics displays four saddle fixed points $Q_i = (\frac{i}{2}, 0)$ with $i \in \mathbb{Z}/4\mathbb{Z}$, so that $\bigcup_{i=1}^2 W^s(Q_{2i}) \cup \{Q_{2i+1}\} = \bigcup_{i=1}^2 W^u(Q_{2i+1}) \cup \{Q_{2i}\} = C$. In particular C consists of four heteroclinic links. Thus we can conclude by applying the next lemma with $k = 2$. \square

Lemma 3.4 (Generalized cap). *Let H be an analytic Hamiltonian defined on a neighborhood V of $C = \mathbb{R}/2\mathbb{Z} \times \{0\}$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ such that C is an union of $2k$ -heteroclinic links: $C = \bigcup_{i \in \mathbb{Z}_k} W^s(Q_{2i}) \cup \{Q_{2i+1}\} = \bigcup_{i \in \mathbb{Z}_k} W^u(Q_{2i+1}) \cup \{Q_{2i}\}$. Then there exists there exists $\delta > 0$ and an analytic map ϕ from $\mathbb{R}/2\mathbb{Z} \times (-\delta, 0]$ to a neighborhood of $\partial\mathbb{D}$ in $\bar{\mathbb{D}}$ such that $H \circ \phi^{-1}$ extends to an analytic function on $\bar{\mathbb{D}}$ which displays exactly $k + 1$ critical points in \mathbb{D} with definite Hessian.*

Proof. We depict the construction for $k = 2$ in Fig. 5 [left-center]. For $k = 1$, this lemma implies §2.4 and 2.5; its proof is similar.

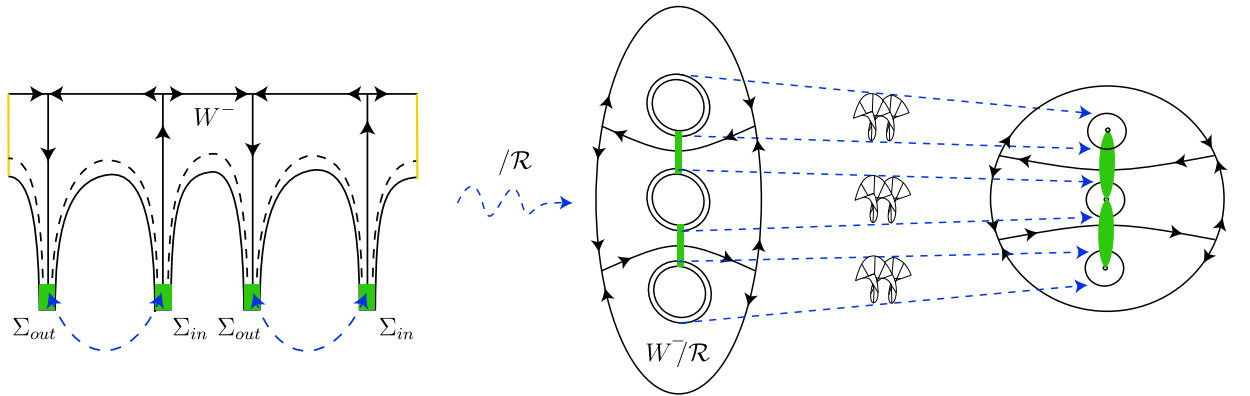


Figure 5: Making an integrable generalized cap by surgery with $k = 2$.

On C the function H must be constant; let us assume it equal to 0. For $\eta > 0$ small, we define:

$$W^- := \{(\theta, r) \in \mathbb{R}/2\mathbb{Z} \times [-\eta, 0] : |H(\theta, r)| \leq \eta^3 \quad \text{and} \quad |r| \leq \eta\}.$$

The boundary of W^- is made by $4k + 1$ curves (see Fig. 3[left]); $2k$ of them form:

$$\Sigma := \{\theta \in \mathbb{R}/2\mathbb{Z} : |H(\theta, r)| \leq \eta^2\} \times \{-\eta\}.$$

