Spreading speeds and one-dimensional symmetry for reaction-diffusion equations

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Abstract

This paper is devoted to the study of the large time dynamics of bounded solutions of reaction-diffusion equations with unbounded initial support in $\mathbb{R}^N$. We first prove a general Freidlin-Gärtner type formula for the spreading speeds of the solutions in any direction. This formula holds under general assumptions on the reaction and for solutions emanating from initial conditions with general unbounded support, whereas most of earlier results were concerned with more specific reactions and compactly supported or almost-planar initial conditions. We also prove some results of independent interest on some conditions guaranteeing the spreading of solutions with large initial support and the link between these conditions and the existence of traveling fronts with positive speed. Furthermore, we show some flattening properties of the level sets of the solutions if initially supported on subgraphs. We also investigate the special case of asymptotically conical-shaped initial conditions. For Fisher-KPP equations, we prove some asymptotic one-dimensional symmetry properties for the elements of the Ω-limit set of the solutions, in the spirit of a conjecture of De Giorgi for stationary solutions of Allen-Cahn equations. Lastly, we show some logarithmic-in-time estimates of the lag of the position of the solutions with respect to that of a planar front with minimal speed, for initial conditions which are supported on subgraphs with logarithmic growth at infinity. The proofs use a mix of ODE and PDE methods, as well as some geometric arguments. The paper also contains some related conjectures and open problems.

Contents

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1 Introduction and main results

In this paper, we are interested in the large time dynamics of solutions of the reaction-diffusion equation

$$\partial_t u = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

with $N \geq 2$ and initial conditions u_0 having unbounded support. More precisely, the reaction term $f : [0,1] \to \mathbb{R}$ is of class $C^1([0,1])$ with

$$f(0) = f(1) = 0,$$

and the initial conditions u_0 are assumed to be characteristic functions $\mathbb{1}_U$ of sets U, i.e.

$$u_0(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus U, \end{cases} \tag{1.2}$$

where the initial support U is an unbounded measurable subset of $\mathbb{R}^N$ (although some results also cover the case of non-empty bounded sets U).[1] This Cauchy problem is well posed

[1] We use the term "initial support U", with an abuse of notation, to refer to the set U in the definition (1.2) of the initial condition u_0. This set U differs in general from the usual support $\operatorname{supp} u_0$ of u_0, which is defined as the complement of the largest open set of $\mathbb{R}^N$ where u_0 is equal to 0 almost everywhere with respect to the Lebesgue measure. However, U coincides with $\operatorname{supp} u_0$ if and only if U is closed and the intersection of U with any non-trivial ball centered at any point of U has a positive Lebesgue measure.

and, given u_0, there is a unique bounded classical solution u of (1.1) such that $u(t,\cdot) \to u_0$ as $t \to 0^+$ in $L^1_{loc}(\mathbb{R}^N)$. For mathematical convenience, we extend f by 0 in $\mathbb{R} \setminus [0,1]$, and the extended function, still denoted f, is then Lipschitz continuous in $\mathbb{R}$.

Instead of initial conditions $u_0 = \mathbb{1}_U$, we could also have considered multiples $\alpha \mathbb{1}_U$ of characteristic functions, with $\alpha > 0$, at the expense of some further assumptions on the reaction term f, or even other more general initial conditions $0 \le u_0 \le 1$ for which the upper level set $\{x \in \mathbb{R}^N : u_0(x) \ge \theta\}$ is at bounded Hausdorff distance from the support $\mathrm{supp}\, u_0$ of u_0, where $\theta \in (0,1)$ is a suitable value depending on f, precisely given by Hypothesis 1.1 below (see Remarks 3.7 and 5.3 below). But we preferred to keep the assumption $u_0 = \mathbb{1}_U$ for the sake of simplicity of the presentation and of readability of the statements, all the more as this case already gives rise to many interesting and non-trivial features, depending on the type and shape of the unbounded set U.

1.1 Two main questions

Due to diffusion, the solution u of (1.1)-(1.2) is smooth at positive times and

$$0 < u < 1 \ \text{ in } (0,+\infty) \times \mathbb{R}^N$$

from the strong parabolic maximum principle, provided the Lebesgue measures of U and $\mathbb{R}^N \setminus U$ are positive. However, from parabolic estimates, at each finite time, u stays close to 1 or 0 in subregions of U or $\mathbb{R}^N \setminus U$ which are far away from ∂U.

One of the objectives of the present work is to describe the location at large time of the regions where u stays close to 1 or 0. How do these regions move and possibly spread in any direction? A fundamental issue is to understand whether and how the solution keeps a memory at large time of its initial support U. A basic question is the following:

Question A. For a given vector $e \in \mathbb{R}^N$ with unit Euclidean norm, is there a *spreading speed* $w(e)$ such that

$$u(t, cte) \to 1 \text{ as } t \to +\infty \text{ for every } 0 \le c < w(e),$$

and

$$u(t, cte) \to 0 \text{ as } t \to +\infty \text{ for every } c > w(e)?$$

Can one find a formula for $w(e)$ and how does $w(e)$ depend on e and the initial support U?

We will answer this question especially in Theorems 1.5 and 1.6 under some general hypotheses, and in Theorem 1.19 and Proposition 1.20 for more specific reactions f, with more precise estimates on the location of the level sets of the solutions in some directions in the latter case. The speed $w(e)$ can possibly be $+\infty$ in some directions e, and this actually occurs in the directions around which U is unbounded, in a sense that will be made precise in Section 1.3. We also provide several counterexamples in Section 3.2.

Another goal of the paper is to have an insight about the profile of the solution around its level sets. With this respect, we investigate two classes of properties: the *flattening* of the level sets, and the *asymptotic one-dimensional symmetry* of the solution. The latter is expressed in terms of the notion of limit set, which is defined as follows: for a given function $u : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$, the set

$$\begin{aligned}\Omega(u) := \big\{ & \psi \in L^\infty(\mathbb{R}^N) \ : \ u(t_n, x_n + \cdot) \to \psi \text{ in } L^\infty_{loc}(\mathbb{R}^N) \text{ as } n \to +\infty, \\ & \text{for some sequences } (t_n)_{n\in\mathbb{N}} \text{ in } \mathbb{R}^+ \text{ diverging to } +\infty \text{ and } (x_n)_{n\in\mathbb{N}} \text{ in } \mathbb{R}^N \big\}\end{aligned} \tag{1.3}$$

is called the Ω-limit set of u.

Roughly speaking, the Ω-limit set contains all possible asymptotic profiles of the function as $t \to +\infty$. Notice that, for any bounded solution u of (1.1), the set $\Omega(u)$ is not empty and is included in $C^2(\mathbb{R}^N)$, from standard parabolic estimates. Motivated by some known results in the literature, the following question naturally arises.

Question B. Let u be a solution to (1.1) emerging from an initial datum $u_0 = \mathbb{1}_U$. Is it true that any function $\psi \in \Omega(u)$ is of the form

$$\psi = \psi(x \cdot e),$$

for some $e \in \mathbb{S}^{N-1}$? [2] If the answer to the question is positive, we then say that u satisfies the *asymptotic one-dimensional symmetry*.

In short, we will first prove, in the case of initial supports U that are subgraphs, some flattening properties in Theorems 1.7 and 1.9, which can be seen as some steps towards a positive answer to Question B. We will later answer Question B and related issues, for some specific reactions f, in Theorems 1.13 and 1.14 and in Corollaries 1.15 and 1.21.

More precisely, for the answer to Question B to possibly be affirmative, some conditions on f and U need to be imposed, as shown by some counter-examples that we exhibit in Section 1.5. We will also review in that section some known positive results which hold in the case where the initial support U is bounded, or when it is at finite Hausdorff distance from a half-space, under some assumptions on f. We will extend such results for a nonlinearity f of the Fisher-KPP type (cf. condition (1.43) below), giving a positive answer to Question B when U fulfills (in particular) a uniform interior ball condition and it is convex, or, more generally, it is at bounded Hausdorff distance from a convex set, see Theorem 1.13 below. These conditions on U are actually a very particular instance of the geometric hypotheses under which we derive our most general result about the asymptotic one-dimensional symmetry, Theorem 1.14 below.

Question B reclaims the De Giorgi conjecture about solutions of the Allen-Cahn equation (that is, stationary solutions of the reaction-diffusion equation $\Delta u + u(1-u)(u-1/2)$, obtained after a change of unknown from the original Allen-Cahn equation), see [8]. We can also wonder whether the following stronger property holds: with the existence of a planar front connecting 1 to 0 with a positive speed (see Hypothesis 1.3 below), for a solution u to (1.1) emerging from an initial datum $u_0 = \mathbb{1}_U$, is it true that $\Omega(u)$ only contains the steady states 0 and 1 and the profile of the critical planar front in certain directions (the existence of such a critical front is then provided by Proposition 1.4 below)?

The situation considered in this paper can be viewed as a counterpart of many papers devoted to the large time dynamics of solutions of (1.1) with initial conditions u_0 that are compactly supported or converge to 0 at infinity. We refer to e.g. [2, 9, 31, 36, 37, 55] for extinction/invasion results in terms of the size and/or the amplitude of the initial condition u_0 for various reaction terms f, and to [9, 11, 34, 35, 42] for general local convergence and quasiconvergence results at large time. For the invading solutions u (that is, those converging to 1 locally uniformly in $\mathbb{R}^N$ as $t \to +\infty$) with localized initial conditions, further estimates on the location and shape at large time of the level sets have been established in [13, 18, 27, 45, 49, 51, 53]. Lastly, equations of the type (1.1) set in unbounded

[2] Throughout the paper, by the formula $\psi = \psi(x \cdot e)$ we mean, with a slight abuse of notation, that there is a function $\Psi : \mathbb{R} \to [0,1]$ such that $\psi(x) = \Psi(x \cdot e)$ for all $x \in \mathbb{R}^N$ where Ψ is necessarily of class $C^2(\mathbb{R})$.

domains Ω instead of $\mathbb{R}^N$ and notions of spreading speeds and persistence/invasion in such domains have been investigated in [5, 50].

1.2 Two main hypotheses and two preliminary results

In this section, we list some notations and hypotheses which are used in the various main results. The hypotheses are expressed in terms of the solutions of (1.1) with more general initial conditions than characteristic functions, or actually in terms of the reaction term f solely. We then discuss the logical link between these hypotheses.

Throughout the paper, "$|\ |$" and "$\cdot$" denote respectively the Euclidean norm and inner product in $\mathbb{R}^N$,

$$B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$$

is the open Euclidean ball of center $x \in \mathbb{R}^N$ and radius $r > 0$, $B_r = B_r(0)$, and $\mathbb{S}^{N-1} = \{e \in \mathbb{R}^N : |e| = 1\}$ is the unit Euclidean sphere of $\mathbb{R}^N$. The distance of a point $x \in \mathbb{R}^N$ from a set $A \subset \mathbb{R}^N$ is given by $\mathrm{dist}(x, A) := \inf\big\{|y - x| : y \in A\big\}$, with the convention $\mathrm{dist}(x, \emptyset) = +\infty$. We also call $(\mathrm{e}_1, \cdots, \mathrm{e}_N)$ the canonical basis of $\mathbb{R}^N$, that is,

$$\mathrm{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$$

for $1 \le i \le N$, where 1 is the ith coordinate of e_i.

Since both 0 and 1 are steady states, the question of the interplay between these two states and the diffusion is intricate. One way to differentiate the roles of 0 and 1 is to assume that the state 1 is more stable than 0, in the sense that it attracts the solutions of (1.1) – not necessarily satisfying (1.2) – that are "large enough" in large balls at initial time.

Hypothesis 1.1. *Invasion occurs for any solution u of (1.1) with a "large enough" initial datum u_0, that is, there exist $\theta \in (0, 1)$ and $\rho > 0$ such that if*

$$\theta\, \mathbb{1}_{B_\rho(x_0)} \le u_0 \le 1 \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

for some $x_0 \in \mathbb{R}^N$, then $u(t, x) \to 1$ as $t \to +\infty$, locally uniformly with respect to $x \in \mathbb{R}^N$.

If f satisfies the following conditions:

$$f > 0 \ \text{ in } (0, 1) \quad \text{and} \quad \liminf_{s \to 0^+} \frac{f(s)}{s^{1+2/N}} > 0, \tag{1.5}$$

then Hypothesis 1.1 is satisfied with any $\theta \in (0, 1)$ and $\rho > 0$, see [2]; this property is known as the *hair trigger effect*. If $f > 0$ in $(0, 1)$ (without any further assumption on the behavior of f at 0^+), then Hypothesis 1.1 is still satisfied with any $\theta \in (0, 1)$, and with $\rho > 0$ large enough (this fact can also be viewed as a particular case of Proposition 1.2 below). Hypothesis 1.1 holds as well if f is of the ignition type, that is,

$$\exists\, \alpha \in (0, 1), \quad f = 0 \text{ in } [0, \alpha] \text{ and } f > 0 \text{ in } (\alpha, 1), \tag{1.6}$$

and θ in Hypothesis 1.1 can be any real number in the interval $(\alpha, 1)$, provided $\rho > 0$ is large enough. For a bistable function f satisfying

$$\exists\, \alpha \in (0, 1), \quad f < 0 \text{ in } (0, \alpha) \text{ and } f > 0 \text{ in } (\alpha, 1), \tag{1.7}$$

Hypothesis 1.1 is equivalent to $\int_0^1 f(s)\,ds > 0$, see [2, 15], and in that case θ in Hypothesis 1.1 can be any real number in $(\alpha, 1)$, provided $\rho > 0$ is large enough. For a tristable function f satisfying

$$\exists\, 0 < \alpha < \beta < \gamma < 1, \quad f < 0 \text{ in } (0,\alpha) \cup (\beta,\gamma) \text{ and } f > 0 \text{ in } (\alpha,\beta) \cup (\gamma,1), \tag{1.8}$$

then it easily follows from [15] that Hypothesis 1.1 is equivalent to the positivity of both integrals $\int_\beta^1 f$ and $\int_0^1 f$, and, for such a function, these positivity conditions are in turn equivalent to the positivity of $\int_t^1 f$ for every $t \in [0,1)$.

More generally speaking, it actually turns out that Hypothesis 1.1 is equivalent to some simple conditions on the function f involving the integrals $\int_t^1 f$ and the positivity of f in a left neighborhood of 1, as the following result shows.[3]

Proposition 1.2. *For a $C^1([0,1])$ function f such that $f(0) = f(1) = 0$, Hypothesis 1.1 is equivalent to the following two conditions simultaneously:*

$$\exists\, \theta \in (0,1), \quad f > 0 \text{ in } [\theta, 1), \tag{1.9}$$

and

$$\forall\, t \in [0,1), \quad \int_t^1 f(s)\,ds > 0. \tag{1.10}$$

Furthermore, θ can be chosen as the same real number in Hypothesis 1.1 and in (1.9).

In particular, Hypothesis 1.1 is satisfied if $f \geq 0$ in $[0,1]$ and if condition (1.9) holds. Notice however that condition (1.9) alone is not enough to guarantee Hypothesis 1.1, since bistable functions f of the type (1.7) satisfy (1.9) but do not satisfy Hypothesis 1.1 as soon as $\int_0^1 f \leq 0$. Similarly, condition (1.10) alone is not enough to guarantee Hypothesis 1.1, since there are $C^1([0,1])$ functions f which vanish at 0 and 1 and satisfy (1.10) but not (1.9): consider for instance f defined by $f(1) = 0$ and $f(s) = s(1-s)^3 \sin^2(1/(1-s))$ for $s \in [0,1)$.

We also point out that Proposition 1.2 implies that Hypothesis 1.1 is independent of the dimension N, whereas, for a function f which is positive in $(0,1)$, the hair trigger effect (that is, the arbitrariness of $\theta \in (0,1)$ and $\rho > 0$ in Hypothesis 1.1) depends on N (for instance, for the function $f(s) = s^p(1-s)$ with $p \geq 1$, Hypothesis 1.1 holds in any dimension $N \geq 1$, but the hair trigger effect holds if and only if $p \leq 1 + 2/N$, see [2]).

In the large time dynamics of the solutions of the Cauchy problem (1.1), a crucial role is played by the *traveling front* solutions connecting the steady states 1 and 0, namely the solutions of the form

$$u(t,x) = \varphi(x \cdot e - ct)$$

with $c \in \mathbb{R}$, $e \in \mathbb{S}^{N-1}$, and

$$0 = \varphi(+\infty) < \varphi(z) < \varphi(-\infty) = 1 \text{ for all } z \in \mathbb{R}. \tag{1.11}$$

The level sets of these solutions are hyperplanes orthogonal to e traveling with the constant speed c in the direction e. If they exist, their profile φ is necessarily decreasing and unique up to shifts, for a given speed c (see also the proof of Lemma 2.1 below for further details on these properties). Most of our main results are derived under the following hypothesis:

[3] To the best of our knowledge, the equivalence stated in Proposition 1.2 is not present in the literature, for general functions f. However, the fact that (1.9)-(1.10) imply Hypothesis 1.1 is contained in [11, Lemma 2.4].

Hypothesis 1.3. *Equation* (1.1) *in* $\mathbb{R}$ *admits a traveling front connecting* 1 *to* 0 *with positive speed* $c_0 > 0$.

Hypothesis 1.3 is fulfilled for instance if $f > 0$ in $(0,1)$, or if f is of the ignition type (1.6), or if f is of the bistable type (1.7) with $\int_0^1 f(s)\,ds > 0$ (in the last two cases, the speed c_0 is unique), see [2, 15, 28]. Hypothesis 1.3 is also satisfied for some functions f having multiple oscillations in the interval $[0,1]$, see the comments on the example (1.8) after Proposition 1.4 below.

It actually turns out that Hypothesis 1.3 is equivalent to the existence of a positive minimal speed c^* of traveling fronts connecting 1 to 0, and that Hypothesis 1.3 also implies Hypothesis 1.1 and further spreading properties for the solutions of (1.1) fulfilling the conditions of Hypothesis 1.1. These facts are expressed by the following.

Proposition 1.4. *Assume Hypothesis* 1.3. *Then equation* (1.1) *in* $\mathbb{R}$ *admits a traveling front connecting* 1 *to* 0 *with minimal speed* c^*, *and* $c^* > 0$. *Furthermore, Hypothesis* 1.1 *is fulfilled and, for any solution* u *as in Hypothesis* 1.1, *there holds that*

$$\forall\, c \in [0, c^*), \quad \min_{|x| \le ct} u(t,x) \to 1 \quad as \;\; t \to +\infty. \tag{1.12}$$

Lastly, for any compactly supported initial condition $0 \le u_0 \le 1$, *the solution* u *of* (1.1) *satisfies*

$$\forall\, c > c^*, \quad \sup_{|x| \ge ct} u(t,x) \to 0 \quad as \;\; t \to +\infty. \tag{1.13}$$

Several comments are in order. Firstly, the minimality of c^* means that (1.1) in $\mathbb{R}$ admits a solution of the form $\varphi(x - c^*t)$ satisfying (1.11), and it does not admit a solution of the same type with c^* replaced by a smaller quantity (notice that, necessarily, $c^* \le c_0$ under the notation of Hypothesis 1.3).

Secondly, the part of Proposition 1.4 asserting that the existence of a traveling front with positive minimal speed c^* yields the spreading properties (1.12)-(1.13), answers affirmatively to Question A of Section 1.1 under Hypothesis 1.3, in the particular case of compactly supported initial data. This can be viewed as a natural extension of some results of the seminal paper [2], which were originally obtained under more specific assumptions on f. Furthermore, we mention that, if there is $\delta > 0$ such that f is nonincreasing in $[0, \delta]$ and in $[1-\delta, 1]$, it has also been known that the existence of a traveling front with positive minimal speed c^* implies Hypothesis 1.1 in this case (see [12, Theorem 1.5] which contains a more general result concerning periodic equations). Proposition 1.4 means that this implication holds without any further assumption on f, and that the existence of a traveling front with positive speed is actually sufficient to get the conclusion.

Thirdly, whereas Proposition 1.4 shows the implication "Hypothesis 1.3 $\Longrightarrow$ Hypothesis 1.1", we point out that the converse implication "Hypothesis 1.1 $\Longrightarrow$ Hypothesis 1.3" is false in general. For instance, consider equation (1.1) in dimension $N = 1$ with a tristable function f satisfying (1.8) and such that $\int_0^\beta f > 0$ and $\int_\beta^1 f > 0$, and let c_1 and c_2 be the unique (positive) speeds of the traveling fronts $\varphi_1(x - c_1 t)$ and $\varphi_2(x - c_2 t)$ connecting β to 0 on the one hand, and 1 to β on the other hand. It follows from [15] that, if $c_1 \ge c_2$, then Hypothesis 1.3 is not satisfied, while Hypothesis 1.1 is (from [15], or from Proposition 1.2). In that case, it turns out that the compactly supported initial conditions u_0 giving rise to invading solutions u as in Hypothesis 1.1 develop into a terrace of two expanding fronts

with speeds c_1 and c_2 (the notions of terraces have been further investigated in more general frameworks in [10, 14, 20, 44]). On the other hand, still with (1.8) and the positivity of $\int_0^\beta f$ and $\int_\beta^1 f$, Hypothesis 1.3 is satisfied if (and, then, only if) $c_1 < c_2$, see [15].

Lastly, under Hypothesis 1.3, Proposition 1.4 gives the exact spreading speed (1.12)-(1.13) of the solutions u satisfying the conditions of Hypothesis 1.1 with compactly supported initial conditions. However, as observed in Remark 2.5 below, under the sole Hypothesis 1.1, the property (1.12) is still fulfilled, for a certain positive speed c^* which nevertheless may not be any speed of a traveling front connecting 1 to 0.

Most of our main results are derived under Hypothesis 1.3, which then yields Hypothesis 1.1 and property (1.12) with the minimal speed c^*. But one result (namely, Theorem 1.9), only requires Hypothesis 1.1 (but not necessarily Hypothesis 1.3).

1.3 A general notion of spreading set and a Freidlin-Gärtner type formula

In this section, under Hypothesis 1.3, we investigate the asymptotic set of spreading for the solutions u of (1.1)-(1.2) with general unbounded sets U containing large enough balls. Such solutions u then converge to 1 as $t \to +\infty$ locally uniformly in $\mathbb{R}^N$, and even satisfy (1.12), with $c^* > 0$ given by Proposition 1.4. But we now want to provide a more precise description of the invasion of the state 0 by the state 1. We point out that the invasion cannot be uniform in all directions in general, since it shall strongly depend on the initial set U. For $e \in \mathbb{S}^{N-1}$, we then look for a quantity $w(e) \in (0, +\infty]$ satisfying

$$\lim_{t\to+\infty} u(t, cte) = \begin{cases} 1 & \text{if } \ 0 \le c < w(e), \\ 0 & \text{if } \ c > w(e), \end{cases} \tag{1.14}$$

and even the stronger condition

$$\begin{cases} \lim\limits_{t\to+\infty} \left(\min\limits_{0\le s\le c} u(t, ste) \right) = 1 & \text{if } \ 0 \le c < w(e), \\ \lim\limits_{t\to+\infty} \left(\sup\limits_{s\ge c} u(t, ste) \right) = 0 & \text{if } \ c > w(e). \end{cases} \tag{1.15}$$

This quantity is referred to as the *spreading speed* and represents the asymptotic speed at which the level sets between 0 and 1 move in the direction e. If it exists, it necessarily satisfies $w(e) \ge c^*$ by Proposition 1.4. However, in contradistinction with the case of compactly supported initial data, the spreading speed may not exist when the support of the initial datum is unbounded, as we show in Proposition 3.8 below.

We will derive a geometric condition on U under which the spreading speed exists and, in addition, it fulfills the following Freidlin-Gärtner type formula:

$$w(e) = \sup_{\substack{\xi\in\mathcal{U}(U) \\ \xi\cdot e\ge 0}} \frac{c^*}{\sqrt{1-(\xi\cdot e)^2}}, \tag{1.16}$$

where $\mathcal{U}(U) \subset \mathbb{S}^{N-1}$ is a suitable set of directions, depending on U.[4] We use the conventions:

$$\begin{cases} w(e) = c^* & \text{if there is no } \xi \in \mathcal{U}(U) \text{ such that } \xi \cdot e \geq 0, \\ w(e) = +\infty & \text{if } e \in \mathcal{U}(U). \end{cases} \tag{1.17}$$

Loosely speaking, $\mathcal{U}(U)$ is the set of directions "around which U is unbounded". Here is the precise definition:

$$\mathcal{U}(U) := \Big\{ \xi \in \mathbb{S}^{N-1} : \lim_{\tau \to +\infty} \frac{\mathrm{dist}(\tau\xi, U)}{\tau} = 0 \Big\}.$$

Its counterpart is the set $\mathcal{B}(U)$ of directions "around which U is bounded", defined by:

$$\mathcal{B}(U) := \Big\{ \xi \in \mathbb{S}^{N-1} : \liminf_{\tau \to +\infty} \frac{\mathrm{dist}(\tau\xi, U)}{\tau} > 0 \Big\}.$$

The sets $\mathcal{B}(U)$ and $\mathcal{U}(U)$ are respectively open and closed relatively to $\mathbb{S}^{N-1}$. The condition $\xi \in \mathcal{B}(U)$ is equivalent to the existence of an open cone $\mathcal{C}$ containing the ray $\mathbb{R}^+\xi = \{\tau\,\xi : \tau > 0\}$ such that $U \cap \mathcal{C}$ is bounded, that is, $\mathbb{R}^+\xi \subset \mathcal{C} \subset (\mathbb{R}^N \setminus U) \cup B_R$ for some $R > 0$. Conversely, for any $\xi \in \mathcal{U}(U)$ and any open cone $\mathcal{C}$ containing the ray $\mathbb{R}^+\xi$, there holds that the set $U \cap \mathcal{C}$ is unbounded.

Our first main result, Theorem 1.5 below, also provides a description of the asymptotic shape of the level sets of a solution u, defined for $\lambda \in (0,1)$ and $t > 0$ by

$$E_\lambda(t) := \big\{ x \in \mathbb{R}^N : u(t,x) > \lambda \big\}. \tag{1.18}$$

This description involves the envelop set of $w(e)$ given in (1.16), i.e.,

$$\mathcal{W} := \big\{ re \,:\, e \in \mathbb{S}^{N-1},\ \ 0 \leq r < w(e) \big\}, \tag{1.19}$$

and is expressed in terms of the Hausdorff distance between some sets involving $E_\lambda(t)$ and $t\mathcal{W}$. The Hausdorff distance is defined, for any pair of subsets $A, B \subset \mathbb{R}^N$, by

$$d_{\mathcal{H}}(A,B) := \max\Big(\sup_{x \in A} \mathrm{dist}(x,B), \sup_{y \in B} \mathrm{dist}(y,A) \Big),$$

with the conventions that

$$d_{\mathcal{H}}(A,\emptyset) = d_{\mathcal{H}}(\emptyset,A) = +\infty \text{ if } A \neq \emptyset \ \text{ and } \ d_{\mathcal{H}}(\emptyset,\emptyset) = 0.$$

Notice that a first relation between $E_\lambda(t)$ and $t\mathcal{W}$ immediately follows from (1.14), provided this formula holds: for any $\lambda \in (0,1)$,

$$\begin{cases} x \in \mathcal{W} \implies tx \in E_\lambda(t) \\ x \notin \overline{\mathcal{W}} \implies tx \notin E_\lambda(t) \end{cases} \quad \text{for large } t,$$

for which reason the set $\mathcal{W}$ (actually its closure) is referred to as the *asymptotic set of spreading* in [48]. Lastly, before stating our first main result, we define the notion of positive-distance-interior U_ρ (with $\rho > 0$) of the set U as

$$U_\rho := \big\{ x \in U : \mathrm{dist}(x, \partial U) \geq \rho \big\}.$$

[4] We call (1.16) a Freidlin-Gärtner type formula since Freidlin and Gärtner [17] were the first to characterize the spreading of solutions of reaction-diffusion equations in $\mathbb{R}^N$ by a variational formula. They were actually concerned with the spreading of solutions of KPP-type reaction-diffusion equations in periodic media with compactly supported initial conditions. Such formulas have been recently extended to more general reaction terms in [48].

Theorem 1.5. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let* $c^* > 0$ *be the minimal speed given by Proposition* 1.4, *let* $\rho > 0$ *be given by Hypothesis* 1.1, *and let* u *be the solution of* (1.1) *with an initial datum* $u_0 = \mathbb{1}_U$, *where* $U \subset \mathbb{R}^N$ *satisfies* $U_\rho \neq \emptyset$ *and*

$$\mathcal{B}(U) \cup \mathcal{U}(U_\rho) = \mathbb{S}^{N-1}. \tag{1.20}$$

Then, letting $w(e)$, *for* $e \in \mathbb{S}^{N-1}$, *be given by* (1.16)-(1.17) *and letting* $\mathcal{W}$ *be its envelop, defined by* (1.19), *the following properties hold:*

(i) *the limits* (1.15) *hold for every* $e \in \mathbb{S}^{N-1}$ *and the map* $e \mapsto w(e) \in [c^*, +\infty]$ *is continuous on* $\mathbb{S}^{N-1}$;

(ii) *for any compact set* $C \subset \mathbb{R}^N$, *there holds that*

$$\begin{cases} \lim\limits_{t \to +\infty} \Big(\min\limits_{x \in C} u(t, tx) \Big) = 1 & \text{if } C \subset \mathcal{W}, \\ \lim\limits_{t \to +\infty} \Big(\max\limits_{x \in C} u(t, tx) \Big) = 0 & \text{if } C \subset \mathbb{R}^N \setminus \overline{\mathcal{W}}; \end{cases} \tag{1.21}$$

(iii) *for any compact set* $K \subset \mathbb{R}^N$ *satisfying* $\overline{K \cap \mathcal{W}} = K \cap \overline{\mathcal{W}}$, *there holds that*

$$d_{\mathcal{H}}\Big(K \cap \frac{1}{t} E_\lambda(t) \,,\, K \cap \mathcal{W} \Big) \to 0 \quad \text{as } t \to +\infty, \quad \text{for all } \lambda \in (0,1). \tag{1.22}$$

Several comments are in order, on formula (1.16) and on the hypotheses and statements (i)-(iii) of Theorem 1.5. Firstly, since $\mathcal{U}(U)$ is closed in $\mathbb{S}^{N-1}$, it follows from formula (1.16) and the convention (1.17) that

$$\begin{cases} w(e) = +\infty & \text{if and only if } e \in \mathcal{U}(U), \\ w(e) > c^* & \text{if and only if there is } \xi \in \mathcal{U}(U) \text{ such that } \xi \cdot e > 0, \\ w(e) = c^* & \text{if and only if there is no } \xi \in \mathcal{U}(U) \text{ such that } \xi \cdot e > 0. \end{cases}$$

In particular, if U is bounded in (1.2), then $\mathcal{U}(U) = \emptyset$, $\mathcal{B}(U) = \mathbb{S}^{N-1}$, hence (1.20) is automatically fulfilled, and Theorem 1.5 – which means that (1.15), (1.21) and (1.22) hold with $w(e) \equiv c^*$ in $\mathbb{S}^{N-1}$ and $\mathcal{W} = B_{c^*}$ – can be viewed in that case as a consequence of Proposition 1.4. Furthermore, (1.22) then holds without reference to any compact set K, that is, $d_{\mathcal{H}}(t^{-1} E_\lambda(t), \mathcal{W}) \to 0$ as $t \to +\infty$.

Secondly, we observe that, for an arbitrary set U satisfying $\mathcal{U}(U) \neq \emptyset$, formula (1.16) with the convention (1.17) can be rephrased in a more geometric way:

$$w(e) \,=\, \frac{c^*}{\mathrm{dist}(e, \mathbb{R}^+ \mathcal{U}(U))} \,=\, \frac{c^*}{\sin \vartheta}\,, \tag{1.23}$$

where $\vartheta \in [0, \pi/2]$ is the minimum between $\pi/2$ and the smallest angle between the direction e and the directions in $\mathcal{U}(U)$ (with the convention $c^*/0 = +\infty$). This immediately implies the continuity of the map $e \mapsto w(e) \in [c^*, +\infty]$ in $\mathbb{S}^{N-1}$ (if $\mathcal{U}(U) = \emptyset$, then the map $e \mapsto w(e)$ is constant equal to c^*, hence continuous in $\mathbb{S}^{N-1}$).[5] Formula (1.23) also reveals

[5] For reaction-diffusion equations in $\mathbb{R}^N$ with spatially periodic coefficients, the spreading speed $w(e)$, in the sense of (1.15), may depend on the direction e, even for spreading solutions u with compactly supported initial conditions. However, the continuity of the map $e \mapsto w(e)$ still holds for monostable, ignition or bistable reactions f, as follows from the Freidlin-Gärtner formula given in [17, 48] and from the (semi)continuity of the minimal or unique speeds of pulsating traveling fronts with respect to the direction [1, 21, 48] (but the continuity of the spreading speeds and even their existence do not hold in general when pulsating waves connecting 1 to 0 do not exist anymore [20]).

that the envelop set $\mathcal{W}$ defined in (1.19) has the following simple geometric expression:

$$\mathcal{W} = \mathbb{R}^+ \mathcal{U}(U) + B_{c^*} \tag{1.24}$$

(actually whenever $\mathcal{U}(U)$ be empty or not, with the convention that $\mathbb{R}^+\emptyset + B_{c^*} = B_{c^*}$). Indeed, on the one hand, if $\mathcal{U}(U) = \emptyset$, then $w(e) = c^*$ for all $e \in \mathbb{S}^{N-1}$, and $\mathcal{W} = B_{c^*}$. On the other hand, if $\mathcal{U}(U) \neq \emptyset$, for any $e \in \mathbb{S}^{N-1}$ and $r \geq 0$ there holds $\mathrm{dist}(re, \mathbb{R}^+\mathcal{U}(U)) = r\ \mathrm{dist}(e, \mathbb{R}^+\mathcal{U}(U)) = rc^*/w(e)$ by (1.23) (using the convention $rc^*/(+\infty) = 0$), and therefore the equivalence between (1.19) and (1.24) follows. Formula (1.24) means that $\mathcal{W}$ is given by the c^*-neighborhood of the positive cone generated by the directions $\mathcal{U}(U)$, and immediately shows that $\mathcal{W}$ is an open set which is either unbounded (when $\mathcal{U}(U) \neq \emptyset$), or it coincides with B_{c^*}.

Thirdly, we point out that property (1.21) contains (1.14), owing to the continuity of the map $e \mapsto w(e)$ in $\mathbb{S}^{N-1}$. It further yields the first line of (1.15) by taking C as the segment between 0 and ce with $0 \leq c < w(e)$. Compared to the first lines of (1.14)-(1.15), the first line of (1.21) provides an additional uniformity with respect to the direction e. Let us also remark that property (1.22) applies in particular with $K = \overline{B_R}$, for an arbitrary $R > 0$, since $\overline{\overline{B_R} \cap \mathcal{W}} = \overline{B_R} \cap \overline{\mathcal{W}}$.

As far as the assumptions of Theorem 1.5 are concerned, we first remind that Hypothesis 1.3 is known to hold in the classical monostable case ($f > 0$ in $(0,1)$), ignition case (1.6), and bistable case (1.7) (in the latter case, with $\int_0^1 f(s)ds > 0$). Hence Theorem 1.5 implies that the convergences (1.15), (1.21) and (1.22) hold in such cases for initial data $u_0 = \mathbb{1}_U$ associated with a set $U \subset \mathbb{R}^N$ satisfying $U_\rho \neq \emptyset$ and (1.20). Moreover, in the case of a positive nonlinearity satisfying (1.5), for which the hair trigger effect holds, it suffices that such geometric conditions on U hold with ρ arbitrarily small.

Let us now comment on the geometric assumption (1.20), which is readily seen to be invariant under rigid transformations of U. Some sufficient conditions for the validity of (1.20) are given in Proposition 3.6 below. Next, it is clear that $\mathcal{U}(U_\rho) \subset \mathcal{U}(U)$ for every $\rho \geq 0$, but the converse is not true in general (for instance, if B' denotes the open Euclidean unit ball of $\mathbb{R}^{N-1}$ and $U = \mathbb{R}^+ \times B'$, then $\mathcal{U}(U) = \mathcal{U}(U_\rho) = \{\mathrm{e}_1\}$ if $\rho \in [0,1]$, while $\mathcal{U}(U_\rho) = \emptyset$ if $\rho > 1$). Here are some sufficient conditions and examples to have that a direction ξ belongs to $\mathcal{U}(U_\rho)$:

- the set $\mathcal{C} \setminus U$ is bounded, for some open cone $\mathcal{C}$ containing the ray $\mathbb{R}^+\xi$, or, more generally, for a half-cylinder $\mathcal{C}$ with axis ξ whose section orthogonal to ξ contains an $(N-1)$-dimensional ball of radius ρ;

- U satisfies the uniform interior sphere condition of radius ρ (that is, for every $p \in \partial U$, there is $a \in U$ such that $|a - p| = \rho$ and $B_\rho(a) \subset U$) and it is *strongly unbounded* in the direction ξ, in the sense that $\mathbb{R}^+\xi \subset U + B_R$ for some $R > 0$;

- $U \cup B_R \supset \bigcup_{n\in\mathbb{N}} B_\rho(n^\alpha \xi)$ for some $R > 0$ and $\alpha > 0$ (observe that when $\alpha > 1$, the distance between two consecutive centers is $|(n+1)^\alpha\xi - n^\alpha\xi| \sim n^{\alpha-1} \to +\infty$ as $n \to \infty$).

A situation which is of particular interest for us is when U is a subgraph (we will focus on the subgraph case in Section 1.4). Let us consider some typical examples of application of Theorem 1.5 in such case. Let us use $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ for the generic notation of a point $x \in \mathbb{R}^N$, and let

$$U = \big\{x \in \mathbb{R}^N : x_N \leq \gamma(x')\big\},$$

with $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$. Assume in this paragraph that γ satisfies

$$\gamma(x') = \alpha\,|x'| + o(|x'|) \quad \text{as} \quad |x'| \to +\infty,$$

for some $\alpha \in \mathbb{R}$, which is for instance the case if $\gamma \in C^1(\mathbb{R}^{N-1})$ and $\nabla\gamma(x') \cdot x'/|x'| \to \alpha$ as $|x'| \to +\infty$. We see that $U_\rho \neq \emptyset$ for any $\rho > 0$ and that

$$\mathcal{B}(U) = \big\{e \in \mathbb{S}^{N-1} : e_N > \alpha|e'|\big\}, \quad \mathcal{U}(U) = \mathcal{U}(U_\rho) = \big\{e \in \mathbb{S}^{N-1} : e_N \leq \alpha|e'|\big\}.$$

Thus (1.20) is fulfilled and Theorem 1.5 entails the validity of (1.15), (1.21) and (1.22) under Hypothesis 1.3. However, the shape of the envelop set $\mathcal{W}$ of $w(e)$ given by (1.19) or (1.24) is completely different depending on the sign of α: if $\alpha > 0$ then $\mathcal{W} = \{x \in \mathbb{R}^N : x_N < \alpha\,|x'| + c^*\sqrt{1+\alpha^2}\}$ – which is simply a translation of the interior of the cone $\mathbb{R}^+\mathcal{U}(U)$ – hence it is non-convex and not C^1. If $\alpha < 0$ then the set $\mathcal{W}$ is given by the c^*-neighborhood of the same cone $\mathbb{R}^+\mathcal{U}(U)$, which now becomes "rounded" in its upper part; indeed in such case $w(e) = c^*$ if $e_N \geq |e'|/|\alpha|$, and the envelop $\mathcal{W}$ is convex and C^1 (but not C^2). If $\alpha = 0$ (which includes the case γ bounded) then $\mathcal{W} = \{x \in \mathbb{R}^N : x_N < c^*\}$ is an half-space, with $w(e) = +\infty$ if $e_N \leq 0$, and $w(e) = c^*/e_N$ if $e_N > 0$.

In the case where γ is locally bounded and satisfies

$$\frac{\gamma(x')}{|x'|} \to -\infty \quad \text{as } |x'| \to +\infty,$$

then $U_\rho \neq \emptyset$ for any $\rho > 0$, $\mathcal{B}(U) = \mathbb{S}^{N-1} \setminus \{-\mathrm{e}_N\}$, and $\mathcal{U}(U) = \mathcal{U}(U_\rho) = \{-\mathrm{e}_N\}$. Hence (1.20) is fulfilled and therefore, under Hypothesis 1.3, Theorem 1.5 implies that (1.15), (1.21) and (1.22) hold with the quantities $w(e)$ having the envelop $\mathcal{W} = \{x \in \mathbb{R}^N : |x'| < c^*,\ x_N \leq 0\} \cup B_{c^*}$. This is a cylinder with a "rounded" top, which is convex and C^1, but not C^2.

Statements (iii) of Theorem 1.5 gives the approximation of $t^{-1}E_\lambda(t)$ by $\mathcal{W}$ *locally* with respect to the Hausdorff distance as $t \to +\infty$. But we point out that this convergence is not *global* in general, that is, the truncation by the compact set K is truly needed for (1.22) to hold and $d_{\mathcal{H}}(t^{-1}E_\lambda(t), \mathcal{W}) \not\to 0$ as $t \to +\infty$ in general, even under the assumptions of Theorem 1.5 (see Proposition 3.12 below and also the discussion at the end of this subsection about the possible lack of convergence of $t^{-1}E_\lambda(t)$). However, the next result provides an asymptotic approximation of $t^{-1}E_\lambda(t)$ by a *family of sets*, namely $t^{-1}U + B_{c^*}$, *globally* with respect to the Hausdorff distance. For this, we do not need the geometric hypothesis (1.20), but rather that the Hausdorff distance between U and its positive-distance-interior U_ρ is finite.

Theorem 1.6. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let* $c^* > 0$ *be the minimal speed given by Proposition* 1.4, *let* $\rho > 0$ *be given by Hypothesis* 1.1, *and let* u *be the solution of* (1.1) *with an initial datum* $u_0 = \mathbb{1}_U$, *where* $U \subset \mathbb{R}^N$ *satisfies* $U_\rho \neq \emptyset$ *and*

$$d_{\mathcal{H}}(U, U_\rho) < +\infty. \tag{1.25}$$

Then, for any $\lambda \in (0,1)$, *there holds that*

$$d_{\mathcal{H}}\big(E_\lambda(t)\,,\, U + B_{c^*t}\big) = o(t) \quad \text{as} \quad t \to +\infty. \tag{1.26}$$

Property (1.26) means that $E_\lambda(t)$ behaves at large time t as the set U thickened by c^*t. A sufficient condition for (1.25) to hold is that the (non-empty) set U fulfills the uniform interior sphere condition of radius ρ: in such case $d_{\mathcal{H}}(U, U_\rho) \leq 2\rho$. In particular, if f satisfies condition (1.5) ensuring the hair trigger effect, then Theorem 1.6 applies to any non-empty set U which is uniformly smooth.

To complete this subsection, we present a list of situations where one or both hypotheses (1.20) and (1.25) of Theorems 1.5 and 1.6 do not hold and the conclusions (1.15), (1.21), (1.22) and (1.26) fail (the examples will then also show that the conditions (1.20) and (1.25) on U cannot be compared in general). We also further discuss the validity of the following convergences:

$$\lim_{t\to+\infty} \frac{1}{t} E_\lambda(t) = \mathcal{W} = \lim_{t\to+\infty} \frac{1}{t} U + B_{c^*}, \tag{1.27}$$

that one may expect to hold but that actually fail in general. This will enlighten the sharpness of our result (1.22). The above convergences are understood with respect to the Hausdorff distance (which, we point out, does not guarantee the uniqueness of the limit). We first observe that, if (1.20) is fulfilled together with the other hypotheses of Theorem 1.5, then (1.22) holds and the limit of $t^{-1}E_\lambda(t)$, if any, must be the set $\mathcal{W}$ (in the sense that the Hausdorff distance between the limit set and $\mathcal{W}$ must be 0). All of the following instances refer to the equation (1.1) with logistic term $f(u) = u(1-u)$, for which Hypothesis 1.3 holds, as well as the hair trigger effect, i.e., Hypothesis 1.1 for any $\theta \in (0,1)$ and $\rho > 0$. Then (1.20) and (1.25) are understood here with $\rho > 0$ arbitrarily small.

- There exists U violating (1.20), but fulfilling (1.25) (hence (1.26) holds), for which (1.14), (1.15), (1.21) and (1.22) all fail, for any function $w : \mathbb{S}^{N-1} \to [0, +\infty]$ and any star-shaped, open set $\mathcal{W} \subset \mathbb{R}^N$, and moreover both limits in (1.27) do not exist (see Proposition 3.8).

- There exists U violating (1.25), but fulfilling (1.20) (hence (1.14), (1.15), (1.21) and (1.22) hold), for which (1.26) fails and the first limit in (1.27) exists whereas the second one does not (see Proposition 3.9).

- There exists U violating both (1.20) and (1.25), for which (1.14), (1.15), (1.21), (1.22) and (1.26) all fail, with $w(e)$ and $\mathcal{W}$ given by (1.16)-(1.17) and (1.19), and the two limits in (1.27) exist but do not coincide (see Proposition 3.10).

- There exists U fulfilling (1.20) and (1.25) (hence (1.14), (1.15), (1.21), (1.22) and (1.26) all hold), for which both limits in (1.27) do not exist and, for all $\lambda \in (0,1)$ and $t > 0$, $d_{\mathcal{H}}(t^{-1}E_\lambda(t), \mathcal{W}) = +\infty$ (see Proposition 3.12).

1.4 The subgraph case and flattening properties

In this section, we focus on the important class of initial conditions which are characteristic functions of subgraphs in $\mathbb{R}^N$. Up to rotation, let us consider graphs in the direction x_N, and initial conditions u_0 given by

$$u_0(x', x_N) = \begin{cases} 0 & \text{if } x_N > \gamma(x'), \\ 1 & \text{otherwise,} \end{cases} \tag{1.28}$$

that is, $u_0 = \mathbb{1}_U$ with

$$U = \big\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \le \gamma(x')\big\}.$$

In all statements involving it, the function $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ is always assumed to be measurable and locally bounded. First of all, from parabolic estimates, one has $u(t, x', x_N) \to 0$ as $x_N \to +\infty$ and $u(t, x', x_N) \to 1$ as $x_N \to -\infty$, locally uniformly in $(t, x') \in [0, +\infty) \times \mathbb{R}^{N-1}$. Furthermore, $u(t, x', x_N)$ is non-increasing with respect to x_N by the parabolic maximum principle, because the initial datum u_0 is, and one actually sees that $\partial_{x_N} u < 0$ in $(0, +\infty) \times \mathbb{R}^N$ by differentiating (1.1) with respect to x_N and applying the strong maximum principle to $\partial_{x_N} u$. As a consequence, one infers that, for every $t > 0$, $x' \in \mathbb{R}^{N-1}$ and $\lambda \in (0, 1)$, there exists a unique value x_N such that $u(t, x', x_N) = \lambda$, which will be denoted $X_\lambda(t, x')$ in the sequel, that is,

$$u(t, x', X_\lambda(t, x')) = \lambda.^6 \tag{1.29}$$

In other words, the sets $E_\lambda(t)$ given in (1.18) are the open subgraphs of $x' \mapsto X_\lambda(t, x')$. Hence Theorems 1.5 and 1.6 applied to this case give some information on the shape of the graphs of $X_\lambda(t, \cdot)$ at large time and large space in terms of the function γ, provided the assumptions of these theorems are fulfilled (some explicit examples were provided in the comments after Theorem 1.5). We are now interested in the *local-in-space* behavior of the graphs of $X_\lambda(t, \cdot)$ at large time. Let us first point out that, because of the asymmetry of the roles of the rest states 0 and 1 (assuming Hypothesis 1.3), the behavior of the graphs of $X_\lambda(t, \cdot)$ will be radically different depending on the profile of the function γ at infinity, and especially on whether the function γ be large enough or not at infinity. This difference is already inherent in the results of the previous subsection. Indeed, in the particular case $\gamma(x') = \alpha\,|x'|$ of the example given after Theorem 1.5, whatever $\alpha \in \mathbb{R}$ may be, the graphs of the functions $X_\lambda(t, \cdot)$ look like the sets $\{x \in \mathbb{R}^N : \mathrm{dist}(x, U) = c^* t\}$ at large time t, as a consequence of Theorem 1.6. If $\alpha > 0$, then for each $t > 0$ the set $\{x \in \mathbb{R}^N : \mathrm{dist}(x, U) = c^* t\}$ is a shift of the graph of γ in the direction x_N and therefore it has a vertex, whereas it is C^1 if $\alpha \le 0$. Of course, for each $t > 0$, in both cases $\alpha > 0$ and $\alpha \le 0$, each level set of u (that is, each graph of $X_\lambda(t, \cdot)$) is at least of class C^2 from the implicit function theorem and the fact that $\partial_{x_N} u < 0$ in $(0, +\infty) \times \mathbb{R}^N$. Nevertheless, the previous observations imply that there should be a difference between the flattening properties of the level sets of u according to the coercivity of the function γ at infinity.

The following result deals with the non-coercive case, i.e., $\limsup_{|x'| \to +\infty} \gamma(x')/|x'| \le 0$.

Theorem 1.7. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let u be the solution of* (1.1) *with an initial datum u_0 given by* (1.28). *If*

$$\limsup_{|x'| \to +\infty} \frac{\gamma(x')}{|x'|} \le 0, \tag{1.30}$$

then, for every $\lambda \in [\theta, 1)$, with $\theta \in (0, 1)$ given by Hypothesis 1.1, *and every basis $(\mathrm{e}'_1, \cdots, \mathrm{e}'_{N-1})$ of $\mathbb{R}^{N-1}$, there holds*

$$\liminf_{t \to +\infty} \Big(\min_{|x'| \le R,\, 1 \le i \le N-1} |\nabla_{x'} X_\lambda(t, x') \cdot \mathrm{e}'_i| \Big) \;\longrightarrow\; 0 \quad \text{as } R \to +\infty, \tag{1.31}$$

[6] The above arguments also easily imply that the function $(\lambda, t, x') \mapsto X_\lambda(t, x')$ is continuous in $(0, 1) \times (0, +\infty) \times \mathbb{R}^{N-1}$.

and even

$$\sup_{x'_0\in\mathbb{R}^{N-1}}\left[\liminf_{t\to+\infty}\left(\min_{x'\in\overline{B'_R(x'_0)},\,1\le i\le N-1}|\nabla_{x'}X_\lambda(t,x')\cdot \mathrm{e}'_i|\right)\right]\ \longrightarrow\ 0\quad \textit{as } R\to+\infty, \tag{1.32}$$

where the function $X_\lambda:(0,+\infty)\times\mathbb{R}^{N-1}\to\mathbb{R}$ *is defined by* (1.29) *and* $B'_R(x'_0)$ *denotes the open Euclidean ball of center* x'_0 *and radius* R *in* $\mathbb{R}^{N-1}$.