For every $i \in \mathbb{Z}_k$, let $\Sigma_{out,i}$ be the component of Σ which intersects $W_{2\eta}^u(Q_{2i})$ and $\Sigma_{in,i}$ be the component of Σ which intersects $W_{2\eta}^s(Q_{2i+1})$. On each $\Sigma_{out,i}$, the restriction $H|_{\Sigma_{out,i}}$ is a diffeomorphism onto $[-\eta^3, \eta^3]$. Thus there is a canonical parametrization of Σ_{out} with $[-\eta^3, \eta^3] \times \mathbb{Z}_k$. Likewise there is a canonical parametrization of Σ_{in} with $[-\eta^3, \eta^3] \times \mathbb{Z}_k$. Let ϕ_H^t the Hamiltonian flow of H . To perform the surgery depicted in Fig. 5 [left-center], we define:

$$\psi : \bigcup_{t \in [0,1]} \phi_H^{-t}(\Sigma_{out}) \rightarrow \bigcup_{t \in [0,1]} \phi_H^t(\Sigma_{in}),$$

such that for $t \in [0, 1]$, the point $(\theta, r) = \phi_H^{-t}(\theta_0, r_0)$ with $(r_0, \theta_0) \in \Sigma_{out}$ parametrized by $(x, i) \in [-\eta^3, \eta^3] \times \mathbb{Z}_k$, is sent by ψ to $\phi_H^{1-t}(\theta'_0, r'_0)$ with $(r'_0, \theta'_0) \in \Sigma_{in}$ parametrized by $(x, i) \in [-\eta^3, \eta^3] \times \mathbb{Z}_k$. Note that the map ψ is analytic and symplectic. Moreover it respects H . Let W^-/\mathcal{R} be equal to W^- quotiented by the equivalence relation \mathcal{R} spanned by:

$$(r, \theta)\mathcal{R}(r', \theta') \quad \text{if} \quad (r, \theta) = \psi(r', \theta'), \quad \forall (r, \theta) \in \bigcup_{t \in [0,1]} \phi_H^{-t}(\Sigma_{out}) \text{ and } (r', \theta') \in \bigcup_{t \in [0,1]} \phi_H^t(\Sigma_{in}).$$

Also as ϕ is an analytic diffeomorphism which sends the intersection of its domain with $\partial W^- \setminus \Sigma$ into $\partial W^- \setminus \Sigma$, the space W^-/\mathcal{R} is an analytic surface with boundary. Moreover, we notice that W^-/\mathcal{R} is equal to the closed disk \mathbb{D} without $k + 1$ disks $(\mathbb{D}_i)_{0 \leq i \leq k}$ as depicted in Fig. 5 [center]:

$$W^-/\mathcal{R} = \mathbb{D} \setminus \left(\bigcup_{i=0}^k \mathbb{D}_i \right)$$

This surgery respects the symplectic form Ω and the Hamiltonian H , which remains analytic on W/\mathcal{R} that we still denote by H .

Now we would like to blow-down the circles $\partial \mathbb{D}_i$ to fixed points P_i . These blow-downs construct $k + 1$ disks depicted in Fig. 5 [Center-Right]. In order to do so, we first observe that on $\bigsqcup_i \partial \mathbb{D}_i$, the function H is locally constant (equal to respectively η^3 , $-\eta^3$ and η^3) but DH does not vanish on these circles. So we can apply Lemma 2.1. For every i , it gives the existence of a neighborhood V_i of $\partial \mathbb{D}_i$ endowed with analytic and symplectic coordinate $\psi_\sigma : V_i \rightarrow \mathbb{R}/\mathbb{Z} \times [0, \delta]$ such that $H \circ \psi_i^{-1}(\theta, r) = h_i(r)$ for an analytic map h_i . So we can perform a blow-down of the circle $\partial \mathbb{D}_i$ to a fixed point P_i . This blow-down sends V_i to a disk of radius $\sqrt{\delta/\pi}$ and on this disk the continuation of the Hamiltonian is equal to $P_i + (x, y) \mapsto h_i(x^2 + y^2)$. As the unique critical points of $H|_{W^-}$ were $(Q_i)_{i \in \mathbb{Z}_{2k}}$, these surgeries creates only $k + 1$ -new critical points at P_i which are all with definite Hessian. \square

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