Notice that, in dimension $N=2$, property (1.31) means that

$$\liminf_{t\to+\infty}\left(\min_{[-R,R]}|\partial_{x_1}X_\lambda(t,\cdot)|\right)\ \longrightarrow\ 0\ \text{ as } R\to+\infty,$$

for every $\lambda\in[\theta,1)$, and that an analogous consideration holds for (1.32).

Roughly speaking, Theorem 1.7 says that the level set of any value $\lambda\in[\theta,1)$ becomes almost flat in some directions along some sequences of points and some sequences of times converging to $+\infty$. We point out that the estimates on $\nabla_{x'}X_\lambda(t,x')$ immediately imply analogous estimates on $\nabla_{x'}u(t,x',X_\lambda(t,x'))$, because

$$\nabla_{x'}u(t,x',X_\lambda(t,x'))=-\partial_{x_N}u(t,x',X_\lambda(t,x'))\nabla_{x'}X_\lambda(t,x'), \tag{1.33}$$

and $\partial_{x_N}u$ is bounded in $[1,+\infty)\times\mathbb{R}^N$ from standard parabolic estimates.[7] Hence, the conclusions (1.31)-(1.32) imply that

$$\liminf_{t\to+\infty}\left(\min_{|x'|\le R,\,1\le i\le N-1}|\nabla_{x'}u(t,x',X_\lambda(t,x'))\cdot \mathrm{e}'_i|\right)\ \longrightarrow\ 0\ \text{ as } R\to+\infty$$

and

$$\sup_{x'_0\in\mathbb{R}^{N-1}}\left[\liminf_{t\to+\infty}\left(\min_{x'\in\overline{B'_R(x'_0)},\,1\le i\le N-1}|\nabla_{x'}u(t,x',X_\lambda(t,x'))\cdot \mathrm{e}'_i|\right)\right]\ \longrightarrow\ 0\ \text{ as } R\to+\infty,$$

for every $\lambda\in[\theta,1)$ and every basis $(\mathrm{e}'_1,\cdots,\mathrm{e}'_{N-1})$ of $\mathbb{R}^{N-1}$. The proof of (1.31)-(1.32) is done by way of contradiction and uses the fact that the level value λ belongs to the interval $[\theta,1)$, where $\theta\in(0,1)$ is given by Hypothesis 1.1. Since the interface between the values 0 and 1 is initially sharp, we expect, as in the one-dimensional case dealt with in [38], that the transition between 0 and 1 has a uniformly bounded width (in the sense of [4]) in the direction x_N if, say, γ is Lipschitz continuous (although the proof of this property in higher dimensions does not extend easily). If so, it would follow from the proof of Theorem 1.7 that (1.31)-(1.32) would then hold for any $\lambda\in(0,1)$. Actually, for a function f which is positive in $(0,1)$, Hypothesis 1.1 is satisfied for any $\theta\in(0,1)$ and it follows from Theorem 1.7 that (1.31)-(1.32) then hold for all $\lambda\in(0,1)$.

We stress that, without the assumption (1.30), the conclusions (1.31)-(1.32) immediately do not hold in general (immediate couterexamples are given by solutions whose level sets are parallel hyperplanes which are not orthogonal to the vector e_N, see Remark 4.5 below). Moreover, if one assumes that $\liminf_{|x'|\to+\infty}\gamma(x')/|x'|\ge 0$ instead of (1.30), the conclusions (1.31)-(1.32) do not hold either in general (counterexamples are given by rotated bistable V-shaped fronts, see Proposition 4.6 (i) for further details). However, with the assumption (1.30), we expect that the liminf of the minimum can be replaced by a limit in (1.31), without any reference to the size R, leading to the following conjecture.

[7] Since u is of class C^1 in $(0,+\infty)\times\mathbb{R}^N$ and the function $(\lambda,t,x')\mapsto X_\lambda(t,x')$ is continuous in $(0,1)\times(0,+\infty)\times\mathbb{R}^{N-1}$, it follows that the function $(\lambda,t,x')\mapsto\nabla_{x'}X_\lambda(t,x')$ is also continuous in $(0,1)\times(0,+\infty)\times\mathbb{R}^{N-1}$.

Conjecture 1.8. *Under the assumptions of Theorem* 1.7*, the conclusion* (1.31) *can be strengthened by the limit*

$$\nabla_{x'} X_\lambda(t,x') \longrightarrow 0 \quad \text{as } t \to +\infty, \text{ locally uniformly in } x' \in \mathbb{R}^{N-1}, \tag{1.34}$$

for every $\lambda \in [\theta, 1)$.

Even for x'-symmetric solutions u, property (1.34) does not hold in general without the assumption (1.30) of Theorem 1.7 (as for (1.31)-(1.32), counterexamples are given by bistable V-shaped fronts, see Proposition 4.6 (ii) for further details). We also point out that, even with the assumption (1.30), property (1.34) does not hold in general *uniformly* with respect to $x' \in \mathbb{R}^{N-1}$ (for instance, in dimension $N = 2$, easy counterexamples are given by nonpositive functions γ with a negative slope as $x_1 \to +\infty$, see Proposition 4.6 (iii) for further details). On the other hand, a strong support to the validity of Conjecture 1.8 is provided by the conclusion of Theorem 1.6. Indeed, it asserts that, for any $\lambda \in (0,1)$, $E_\lambda(t) \sim U + B_{c^* t}$ for t large, in the sense of (1.26), and one can check that condition (1.30) entails that the exterior unit normals to the set $U + B_{c^* t}$ at the points $(x', x_N) \in \partial(U + B_{c^* t})$ (whenever they exist) approach the vertical direction $\mathrm{e}_N = (0, \cdots, 0, 1)$ as $t \to +\infty$, locally uniformly with respect to $x' \in \mathbb{R}^{N-1}$. Hence the same is expected to hold for the sets $E_\lambda(t)$, which is what (1.34) asserts. This kind of arguments can be made rigorous, building on the results of the previous subsection, and lead to a weaker version of Conjecture 1.8, see Proposition 4.3 below. Two other weaker statements than Conjecture 1.8 are derived in the case where f fulfills (1.5) or the more restrictive Fisher-KPP condition (1.43) below, see Proposition 4.4 and Corollary 5.6 respectively.

As for the full Conjecture 1.8, the following result shows that, under (1.30) and the other assumptions of Theorem 1.7, the conclusion (1.34) holds in the case of initial conditions u_0 having an asymptotically x'-symmetric conical support. Notice also that the following result uses the weaker Hypothesis 1.1 instead of Hypothesis 1.3.

Theorem 1.9. *Assume that Hypothesis* 1.1 *holds. Let u be the solution of* (1.1) *with an initial datum u_0 given by* (1.28)*, where the function γ satisfies one of the following assumptions:*

(i) *either γ is of class C^1 outside a compact set and there is $\ell \geq 0$ such that*

$$\begin{cases} \gamma'(x_1) \to \mp \ell \quad \text{as } x_1 \to \pm\infty & \text{if } N = 2, \\ \nabla\gamma(x') = -\ell \, \dfrac{x'}{|x'|} + O(|x'|^{-1-\eta}) \quad \text{as } |x'| \to +\infty, \text{ for some } \eta > 0, & \text{if } N \geq 3; \end{cases} \tag{1.35}$$

(ii) *or γ is continuous outside a compact set and $\gamma(x')/|x'| \to -\infty$ as $|x'| \to +\infty$;*

(iii) *or $\gamma(x') = \Gamma(|x' - x'_0|)$ outside a compact set, for some $x'_0 \in \mathbb{R}^{N-1}$ and some continuous nonincreasing function $\Gamma : \mathbb{R}^+ \to \mathbb{R}$;*

(iv) *or $\gamma(x') = \Gamma(|x' - x'_0|)$ outside a compact set, for some $x'_0 \in \mathbb{R}^{N-1}$ and some C^1 function $\Gamma : \mathbb{R}^+ \to \mathbb{R}$ such that $\Gamma'(r) \to 0$ as $r \to +\infty$.*

Then, for every $\lambda_0 \in (0,1)$, there holds that

$$\nabla_{x'} X_\lambda(t,x') \longrightarrow 0 \quad \text{as } t \to +\infty, \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } \lambda \in (0, \lambda_0] \tag{1.36}$$

and moreover

$$\nabla_{x'} u(t, x', x_N) \longrightarrow 0 \quad \text{as } t \to +\infty, \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } x_N \in \mathbb{R}. \tag{1.37}$$

In dimension $N = 3$, by writing $\gamma(x') = \widetilde{\gamma}(r, \vartheta)$ in the standard polar coordinates, condition (1.35) means that $\partial_r \widetilde{\gamma}(r, \vartheta) = -\ell + O(r^{-1-\eta})$ and $\partial_\vartheta \widetilde{\gamma}(r, \vartheta) = O(r^{-\eta})$ as $r \to +\infty$.

It is easy to see that, even under Hypothesis 1.3 (which is stronger than Hypothesis 1.1), if (1.35) holds with $\ell > 0$, then the convergence in (1.36) cannot be uniform with respect to $x' \in \mathbb{R}^{N-1}$, see Proposition 4.6 (iv) for further details. In other words, if the initial interface between the states 0 and 1 has a non-zero slope at infinity, then the level sets cannot become uniformly flat at large time. This observation naturally leads to the following conjecture.

Conjecture 1.10. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let $\theta \in (0,1)$ be given by Hypothesis* 1.1 *and let u be the solution of* (1.1) *with an initial datum u_0 given by* (1.28). *If*

$$\lim_{|x'| \to +\infty} \nabla \gamma(x') = 0, \tag{1.38}$$

then, for every $\lambda_0 \in (0,1)$,

$$\nabla_{x'} X_\lambda(t, x') \longrightarrow 0 \quad \text{as } t \to +\infty, \text{ uniformly in } x' \in \mathbb{R}^{N-1} \text{ and in } \lambda \in (0, \lambda_0] \tag{1.39}$$

and moreover

$$\nabla_{x'} u(t, x) \longrightarrow 0 \quad \text{as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}^N. \tag{1.40}$$

Properties (1.39)-(1.40) obviously hold if γ is constant. Furthermore, if condition (1.38) is replaced by the boundedness of γ, then, at least for some classes of functions f, properties (1.39) (with $\lambda \in (0, \lambda_0]$ replaced by $\lambda \in [a, b]$, for some fixed $0 < a \leq b < 1$) and (1.40) hold: more precisely, if the function f is of the bistable type (1.7), these properties follow from some results in [3, 15], and the same conclusions hold for more general functions f of the multistable type [43] or for KPP type functions f satisfying (1.43) below or slightly weaker conditions, see [3, 7, 52]. Further estimates on the exact position of the level sets X_λ in the bistable or KPP cases have been established in [32, 33, 47]. We can also relax the boundedness of γ for the validity of the conclusion (1.40) in the KPP case, see Corollary 1.15 below.

However, by considering some functions γ with large local oscillations at infinity, it turns out that both conclusions of Conjecture 1.10 cannot hold if (1.38) is replaced by the weaker condition $\lim_{|x'| \to +\infty} \gamma(x')/|x'| = 0$, as the following result shows.

Proposition 1.11. *Conjecture* 1.10 *fails in general if assumption* (1.38) *is replaced by*

$$\nabla \gamma \in L^\infty(\mathbb{R}^{N-1}) \quad \text{and} \quad \lim_{|x'| \to +\infty} \frac{\gamma(x')}{|x'|} = 0. \tag{1.41}$$

To complete this section, let us point out that, under the assumptions of Theorems 1.5 and 1.7, the solution u of (1.1) with (1.28) propagates with speed c^* in the direction x_N, owing to Theorem 1.5, that is, $w(\mathrm{e}_N) = c^*$ in (1.15). We conjecture that the solution u then locally converges along its level sets to the front profile φ with speed c^*.

Conjecture 1.12. *Under the assumptions of Theorems* 1.5 *and* 1.7, *there holds, for every $\lambda \in (0,1)$, for every sequence $(t_n)_{n \in \mathbb{N}}$ converging to $+\infty$, and for every bounded sequence $(x'_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{N-1}$,*

$$u(t_n + t, x'_n + x', X_\lambda(t_n, x'_n) + x_N) \longrightarrow \varphi(x_N - c^* t + \varphi^{-1}(\lambda)) \quad \text{as } n \to +\infty, \tag{1.42}$$

in $C^{1;2}_{loc}(\mathbb{R}_t \times \mathbb{R}^{N-1}_{x'})$ and uniformly with respect to $x_N \in \mathbb{R}$. If one further assumes (1.38), *then the above limit holds for every sequence $(x'_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{N-1}$, bounded or not.*

As before, by Proposition 1.11, the second conclusion does not hold in general if assumption (1.38) is replaced by $\lim_{|x'|\to+\infty}\gamma(x')/|x'|=0$. On the other hand, Conjecture 1.12, and especially its second part, holds if γ is bounded, for some classes of functions f, see [3, 32, 33, 43, 47].

1.5 The asymptotic one-dimensional symmetry

Let us present our asymptotic symmetry results, related to Question B in Section 1.1. They concern the case where f is of the Fisher-KPP type [16, 28], that is,

$$f(0)=f(1)=0,\ \ f(s)>0 \text{ for all } s\in(0,1),\ \text{ and } s\mapsto\frac{f(s)}{s} \text{ is nonincreasing in } [0,1]. \quad (1.43)$$

In this case the *hair trigger effect* holds [2], i.e., Hypothesis 1.1 is fulfilled for any $\theta,\rho>0$, moreover Hypothesis 1.3 also holds and the minimal speed with the properties stated in Proposition 1.4 is explicit: $c^*=2\sqrt{f'(0)}$ [2, 28]. We use again the notion of positive-distance-interiors of a set $U\subset\mathbb{R}^N$, that is,

$$U_\delta:=\big\{x\in U:\mathrm{dist}(x,\partial U)\geq\delta\big\},\quad \delta>0.$$

Theorem 1.13. *Assume that f is of the Fisher-KPP type* (1.43)*. Let u be the solution of* (1.1) *with an initial datum $u_0=\mathbb{1}_U$ such that $U\subset\mathbb{R}^N$ satisfies*

$$\exists\,\delta>0,\quad d_{\mathcal{H}}(U,U_\delta)<+\infty. \quad (1.44)$$

Assume moreover that U is convex, or more generally, that there exists a convex set $U'\subset\mathbb{R}^N$ satisfying $d_{\mathcal{H}}(U,U')<+\infty$. Then, any function $\psi\in\Omega(u)$ is of the form $\psi=\psi(x\cdot e)$, for some $e\in\mathbb{S}^{N-1}$.

Theorem 1.13 extends the asymptotic one-dimensional symmetry property for the Fisher-KPP equation, known to hold when U is bounded, as a consequence of [27], as well as when U is the subgraph of a bounded function, by [3, 7, 52]. Condition (1.44) means that there exists some $R>0$ such that, for any $x\in U$, there is a ball $B_\delta(x_0)\subset U$ with $|x-x_0|<R$. It is fulfilled in particular if U satisfies a uniform interior ball condition. One can show that, in dimension $N=2$, for a convex set U, property (1.44) is equivalent to require that U has nonempty interior.

The idea of the proof of Theorem 1.13 consists in reducing to a case where it is possible to apply the reflection argument of Jones [27]. This is achieved by an approximation of the solution through a suitable truncation of its initial support. In order to control the error, we exploit a new type of supersolutions initially supported in exterior domains, which are also one of the key ingredients behind the results of Section 1.3.

As a matter of fact, the convexity (or convex proximity) assumption on U in Theorem 1.13 is a very special case of a geometric hypothesis that we now introduce. For a given nonempty set $U\subset\mathbb{R}^N$ and a given point $x\in\mathbb{R}^N$, we let π_x denote the set of orthogonal projections of x onto $\overline{U}$, i.e.,

$$\pi_x=\big\{\xi\in\overline{U}:|x-\xi|=\mathrm{dist}(x,\overline{U})\big\}, \quad (1.45)$$

and, for $x\notin\overline{U}$, we set

$$\mathcal{O}(x):=\sup_{\substack{\xi\in\pi_x\\ y\in U\setminus\{\xi\}}}\frac{x-\xi}{|x-\xi|}\cdot\frac{y-\xi}{|y-\xi|}, \quad (1.46)$$

with the convention that $\mathcal{O}(x) = -\infty$ if $U = \emptyset$ or U is a singleton (otherwise $-1 \leq \mathcal{O}(x) \leq 1$). Namely, when $-1 \leq \mathcal{O}(x) \leq 1$, the quantity $2\arccos(\mathcal{O}(x))$ is the infimum among all $\xi \in \pi_x$ of the opening of the largest circular cone intersecting U having apex ξ and axis $x - \xi$. Here is our more general asymptotic symmetry result.

Theorem 1.14. *Assume that f is of the Fisher-KPP type* (1.43). *Let u be the solution of* (1.1) *with an initial datum $u_0 = \mathbb{1}_U$ such that $U \subset \mathbb{R}^N$ satisfies* (1.44) *and moreover*

$$\lim_{R \to +\infty} \left(\sup_{x \in \mathbb{R}^N,\, \mathrm{dist}(x,U)=R} \mathcal{O}(x) \right) \leq 0. \tag{1.47}$$

Then, any function $\psi \in \Omega(u)$ is of the form $\psi = \psi(x \cdot e)$, for some $e \in \mathbb{S}^{N-1}$.

It is understood that the left-hand side in condition (1.47) is equal to $-\infty$ (hence the condition is fulfilled) if $\sup_{x \in \mathbb{R}^N} \mathrm{dist}(x, U) < +\infty$ (and indeed in such case the asymptotic one-dimensional symmetry trivially holds because condition (1.44) yields that $u(t, x) \to 1$ uniformly in $x \in \mathbb{R}^N$ as $t \to +\infty$). We remark that the limit in (1.47) always exists, because the involved quantity is nonincreasing with respect to R, see Lemma 5.2 below.

Theorem 1.14 yields Theorem 1.13 because, firstly, convex sets satisfy $\mathcal{O}(x) \leq 0$ for every $x \notin \overline{U}$ (actually, they are characterized by such condition in the class of closed sets) and, secondly, if (1.47) holds for a given set, then it holds true for any set at finite Hausdorff distance from it, as stated in Lemma 5.2. However, there are also other sets satisfying (1.47), such as the subgraphs of functions γ with *vanishing global mean*, i.e.

$$\frac{\gamma(x') - \gamma(y')}{|x' - y'|} \longrightarrow 0 \quad \text{as} \quad |x' - y'| \to +\infty. \tag{1.48}$$

As a byproduct of the proof of Theorem 1.14, we characterize the set of directions e in which u becomes asymptotically one-dimensional. Namely, we prove that

$$\mathcal{E} := \{e \in \mathbb{S}^{N-1} \ : \ \exists\, \psi \in \Omega(u),\ \psi = \psi(x \cdot e) \text{ is non-constant}\}$$

coincides with the set of limits $(x_n - \xi_n)/|x_n - \xi_n|$ as $n \to +\infty$, for all sequences $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^N$ and $(\xi_n \in \pi_{x_n})_{n \in \mathbb{N}}$ such that $|x_n - \xi_n| = \mathrm{dist}(x_n, U) \to +\infty$. In particular, if U is of class C^1, then the set $\mathcal{E}$ is contained in the closure of the set of the unit normal vectors to U; if U is convex then $\mathcal{E}$ coincides with the closure of the set of the unit normal vectors to the supporting hyperplanes of U. Moreover, when $u_0 = \mathbb{1}_U$ with U given by the subgraph of a function γ satisfying (1.30), it is possible to show that $(x_n - \xi_n)/|x_n - \xi_n| \to \mathrm{e}_N = (0, \cdots, 0, 1)$ whenever $x_n/|x_n| \to \mathrm{e}_N$ as $n \to +\infty$, see Theorem 5.5 below. One then infers in particular that, in such case, the solution u satisfies

$$\nabla_{x'} u(t, x', x_N) \to 0 \quad \text{as} \quad t \to +\infty, \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } x_N \in \mathbb{R}.$$

Observe that this is weaker than the property (1.34) stated in Conjecture 1.8, but it would be equivalent if one knew that $\partial_{x_N} u$ stays bounded away from zero on the level set corresponding to the value λ. This actually necessarily occurs along some sequence of times, and therefore we infer that the convergence in (1.34) holds along such sequence and locally in x', see Corollary 5.6 below. Finally, when the condition (1.30) is strengthened by the requirement that γ has vanishing global mean in the sense of (1.48), then $\mathcal{E} = \{\mathrm{e}_N\}$.

Corollary 1.15. *Assume that f is of the Fisher-KPP type* (1.43)*. Let u be the solution of* (1.1) *with an initial condition u_0 of the type* (1.28)*, where γ is a continuous function satisfying* (1.48)*. Then, any function $\psi \in \Omega(u)$ is of the form $\psi = \psi(x_N)$.*

The conclusion of Corollary 1.15 can be equivalently rephrased as

$$\nabla_{x'} u(t,x) \to 0 \quad \text{as} \quad t \to +\infty, \quad \text{uniformly with respect to } x.$$

Hence this result yields that property (1.40) of Conjecture 1.10 holds in the Fisher-KPP case, even if the hypothesis (1.38) is relaxed by (1.48). A way to interpret Corollary 1.15 is that the oscillations of the initial datum are "damped" as time goes on through some kind of averaging process.

Remark 1.16. One can wonder whether the reciprocal of Theorem 1.14 is true in the following sense: if the asymptotic one-dimensional symmetry holds for a solution u of (1.1) with initial datum $\mathbb{1}_U$ and U satisfying (1.44), does necessarily U fulfill (1.47)? The answer is immediately negative in general: take for instance U given by

$$U = \bigcup_{n \in \mathbb{N}} [2^n, 2^n + 1] \times \mathbb{R}^{N-1},$$

which satisfies (1.44) but not (1.47), while any element of $\Omega(u)$ necessarily depends on the variable x_1 only. However, the question still makes sense if one adds another condition on U, for instance the connectivity. In this case, the question is open.

Towards a De Giorgi-type conjecture for reaction-diffusion equations

To complete this section, for more general reactions f than the Fisher-KPP case (1.43), we state two conjectures suggesting a positive answer to Question B addressed in Section 1.1, that is, the one-dimensional symmetry of the elements of the Ω-limit set of solutions to (1.1) with initial conditions $u_0 = \mathbb{1}_U$. Actually, for the answer to Question B to be affirmative, some conditions on U need to be imposed.

Firstly, let us recall in which cases Question B has a positive answer:

- when U is bounded and has a non-empty positive-distance-interior U_ρ, with f satisfying Hypothesis 1.1 and $\rho > 0$ given there, by [27];
- when U is between two half-spaces and f is of the bistable type (1.7) or of the Fisher-KPP type (1.43), by [3, 7, 15, 52];
- in the Fisher-KPP case (1.43), when U satisfies (1.44) and when it is at bounded Hausdorff distance from a convex set or more generally speaking when (1.47) holds, from Theorem 1.14.

Secondly, cases in which Question B has a negative answer are:

- when U is "V-shaped", and f is of the bistable type (1.7) with positive integral over $[0,1]$ as follows from [23, 24, 39, 46] (see Proposition 4.6) or f is of the Fisher-KPP type (1.43) as follows from [7, 25];

- as follows from the previous item, in the Fisher-KPP or bistable cases (1.43) or (1.7) with positive integral over $[0,1]$, when there exist two sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ and $(R_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ such that $|x_n|\to+\infty$ and $R_n\to+\infty$ as $n\to+\infty$ and $U\cap B_{R_n}(x_n)$ is "V-shaped" (notice that a set U with such property can also be given by the subgraph of a function γ satisfying (1.41));

- when f is of the bistable type (1.7) with positive integral over $[0,1]$ and $U=B_\delta$, for a suitable choice of δ which guarantees that the emerging solution u converges to a ground state as $t\to+\infty$, in the sense that $u(t,\cdot)\to u_\infty$ as $t\to+\infty$ in $C^2(\mathbb{R}^N)$, with $u_\infty>0$ in $\mathbb{R}^N$ and $u_\infty(x)\to 0$ as $|x|\to+\infty$ (the existence and uniqueness of such value $\delta>0$ is proved in [9, 41], and necessarily $\delta<\rho$, where ρ is any positive real number as in Hypothesis 1.1, which is satisfied in the bistable case (1.7) with positive integral over $[0,1]$);

- when f is of the bistable type (1.7) with positive integral over $[0,1]$ and

$$U=B_R\cup\bigcup_{n\in\mathbb{N}}B_\delta(2^n e),$$

  for any $R>0$ and $e\in\mathbb{S}^{N-1}$, and $\delta>0$ is such that the solution with initial datum $\mathbb{1}_{B_\delta}$ converges to a ground state (that is, a similar phenomenon as in the above counter-example can occur even with $u(t,\cdot)\to 1$ as $t\to+\infty$).

Thirdly, cases where our results and previous conjectures suggest a positive answer to Question B are:

- when U is asymptotic to a convex cone, suggested by Theorem 1.9;

- when u_0 is given by (1.28) with γ satisfying (1.38), suggested by Conjecture 1.10.

These observations lead us to formulate the following De Giorgi type conjecture for the solutions of the reaction-diffusion equation (1.1).

Conjecture 1.17. *Assume that Hypothesis* 1.1 *holds for some $\rho>0$. Let u be the solution to* (1.1) *with an initial datum $u_0=\mathbb{1}_U$ such that $U\subset\mathbb{R}^N$ satisfies* (1.25) *and* (1.47). *Then, for any function $\psi\in\Omega(u)$, there exists $e\in\mathbb{S}^{N-1}$ such that $\psi=\psi(x\cdot e)$.*

Another conjecture is that of the stability of the asymptotic one-dimensional symmetry with respect to bounded perturbations of the initial support.

Conjecture 1.18. *Assume that Hypothesis* 1.1 *holds for some $\rho>0$. Let u be the solution to* (1.1) *with an initial datum $\mathbb{1}_U$ such that $U\subset\mathbb{R}^N$ satisfies* (1.25) *and u satisfies the asymptotic one-dimensional symmetry. If $U'\subset\mathbb{R}^N$ satisfies* (1.25) *and $d_{\mathcal{H}}(U',U)<+\infty$, then the solution u' to* (1.1) *with initial datum $\mathbb{1}_{U'}$ still satisfies the asymptotic one-dimensional symmetry.*

1.6 The logarithmic lag in the KPP case

We are still concerned here with f satisfying the Fisher-KPP condition (1.43). We recall that Hypotheses 1.1 and 1.3 are fulfilled, and the minimal speed c^* of traveling fronts

connecting 1 to 0 is given by $c^* = 2\sqrt{f'(0)}$. It is known that in the one-dimensional case, the solution u of (1.1) with initial condition $u_0 = \mathbb{1}_{\mathbb{R}^-}$ is such that

$$\sup_{x\in\mathbb{R}} \Big| u(t,x) - \varphi\Big(x - c^*t + \frac{3}{c^*}\,\log t + x_0\Big)\Big| \to 0 \text{ as } t \to +\infty,$$

for some $x_0 \in \mathbb{R}$. Hence, there is a lag by $(3/c^*)\log t$ of the position of the level sets of u behind the position c^*t given by the spreading speed. The first proof of this logarithmic lag was given in [7], and further results and other proofs have been given in [26, 30, 40, 52]. In dimension $N = 2$, for initial conditions trapped between two shifts of $\mathbb{1}_{\mathbb{R}\times\mathbb{R}^-}$, then

$$\sup_{(x_1,x_2)\in\mathbb{R}^2} \Big| u(t,x_1,x_2) - \varphi\Big(x_2 - c^*t + \frac{3}{c^*}\,\log t + a(t,x_1)\Big)\Big| \to 0 \text{ as } t \to +\infty,$$

for some bounded function a, see [47]. In any dimension $N \geq 2$, for nonnegative compactly supported initial conditions $0 \not\equiv u_0 \leq 1$, then

$$\sup_{x\in\mathbb{R}^N\setminus\{0\}} \Big| u(t,x) - \varphi\Big(|x| - c^*t + \frac{N+2}{c^*}\,\log t + a\Big(\frac{x}{|x|}\Big)\Big)\Big| \to 0 \ \text{ as } t \to +\infty,$$

for some Lipschitz continuous function a defined in $\mathbb{S}^{N-1}$, see [13, 18, 45]. Notice that $N+2 = 3+(N-1)$ corresponds to an additional lag by $((N-1)/c^*)\log t$, compared with the 1-dimensional case, which is due to the curvature of the level sets inherited from the fact that the initial condition is compactly supported.

Let us now consider the case of a solution to (1.1) with an initial condition given by (1.28) with γ bounded from above, and investigate the lag between the position of the level sets of u behind c^*t in the direction x_N. By comparison, we know that, up to an additive constant, the lag is between $(3/c^*)\log t$, which is the lag in the 1-dimensional case, and $((N+2)/c^*)\log t$, which is the lag in the case of compactly supported initial conditions: namely, for every $\lambda \in (0,1)$ and $x' \in \mathbb{R}^{N-1}$, under the notations (1.29), the lag $c^*t - X_\lambda(t,x')$ satisfies

$$\frac{3}{c^*}\,\log t + O(1) \leq c^*t - X_\lambda(t,x') \leq \frac{N+2}{c^*}\,\log t + O(1) \quad \text{as } t \to +\infty. \tag{1.49}$$

But it is not clear in principle whether or not this lag is equal to one of these bounds or whether it takes intermediate values. Our first main result of this section states that, for an initial condition u_0 satisfying (1.28) with $-\gamma$ sufficiently large at infinity, the lag coincides with the above upper bound, that is, the position of the level sets of u in the direction x_N is the same as when the initial condition is compactly supported.

Theorem 1.19. *Assume that f is of the Fisher-KPP type* (1.43) *and let u be the solution of* (1.1) *with an initial condition u_0 satisfying* (1.28). *If*

$$\limsup_{|x'|\to+\infty} \frac{\gamma(x')}{\log(|x'|)} < -\frac{2(N-1)}{c^*}, \tag{1.50}$$

then

$$X_\lambda(t,x') = c^*t - \frac{N+2}{c^*}\,\log t + O(1) \quad \text{as } t \to +\infty, \tag{1.51}$$

locally uniformly with respect to $\lambda \in (0,1)$ and $x' \in \mathbb{R}^{N-1}$, and the inequality "$\leq$" holds true in the above formula locally uniformly in $\lambda \in (0,1)$ and uniformly in $x' \in \mathbb{R}^{N-1}$.

If the upper bound for γ in (1.50) is relaxed, we expect the lag of the solution with respect to the critical front to differ from the one associated with compactly supported initial data, that is $((N+2)/c^*)\log t$. We derive the following lower bound for the lag.

Proposition 1.20. *Assume that f is of the Fisher-KPP type* (1.43) *and let u be the solution of* (1.1) *with an initial condition u_0 satisfying* (1.28). *If there is $\sigma \geq -(N-1)$ such that*

$$\limsup_{|x'|\to+\infty} \frac{\gamma(x')}{\log|x'|} \leq \frac{2\sigma}{c^*}, \tag{1.52}$$

then, for any $\lambda \in (0,1)$,

$$X_\lambda(t,x') \leq c^* t - \frac{3-\sigma}{c^*}\log t + o(\log t) \quad \text{as} \quad t \to +\infty, \tag{1.53}$$

locally uniformly with respect to $x' \in \mathbb{R}^{N-1}$.

Property (1.53) means that the lag $c^*t - X_\lambda(t,x')$ is at least $((3-\sigma)/c^*)\log t + o(\log t)$ as $t \to +\infty$. We point out that this holds even for positive σ. We conjecture that, if the limsup is replaced by a limit in (1.52) and the inequality by an equality, then the lag should precisely be

$$c^*t - X_\lambda(t,x') = \frac{3-\sigma}{c^*}\log t + o(\log t) \quad \text{as } t \to +\infty,$$

for every $\lambda \in (0,1)$ and $x' \in \mathbb{R}^{N-1}$. When $\sigma = 0$, this formula would be coherent with the 1-dimensional lag. This formula would also mean that the constant $-2(N-1)/c^*$ in (1.50) would be optimal for the lag to be equivalent to that of solutions with compactly supported initial conditions. Lastly, it would provide a continuum of lags with logarithmic factors ranging in the whole half-line $(-\infty, (N+2)/c^*]$. In particular, solutions with initial conditions of the type (1.28) with $\gamma(x') \sim (6/c^*)\log|x'|$ as $|x'| \to +\infty$ would have no logarithmic lag, i.e., the same position c^*t along the x_N-axis as that of the planar front moving in the direction e_N, up to a $o(\log t)$ term as $t \to +\infty$. While $\gamma(x') \sim \kappa \log|x'|$ as $|x'| \to +\infty$ for some $\kappa > (6/c^*)$, would lead to a negative logarithmic lag, i.e., the position of the solution would be ahead of that of the front by a logarithmic-in-time term (observe that the term is linear in time when $\gamma(x') \sim \alpha|x'|$ as $|x'| \to +\infty$ with $\alpha > 0$, according to formula (1.16)).

More precise estimates of the position of the solutions with initial conditions of the type (1.28) and functions γ having some logarithmic or more general asymptotics at infinity will be the purpose of a following paper.

Theorem 1.19 allows us to prove the conjecture about the flattening of the level sets under the hypotheses of that theorem.

Corollary 1.21. *Assume that f is of the Fisher-KPP type* (1.43) *and let u be the solution of* (1.1) *with an initial condition u_0 satisfying* (1.28) *and* (1.50). *Then the following hold:*

(i) *the conclusion* (1.34) *of Conjecture* 1.8 *holds, and even locally uniformly with respect to $\lambda \in (0,1)$, that is,*

$$\nabla_{x'} X_\lambda(t,x') \to 0 \quad \text{as } t \to +\infty, \text{ locally uniformly in } x' \in \mathbb{R}^{N-1} \text{ and } \lambda \in (0,1);$$

(ii) *for any* $\lambda \in (0,1)$ *and* $x'_0 \in \mathbb{R}^{N-1}$, *the function*

$$\widetilde{u}(t,x',x_N) := \lim_{s\to+\infty} u(s+t,x',X_\lambda(s,x'_0)+x_N),$$

which exists (*up to subsequences*) *locally uniformly with respect to* $(t,x',x_N) \in \mathbb{R}\times\mathbb{R}^N$, *is independent of* x' *and satisfies*

$$\lim_{x_N\to-\infty} \widetilde{u}(t,x_N+c^*t) = 1, \qquad \lim_{x_N\to+\infty} \widetilde{u}(t,x_N+c^*t) = 0,$$

uniformly with respect to $t \in \mathbb{R}$.

Corollary 1.21 shows that, in the large time limit, the solution approaches a one-dimensional entire solution whose level sets move in the direction e_N with an average velocity equal to the minimal speed c^*. It is then natural to expect that $\widetilde{u}(t,x_N) = \varphi(x_N - c^*t + \varphi^{-1}(\lambda))$ for all $(t,x_N) \in \mathbb{R}^2$, where φ is the front connecting 1 and 0 with minimal speed c^*. That would correspond to property (1.42) in Conjecture 1.12. By comparison and some arguments based on the number of intersections of solutions to (1.1) in dimension 1, it can be shown that $\widetilde{u}(t,x_N) \geq \varphi(x_N - c^*t + \zeta)$ in $\mathbb{R}^2$, for some $\zeta \in \mathbb{R}$. But the proof of (1.42) would still require additional arguments.

Outline of the paper. Propositions 1.2 and 1.4 as well as some other auxiliary results on planar fronts are shown in Section 2. Section 3 contains the proofs of Theorems 1.5 and 1.6 on the spreading results for general initial support U, which make use of a new type of supersolutions constructed using the results of the previous section; we also exhibit some counter-examples when the hypotheses (1.20) and (1.25) of Theorems 1.5 and 1.6 do not hold. Section 4 is concerned with the case of initial conditions u_0 that are characteristic functions of subgraphs. The proofs of Theorems 1.7 and 1.9 on the local flatness of the level sets of the solution u at large time in the general and the conical cases are carried out, as well as that of other flatness results and some counterexamples, such as Proposition 1.11, on the non-global flatness in general, even under condition (1.41). Section 5 contains the proofs of Theorems 1.13, 1.14 and Corollary 1.15 on the asymptotic one-dimensional symmetry in the Fisher-KPP case (1.43). Lastly, Section 6 is devoted to the proof of Theorem 1.19, Proposition 1.20 and Corollary 1.21 on the logarithmic lag for solutions of KPP type equations.

2 Preliminary considerations on planar fronts and the proof of Propositions 1.2 and 1.4

This section is devoted to the proof of Propositions 1.2 and 1.4, together with other auxiliary results on planar fronts. We start with the proof of Proposition 1.2 on the equivalence between Hypothesis 1.1 and some simple conditions on the function f.

Proof of Proposition 1.2. The fact that (1.9)-(1.10) imply Hypothesis 1.1 is contained in [11, Lemma 2.4].

Conversely, let us assume that Hypothesis 1.1 holds, for some $\theta \in (0,1)$ and $\rho > 0$. If there would exist $\theta' \in [\theta,1)$ such that $f(\theta') \leq 0$, then the solution u of (1.1) with initial condition $u_0 = \theta\,\mathbb{1}_{B_\rho}$ would be such that $u(t,\cdot) \leq \theta'$ in $\mathbb{R}^N$ for all $t \geq 0$, by the maximum principle, and then this solution would not converge to 1 as $t \to +\infty$ locally uniformly

in $\mathbb{R}^N$. Therefore, one necessarily has $f > 0$ in $[\theta, 1)$, that is, (1.9) holds (with the same value θ as in Hypothesis 1.1).

Still with Hypothesis 1.1, assume now by way of contradiction that (1.10) does not hold. From (1.9), there is then $\alpha \in [0, 1)$ such that

$$\int_\alpha^1 f(s)\,ds = 0 < \int_t^1 f(s)\,ds \ \text{ for all } t \in (\alpha, 1).$$

Thus, $f(\alpha) \le 0$ and $\int_\alpha^t f(s)\,ds < 0$ for all $t \in (\alpha, 1)$. Two cases may then occur: either $f(\alpha) < 0$, or $f(\alpha) = 0$. If $f(\alpha) < 0$, it then follows from standard elementary arguments that there is an even $C^2(\mathbb{R})$ function $\phi : \mathbb{R} \to [\alpha, 1)$ such that $\phi(0) = \alpha$, $\phi(\pm\infty) = 1$, $\phi' < 0$ in $(-\infty, 0)$, and $\phi'' + f(\phi) = 0$ in $\mathbb{R}$. Similarly, if $f(\alpha) = 0$, there is a $C^2(\mathbb{R})$ function $\widetilde{\phi} : \mathbb{R} \to (\alpha, 1)$ such that $\widetilde{\phi}(-\infty) = 1$, $\widetilde{\phi}(+\infty) = \alpha$, $\widetilde{\phi}' < 0$ in $\mathbb{R}$, and $\widetilde{\phi}'' + f(\widetilde{\phi}) = 0$ in $\mathbb{R}$. Let us then define

$$\overline{\phi} := \phi \text{ if } f(\alpha) < 0, \ \text{ and } \ \overline{\phi} := \widetilde{\phi} \text{ if } f(\alpha) = 0.$$

Consider now the solution u of (1.1) with initial condition $u_0 = \theta\,\mathbb{1}_{B_\rho}$. Hypothesis 1.1 implies that $u(t, x) \to 1$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$. But, since $\overline{\phi}(-\infty) = 1 > \theta$, there is $A > 0$ large enough such that

$$0 \le u_0(x) \le \overline{\phi}(x_1 - A) < 1 \text{ for all } x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$

and, since the function $\mathbb{R}^N \ni x \mapsto \overline{\phi}(x_1 - A)$ is a steady solution of (1.1), the parabolic comparison principle implies that $u(t, x) \le \overline{\phi}(x_1 - A) < 1$ for all $t \ge 0$ and $x \in \mathbb{R}^N$. The limit as $t \to +\infty$ leads to a contradiction. To sum up, one has shown that Hypothesis 1.1 implies both (1.9) and (1.10). □

The next two lemmas are part of the proof of Proposition 1.4. The first one shows that Hypothesis 1.3 implies Hypothesis 1.1.

Lemma 2.1. *Hypothesis* 1.3 *implies* (1.9)-(1.10) (*hence Hypothesis* 1.1).

Proof. Let us first consider any traveling front $\varphi(x \cdot e - ct)$ for (1.1) connecting 1 to 0, for some $e \in \mathbb{S}^{N-1}$ and $c \in \mathbb{R}$, that is, $\varphi : \mathbb{R} \to (0, 1)$ solves $\varphi''(z) + c\varphi'(z) + f(\varphi(z)) = 0$ in $\mathbb{R}$ with $\varphi(-\infty) = 1 > \varphi(z) > 0 = \varphi(+\infty)$ for all $z \in \mathbb{R}$. Let us first shortly check that φ is decreasing, in the case $c \ge 0$ (the case $c < 0$ can be dealt with similarly; the monotonicity of φ can also be deduced from phase-plane analysis). Assume for a moment that φ is not non-increasing. Then there are $x_m < x_M < y \in \mathbb{R}$ such that $\varphi(x_M) > \varphi(x_m) = \varphi(y)$, and x_m (resp. x_M) is a local minimum (resp. maximum) of φ. Integrating the equation $\varphi'' + c\varphi' + f(\varphi) = 0$ against φ' over $[x_m, y]$ leads to:

$$c\int_{x_m}^y (\varphi')^2 = -\frac{(\varphi'(y))^2}{2},$$

hence $\varphi'(y) = 0$, $c = 0$, and φ is periodic by the Cauchy-Lipschitz theorem (since $\varphi(x_m) = \varphi(y)$ and $\varphi'(x_m) = \varphi'(y) = 0$). This is impossible. Therefore, φ is non-increasing and, by differentiating the equation $\varphi'' + c\varphi' + f(\varphi) = 0$ and applying the strong maximum principle to φ', one gets that $\varphi' < 0$ in $\mathbb{R}$. From standard elliptic estimates, one also has that $\varphi'(\pm\infty) = \varphi''(\pm\infty) = 0$. Now, the function $p : s \in (0, 1) \mapsto \varphi'(\varphi^{-1}(s)) < 0$ obeys

$p'(s)p(s) = -cp(s) - f(s)$ for all $s \in (0,1)$ with $p(0^+) = p(1^-) = 0$ and it easily follows that p is unique (for a given c), hence φ is unique up to shifts (for a given c).

Next, let $c_0 > 0$ be given as in Hypothesis 1.3, that is, $c_0 > 0$ is the speed of a traveling front $\varphi_0(x - c_0 t)$ of (1.1) in $\mathbb{R}$, connecting 1 to 0. Integrating the equation $\varphi_0'' + c_0\varphi_0' + f(\varphi_0) = 0$ against φ_0' over the interval $(-\infty, x)$, for any $x \in \mathbb{R}$, leads to:

$$\frac{(\varphi_0'(x))^2}{2} + \int_{-\infty}^{x} c_0(\varphi_0'(z))^2\, dz = \int_{\varphi_0(x)}^{1} f(s)\, ds.$$

By the arbitrariness of x and the positivity of c_0, we deduce that $\int_t^1 f(s)\, ds > 0$ for all $t \in (0,1)$ and, taking the limit $x \to +\infty$, we get

$$\int_{-\infty}^{+\infty} c_0(\varphi_0'(z))^2\, dz = \int_0^1 f(s)\, ds, \tag{2.1}$$

hence $\int_0^1 f(s)\, ds$ is positive too. This show (1.10).

Let us finally show that (1.9) is satisfied as well. Assume by contradiction that this property does not hold, that is, that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 1 such that $f(t_n) \le 0$ for all $n \in \mathbb{N}$. Together with (1.10), it follows that there exists another sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 1 such that $f(\sigma_n) = 0$ for all $n \in \mathbb{N}$. We deduce in particular that $f'(1) = 0$. For $n \in \mathbb{N}$, consider the function

$$\psi : z \mapsto \psi(z) := \sigma_n + e^{-c_0 z/2},$$

where $c_0 > 0$ is, as in the previous paragraph, given by Hypothesis 1.3. With f being extended for convenience by 0 in $(1, +\infty)$, we have that

$$\begin{aligned}\psi''(z) + c_0\psi'(z) + f(\psi(z)) &= -\frac{c_0^2}{4}\, e^{-c_0 z/2} + f(\sigma_n) + \int_{\sigma_n}^{\psi(z)} f'(s)\, ds \\ &\le -\frac{c_0^2}{4}\, e^{-c_0 z/2} + \left(\max_{s\in[\sigma_n,1]} f'(s)\right) e^{-c_0 z/2}\end{aligned}$$

for all $z \in \mathbb{R}$. Taking n large enough, we find that the right-hand side is negative, that is, ψ is a strict supersolution of the equation satisfied by the front profile φ_0. By suitable translation, one can reduce to the case where ψ "touches from above" φ_0, i.e., $\min_{\mathbb{R}}(\psi - \varphi_0) = 0$. This contradicts the elliptic strong maximum principle. Therefore, (1.9) is satisfied too.

As a conclusion, one has shown that Hypothesis 1.3 implies both (1.10) and (1.9), hence it implies Hypothesis 1.1 from Proposition 1.2. □

The next lemma shows the equivalence between the existence of a positive speed of traveling front connecting 1 to 0 and the existence of a minimal positive speed.

Lemma 2.2. *There is a traveling front with positive speed c_0 connecting 1 to 0 if and only if there is a traveling front with positive minimal speed c^* connecting 1 to 0.*

Proof. We obviously only have to show the "only if" part. So, let us assume that there is a traveling front $\varphi_0(x - c_0 t)$ for (1.1) in $\mathbb{R}$, connecting 1 and 0 in the sense of (1.11), with positive speed c_0 (that is, we assume Hypothesis 1.3). Making use of the existence of the $[0,1]$-*minimal system of waves*, provided by [54, Theorem 3.6], it follows from our hypothesis that the minimal system of waves reduces to a single front solution $\varphi^*(x - c^* t)$ connecting 1 to 0, for some $c^* \in \mathbb{R}$. We further know from [54] that c^* is the minimal speed of traveling fronts connecting 1 to 0, see also [44, Theorem 2.4]. Since $\int_0^1 f(s)ds > 0$ by (1.10) and Lemma 2.1, formula (2.1) applied with (c^*, φ^*) instead of (c_0, φ_0) yields $c^* > 0$. □

The last step before the proof of Proposition 1.4 is a key-result on the existence of front profiles in finite or semi-infinite intervals, for some speeds smaller or larger than the minimal speed c^*, under Hypothesis 1.3. Before stating the result (which is also used in the proof of Theorems 1.5 and 1.6), we first recall that f is extended by 0 outside $[0,1]$, and that a planar front connecting 1 to 0 with speed $c \in \mathbb{R}$ is a solution $\varphi(x-ct)$ of (1.1) in $\mathbb{R}$, that is,

$$\varphi'' + c\varphi' + f(\varphi) = 0 \tag{2.2}$$

in $\mathbb{R}$, satisfying $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$ (it is also understood that $0 < \varphi < 1$ in $\mathbb{R}$). Equation (2.2) is equivalent to the system of ODEs

$$\begin{cases} q' = p \\ p' = -cp - f(q). \end{cases} \tag{2.3}$$

In the phase plane, this system generates orbits $(q(x), p(x))$ which, as long as $p \neq 0$, can be parameterized as the graph of a function $p = p(q)$ which solves

$$\frac{dp}{dq} = -c - \frac{f(q)}{p}. \tag{2.4}$$

From the arguments used in the proof of Lemma 2.1, a planar front φ connecting 1 to 0 corresponds to a heteroclinic connection between the stationary points $(0,0)$ and $(1,0)$ for (2.3), with $p < 0$ along the orbit, that is, $\varphi' < 0$ in $\mathbb{R}$. We always restrict to orbits of (2.3) contained in $[0,1]\times(-\infty,0]$ and moreover, if they contain a point $(q_0,0)$ for some $q_0 \in (0,1)$, we extend them by $(q_0,1]\times\{0\}$. Hypothesis 1.3 and Lemma 2.2 translate to the existence of a heteroclinic connection between $(0,0)$ and $(1,0)$ for (2.3) when $c = c^*$ and the nonexistence of such connection when $c < c^*$. We also remember that, by Lemma 2.1, Hypothesis 1.3 implies (1.10), hence in particular

$$\int_0^1 f(s)ds > 0. \tag{2.5}$$

The following result asserts that, when c is slightly below the threshold c^* for the existence of a heteroclinic connection for (2.3), one can find a trajectory joining $(0,1)\times\{0\}$ to $\{0\}\times\mathbb{R}^-$, whereas, for c above that threshold, there exists a trajectory joining $\{1\}\times\mathbb{R}^-$ to $\{(0,0)\}$.

Proposition 2.3. *Assume that Hypothesis* 1.3 *holds, and let* $c^* > 0$ *be given by Lemma* 2.2. *Then the following properties hold:*

(i) *there is* $\eta \in (0,c^*)$ *such that, for any* $c \in [c^*-\eta, c^*)$, *there exists a* C^2 *decreasing function* $\underline{\varphi}$ *defined in some interval* $[a,b]$ *with* $a < b \in \mathbb{R}$, *satisfying* (2.2) *in* $[a,b]$ *together with*

$$0 < \underline{\varphi}(a) < 1, \qquad \underline{\varphi}'(a) = 0, \qquad \underline{\varphi}(b) = 0;$$

(ii) *for any* $c > c^*$, *there exists a* $C^2(\mathbb{R})$ *decreasing positive function* $\overline{\varphi}$, *satisfying* (2.2) *in* $\mathbb{R}$ *together with*

$$\overline{\varphi}(0) = 1, \quad \overline{\varphi}(+\infty) = 0, \quad 0 < m^{-1}\overline{\varphi} \le -\overline{\varphi}' \le m\,\overline{\varphi} \text{ in } \mathbb{R}, \quad \overline{\varphi}'' \ge 0 \text{ in } [b,+\infty), \tag{2.6}$$

for some $m > 1$ *and* $b > 0$.

Proof. We use several results from [2], which, combined with Hypothesis 1.3, provide the desired properties. In [2], a critical speed $\widehat{c}$ is constructed. We now recall its construction and show that it coincides with the quantity $c^* > 0$ given in Lemma 2.2.

First, for any given $c \in \mathbb{R}$, one lets p_c denote the (uniform) limit in $[0,1]$ as $\varepsilon \searrow 0$ of the parameterization $p = p_{c,\varepsilon}(q)$ of the trajectory emerging from the regular point $(0, -\varepsilon)$ (that is, $p_{c,\varepsilon}$ solves (2.4) with $p_{c,\varepsilon}(0) = -\varepsilon$). It may happen that $p_c \equiv 0$ in $[0,1]$. Otherwise, as long as p_c is negative, it parameterizes a trajectory of (2.3) whence it corresponds to a solution of (2.2) in some interval of $\mathbb{R}$. It follows from [2, Proposition 4.2] that $p_c(1) < 0$ for c large enough. Then the critical speed $\widehat{c}$ is defined as the infimum of c such that $p_c(1) < 0$. Since (2.5) holds under Hypothesis 1.3, [2, Proposition 4.3] yields $\widehat{c} > 0$.

Let us show that $\widehat{c} \le c^*$. Let $p^*(q)$ be the parameterization of a front with (minimal) speed c^* given by Lemma 2.2. Then $p^* < 0$ in $(0,1)$ and (its continuous extension to $[0,1]$) satisfies $p^*(0) = p^*(1) = 0$. One readily sees that $c^* \ge 2\sqrt{f'(0)}$ if $f'(0) > 0$ (because otherwise $(0,0)$ is a focus for (2.3) and no trajectory can converge toward it). By construction, $p_{c^*} \le p^*$ in $[0,1]$ (trajectories cannot cross each other), hence $p_{c^*} < 0$ in $(0,1)$. Then, [2, Lemma 4.2] yields $p_c(1) < p_{c^*}(1) \le p^*(1) = 0$ for $c > c^*$, that is, $\widehat{c} \le c^*$.

Let us now show that $\widehat{c} \ge c^*$. Assume by contradiction that $\widehat{c} < c^*$. Then for any $c \in (\widehat{c}, c^*)$ there holds $p_c(1) < 0 = p^*(1)$. If by contradiction $p_c(q_1) > p^*(q_1)$ for some $q_1 \in (0,1)$ then there would exist $q_2 \in (q_1, 1)$ such that

$$p_c(q_2) = p^*(q_2), \qquad \frac{dp_c}{dq}(q_2) \le \frac{dp^*}{dq}(q_2),$$

which is impossible due to (2.4). This means that $p_c \le p^*$ in $[0,1]$, and thus $p_{\widehat{c}} \le p^*$ in $[0,1]$ by continuity, see the Remark after [2, Proposition 4.5]. In particular, $p_{\widehat{c}} < 0$ in $(0,1)$. On the other hand, $p_{\widehat{c}}(1) = 0$ by the definition of $\widehat{c}$, and again by continuity. We conclude that $p_{\widehat{c}}$ parameterizes an orbit corresponding to a front connecting 1 to 0 with speed $\widehat{c}$. This contradicts Lemma 2.2. We have thereby shown that $\widehat{c} = c^*$.

We can now prove statement (i). Take $0 < c < c^*$. If $f'(0) > 0$ and $c < 2\sqrt{f'(0)}$ then the existence of the solution $\underline{\varphi}$ simply follows from the fact that $(0,0)$ is a focus for the system (2.3). Otherwise, it follows from [2, Lemma 4.2] and the inequality $p_{c^*} < 0$ in $(0,1)$ that $0 \ge p_c > p_{c^*}$ in $(0,1)$, and moreover that p_c cannot be negative in the whole interval $(0,1)$, because otherwise p_c would provide a front connecting 1 to 0 with speed c, contradicting Lemma 2.2. This means that p_c vanishes somewhere in $(0,1)$. Consider $\theta \in (0,1)$ provided by (1.9) (from Lemma 2.1). By continuity, there is $\eta \in (0, c^*)$ such that $p_c(\theta) < 0$ if $c \in [c^* - \eta, c^*)$. For any such c, there is then a unique $q_c \in (\theta, 1)$ such that $p_c(q_c) = 0$ and $p_c < 0$ in $(0, q_c)$ (and then also $p_c = 0$ in $[q_c, 1]$). Take $\widetilde{\theta} \in (q_c, 1)$ and let $(\widetilde{q}, \widetilde{p})$ be the solution of (2.3) emerging from the regular point $(\widetilde{\theta}, 0)$. Since $f > 0$ in $[\theta, \widetilde{\theta}]$ by (1.9), it follows from (2.4) that the parameterization $\widetilde{p}(q)$ of the trajectory of $(\widetilde{q}, \widetilde{p})$ is strictly negative in $[\theta, \widetilde{\theta})$, whence in particular $\widetilde{p}(q_c) > p_c(q_c) = 0$. Thus, by definition of p_c, for ε sufficiently small there holds that $p_{c,\varepsilon}(q_c) < \widetilde{p}(q_c)$ and therefore, since trajectories cannot cross each other, we deduce $\widetilde{p} \le p_{c,\varepsilon} \le p_c \le 0$ in $(0, \widetilde{\theta}]$. It follows that the first zero of $p_{c,\varepsilon}$ is in $[\theta, \widetilde{\theta}]$, hence it corresponds to a regular point. The parameterization $p_{c,\varepsilon}$ provides the desired subsolution $\underline{\varphi}$ (up to translation).

We finally prove (ii). For $c > c^*$, the function p_c satisfies $p_c(1) < 0$ by the definition of $\widehat{c} = c^*$. We also extend p_c in $(1, +\infty)$ by setting $p_c(q) = p_c(1) - c(q-1)$ for $q > 1$. We then define $\overline{\varphi}$ as the positive solution of (1.1) associated with this trajectory, for which $\overline{\varphi}(0) = 1$. Recalling that f is extended to 0 in $(1, +\infty)$, we have that $\overline{\varphi}(z) = Ae^{-cz} + 1 - A$ for $z < 0$,

for some $A > 0$. The existence of $m > 1$ such that $|\overline{\varphi}'| \leq m\overline{\varphi}$ follows from elliptic estimates and Harnack's inequality. Next, by [2, Proposition 4.1], $p_c(q)/q \leq -c/2$ in a right neighborhood of 0, that is, $\overline{\varphi}'(z) \leq -(c/2)\overline{\varphi}(z)$ for z large enough. From this we deduce on the one hand that, up to increasing m if need be, $\overline{\varphi}' \leq -m^{-1}\overline{\varphi}$ in $\mathbb{R}$. On the other hand, we infer that, for any $\nu > 0$,

$$\overline{\varphi}'' \geq \frac{c^2}{2}\overline{\varphi} - f(\overline{\varphi}) \geq \Big(\frac{c^2}{2} - f'(0) - \nu\Big)\overline{\varphi},$$

for all z large enough. Thus, recalling that $c > c^* \geq 2\sqrt{\max(f'(0), 0)}$, we get that $\varphi'' > 0$ in some interval $[b, +\infty)$ with $b > 0$. □

Putting together the previous results allows one to derive Proposition 1.4 about the asymptotic speed of spreading for the solutions of (1.1).

Proof of Proposition 1.4. Since we here assume Hypothesis 1.3, Lemma 2.2 yields the existence of a positive minimal speed c^* of traveling fronts connecting 1 to 0, and Lemma 2.1 shows that (1.9)-(1.10) and Hypothesis 1.1 are fulfilled, for some $\theta \in (0, 1)$ and $\rho > 0$. Therefore, any solution u as in Hypothesis 1.1 spreads, in the sense that $u(t, \cdot) \to 1$ as $t \to +\infty$ locally uniformly in $\mathbb{R}^N$, and using the function $\underline{\varphi}$ provided by Proposition 2.3 (i), exactly as in the proof of [2, Theorem 5.3], one shows that such a spreading solution u satisfies (1.12). Assume now that the initial condition u_0 of (1.1) is compactly supported. As in the proof of [2, Theorem 5.1], it follows from Proposition 2.3 (ii) that (1.13) holds. □

Remark 2.4. Under Hypothesis 1.3, if the *hair trigger effect* holds then Hypothesis 1.1 and properties (1.12)-(1.13) hold (with $c^* > 0$ given by Proposition 1.4) for any compactly supported initial datum $0 \leq u_0 \leq 1$ such that $u_0 > 0$ on a set of positive measure.

Remark 2.5. Under the sole Hypothesis 1.1, the property (1.12) of Proposition 1.4 is still fulfilled, for a certain positive speed c^*. Indeed, if u_0 is as in Hypothesis 1.1 and if v denotes the solution to (1.1) with initial condition $v_0 = \theta\, \mathbb{1}_{B_\rho(x_0)}$, then there exists $T > 0$ such that $1 \geq u(T, \cdot + y) \geq v(T, \cdot + y) \geq v_0$ in $\mathbb{R}^N$ for every $|y| \leq 1$. Hence, iterating and using the comparison principle, one finds $1 \geq u(kT + t, \cdot + ky) \geq v(kT + t, \cdot + ky) \geq v(t, \cdot)$ in $\mathbb{R}^N$ for all $k \in \mathbb{N}$, $t \geq 0$, and $|y| \leq 1$. Since $v(t, \cdot) \to 1$ locally uniformly as $t \to +\infty$, one readily infers that $\min_{|x| \leq ct} u(t, x) \to 1$ as $t \to +\infty$, for every $c \in [0, 1/T)$.

3 General support and the Freidlin-Gärtner formula: proofs of Theorems 1.5 and 1.6

Theorems 1.5 and 1.6 are shown in Section 3.1, together with Proposition 3.6 providing some sufficient conditions for the validity of the hypothesis (1.20). Section 3.2 is devoted to some counter-examples of Theorems 1.5 and 1.6 when the assumptions (1.20) or (1.25) are violated.

3.1 The asymptotic speed of spreading: proofs of Theorems 1.5 and 1.6

There are two main parts in the proof of Theorems 1.5 and 1.6. On the one hand, lower bounds for the spreading speeds of upper level sets of the solutions can be established

from the assumptions (1.20) or (1.25), together with Proposition 1.4. On the other hand, upper bounds for the spreading speeds will follow from the derivation of upper bounds for solutions vanishing initially in large balls, which are obtained through a new type of supersolution. We start with the following lemma, which is a straightforward consequence of Proposition 1.4.

Lemma 3.1. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let* $c^* > 0$ *be given by Proposition* 1.4, *and let* u *be a solution of* (1.1) *emerging from an initial datum* $u_0 = \mathbb{1}_U$, *with a set* $U \subset \mathbb{R}^N$ *satisfying* $U_\rho \neq \emptyset$, *where* $\rho > 0$ *is given by Hypothesis* 1.1. *Then, there holds that*

$$\forall\, c \in (0, c^*), \quad \inf_{x \in U_\rho + B_{ct}} u(t,x) \to 1 \quad \text{as} \quad t \to +\infty.$$

Proof. Let v be the solution to (1.1) emerging from the initial datum $v_0 = \mathbb{1}_{B_\rho}$. Take $c \in (0, c^*)$ and $\lambda < 1$. By Proposition 1.4, there exists $T > 0$ such that

$$\forall\, t \geq T, \quad \forall\, x \in B_{ct}, \quad v(t,x) > \lambda.$$

Now, for any $x_0 \in U_\rho$, there holds that $u_0 \geq v_0(\cdot - x_0)$ in $\mathbb{R}^N$ and therefore, by the parabolic comparison principle,

$$\forall\, t \geq T, \quad \forall\, x \in B_{ct}(x_0), \quad u(t,x) \geq v(t, x - x_0) > \lambda.$$

This is true for any $x_0 \in U_\rho$, hence the result follows from the arbitrariness of $\lambda < 1$. □

The following key result provides us with a family of supersolutions that will be used for the proof of both Theorems 1.5 and 1.6 to get upper bounds for the spreading speeds. In the KPP case, such supersolutions could be obtained as the sums of solutions. In order to handle the general case, we construct some radially symmetric supersolutions which retract with a speed $c > c^*$. Such supersolutions are obtained by modulating the function $\overline{\varphi}$ provided by Proposition 2.3 (*ii*). More precisely, by a (generalized) supersolution ψ to (1.1) in $[0,T] \times \mathbb{R}^N$ with $T > 0$, we mean a continuous function $\psi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that, if a solution u to (1.1) satisfies $0 \leq u_0 \leq \psi(0,\cdot)$ in $\mathbb{R}^N$, then $u(t,\cdot) \leq \psi(t,\cdot)$ in $\mathbb{R}^N$ for all $t \in [0,T]$. We recall that f is extended by 0 in $\mathbb{R} \setminus [0,1]$.

Proposition 3.2. *Assume that Hypothesis* 1.3 *holds, and let* $c^* > 0$ *be given by Proposition* 1.4. *Then, for any* $c > c^*$ *and* $\lambda > 0$, *there exist* $R > 0$ (*depending on* f, N, c *and* λ) *and a family of functions* $(v^T)_{T>0}$ *such that, for each* $T > 0$, v^T *is a positive supersolution to* (1.1) *in* $[0,T] \times \mathbb{R}^N$ *and satisfies*

$$\begin{cases} v^T(0,x) \geq 1, & \forall\, |x| \geq R + cT, \\ v^T(t,0) < \lambda, & \forall\, t \in [0,T]. \end{cases} \tag{3.1}$$

Proof. We start with constructing the desired family of supersolutions in dimension 1. We then use them to construct radially symmetric supersolutions in higher dimension. But before doing so, we use some auxiliary notations. For any

$$c' > c'' > c^*,$$

consider the function $\overline{\varphi}$ provided by Proposition 2.3 (ii) associated with c''. Let $m > 1$ be given by (2.6), and $s_0 \in (0,1)$ be such that

$$\forall\, s \in (0, s_0), \quad |f(s) - f'(0)s| \le \frac{c' - c''}{4m}\, s. \tag{3.2}$$

Call then $Z > 0$ the quantity where $\overline{\varphi}(Z) = s_0$. For $\beta > 0$ we define

$$\psi(z) := \overline{\varphi}(z)\, e^{-\beta(z-Z)} \ \text{ for } z \in \mathbb{R}.$$

This function ψ is of class $C^2(\mathbb{R})$ and it satisfies in $\mathbb{R}$

$$-\psi'' - (c' - \beta)\psi' = \big(f(\overline{\varphi}) - (c' - 3\beta - c'')\overline{\varphi}' + \beta(c' - 2\beta)\overline{\varphi}\big)\, e^{-\beta(z-Z)}.$$

Then, because of (2.6), we can choose $\beta \in (0, c' - c'')$ small enough so that

$$-\psi'' - (c' - \beta)\psi' > f(\overline{\varphi})\, e^{-\beta(z-Z)} + \frac{c' - c''}{2m}\psi. \tag{3.3}$$

With $b > 0$ as in (2.6), we also choose arbitrarily large real numbers L and R' such that

$$L > \max\Big(Z + \frac{\log 2}{\beta}, b\Big) \ \text{ and } \ R' > \frac{N-1}{\beta}. \tag{3.4}$$

Step 1: the 1*-dimensional case.* Our goal is to connect $\overline{\varphi}$ with its reflection $\overline{\varphi}(-\cdot)$, by using an even function which is steeper than $\overline{\varphi}$ at some point. This will be achieved through the function ψ defined above. Then, the symmetrized function $\psi(x - c't) + \psi(-x - c't)$ will be a supersolution where it is smaller than s_0, and we take the minimum between the functions $\overline{\varphi}(x - c't)$ and $\psi(x - c't) + \psi(-x - c't)$, which will be a (generalized) supersolution for $x \le 0$. Next, we want the minimum to be achieved by the latter function at $t = 0$, so that we can extend the supersolution to the whole line by even reflection.

More precisely, we consider an arbitrary $T > 0$ and we call

$$\begin{cases} v_1(t,r) := \overline{\varphi}(r - c'(t-T) + L), \\ v_2(t,r) := \psi(r - c'(t-T) + L) + \psi(-r - c'(t-T) + L), \\ v(t,r) \ := \min\big(v_1(t,r), v_2(t,r)\big), \end{cases} \tag{3.5}$$

for $(t,r) \in [0,T] \times \mathbb{R}$. These functions are positive. Moreover, we see that

$$\partial_t v_1(t,r) - \partial_{rr} v_1(t,r) - f(v_1(t,r)) = (c'' - c')\,\overline{\varphi}'(r - c'(t-T) + L) > 0 \ \text{ in } [0,T] \times \mathbb{R}$$

since $c'' < c'$ and $\overline{\varphi}' < 0$ in $\mathbb{R}$, hence v_1 is a supersolution to (1.1) in $[0,T] \times \mathbb{R}$.

The definition of ψ and the positivity of $\overline{\varphi}$ also imply that

$$\forall\, 0 \le t \le T,\ \forall\, r \le c'(t-T) - L + Z, \quad v_2(t,r) > v_1(t,r).$$

This means that if there exists $(\bar t, \bar r) \in [0,T] \times (-\infty, 0]$ where $v(\bar t, \bar r) = v_2(\bar t, \bar r)$, then necessarily $\bar r > c'(\bar t - T) - L + Z$. Together with the fact that $\overline{\varphi}$ is decreasing, it follows that, for all $(\bar t, \bar r) \in [0,T] \times (-\infty, 0]$,

$$v(\bar t, \bar r) = v_2(\bar t, \bar r) \implies \begin{cases} 0 < v_2(\bar t, \bar r) \le v_1(\bar t, \bar r) = \overline{\varphi}(\bar r - c'(\bar t - T) + L) < s_0, \\ 0 < \overline{\varphi}(-\bar r - c'(\bar t - T) + L) \le \overline{\varphi}(\bar r - c'(\bar t - T) + L) < s_0. \end{cases}$$

On the other hand, by (3.3) and the negativity of ψ' we have that

$$\begin{aligned}\partial_t v_2(\bar t,\bar r)-\partial_{rr}v_2(\bar t,\bar r)-\beta|\partial_r v_2(\bar t,\bar r)| \;>\; & f(\overline{\varphi}(\bar r-c'(\bar t-T)+L))\,e^{-\beta(\bar r-c'(\bar t-T)+L-Z)}\\ & +f(\overline{\varphi}(-\bar r-c'(\bar t-T)+L))\,e^{-\beta(-\bar r-c'(\bar t-T)+L-Z)}\\ & +\frac{c'-c''}{2m}\,v_2(\bar t,\bar r).\end{aligned}$$

Hence, estimating $f(\overline{\varphi}(\pm\bar r-c'(\bar t-T)+L))$ by (3.2), and then using (3.2) again, we eventually derive

$$\partial_t v_2-\partial_{rr}v_2-\beta|\partial_r v_2|>f(v_2)\ \text{ in }\big\{(t,r)\in[0,T]\times(-\infty,0]:v(t,r)=v_2(t,r)\big\}. \tag{3.6}$$

At the point $r=0$ we compute, for $0\le t\le T$,

$$v_2(t,0)=2\,\overline{\varphi}(-c'(t-T)+L)\,e^{-\beta(-c'(t-T)+L-Z)}\le 2\,v_1(t,0)\,e^{-\beta(L-Z)}.$$

Since $\beta(L-Z)>\log 2$ by (3.4), we have that

$$\forall\, t\in[0,T],\quad v(t,0)=v_2(t,0)<v_1(t,0)=\overline{\varphi}(-c'(t-T)+L)\le\overline{\varphi}(L). \tag{3.7}$$

Observe that the function $r\mapsto v_2(t,r)$ is even and that $v(t,r)$ is equal to $v_2(t,r)$ and then symmetric with respect to r in a neighborhood of $r=0$, for each $0\le t\le T$. Hence

$$\forall\, t\in[0,T],\quad \partial_r v(t,0)=\partial_r v_2(t,0)=0. \tag{3.8}$$

Remember also that v_1 is a supersolution to (1.1) in $[0,T]\times\mathbb{R}$. All these facts imply that, if we restrict $v(t,r)$ to $r\le 0$ and we take its even reflection around $r=0$, we obtain a (generalized) supersolution to (1.1) in $[0,T]\times\mathbb{R}$.

We also see that, for every $0\le t\le T$,

$$\begin{aligned}\partial_{rr}v(t,0)&=\partial_{rr}v_2(t,0)=2\,\psi''(-c'(t-T)+L)\\ &=2\,\big(\overline{\varphi}''(-c'(t-T)+L)-2\beta\overline{\varphi}'(-c'(t-T)+L)+\beta^2\overline{\varphi}(-c'(t-T)+L)\big)\\ &\qquad\qquad\times e^{-\beta(-c'(t-T)+L-Z)}\\ &>0\end{aligned} \tag{3.9}$$

since $\overline{\varphi}'<0$, $\overline{\varphi}>0$ and $L>b$ by (3.4), where $b>0$ from (2.6) is such that $\overline{\varphi}''\ge 0$ in $[b,+\infty)$.

Step 2: the case of dimension $N\ge 2$. Consider the function v defined before. With $R'>0$ given by (3.4), we define, for $T>0$, a continuous function v^T in $[0,T]\times\mathbb{R}^N$ as follows:

$$v^T(t,x):=\begin{cases}v(t,0) & \text{if } t\in[0,T] \text{ and } |x|\le R',\\ v(t,R'-|x|) & \text{if } t\in[0,T] \text{ and } |x|>R'.\end{cases}$$

We want to show that v^T is a (generalized) supersolution to (1.1) in $[0,T]\times\mathbb{R}^N$.

We start with checking this in the region $|x|>R'$. Recall that v is defined in (3.5) as the minimum between v_1 and v_2. A direct computation reveals that the function $v_1(t,R'-|x|)<1$ fulfills

$$\begin{aligned}&(\partial_t-\Delta)\big(v_1(t,R'-|x|)\big)-f(v_1(t,R'-|x|))\\ &\quad=\partial_t v_1(t,R'-|x|)-\partial_{rr}v_1(t,R'-|x|)+\frac{N-1}{|x|}\partial_r v_1(t,R'-|x|)-f(v_1(t,R'-|x|))\\ &\quad=\left(c''-c'+\frac{N-1}{|x|}\right)\overline{\varphi}'(R'-|x|-c'(t-T)+L)\\ &\quad>0\end{aligned}$$

for all $t \in [0,T]$ and $|x| > R'$, since $\overline{\varphi}' < 0$ in $\mathbb{R}$ and $c''-c'+(N-1)/R' < 0$ by (3.4) together with $0 < \beta < c'-c''$. We now turn to the function v_2. At any point $(\bar{t},\bar{x}) \in [0,T]\times(\mathbb{R}^N\setminus\overline{B_{R'}})$ such that $v(\bar{t},R'-|\bar{x}|) = v_2(\bar{t},R'-|\bar{x}|)$, we deduce from (3.4) and (3.6) that

$$\begin{aligned}(\partial_t-\Delta)\big(v_2(t,R'-|x|)\big)\big|_{\bar{t},\,\bar{x}} - f(v_2(\bar{t},R'-|\bar{x}|)) \\ > \beta\,|\partial_r v_2(\bar{t},R'-|\bar{x}|)| + \frac{N-1}{|\bar{x}|}\,\partial_r v_2(\bar{t},R'-|\bar{x}|) \geq 0.\end{aligned}$$

We have thereby shown that v^T is a supersolution to (1.1) outside the ball $\overline{B_{R'}}$. Observe now that (3.8) implies that v^T is $W^{2,\infty}$ with respect to x in a neighborhood of $[0,T]\times\partial B_{R'}$ (relatively to $[0,T]\times\mathbb{R}^N$). Next, for $t \in [0,T]$ and $|x| < R'$, using (3.6) and (3.9) we get

$$\partial_t v^T(t,x) - \Delta v^T(t,x) = \partial_t v(t,0) > \partial_{rr} v(t,0) + f(v(t,0)) > f(v^T(t,x)).$$

Summing up, the function v^T is a positive supersolution to (1.1) in $[0,T]\times\mathbb{R}^N$. We further see from the definition of v and from (2.6) and (3.7) that v^T satisfies

$$\begin{cases} v^T(0,x) \geq 1, & \forall\,|x| \geq R'+L+c'T, \\ v^T(t,0) < \overline{\varphi}(L), \forall\, t \in [0,T]. \end{cases} \tag{3.10}$$

Step 3: conclusion. Consider any $c > c^*$ and $\lambda > 0$. Let any c' and c'' be such that $c > c' > c'' > c^*$, and let s_0, Z, β, L and R' be the positive parameters (depending on f, N, c' and c'', hence on f, N and c, since c' and c'' depend on c and c^* while c^* depends on f only) given as in (3.2)-(3.4). Without loss of generality, one can also assume now that L is large enough (depending also on λ) so that $\overline{\varphi}(L) \leq \lambda$. Let $(v^T)_{T>0}$ be the functions defined as in Step 2 above. Since $c > c'$, the conclusion (3.1) with $R = R'+L > 0$ follows from (3.10). The proof of Proposition 3.2 is thereby complete. □

With Lemma 3.1 and Proposition 3.2 in hand, the proof of Theorem 1.6 easily follows.

Proof of Theorem 1.6. Fix $c \in (0,c^*)$, where $c^* > 0$ is given by Proposition 1.4. It follows from the assumption (1.25) that, for given $c' \in (c,c^*)$, the inclusion $U + B_{ct} \subset U_\rho + B_{c't}$ holds for $t > 0$ sufficiently large (depending on c,c'). Thus, Lemma 3.1 implies that $\inf_{x\in U+B_{ct}} u(t,x) \to 1$ as $t \to +\infty$. Therefore, for any $\lambda \in (0,1)$, there holds that

$$U + B_{ct} \subset E_\lambda(t),$$

for t sufficiently large. Since this holds for each $c \in (0,c^*)$, one infers that

$$\sup_{x\in U+B_{c^*t}} \mathrm{dist}(x,E_\lambda(t)) = o(t) \quad \text{as } t \to +\infty. \tag{3.11}$$

Conversely, by taking any $c' > c^*$ and $\lambda \in (0,1)$, we will show that

$$\big\{x \in \mathbb{R}^N : \mathrm{dist}(x,U) \geq c't\big\} \subset \mathbb{R}^N \setminus E_\lambda(t), \tag{3.12}$$

for t sufficiently large. To do so, consider any $c \in (c^*,c')$, and let $R > 0$ and $(v^T)_{T>0}$ be given by Proposition 3.2. Denote $t_0 = R/(c'-c) > 0$, and consider any $t \geq t_0$. For any $x_0 \in \mathbb{R}^N$ such that $\mathrm{dist}(x_0,U) \geq c't$, one has $B_{c't}(x_0) \subset \mathbb{R}^N\setminus U$, hence $u_0 \leq \mathbb{1}_{\mathbb{R}^N\setminus B_{c't}(x_0)}$ and $u_0 \leq v^t(0,\cdot-x_0)$ in $\mathbb{R}^N$ by (3.1) (observe that $c't \geq R+ct$). By the maximum principle, the fact that v^t is a supersolution imply in particular that $u(t,\cdot) \leq v^t(t,\cdot-x_0)$ in $\mathbb{R}^N$, hence

$u(t,x_0) \le v^t(t,0) < \lambda$ by (3.1), whence $x_0 \in \mathbb{R}^N \setminus E_\lambda(t)$. Therefore, one has shown (3.12) for all $t \ge t_0$, that is, $E_\lambda(t) \subset U + B_{c't}$ for all $t \ge t_0$. Since this holds for each $c' > c^*$, one infers that

$$\sup_{x \in E_\lambda(t)} \operatorname{dist}(x, U + B_{c^* t}) = o(t) \quad \text{as } t \to +\infty.$$

Together with (3.11), this gives (1.26). □

Before turning to the proof of Theorem 1.5, we derive two auxiliary lemmas. The first one follows from simple geometric considerations.

Lemma 3.3. *Let U be a non-empty subset of $\mathbb{R}^N$ satisfying (1.20) for some $\rho > 0$. Then for every $e \in \mathcal{B}(U)$, there holds that*

$$\liminf_{\tau \to +\infty} \frac{\operatorname{dist}(\tau e, U)}{\tau} = \inf_{\substack{\xi \in \mathcal{U}(U) \\ \xi \cdot e \ge 0}} \sqrt{1 - (\xi \cdot e)^2} > 0, \tag{3.13}$$

with the convention that the right-hand side is 1 if there is no $\xi \in \mathcal{U}(U)$ satisfying $\xi \cdot e \ge 0$.

Proof. Call

$$\bar\delta := \liminf_{\tau \to +\infty} \frac{\operatorname{dist}(\tau e, U)}{\tau}, \qquad m := \inf_{\substack{\xi \in \mathcal{U}(U) \\ \xi \cdot e \ge 0}} \sqrt{1 - (\xi \cdot e)^2}.$$

The definition of $\mathcal{B}(U)$ yields $\bar\delta > 0$.

We start with proving $\bar\delta \le m$. If there exists no $\xi \in \mathcal{U}(U)$ satisfying $\xi \cdot e > 0$ then $m = 1 \ge \bar\delta$ because $U \ne \emptyset$. Suppose now that there is $\xi \in \mathcal{U}(U)$ such that $\xi \cdot e > 0$. The definition of $\mathcal{U}(U)$ yields the existence of a family of points $(x_\tau)_{\tau > 0}$ in U such that

$$\Big|\xi - \frac{x_\tau}{\tau}\Big| \to 0 \quad \text{as } \tau \to +\infty.$$

It follows that

$$\bar\delta \le \lim_{\tau \to +\infty} \frac{\left|\frac{\tau}{\xi \cdot e} e - x_\tau\right|}{\frac{\tau}{\xi \cdot e}} = \lim_{\tau \to +\infty} \Big|e - \frac{\xi \cdot e}{\tau} x_\tau\Big| = |e - (\xi \cdot e)\xi| = \sqrt{1 - (\xi \cdot e)^2}.$$

Since this holds for any $\xi \in \mathcal{U}(U)$ such that $\xi \cdot e > 0$, the inequality $\bar\delta \le m$ follows.

Let us pass now to the proof of the reverse inequality $m \le \bar\delta$. Remember first that $0 \le \bar\delta \le 1$. If $\bar\delta = 1$ it trivially holds because $m \le 1$. Suppose that $\bar\delta < 1$. There exist a positive sequence $(\tau_n)_{n \in \mathbb{N}}$ diverging to $+\infty$ and a sequence of points $(x_n)_{n \in \mathbb{N}}$ in U satisfying

$$\Big|e - \frac{x_n}{\tau_n}\Big| < \bar\delta + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Because $\bar\delta < 1$, we see that $|x_n| \to +\infty$ as $n \to +\infty$, hence we can assume that $x_n \ne 0$ for all $n \in \mathbb{N}$. We further deduce from the above inequality that

$$\frac{(x_n \cdot e)^2}{|x_n|^2} \ge 2\frac{x_n \cdot e}{\tau_n} - \frac{|x_n|^2}{\tau_n^2} = 1 - \Big|e - \frac{x_n}{\tau_n}\Big|^2 > 1 - \Big(\bar\delta + \frac{1}{n}\Big)^2.$$

hence, calling $\xi_n := x_n/|x_n|$,

$$\xi_n \cdot e \ge \sqrt{1 - \Big(\bar\delta + \frac{1}{n}\Big)^2}.$$

Thus, the limit $\overline{\xi} \in \mathbb{S}^{N-1}$ of a converging subsequence of $(\xi_n)_{n\in\mathbb{N}}$ satisfies $\overline{\xi}\cdot e \geq \sqrt{1-\bar{\delta}^2}$. Furthermore, there holds that

$$\frac{\big||x_n|\overline{\xi}-x_n\big|}{|x_n|} = |\overline{\xi}-\xi_n| \to 0 \quad \text{as} \quad n\to+\infty,$$

which means that $\overline{\xi}\notin\mathcal{B}(U)$. Hence, by (1.20), $\overline{\xi}\in\mathcal{U}(U_\rho)\subset\mathcal{U}(U)$. We eventually derive

$$\sup_{\xi\in\mathcal{U}(U)} \xi\cdot e \geq \sqrt{1-\bar{\delta}^2} > 0,$$

that is, $\bar{\delta}\geq m$. The proof of Lemma 3.3 is thereby complete. □

Thanks to the above geometric lemma, using the family of supersolutions provided by Proposition 3.2 one can derive an upper bound for the spreading speeds in a cone of directions around $\mathcal{B}(U)$.

Lemma 3.4. *Assume that Hypothesis* 1.3 *holds, and let* $c^*>0$ *be given by Proposition* 1.4. *Let* u *be a solution of* (1.1) *with an initial datum* $u_0=\mathbb{1}_U$, *where the non-empty set* $U\subset\mathbb{R}^N$ *satisfies* (1.20) *for some* $\rho>0$. *Assume that there exists* $e\in\mathcal{B}(U)$. *Then, for any* $w>w(e)$, *where* $w(e)$ *is given by* (1.16), *in the cone*

$$\mathcal{C} := \bigcup_{\tau>1} B_{c^*(\tau-1)}(\tau we),$$

there holds that

$$\sup_{x\in\mathcal{C}} u(t,tx)\to 0 \quad \textit{as} \quad t\to+\infty.$$

Proof. Consider U, u, ρ, e, w and $\mathcal{C}$ as in the statement. Because $w>w(e)>0$, there exists a real number k satisfying

$$0<\frac{c^*}{w}<k<\frac{c^*}{w(e)} = \inf_{\substack{\xi\in\mathcal{U}(U)\\ \xi\cdot e\geq 0}} \sqrt{1-(\xi\cdot e)^2}. \tag{3.14}$$

Then, by Lemma 3.3, we can find $\tau_1>0$ such that

$$\forall\,\tau\geq\tau_1, \quad \operatorname{dist}(\tau e, U)\geq k\tau. \tag{3.15}$$

Take $c\in(c^*,kw)$ and $\lambda>0$. Consider the associated constant $R>0$ and the functions $(v^T)_{T>0}$ provided by Proposition 3.2. By (3.15), for all $T>0$, there holds that

$$\forall\,\tau>\tau_1+\frac{R+cT}{k}, \quad \forall\, y\in B_{k\tau-R-cT}(\tau e), \quad \operatorname{dist}(y,U)\geq R+cT,$$

hence, for τ and y as above, u is less than or equal to the positive function $v^T(\cdot,\cdot-y)$ at time 0, due to (3.1), and therefore the comparison principle yields $u(T,y)\leq v^T(T,0)<\lambda$. We rewrite this inequality using $t=T$, $s=\tau/T$, $x=y/T$, that is,

$$\forall\, t>0, \quad \forall\, s>\frac{c}{k}+\frac{1}{t}\Big(\tau_1+\frac{R}{k}\Big), \quad \forall\, x\in B_{ks-c-R/t}(se), \quad u(t,tx)<\lambda. \tag{3.16}$$

If we show that, for t sufficiently large, any point $x \in \mathcal{C}$ can be written in the above form, the lemma is proved, due to the arbitrariness of λ. Consider $x \in \mathcal{C}$. We write it as follows:

$$x = s_x e + y_x, \quad \text{with } s_x > w \text{ and } |y_x| < c^*\Big(\frac{s_x}{w} - 1\Big). \tag{3.17}$$

Then, recalling that $c < kw$, we can find $t_1 > 0$ (depending on c, k, w, τ_1, R, but not on x), such that

$$\forall\, t > t_1, \quad s_x > w > \frac{c}{k} + \frac{1}{t}\Big(\tau_1 + \frac{R}{k}\Big).$$

Next, using $c^*/w < k$ and $s_x > w$ in (3.17), we infer that

$$|y_x| - k s_x < \Big(\frac{c^*}{w} - k\Big) s_x - c^* < \Big(\frac{c^*}{w} - k\Big) w - c^* = -kw.$$

Then, because $c < kw$, we can find $t_2 \in [t_1, +\infty)$ (depending on t_1, c, k, w, R, but not on x), such that $-kw < -c - R/t$ for all $t > t_2$, hence $|y_x| < k s_x - c - R/t$ for $t > t_2$. We have shown that x, $s = s_x$ fulfill the inclusion and inequality in (3.16), hence the proof is concluded. □

Remark 3.5. The conclusion of Lemma 3.4 holds good when the assumption (1.20) is replaced by

$$\liminf_{\tau \to +\infty} \frac{\operatorname{dist}(\tau e, U)}{\tau} \geq 1$$

(that is, $\operatorname{dist}(\tau e, U) \sim \tau$ as $\tau \to +\infty$ since $\limsup_{\tau \to +\infty} \operatorname{dist}(\tau e, U)/\tau \leq 1$ as soon as U is not empty). Indeed, in that case, one has $e \in \mathcal{B}(U)$ and $\mathcal{U}(U) \subset \{e' \in \mathbb{S}^{N-1} : e' \cdot e \leq 0\}$, hence $w(e) = c^*$. Therefore, k in (3.14) is now chosen so that $c^*/w < k < 1$ and (3.15) is satisfied for τ_1 large enough. The rest of the proof after (3.15) is identical.

With the previous results in hand, we are now in position to prove Theorem 1.5.

Proof of Theorem 1.5. We prove the statements of the theorem in a different order.

Statement (ii). First of all, we deduce from (1.20) that

$$\mathcal{U}(U) \supset \mathcal{U}(U_\rho) = \mathbb{S}^{N-1} \setminus \mathcal{B}(U) \supset \mathcal{U}(U),$$

that is, $\mathcal{U}(U_\rho) = \mathcal{U}(U)$. Next, fix a compact set C included in $\mathcal{W}$, where $\mathcal{W}$ is given by the equivalent formulas (1.19) or (1.24). For any $\xi \in \mathcal{U}(U) = \mathcal{U}(U_\rho)$ (if it exists), and any $\tau > 0$ and $0 < c' < c < c^*$, one has, by the definition of $\mathcal{U}(U_\rho)$,

$$\frac{1}{t} \operatorname{dist}(t\tau\xi, U_\rho) \to 0 \quad \text{as } \ t \to +\infty,$$

hence $B_{c't}(t\tau\xi) \subset U_\rho + B_{ct}$ for t sufficiently large. It then follows from Lemma 3.1 that

$$\inf_{x \in B_{c'}(\tau\xi)} u(t, tx) \to 1 \quad \text{as } \ t \to +\infty. \tag{3.18}$$

Observe that the above limit holds good when $\tau = 0$ (without any reference to ξ) due to Proposition 1.4 and the fact that u_0 fulfills (1.4) for some $x_0 \in \mathbb{R}^N$, because $U_\rho \neq \emptyset$. Moreover, from (1.24), any point $x \in \mathcal{W}$ is contained either in $B_{c'_x}$ or in $B_{c'_x}(\tau_x \xi_x)$, for certain $c'_x \in (0, c^*)$, $\xi_x \in \mathcal{U}(U)$, and $\tau_x > 0$. Then, by compactness, C can be covered by a

finite number of such balls and therefore, since (3.18) holds in each one of them, the first limit in (1.21) follows.

Consider now a compact set C included in $\mathbb{R}^N \setminus \overline{\mathcal{W}}$. Any point $y \in C$ is such that $e := y/|y|$ satisfies $w(e) < |y| < +\infty$, hence necessarily $e \in \mathcal{B}(U)$, because otherwise (1.20) would yield $e \in \mathcal{U}(U_\rho) = \mathcal{U}(U)$ and then $w(e) = +\infty$. As a consequence, for an arbitrary $\lambda > 0$, applying Lemma 3.4 with $w \in (w(e), |y|)$, we infer the existence of an open neighborhood $\mathcal{C}_y$ of y and of some $t_y > 0$ such that

$$\forall\, t > t_y,\ \forall x \in \mathcal{C}_y, \quad u(t, tx) < \lambda.$$

By a covering argument we can find $t_C > 0$ such that

$$\forall\, t > t_C,\ \forall x \in C, \quad u(t, tx) < \lambda.$$

This concludes the proof of (1.21).

Statement (i). The continuity of $w : \mathbb{S}^{N-1} \to [c^*, +\infty]$ is provided by formula (1.23), since so is the map $e \mapsto \mathrm{dist}(e, \mathbb{R}^+\mathcal{U}(U))$ (when $\mathcal{U}(U) = \emptyset$, then the map $e \mapsto w(e) \equiv c^*$ is obviously continuous too!). The first limit in (1.15) is a particular instance of the first limit in (1.21). The second one only involves the directions e for which $w(e) < +\infty$, whence, by (1.20), the ones in $\mathcal{B}(U)$. But then the second limit in (1.15) immediately follows by applying Lemma 3.4 with $w \in (w(e), c)$.

Statement (iii). Consider $\lambda \in (0, 1)$ and a compact set $K \subset \mathbb{R}^N$ satisfying $\overline{K \cap \mathcal{W}} = K \cap \overline{\mathcal{W}}$. Take $\varepsilon > 0$. For $\eta > 0$, we define the following subset of $K \cap \mathcal{W}$:

$$K_\eta := K \cap \big\{re \,:\, e \in \mathbb{S}^{N-1},\ \ 0 \le r \le w(e) - \eta\big\}.$$

From the continuity of w, we deduce that K_η is a compact set included in the open set $\mathcal{W}$. On the one hand, since $\overline{K \cap \mathcal{W}}$ is compact, using a covering argument one can find $\eta > 0$ small enough such that

$$K \cap \mathcal{W} \subset \overline{K \cap \mathcal{W}} \subset K_\eta + B_\varepsilon.$$

On the other hand, by the first line of (1.21) applied with $C = K_\eta$, we infer that, for t larger than some $T > 0$ depending on η, there holds that $K_\eta \subset t^{-1}E_\lambda(t)$ and therefore $K_\eta \subset K \cap t^{-1}E_\lambda(t)$. Combining these inclusions one then gets

$$\forall\, t > T, \quad K \cap \mathcal{W} \subset \Big(K \cap \frac{1}{t}E_\lambda(t)\Big) + B_\varepsilon. \tag{3.19}$$

Consider now, for $\sigma > 0$, the set

$$K'_\sigma := K \cap \big\{re \,:\, e \in \mathbb{S}^{N-1},\ \ r \ge w(e) + \sigma\big\}.$$

By the continuity of w, this is a compact set contained in $\mathbb{R}^N \setminus \overline{\mathcal{W}}$. Let us check that

$$K \setminus K'_\sigma \subset \big(K \cap \overline{\mathcal{W}}\big) + B_\varepsilon, \tag{3.20}$$

for all $\sigma > 0$ small enough. Assume by contradiction that this is not the case. Then we can find a sequence $(r_n e_n)_{n\in\mathbb{N}}$ in $K \setminus \big((K \cap \overline{\mathcal{W}}) + B_\varepsilon\big)$ with $(e_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-1}$ and $(r_n)_{n\in\mathbb{N}}$ bounded and satisfying $r_n < w(e_n) + 1/n$ for all $n \in \mathbb{N}$. Thus, up to subsequences, $(e_n)_{n\in\mathbb{N}}$ converges to some $e \in \mathbb{S}^{N-1}$ and then, by the continuity of w, $(r_n)_{n\in\mathbb{N}}$ converges to

some $r \leq w(e)$ (whenever $w(e)$ be finite or not). This means that $re \in K \cap \overline{\mathcal{W}}$ and therefore $r_n e_n \in (K \cap \overline{\mathcal{W}}) + B_\varepsilon$ for n large, a contradiction. We can then choose $\sigma > 0$ such that (3.20) holds. Applying the second line of (1.21) with $C = K'_\sigma$, we can find $\tau > 0$ such that

$$\forall\, t > \tau, \quad K'_\sigma \cap \frac{1}{t} E_\lambda(t) = \emptyset,$$

whence

$$\forall\, t > \tau, \quad K \cap \frac{1}{t} E_\lambda(t) \subset K \setminus K'_\sigma.$$

Using the inclusion (3.20) and recalling that $K \cap \overline{\mathcal{W}} = \overline{K \cap \mathcal{W}}$, one finds that $K \setminus K'_\sigma \subset \overline{K \cap \mathcal{W}} + B_\varepsilon$, and therefore

$$\forall\, t > \tau, \quad K \cap \frac{1}{t} E_\lambda(t) \subset \overline{K \cap \mathcal{W}} + B_\varepsilon = (K \cap \mathcal{W}) + B_\varepsilon.$$

This property, together with (3.19), yields the desired result (1.22), owing to the arbitrariness of $\varepsilon > 0$. □

The last result of this section provides a list of conditions for a set $U \subset \mathbb{R}^N$ to satisfy property (1.20). As we will see in the examples listed in Section 3.2, conditions (1.20) and (1.25) cannot be compared. However, condition (1.25) together with certain additional properties imply (1.20), as the following result shows.

Proposition 3.6. *For a set $U \subset \mathbb{R}^N$, property* (1.20) *holds if U satisfies* (1.25) *together with one of the following conditions:*

- *either U is star-shaped with respect to some point $x_0 \in \mathbb{R}^N$;*
- *or there exists $U' \subset \mathbb{R}^N$ satisfying*

$$\mathcal{B}(U') \cup \mathcal{U}(U') = \mathbb{S}^{N-1} \tag{3.21}$$

  and $d_{\mathcal{H}}(U, U') < +\infty$;

- *or there exists $U' \subset \mathbb{R}^N$ satisfying* (3.21) *and*

$$\frac{d_{\mathcal{H}}(U \cap B_R, U' \cap B_R)}{R} \longrightarrow 0 \quad \textit{as} \quad R \to +\infty. \tag{3.22}$$

Proof. First of all, using (1.25) one sees that, for any $\xi \in \mathcal{U}(U)$,

$$\frac{\operatorname{dist}(\tau\xi, U_\rho)}{\tau} \leq \frac{\operatorname{dist}(\tau\xi, U) + d_{\mathcal{H}}(U, U_\rho)}{\tau} \to 0 \quad \text{as} \quad \tau \to +\infty,$$

that is, $\xi \in \mathcal{U}(U_\rho)$. Thus, it is sufficient to show (1.20) with $\mathcal{U}(U)$ instead of $\mathcal{U}(U_\rho)$.

Consider the case where U is star-shaped. Since properties (1.25) and (1.20) are invariant under rigid transformations of the coordinate system, we can assume without loss of generality that U is star-shaped with respect to the origin. Suppose that there exists $\xi \in \mathbb{S}^{N-1} \setminus \mathcal{B}(U)$ (otherwise property (1.20) trivially holds). This means that there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ diverging to $+\infty$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in U such

that $|\tau_n\xi - x_n|/\tau_n \to 0$ as $n \to +\infty$. Then, for any $0 < \tau \leq \tau_n$, since $\frac{\tau}{\tau_n} x_n \in U$ because U is star-shaped with respect to the origin, one finds

$$\frac{\mathrm{dist}(\tau\xi, U)}{\tau} \leq \frac{|\tau\xi - \frac{\tau}{\tau_n} x_n|}{\tau} = \left|\xi - \frac{1}{\tau_n} x_n\right| \to 0 \quad \text{as} \quad n \to +\infty,$$

hence $\xi \in \mathcal{U}(U)$. This shows that $\mathcal{B}(U) \cup \mathcal{U}(U) = \mathbb{S}^{N-1}$ and, as already emphasized, this proves the statement in this case.

Consider now the hypotheses of the second case, with $U' \subset \mathbb{R}^N$ satisfying (3.21) and $d_{\mathcal{H}}(U, U') < +\infty$. Then there holds that

$$\mathcal{B}(U') = \mathcal{B}(U) \quad \text{and} \quad \mathcal{U}(U') = \mathcal{U}(U), \tag{3.23}$$

hence $\mathcal{B}(U) \cup \mathcal{U}(U) = \mathbb{S}^{N-1}$ and, as above, (1.20) then follows.

Let us check that the same conclusions (3.23) are true when U' satisfies (3.21)-(3.22). We call

$$D_R := d_{\mathcal{H}}(U \cap B_R, U' \cap B_R).$$

For $\xi \in \mathbb{S}^{N-1}$ and $\tau > 0$, there exists $x_\tau \in U$ such

$$|\tau\xi - x_\tau| < \mathrm{dist}(\tau\xi, U) + 1,$$

then in particular

$$|x_\tau| < \tau + \mathrm{dist}(\tau\xi, U) + 1 \leq \tau + |\tau\xi - x_1| + 1 \leq 2\tau + |x_1| + 1. \tag{3.24}$$

Moreover, we can find $x'_\tau \in U' \cap B_{|x_\tau|+1}$ for which $|x'_\tau - x_\tau| < D_{|x_\tau|+1} + 1$. It follows that

$$\mathrm{dist}(\tau\xi, U') \leq |\tau\xi - x'_\tau| \leq \mathrm{dist}(\tau\xi, U) + 1 + D_{|x_\tau|+1} + 1.$$

By (3.22) and (3.24) one then deduces the inequality

$$\mathrm{dist}(\tau\xi, U') \leq \mathrm{dist}(\tau\xi, U) + o(\tau) \quad \text{as} \quad \tau \to +\infty,$$

and then $|\,\mathrm{dist}(\tau\xi, U') - \mathrm{dist}(\tau\xi, U)| = o(\tau)$ as $\tau \to +\infty$ by switching the roles of U and U'. From this, the equivalences (3.23) immediately follow, and one concludes as in the previous paragraph. □

Remark 3.7. The conclusions of Theorems 1.5 and 1.6 still hold for the solutions to (1.1) with initial conditions more general than characteristic functions. To be more precise, firstly, if Hypothesis 1.3 is satisfied, if $c^* > 0$ is the minimal speed given by Proposition 1.4, if $\theta \in (0, 1)$ and $\rho > 0$ are given by Hypothesis 1.1, and if u is a solution to (1.1) such that

$$\{u_0 \geq \theta\}_\rho \neq \emptyset, \;\; \mathcal{B}\big(\{u_0 \geq \theta\}\big) \cup \mathcal{U}\big(\{u_0 \geq \theta\}_\rho\big) = \mathbb{S}^{N-1}, \;\; \text{and} \;\; d_{\mathcal{H}}\big(\mathrm{supp}\, u_0, \{u_0 \geq \theta\}\big) < +\infty,$$

then the conclusions (i), (ii) and (iii) of Theorem 1.5 hold, with $\mathcal{U}(U)$ replaced by $\mathcal{U}\big(\{u_0 \geq \theta\}\big)$ in the definitions (1.16)-(1.17) of $w(e)$. Indeed, it is easy to see that $\mathcal{B}(\mathrm{supp}\, u_0) = \mathcal{B}\big(\{u_0 \geq \theta\}\big)$, that $\mathcal{U}(\mathrm{supp}\, u_0) = \mathcal{U}\big(\{u_0 \geq \theta\}\big)$, that Lemma 3.1 holds with U replaced by $\{u_0 \geq \theta\}$, and that Lemmas 3.3 and 3.4 hold as well with $\mathcal{B}(U)$ replaced by $\mathcal{B}(\mathrm{supp}\, u_0)$ in both statements and U replaced by $\mathrm{supp}\, u_0$ in (3.13). Meanwhile, Proposition 3.2 is kept unchanged. Secondly, if Hypothesis 1.3 is satisfied, if $c^* > 0$ is the minimal

speed given by Proposition 1.4, if $\theta \in (0,1)$ and $\rho > 0$ are given by Hypothesis 1.1, and if u is a solution to (1.1) such that

$$\{u_0 \geq \theta\}_\rho \neq \emptyset, \quad d_{\mathcal{H}}\big(\operatorname{supp} u_0, \{u_0 \geq \theta\}_\rho\big) < +\infty \quad \text{and} \quad d_{\mathcal{H}}\big(\operatorname{supp} u_0, \{u_0 \geq \theta\}\big) < +\infty,$$

then, for any $\lambda \in (0,1)$,

$$d_{\mathcal{H}}\big(E_\lambda(t)\,,\, \operatorname{supp} u_0 + B_{c^*t}\big) = o(t) \text{ and } d_{\mathcal{H}}\big(E_\lambda(t)\,,\, \{u_0 \geq \theta\} + B_{c^*t}\big) = o(t) \quad \text{as} \quad t \to +\infty.$$

Indeed, $\mathcal{B}(\operatorname{supp} u_0) = \mathcal{B}\big(\{u_0 \geq \theta\}\big)$, $\mathcal{U}(\operatorname{supp} u_0) = \mathcal{U}\big(\{u_0 \geq \theta\}\big)$, and Lemma 3.1 holds with U replaced by $\{u_0 \geq \theta\}$, while Proposition 3.2 is kept unchanged.

3.2 Counter-examples

In this section, we show some counter-examples to Theorems 1.5 and 1.6 and to the formula (1.27) when the assumptions (1.20) or (1.25) are not satisfied. In all the counter-examples, we consider the function $f(s) = s(1-s)$ for $s \in [0,1]$. Hence, Hypothesis 1.1 (with any $\theta \in (0,1)$ and $\rho > 0$) and Hypothesis 1.3 hold, and the minimal speed of planar traveling fronts connecting 1 and 0 is equal to $c^* = 2$, see [2, 28].

Proposition 3.8. *Let u be the solution to (1.1) with $f(s) = s(1-s)$ and initial datum $u_0 = \mathbb{1}_U$, where*

$$U = \bigcup_{n \in \mathbb{N}} \overline{B_{2^n+1}} \setminus B_{2^n-1}.$$

The set U does not fulfill (1.20) for any $\rho > 0$, but fulfills (1.25) for any $\rho \leq 1$ (hence, (1.26) holds). Moreover, (1.14), (1.15), (1.21) and (1.22) all fail, for any function $w : \mathbb{S}^{N-1} \to [0, +\infty]$ and any open set $\mathcal{W} \subset \mathbb{R}^N$ which is star-shaped with respect to the origin, and both limits in (1.27) do not exist.

Proof. On the one hand, the intersection of U with any ray $\mathbb{R}^+ e$, $e \in \mathbb{S}^{N-1}$, is unbounded, hence $\mathcal{B}(U) = \emptyset$. On the other hand, for any $e \in \mathbb{S}^{N-1}$, the formula

$$\operatorname{dist}(3 \times 2^n e, U) = 2^n - 1 \tag{3.25}$$

shows that $\mathcal{U}(U) = \emptyset$ too. Therefore (1.20) is not satisfied. Let us check that formula (1.14) (hence the stronger one (1.15)) does not hold in any given direction $e \in \mathbb{S}^{N-1}$, with any $w(e) \in [0, +\infty]$. Indeed, on the one hand, by Lemma 3.1

$$\lim_{t \to +\infty} u(t, 2^n e) = 1 \quad \text{uniformly with respect to } n. \tag{3.26}$$

Thus, if (1.14) were satisfied for some $e \in \mathbb{S}^{N-1}$, one would necessarily have $w(e) = +\infty$. On the other hand, given $\lambda \in (0,1)$ and $c = 2c^*$, consider the family of functions $(v^T)_{T>0}$ and the associated $R > 0$ provided by Proposition 3.2. For any $n \in \mathbb{N}$ satisfying $n > \log_2(R+1)$, we call $T_n := (2^n - 1 - R)/(2c^*) > 0$ and deduce from the first property in (3.1) that

$$\forall\, |x| \geq 2^n - 1, \quad v^{T_n}(0,x) \geq 1,$$

and therefore, because of (3.25), $v^{T_n}(0,\cdot) \geq u_0(\cdot + 3 \times 2^n e)$ in $\mathbb{R}^N$, for every $e \in \mathbb{S}^{N-1}$. Thus, the comparison principle together with the second property in (3.1) entail

$$\forall\, t \leq T_n, \quad u(t, 3 \times 2^n e) \leq v^{T_n}(t,0) < \lambda < 1,$$

for every $e \in \mathbb{S}^{N-1}$. Calling $\tau_n := 2^{n-2}/c^*$, we have that $\tau_n < T_n$ for n large enough, hence we get

$$\limsup_{n\to+\infty} u(\tau_n, 12\, c^*\tau_n e) \le \lambda < 1, \tag{3.27}$$

for every $e \in \mathbb{S}^{N-1}$. Consequently, if (1.14) were satisfied for some $e \in \mathbb{S}^{N-1}$, one would necessarily have $w(e) \le 12\, c^*$, a contradiction with $w(e) = +\infty$. In conclusion, formula (1.14) and then formula (1.15) do not hold in any direction $e \in \mathbb{S}^{N-1}$, for any $w(e) \in [0, +\infty]$.

The set $\mathcal{W}$ given by (1.19) with $w(e)$ as in (1.16)-(1.17) is actually equal to B_{c^*}. We will see that (1.21) and (1.22) fail with $\mathcal{W} = B_{c^*}$, as well as with any open set $\mathcal{W}$ which is star-shaped with respect to the origin. So assume now by way of contradiction that there exists an open set $\mathcal{W} \subset \mathbb{R}^N$ which is star-shaped with respect tot the origin and for which either (1.21) or (1.22) hold. Because of (3.27), the first condition in (1.21) in one case, or (1.22) in the other case, imply that $\{12\, c^*e\} \notin \mathcal{W}$, for any $e \in \mathbb{S}^{N-1}$. Hence, being star-shaped, $\mathcal{W}$ satisfies $\mathcal{W} \subset B_{12\, c^*}$. But we also know that, by (3.26), $u(2^n/\sigma, 2^n e) \to 1$ as $n \to +\infty$ for any $\sigma > 0$. Taking $\sigma > 12\, c^*$, the second line of (1.21) is violated by $C = \{\sigma e\}$ and moreover, for given $\lambda \in (0,1)$, $d_{\mathcal{H}}(\{\sigma e\} \cap \frac{1}{t}\, E_\lambda(t)\,,\, \{\sigma e\} \cap \mathcal{W}) = +\infty$ for $t = 2^n/\sigma$ and n large enough (depending on λ), that is, (1.22) fails too. We have reached a contradiction in both cases.

Finally, for $n \in \mathbb{N}$ and $e \in \mathbb{S}^{N-1}$, calling $t_n := 2^{n-1}/c^*$, we rewrite (3.25) as

$$\mathrm{dist}(6c^*e\, t_n, U) = 2c^*\, t_n - 1.$$

We deduce that

$$B_{c^*-1/t_n}(6c^*e) \subset \frac{1}{t_n}\left\{x \in \mathbb{R}^N : \mathrm{dist}(x, U) > c^*\, t_n\right\}$$

and therefore if $d_{\mathcal{H}}(t^{-1}U + B_{c^*}, \mathcal{W}') \to 0$ as $t \to +\infty$, for some set $\mathcal{W}'$, then necessarily $6c^*e \notin \overline{\mathcal{W}'}$. But we see that, for $s_n := 2^n/(6c^*)$, there holds that

$$6c^*e \in \frac{1}{s_n}U,$$

and thus $d_{\mathcal{H}}(t^{-1}U + B_{c^*}, \mathcal{W}') \to 0$ as $t \to +\infty$ would imply $6c^*e \in \overline{\mathcal{W}'}$. This shows that the second limit in (1.27) does not exist, whence the first limit does not exist either, thanks to (1.26) (notice also that (1.25) is satisfied for any $\rho \in (0,1]$, hence (1.26) holds thanks to Theorem 1.6). □

The second counter-example is the counterpart of Proposition 3.8, with a set U fulfilling (1.20) but not (1.25).

Proposition 3.9. *Let u be the solution to* (1.1) *with $f(s) = s(1-s)$ and initial datum $u_0 = \mathbb{1}_U$, where $U = U_1 \cup U_2$ and*

$$\begin{cases} U_1 = \{x \in \mathbb{R}^N : x_1 \ge 0 \text{ and } x_2^2 + \cdots + x_N^2 \le 1\}, \\ U_2 = \{x \in \mathbb{R}^N : x_1 \ge 0 \text{ and } (x_2 - \sqrt{x_1})^2 + x_3^2 + \cdots + x_N^2 \le e^{-x_1^2}\}. \end{cases} \tag{3.28}$$

The set U does not fulfill (1.25) *for any $\rho > 0$, but fulfills* (1.20) *for $0 < \rho \le 1$ (hence,* (1.14), (1.15), (1.21), (1.22) *hold). Moreover,* (1.26) *fails and the first limit in* (1.27) *exists and is equal to $\mathcal{W}$, whereas the second one does not exist.*

Proof. First of all, it is immediate to see that $\mathcal{U}(U) = \mathcal{U}(U_\rho) = \{\mathrm{e}_1\}$ for any $0 < \rho \le 1$, with $\mathrm{e}_1 = (1, 0, \cdots, 0)$, while $\mathcal{B}(U) = \mathbb{S}^{N-1} \setminus \{\mathrm{e}_1\}$. In particular, the assumption (1.20) is fulfilled for any $0 < \rho \le 1$. Observe also that (1.25) is not fulfilled, for any $\rho > 0$. The set $\mathcal{W}$ given by the equivalent formulas (1.19) and (1.24) is equal to the rounded half-cylinder $\mathcal{W} = \mathbb{R}^+\mathrm{e}_1 + B_{c^*}$. It is not hard to see that the second limit in (1.27) does not exist, owing to the presence of U_2 in the definition of U.

It turns out that the presence of U_2 in the definition of U does not affect the asymptotic of $E_\lambda(t)$ as $t \to +\infty$. To see this we observe that, since the function f vanishes at 0 and 1 and is concave, the maximum principle yields

$$0 \le \max(v_1, v_2) \le u \le \min(v_1 + v_2, 1) \ \text{ in } [0, +\infty) \times \mathbb{R}^N, \tag{3.29}$$

where v_i solves (1.1) with initial condition $v_i(0, \cdot) = \mathbb{1}_{U_i}$, for $i = 1, 2$. Let us call $E^i_\lambda(t) := \{x \in \mathbb{R}^N : v_i(t, x) > \lambda\}$. Using the comparison with the linearized equation $\partial_t w = \Delta w + w$ and the explicit solution for the latter, one can check that $v_2(1, x)$ has a Gaussian decay for $|x| \to +\infty$. It then follows from the standard theory that properties (1.12)-(1.13) hold for v_2, hence $d_{\mathcal{H}}(t^{-1}E^2_\lambda(t), B_{c^*}) \to 0$ as $t \to +\infty$ for any $\lambda \in (0, 1)$. On the other hand, the set U_1 given in (3.28) fulfills both (1.20) and (1.25) with $0 < \rho \le 1$, hence the conclusions of Theorems 1.5 and 1.6 hold for v_1. In particular, $t^{-1}d_{\mathcal{H}}(E^1_\lambda(t), U_1 + B_{c^*t}) \to 0$ as $t \to +\infty$ for any $\lambda \in (0, 1)$. Together with (3.29), one infers that, for any $\lambda \in (0, 1)$,

$$\frac{1}{t}\, d_{\mathcal{H}}(E_\lambda(t), U_1 + B_{c^*t}) \to 0 \ \text{ as } t \to +\infty. \tag{3.30}$$

As a consequence, $d_{\mathcal{H}}(t^{-1}E_\lambda(t), \mathcal{W}) \to 0$ as $t \to +\infty$ for any $\lambda \in (0, 1)$ (that is, the first limit in (1.27) exists and is equal to $\mathcal{W}$). But since $d_{\mathcal{H}}(U + B_{c^*t}, U_1 + B_{c^*t}) = +\infty$ for all $t > 0$, (3.30) also implies that, for any fixed $\lambda \in (0, 1)$, $d_{\mathcal{H}}(E_\lambda(t), U + B_{c^*t}) = +\infty$ for all t large enough, hence (1.26) fails. □

We now exhibit an example where all the conclusions of Theorems 1.5 and 1.6 fail and moreover the two limits in (1.27) exist but they do not coincide.

Proposition 3.10. *Let u be the solution to* (1.1) *with $f(s) = u(1 - s)$ and initial datum $u_0 = \mathbb{1}_U$, where*

$$U = \big\{x \in \mathbb{R}^N : |x_N| \le e^{-|x'|^2}\big\},$$

which does not fulfill (1.20) *or* (1.25)*, for any $\rho > 0$. Then* (1.14)*,* (1.15)*,* (1.21)*,* (1.22) *and* (1.26) *all fail with $w(e)$ and $\mathcal{W}$ given by* (1.16)-(1.17) *and* (1.19)*, and the two limits in* (1.27) *exist but do not coincide.*

Proof. We have that $\mathcal{B}(U) = \{e \in \mathbb{S}^{N-1} : e_N \ne 0\}$ and that $\mathcal{U}(U) = \{e \in \mathbb{S}^{N-1} : e_N = 0\}$ and $\mathcal{U}(U_\rho) = \emptyset$ for any $\rho > 0$. Hence $\mathcal{B}(U) \cup \mathcal{U}(U_\rho) \ne \mathbb{S}^{N-1}$. The set $\mathcal{W}$ defined in the equivalent formulations (1.19) and (1.24) is given by the slab

$$\mathcal{W} = \big\{x \in \mathbb{R}^N : |x_N| < c^*\big\},$$

and it is readily seen that

$$d_{\mathcal{H}}(t^{-1}U + B_{c^*}, \mathcal{W}) \to 0 \ \text{ as } t \to +\infty.$$

However, as for the function v_2 in the proof of Proposition 3.9, one has

$$d_{\mathcal{H}}(t^{-1}E_\lambda(t), B_{c^*}) \to 0 \ \text{ as } t \to +\infty, \tag{3.31}$$

for any $\lambda \in (0,1)$. Namely, the first limit in (1.27) exists and coincides with B_{c^*}, and then it is not equal to $\mathcal{W}$ (in the sense of the Hausdorff distance). We further deduce from (3.31) that (1.21) and (1.22) fail (just taking $C = K = \overline{B_c} \cap \{x \in \mathbb{R}^N : x_N = 0\}$ with $c > c^*$), as well as (1.26), because $d_{\mathcal{H}}(B_{c^*}, t^{-1}U + B_{c^*}) = +\infty$ for all $t > 0$. Lastly, (3.31) implies that the first lines of (1.14) and (1.15) do not hold (because $w(e)$ given by (1.16) satisfies $w(e) = +\infty$ for any $e \in \mathbb{S}^{N-1}$ with $e_N = 0$). □

Remark 3.11. The example given in Proposition 3.10 further reveals that condition (1.20) cannot be relaxed by replacing $\mathcal{U}(U_\rho)$ with $\mathcal{U}(U)$, and moreover that, without (1.20), formulas (1.14) and (1.15) can hold with some $w(e)$ which is not given by (1.16).

We conclude this section by showing that (1.27) may fail even when the hypotheses of Theorems 1.5 and 1.6 are fulfilled.

Proposition 3.12. *Let u be the solution to* (1.1) *with $f(s) = s(1-s)$ and initial datum $u_0 = \mathbb{1}_U$, where*

$$U = \big\{x \in \mathbb{R}^N : x_N \le \sqrt{|x'|}\big\},$$

which fulfills both (1.20) *and* (1.25) *for any $\rho > 0$ (hence* (1.14), (1.15), (1.21), (1.22) *and* (1.26) *all hold). Then both limits in* (1.27) *do not exist and there holds that*

$$\forall\, \lambda \in (0,1),\ \forall\, t > 0, \quad d_{\mathcal{H}}\Big(\frac{1}{t}\, E_\lambda(t)\,,\, \mathcal{W}\Big) = +\infty.$$

Proof. It is immediate to see that $\mathcal{U}(U) = \mathcal{U}(U_\rho) = \{e \in \mathbb{S}^{N-1} : e_N \le 0\}$ for any $\rho > 0$, and that $\mathcal{B}(U) = \{e \in \mathbb{S}^{N-1} : e_N > 0\}$, whence (1.20) holds. It is also clear that (1.25) holds. We see from (1.24) that $\mathcal{W} = \{x \in \mathbb{R}^N : x_N < c^*\}$.

Next, the functions

$$(t,x) \mapsto u\Big(t, x + n\mathrm{e}_1 + \frac{\sqrt{n}}{2}\mathrm{e}_N\Big)$$

converge, as $n \to +\infty$, to the constant solution $\widetilde{u}(t,x) \equiv 1$, locally uniformly in $t \ge 0$, $x \in \mathbb{R}^N$. This shows that $d_{\mathcal{H}}(t^{-1}E_\lambda(t), \mathcal{W}) = +\infty$, for any $\lambda \in (0,1)$ and $t > 0$. But then the limit $\lim_{t\to+\infty} t^{-1}E_\lambda(t)$ cannot exist, because if it does, it must coincide with $\mathcal{W}$ (in the sense of the Hausdorff distance) due to (1.22). Then the limit $\lim_{t\to+\infty} t^{-1}U + B_{c^*}$ does not exist either, owing to (1.26). □

4 The subgraph case: proofs of Theorems 1.7 and 1.9, and Proposition 1.11

Section 4.1 is devoted to the proof of Theorem 1.7 about the flatness property of the level sets of solutions at large time if the initial support is below a graph which is not coercive at infinity. Section 4.2 contains the proofs of other flatness results and weaker versions of Conjecture 1.8. In Sections 4.3 and 4.4, we respectively show Theorem 1.9 on the case of conical initial support, and Proposition 1.11 on the counterexample to the global flatness of the level sets even if the initial support is asymptotically flat.

4.1 Proof of Theorem 1.7

We start with two auxiliary lemmas that will be used in the proof of Theorem 1.7 and in Sections 4.2 and 5.2 below.

Lemma 4.1. *Assume that Hypothesis* 1.3 *holds* (*hence Hypothesis* 1.1 *as well*). *Let* $c^* > 0$ *be the minimal speed given by Proposition* 1.4, *and let* u *be a solution of* (1.1) *with an initial datum* u_0 *given by* (1.28) *with* γ *satisfying* (1.30). *Then, for every* $\lambda \in (0,1)$ *and every* $x'_0 \in \mathbb{R}^{N-1}$, *there holds that*

$$X_\lambda(t, x'_0) = c^* t + o(t) \quad as \ \ t \to +\infty, \tag{4.1}$$

and moreover

$$\forall\, \alpha > 0, \qquad \max_{x' \in \overline{B'_{\alpha t}(x'_0)}} X_\lambda(t, x') \le c^* t + o(t) \quad as \ \ t \to +\infty. \tag{4.2}$$

Proof. Since hypothesis (1.30) on the initial datum u_0 is invariant by translation of the coordinate system of $\mathbb{R}^{N-1}$, we can restrict to the case $x'_0 = 0$. Fix $\lambda \in (0,1)$. Because u_0 is given by (1.28) with $\gamma \in L^\infty_{loc}(\mathbb{R}^{N-1})$, there is $x_0 \in \mathbb{R}^N$ such that $1 \ge u_0 \ge \mathbb{1}_{B_\rho(x_0)}$ in $\mathbb{R}^N$, with $\rho > 0$ given by Hypothesis 1.1. Property (1.12) of Proposition 1.4 and the monotonicity of $u(t,x)$ with respect to x_N then imply that

$$\liminf_{t \to +\infty} \frac{X_\lambda(t,0)}{t} \ge c^*.$$

It remains to show (4.2). Together with the previous formula, (4.2) will then yield (4.1). To show (4.2), we will make use of Remark 3.5. One readily checks that (1.30) implies that, for the set $U = \operatorname{supp} u_0$, there holds that

$$\frac{\operatorname{dist}(\tau \mathrm{e}_N, U)}{\tau} \to 1 \ \ \text{as } \tau \to +\infty,$$

and $\{e \in \mathbb{S}^{N-1} : e \cdot \mathrm{e}_N > 0\} \subset \mathcal{B}(U)$. It follows that $w(\mathrm{e}_N)$ given by (1.16)-(1.17) is equal to c^*, hence Remark 3.5 entails that

$$\sup_{x \in \mathcal{C}_w} u(t, tx) \to 0 \quad \text{as } \ t \to +\infty \tag{4.3}$$

holds for any $w > c^*$, where

$$\mathcal{C}_w := \bigcup_{\tau > 1} B_{c^*(\tau-1)}(\tau w \mathrm{e}_N).$$

Take $\alpha > 0$ and any $c > c^*$. For given $\tau > 1$, we call

$$w_\tau := c^* + \frac{c - c^*}{2\tau},$$

and we compute, for $x' \in \overline{B'_\alpha}$,

$$\begin{aligned} |(x', c) - \tau w_\tau \mathrm{e}_N|^2 &\le \alpha^2 + \Big(\tau c^* - \frac{c + c^*}{2}\Big)^2 = \alpha^2 + \Big(c^*(\tau - 1) - \frac{c - c^*}{2}\Big)^2 \\ &= \alpha^2 + \big(c^*(\tau-1)\big)^2 + \Big(\frac{c - c^*}{2}\Big)^2 - (c - c^*)c^*(\tau - 1). \end{aligned}$$

We can then choose $\tau > 1$ large enough in such a way that $|(x', c) - \tau w_\tau \mathrm{e}_N| < c^*(\tau - 1)$ for all $x' \in \overline{B'_\alpha}$, and therefore $\overline{B'_\alpha} \times \{c\} \subset \mathcal{C}_{w_\tau}$. We deduce from (4.3) that

$$\max_{x' \in \overline{B'_\alpha}} u(t, tx', ct) \to 0 \quad \text{as} \quad t \to +\infty,$$

whence $\max_{x' \in \overline{B'_{\alpha t}}} X_\lambda(t, x') \le ct$ for t sufficiently large. Property (4.2) then follows by the arbitrariness of $c > c^*$. □

Lemma 4.2. *Assume that Hypothesis* 1.3 *holds, hence Hypothesis* 1.1 *as well, for some* $\theta \in (0,1)$ *and* $\rho > 0$. *Let* u *be a solution of* (1.1) *with an initial datum* u_0 *given by* (1.28) *with* γ *satisfying* (1.30), *and let* $E_\theta(t)$ *be the upper level set* $\{x \in \mathbb{R}^N : u(t,x) > \theta\}$ *and* $(X_\lambda)_{\lambda \in (0,1)}$ *be the functions given by* (1.29). *Then, for every* $\lambda \in (0,1)$ *and* $\omega > 0$, *there exists* $\bar{R} > 0$ *such that*

$$\forall\, x'_0 \in \mathbb{R}^{N-1}, \quad \liminf_{t \to +\infty} \left(\sup_{x' \in \partial B'_{\bar{R}}(x'_0)} \mathrm{dist}\left((x', X_\lambda(t, x'_0) + \omega \bar{R}), \mathbb{R}^N \setminus E_\theta(t) \right) \right) \le \rho.$$

Proof. Fix a real number c such that

$$\frac{c^*}{\sqrt{1+\omega^2}} < c < c^*. \tag{4.4}$$

Let v be the solution of (1.1) with initial condition $v_0 := \theta \mathbb{1}_{B_\rho}$. By Proposition 1.4, the function v spreads with the speed c^*. In particular, we can find $T > 0$ such that

$$\min_{|x| \le cT} v(T, x) \ge \lambda. \tag{4.5}$$

Call

$$\bar{R} := \frac{\omega}{\sqrt{1+\omega^2}}\, cT.$$

For all $y' \in \mathbb{R}^{N-1}$ such that $|y'| = \bar{R}$, we compute

$$\left| (0, cT\sqrt{1+\omega^2}) - (y', \omega \bar{R}) \right| = cT \sqrt{\frac{\omega^2}{1+\omega^2} + \left(\sqrt{1+\omega^2} - \frac{\omega^2}{\sqrt{1+\omega^2}} \right)^2} = cT.$$

It follows that

$$v\big(T, (0, cT\sqrt{1+\omega^2}) - (y', \omega \bar{R})\big) \ge \lambda. \tag{4.6}$$

We now use the contradictory assumption. Namely, there exist $x'_0 \in \mathbb{R}^{N-1}$ and $\tau > 0$ such that

$$\forall\, t \ge \tau, \qquad \sup_{x' \in \partial B'_{\bar{R}}(x'_0)} \mathrm{dist}\left((x', X_\lambda(t, x'_0) + \omega \bar{R}), \mathbb{R}^N \setminus E_\theta(t) \right) > \rho.$$

Because condition (1.30) is invariant by translation of the coordinate system of $\mathbb{R}^{N-1}$, we can assume without loss of generality that $x'_0 = 0$. Namely, for any $t \ge \tau$, there exists a point $y'_t \in \mathbb{R}^{N-1}$ with $|y'_t| = \bar{R}$ such that

$$u(t,x) > \theta \quad \text{for all } x \in B_\rho(y'_t, X_\lambda(t,0) + \omega \bar{R}).$$

This means that, for all $t \geq \tau$,

$$u\big(t, x + (0, X_\lambda(t,0))\big) \geq \theta \mathbb{1}_{B_\rho(y'_t, \omega\bar{R})}(x) = v_0(x - (y', \omega\bar{R})),$$

hence, by comparison, thanks to (4.6) one infers

$$u\big(t+T, (0, cT\sqrt{1+\omega^2}) + (0, X_\lambda(t,0))\big) \geq \lambda.$$

We have thereby shown that

$$\forall\, t \geq \tau, \quad X_\lambda(t+T,0) \geq X_\lambda(t,0) + cT\sqrt{1+\omega^2},$$

hence, by iteration,

$$\forall\, n \in \mathbb{N}, \quad X_\lambda(\tau + nT, 0) \geq X_\lambda(\tau, 0) + cnT\sqrt{1+\omega^2}.$$

Therefore,

$$\limsup_{t\to+\infty} \frac{X_\lambda(t,0)}{t} \geq \limsup_{n\to+\infty} \frac{X_\lambda(\tau+nT,0)}{\tau+nT} \geq \sqrt{1+\omega^2}\, c,$$

which is larger than c^* by the choice of c. This is in contradiction with Lemma 4.1. □

Proof of Theorem 1.7. Throughout the proof, one assumes Hypothesis 1.3. Hence Hypothesis 1.1 is satisfied too, by Proposition 1.4. Let $\theta \in (0,1)$ and $\rho > 0$ be given by Hypothesis 1.1, and let $c^* > 0$ be given by Proposition 1.4. Let u be a solution to (1.1), with an initial condition u_0 given by (1.28), where $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$ satisfies (1.30). The functions $X_\lambda : (0,+\infty) \times \mathbb{R}^{N-1} \to \mathbb{R}$ are given by (1.29), for all $\lambda \in (0,1)$.

We will show (1.32), which yields (1.31). To show (1.32), we argue by way of contradiction. Namely, by assuming that (1.32) does not hold for some $\lambda \in [\theta, 1)$ and some basis $(\mathrm{e}'_1, \cdots, \mathrm{e}'_{N-1})$ of $\mathbb{R}^{N-1}$, one will show that $u(T_n, x_n) = \lambda$ and $u(T_n + \tau_n, \xi_n) \geq \lambda$ for some sequences of large times (T_n) and (τ_n) and points (x_n) and (ξ_n) of $\mathbb{R}^N$ with the same projections on $\mathbb{R}^{N-1}$ and such that the difference $(\xi_n - x_n) \cdot \mathrm{e}_N$ is large compared to $c^*\tau_n$. That will eventually lead to a spreading speed larger than c^* in the direction e_N, and then to a contradiction, thanks to Lemma 4.1.

Notice that the conclusion (1.32) could also be easily viewed as a consequence of Lemma 4.2 in dimension $N = 2$. The arguments used below in the general case $N \geq 2$ are actually more involved, and first require some notations.

Step 1: some notations. In the sequel, we fix a basis $(\mathrm{e}'_1, \cdots, \mathrm{e}'_{N-1})$ of $\mathbb{R}^{N-1}$. The desired property (1.32) is invariant by multiplying any vector e'_i by any factor $\alpha_i \in \mathbb{R}^*$. Therefore, without loss of generality, one can assume in the sequel that each vector e'_i has unit norm in $\mathbb{R}^{N-1}$, that is,

$$|\mathrm{e}'_i| = 1 \ \text{ for each } 1 \leq i \leq N-1.$$

Observe that, for any $\varepsilon = (\varepsilon_i)_{1\leq i\leq N-1} \in \{-1,1\}^{N-1}$, one can choose a point $y'_\varepsilon \in \mathbb{R}^{N-1}$ such that

$$\overline{B'_\rho(y'_\varepsilon)} \subset \Big\{ x' = \sum_{i=1}^{N-1} t_{i,\varepsilon,x'} \varepsilon_i \mathrm{e}'_i : t_{i,\varepsilon,x'} \in \mathbb{R}^+ \Big\},$$

where one recalls that the notation $B'_r(y')$ stands for the open Euclidean ball in $\mathbb{R}^{N-1}$ of center $y' \in \mathbb{R}^{N-1}$ and radius $r > 0$. In the above formula, for any $x' \in \mathbb{R}^{N-1}$ and any $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{N-1}) \in \{-1,1\}^{N-1}$, the real numbers $t_{1,\varepsilon,x'}, \ldots, t_{N-1,\varepsilon,x'}$ denote the

(unique) coordinates of x' in the basis $(\varepsilon_1 \mathrm{e}'_1, \cdots, \varepsilon_{N-1}\mathrm{e}'_{N-1})$. One then defines a positive real number ρ' by

$$\rho' = \max_{\varepsilon\in\{-1,1\}^{N-1},\, x'\in\overline{B'_\rho(y'_\varepsilon)},\, 1\le i\le N-1} t_{i,\varepsilon,x'}. \tag{4.7}$$

Step 2: the proof of (1.32). In addition to the basis $(\mathrm{e}'_1, \cdots, \mathrm{e}'_{N-1})$ of $\mathbb{R}^{N-1}$, we now fix any $\lambda \in [\theta, 1)$. Assume by way of contradiction that (1.32) does not hold. Since the quantities involved in (1.32) are nonnegative and nonincreasing with respect to $R > 0$, there exists then $\omega > 0$ such that

$$\forall\, R > 0, \quad \sup_{x'_0\in\mathbb{R}^{N-1}} \left[\liminf_{t\to+\infty} \Big(\min_{x'\in\overline{B'_R(x'_0)},\, 1\le i\le N-1} |\nabla_{x'} X_\lambda(t,x')\cdot \mathrm{e}'_i| \Big) \right] \ge 3\,\omega. \tag{4.8}$$

We now fix a real number c such that

$$\frac{c^*}{\sqrt{1+\omega^2}} < c < c^*. \tag{4.9}$$

Let v be the solution of (1.1) with initial condition $v_0 := \theta\, \mathbb{1}_{B_\rho}$. By Proposition 1.4, the function v spreads with the speed c^*. In particular, there is $T > 0$ such that

$$\min_{|x|\le ct} v(t,x) \ge \lambda \ \text{ for all } t \ge T. \tag{4.10}$$

Let us now consider any $n \in \mathbb{N}$ and apply (4.8) with $R = n + (N-1)\rho' > 0$, with $\rho' > 0$ given in (4.7). There is then a point $x'_n \in \mathbb{R}^{N-1}$ such that

$$\liminf_{t\to+\infty} \Big(\min_{x'\in\overline{B'_{n+(N-1)\rho'}(x'_n)},\, 1\le i\le N-1} |\nabla_{x'} X_\lambda(t,x')\cdot \mathrm{e}'_i| \Big) \ge 2\omega.$$

Since the function X_λ is at least of class C^1 in $(0,+\infty)\times\mathbb{R}^{N-1}$ from the implicit function theorem and the negativity of $\partial_{x_N} u$ in $(0,+\infty)\times\mathbb{R}^N$, it follows by continuity that there exist $T_n > 0$ and $\varepsilon_n = (\varepsilon_{n,i})_{1\le i\le N-1} \in \{-1,1\}^{N-1}$ such that

$$\nabla_{x'} X_\lambda(t,x')\cdot(\varepsilon_{n,i}\mathrm{e}'_i) \ge \omega \ \text{ for all } t \ge T_n,\ x' \in \overline{B'_{n+(N-1)\rho'}(x'_n)} \text{ and } 1 \le i \le N-1. \tag{4.11}$$

One then infers from the fundamental theorem of calculus and from the definitions of y'_{ε_n} and ρ' in Step 2, that

$$X_\lambda(T_n, x'_n + n\,\varepsilon_{n,1}\,\mathrm{e}'_1) \ge X_\lambda(T_n, x'_n) + \omega\, n$$

and then, for any $x' \in \overline{B'_\rho(y'_{\varepsilon_n})}$,

$$\begin{aligned} X_\lambda(T_n, x'_n + n\,\varepsilon_{n,1}\,\mathrm{e}'_1 + x') &\ge X_\lambda(T_n, x'_n + n\,\varepsilon_{n,1}\,\mathrm{e}'_1) + \sum_{i=1}^{N-1} \omega\, \underbrace{t_{i,\varepsilon_n,x'}}_{\ge 0} \\ &\ge X_\lambda(T_n, x'_n) + \omega\, n. \end{aligned} \tag{4.12}$$

Call

$$z'_n = x'_n + n\,\varepsilon_{n,1}\,\mathrm{e}'_1 + y'_{\varepsilon_n} \in \mathbb{R}^{N-1} \ \text{ and } z_n = \big(z'_n, X_\lambda(T_n, x'_n) + \omega\, n - \rho\big) \in \mathbb{R}^N. \tag{4.13}$$

For any $x = (x', x_N) \in \overline{B_\rho(z_n)}$, there holds $x' \in \overline{B'_\rho(z'_n)} = x'_n + n\,\varepsilon_{n,1}\,\mathrm{e}'_1 + \overline{B'_\rho(y'_{\varepsilon_n})}$ and $x_N \le X_\lambda(T_n, x'_n) + \omega\, n$, hence

$$X_\lambda(T_n, x') \ge X_\lambda(T_n, x'_n) + \omega\, n \ge x_N$$

by (4.12). From the definition (1.29) of X_λ and the fact that u is decreasing with respect to x_N in $(0, +\infty) \times \mathbb{R}^N$, one then infers that

$$u(T_n, \cdot) \ge \lambda \ge \theta \ \text{ in } \overline{B_\rho(z_n)}.$$

Hence, $u(T_n, \cdot) \ge v_0(\cdot - z_n)$ in $\mathbb{R}^N$, and

$$u(T_n + t, \cdot) \ge v(t, \cdot - z_n) \ \text{ in } \mathbb{R}^N \text{ for all } t > 0 \tag{4.14}$$

from the maximum principle.

In addition to (4.13), let us now introduce a few other notations, for each $n \in \mathbb{N}$. Call

$$x_n = (x'_n, X_\lambda(T_n, x'_n)) \in \mathbb{R}^N,\ \xi_n = \Big(x'_n, X_\lambda(T_n, x'_n) + |x_n - z_n|\,\frac{\sqrt{1+\omega^2}}{\omega}\Big) \in \mathbb{R}^N, \tag{4.15}$$

and

$$\tau_n = \frac{|\xi_n - z_n|}{c}.$$

Remember that the sequence $(|y'_{\varepsilon_n}|)_{n\in\mathbb{N}}$ takes only a finite number of values, and is therefore bounded. It is then easy to check from (4.13) and (4.15) that

$$|x_n - z_n| \sim n\sqrt{1+\omega^2},\ \ |x_n - \xi_n| \sim n\,\frac{1+\omega^2}{\omega},\ \ |\xi_n - z_n| \sim n\,\frac{\sqrt{1+\omega^2}}{\omega},\ \ \tau_n \sim n\,\frac{\sqrt{1+\omega^2}}{c\,\omega},$$

as $n \to +\infty$. In other words, the angle between the segments $[z_n, x_n]$ and $[z_n, \xi_n]$ is almost right, and then the angle between the segments $[x_n, z_n]$ and $[x_n, \xi_n]$ is almost $\arccos(\omega/\sqrt{1+\omega^2}) = \pi/2 - \arctan\omega$. As a consequence, $\tau_n \to +\infty$ as $n \to +\infty$, and

$$\frac{|x_n - z_n|}{\tau_n}\,\frac{\sqrt{1+\omega^2}}{\omega} \to c\sqrt{1+\omega^2} > c^* \ \text{ as } n \to +\infty,$$

by (4.9). We can then fix $n_0 \in \mathbb{N}$ such that

$$\tau_{n_0} \ge T \ \text{ and } \ \frac{|x_{n_0} - z_{n_0}|}{\tau_{n_0}}\,\frac{\sqrt{1+\omega^2}}{\omega} > c^*, \tag{4.16}$$

with $T > 0$ defined in (4.10).

Lastly, (4.10) and (4.14) yield

$$u\Big(T_{n_0} + \tau_{n_0}, x'_{n_0}, X_\lambda(T_{n_0}, x'_{n_0}) + |x_{n_0} - z_{n_0}|\,\frac{\sqrt{1+\omega^2}}{\omega}\Big) = u(T_{n_0} + \tau_{n_0}, \xi_{n_0}) \ge v(\tau_{n_0}, \xi_{n_0} - z_{n_0}) \ge \lambda,$$

hence $X_\lambda(T_{n_0} + \tau_{n_0}, x'_{n_0}) \ge X_\lambda(T_{n_0}, x'_{n_0}) + |x_{n_0} - z_{n_0}|\,\sqrt{1+\omega^2}/\omega$. Starting again from (4.11) (applied with $n = n_0$) and repeating the above arguments, one infers that

$$u\Big(T_{n_0} + 2\tau_{n_0}, x'_{n_0}, X_\lambda(T_{n_0}, x'_{n_0}) + 2\,|x_{n_0} - z_{n_0}|\,\frac{\sqrt{1+\omega^2}}{\omega}\Big) \ge \lambda$$

and $X_\lambda(T_{n_0}+2\tau_{n_0},x'_{n_0}) \geq X_\lambda(T_{n_0},x'_{n_0}) + 2\,|x_{n_0}-z_{n_0}|\,\sqrt{1+\omega^2}/\omega$. By an immediate induction, there holds

$$X_\lambda(T_{n_0}+k\tau_{n_0},x'_{n_0}) \geq X_\lambda(T_{n_0},x'_{n_0}) + k\,|x_{n_0}-z_{n_0}|\,\frac{\sqrt{1+\omega^2}}{\omega}$$

for all $k \in \mathbb{N}$. Therefore,

$$\limsup_{t\to+\infty} \frac{X_\lambda(t,x'_{n_0})}{t} \geq \frac{|x_{n_0}-z_{n_0}|}{\tau_{n_0}}\,\frac{\sqrt{1+\omega^2}}{\omega} > c^*$$

by (4.16). One has finally reached a contradiction with Lemma 4.1, and the proof of Theorem 1.7 is thereby complete. □

4.2 Weaker versions of Conjecture 1.8 and counterexamples

We here derive some counterexamples to the conclusions (1.31)-(1.32) of Theorem 1.7 when the non-coercivity assumption (1.30) is not fulfilled, as well as two weaker versions of Conjecture 1.8. The first one is a result which, compared to the conclusion (1.31) of Theorem 1.7, provides a refined upper bound for $\nabla_{x'}X_\lambda(x',t)$ for every sequence of times $t\to+\infty$, at the price of taking the minimum on sets of x' growing linearly in time.

Proposition 4.3. *Under the same assumptions and with u as in Theorem* 1.7*, for any $\lambda \in (0,1)$ and $\alpha>0$, there holds that*

$$\min_{|x'|\leq\alpha t} |\nabla_{x'}X_\lambda(t,x')| \;\longrightarrow\; 0 \quad \text{as } t\to+\infty. \tag{4.17}$$

Proof. Take $\lambda\in(0,1)$ and $\alpha>0$. Fix $\varepsilon>0$ and, for $t>0$, define the function $Y_t:\mathbb{R}^{N-1}\to\mathbb{R}$ by

$$Y_t(x') := X_\lambda(t,x') - \frac{\varepsilon}{t}|x'|^2.$$

It follows, on the one hand, that $Y_t(0) = c^*t+o(t)$ as $t\to+\infty$, thanks to Lemma 4.1. On the other hand, (4.2) yields, for $|x'|=\alpha t$, $Y_t(x') \leq (c^*-\varepsilon\alpha^2)t+o(t)$ as $t\to+\infty$. This shows that, for t large enough, depending on α and ε, Y_t has a local maximum at some ξ'_t with $|\xi'_t|<\alpha t$, and thus there holds that

$$|\nabla_{x'}X_\lambda(t,\xi'_t)| = 2\frac{\varepsilon}{t}|\xi'_t| < 2\alpha\varepsilon.$$

This concludes the proof by the arbitrariness of ε. □

The second weaker version of Conjecture 1.8, which nevertheless gives a more precise conclusion than the properties (1.31)-(1.32) of Theorem 1.7, is concerned with positive functions f of the type (1.5).

Proposition 4.4. *Assume that f satisfies* (1.5) *and let u be a solution of* (1.1) *with an initial datum u_0 given by* (1.28)*, where γ satisfies* (1.30)*. Then, for every $\lambda\in(0,1)$, there holds that*

$$\liminf_{t\to+\infty}\Big(\min_{|x'|\leq R}|\nabla_{x'}X_\lambda(t,x')|\Big) \;\longrightarrow\; 0 \quad \text{as } R\to+\infty,$$

and even

$$\sup_{x'_0\in\mathbb{R}^{N-1}}\Big[\liminf_{t\to+\infty}\Big(\min_{x'\in\overline{B'_R(x'_0)}}|\nabla_{x'}X_\lambda(t,x')|\Big)\Big] \;\longrightarrow\; 0 \quad \text{as } R\to+\infty. \tag{4.18}$$

Proof. Take $\lambda \in (0,1)$ and $\omega > 0$. Recall that condition (1.5) ensures the validity of Hypothesis 1.1 for any $\theta \in (0,1)$ and $\rho > 0$, see [2]. We take in particular $\theta = \lambda/2$ and $\rho > 0$ such that

$$\forall\, t \geq 1,\ \forall\, |x-y| \leq 2\rho, \quad |u(t,x) - u(t,y)| \leq \frac{\lambda}{2}, \tag{4.19}$$

which is a possible by interior parabolic estimates. Consider then the positive number $\bar{R}$ given by Lemma 4.2, associated with such $\theta = \lambda/2$ and $\rho > 0$, and also λ, ω. Take $x_0' \in \mathbb{R}^{N-1}$. Then by Lemma 4.2 there exists a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ diverging to $+\infty$ such that, for every $n \in \mathbb{N}$ and every $x' \in \partial B'_{\bar{R}}(x_0')$, we can find $y_n \in \mathbb{R}^N$ with the properties

$$|y_n - (x', X_\lambda(t_n, x_0') + \omega\bar{R})| \leq 2\rho \quad \text{and} \quad u(t_n, y_n) \leq \theta = \frac{\lambda}{2}.$$

It is not restrictive to assume that the $(t_n)_{n\in\mathbb{N}}$ are larger than 1, hence we derive from (4.19)

$$\forall\, n \in \mathbb{N},\ \forall\, x' \in \partial B'_{\bar{R}}(x_0'), \quad u(t_n, x', X_\lambda(t_n, x_0') + \omega\bar{R}) \leq \lambda,$$

that is,

$$\forall\, n \in \mathbb{N},\ \forall\, x' \in \partial B'_{\bar{R}}(x_0'), \quad X_\lambda(t_n, x') \leq X_\lambda(t_n, x_0') + \omega\bar{R}. \tag{4.20}$$

We now deduce from this a bound on $\nabla_{x'} X_\lambda(t_n, \cdot)$ at some point. Namely, for $n \in \mathbb{N}$, we consider the function $Y_n : \mathbb{R}^{N-1} \to \mathbb{R}$ defined by

$$Y_n(x') := X_\lambda(t_n, x') - \frac{\omega}{\bar{R}}|x' - x_0'|^2.$$

It follows from (4.20) that $Y_n(x_0') = X_\lambda(t_n, x_0') \geq \max_{\partial B'_{\bar{R}}(x_0')} Y_n$, hence the maximum of Y_n in $\overline{B'_{\bar{R}}(x_0')}$ is attained at some $\xi_n' \in B'_{\bar{R}}(x_0')$. We infer that

$$|\nabla_{x'} X_\lambda(t_n, \xi_n')| = 2\frac{\omega}{\bar{R}}|\xi_n' - x_0'| < 2\omega.$$

In the end, we have shown that, for any $x_0' \in \mathbb{R}^{N-1}$,

$$\liminf_{t\to+\infty} \Big(\min_{x' \in \overline{B'_{\bar{R}}(x_0')}} |\nabla_{x'} X_\lambda(t, x')| \Big) \leq 2\omega,$$

hence

$$\liminf_{t\to+\infty} \Big(\min_{x' \in \overline{B'_{R}(x_0')}} |\nabla_{x'} X_\lambda(t, x')| \Big) \leq 2\omega$$

for all $R \geq \bar{R}$. By the arbitrariness of $\omega > 0$, and recalling that $\bar{R}$ depends on λ and ω but not on x_0', we conclude that (4.18) holds. □

To complete this section, we list some counterexamples to Theorem 1.7 when the condition (1.30) is not satisfied. We start with the following immediate remark.

Remark 4.5. Without the assumption (1.30), the conclusions (1.31)-(1.32) of Theorem 1.7 immediately do not hold in general. For instance, if $\gamma(x') = x' \cdot e'$ for some nonzero vector $e' \in \mathbb{R}^{N-1}$, then by uniqueness the solution u of (1.1) with initial condition given by (1.28) only depends on the variables t and $x_N - x' \cdot e'$, and is still decreasing with respect to x_N in $(0,+\infty) \times \mathbb{R}^N$. Therefore, for every $t > 0$ and $\lambda \in (0,1)$, the level set $\{x \in \mathbb{R}^N : u(t,x) = \lambda\}$ is an affine hyperplane of $\mathbb{R}^N$ orthogonal to the vector $(e', -1)$, that is, $X_\lambda(t, x') = x' \cdot e' + c_{\lambda,t}$ for some $c_{\lambda,t} \in \mathbb{R}$, hence $\nabla_{x'} X_\lambda(t, x') = e'$ and (1.31)-(1.32) do not hold, for any basis $(\mathrm{e}_1', \cdots, \mathrm{e}_{N-1}')$ of $\mathbb{R}^{N-1}$ such that $\mathrm{e}_i' \cdot e' \neq 0$ for every $1 \leq i \leq N-1$.

In the following proposition, we show that the flatness properties (1.31)-(1.32), (1.34) and (1.36) of Theorems 1.7, 1.9 and Conjecture 1.8 do not hold in general when the assumptions (1.30) or (1.35) are modified, or do not hold uniformly as far as (1.34) and (1.36) are concerned.

Proposition 4.6. *The following properties hold:*

(i) *if one assumes that* $\liminf_{|x'|\to+\infty} \gamma(x')/|x'| \geq 0$ *instead of* (1.30)*, the conclusions* (1.31)-(1.32) *of Theorem* 1.7 *do not hold in general;*

(ii) *even for* x'*-symmetric solutions* u*, the conclusion* (1.34) *of Conjecture* 1.8 *does not hold in general without the assumption* (1.30)*;*

(iii) *even with the assumption* (1.30)*, the conclusion* (1.34) *of Conjecture* 1.8 *does not hold in general uniformly with respect to* $x' \in \mathbb{R}^{N-1}$*;*

(iv) *if* $\ell > 0$ *in condition* (1.35)*, then the conclusion* (1.36) *of Theorem* 1.9 *cannot be uniform with respect to* $x' \in \mathbb{R}^{N-1}$*.*

Proof. (i) To see it, consider for instance a bistable function f satisfying (1.7) with

$$f'(0) < 0, \quad f'(1) < 0 \ \text{ and } \ \int_0^1 f(s)ds > 0. \tag{4.21}$$

In that case, there is a unique up to shift decreasing function $\varphi : \mathbb{R} \to (0,1)$ and a unique speed $c^* > 0$ such that $\varphi(x - c^*t)$ is a traveling front connecting 1 to 0 for (1.1). Hence, Hypothesis 1.3 is fulfilled. Consider now (1.1) in dimension $N = 2$. For any angle $\beta \in (0, \pi/2)$, it is known that there is a V-shaped function $\phi : \mathbb{R}^2 \to (0,1)$ such that

$$\phi\Big(x_1, x_2 - \frac{c^*}{\sin\beta}\, t\Big)$$

is a traveling front solving (1.1), such that ϕ is even in x_1 and, for every $\lambda \in (0,1)$, there exists an even function $\gamma_\lambda \in C^1(\mathbb{R})$ for which there holds

$$\begin{cases} \big\{(x_1,x_2) \in \mathbb{R}^2 : \phi(x_1,x_2) = \lambda\big\} = \big\{(x_1,x_2) \in \mathbb{R}^2 : x_2 = \gamma_\lambda(x_1)\big\}, \\ \gamma_\lambda'(x_1) \to \pm\dfrac{1}{\tan\beta} \ \text{ as } x_1 \to \pm\infty, \\ \phi(x_1,x_2) \to 0 \ (\text{resp. } \to 1) \text{ as } x_2 - \gamma_\lambda(x_1) \to +\infty \ (\text{resp. as } x_2 - \gamma_\lambda(x_1) \to -\infty), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{uniformly in } x_1 \in \mathbb{R}, \end{cases} \tag{4.22}$$

see [23, 24, 39]. Moreover,

$$\sup_{a \leq \lambda \leq b,\, x_1 \in \mathbb{R}} \partial_{x_2}\phi(x_1, \gamma_\lambda(x_1)) < 0$$

for every $0 < a \leq b < 1$, and the function ϕ is decreasing in every direction $(\cos\omega, \sin\omega)$ with $|\omega - \pi/2| \leq \beta$. Consider now any angle $\vartheta \in (0, \beta)$, let $\mathcal{R}$ be the rotation of angle ϑ, and let u be the solution of (1.1) with initial condition (1.28) and γ defined by

$$\{(x_1,x_2) \in \mathbb{R}^2 : x_2 = \gamma(x_1)\} = \mathcal{R}\big(\{(x_1,x_2) \in \mathbb{R}^2 : x_2 = \gamma_{1/2}(x_1)\}\big).^8$$

[8] The function γ is well defined in $\mathbb{R}$ since ϕ is decreasing in the direction $(\cos(\pi/2 - \vartheta), \sin(\pi/2 - \vartheta))$.

Notice in particular that (1.30) is not fulfilled. Instead, one has

$$\liminf_{|x'|\to+\infty} \frac{\gamma(x')}{|x'|} > 0.$$

It follows from applications of some results of [46] that the solution U of (1.1) with initial condition $\mathbb{1}_{\{x_2\le\gamma_{1/2}(x_1)\}}$ satisfies

$$U(t,x_1,x_2) - \phi\Big(x_1, x_2 - \frac{c^*}{\sin\beta}\,t + a\Big) \to 0 \ \text{ as } t\to+\infty \text{ in } C^2(\mathbb{R}^2),$$

for some $a\in\mathbb{R}$. Since $u(t,x_1,x_2) = U(t,\mathcal{R}^{-1}(x_1,x_2))$ for all $t\ge0$ and $(x_1,x_2)\in\mathbb{R}^2$ and since $\gamma_\lambda'(x_1)\to\pm1/\tan\beta>0$ as $x_1\to\pm\infty$, one then infers that, for every $\lambda\in(0,1)$,

$$\partial_{x_1}X_\lambda(t,x_1) \to \frac{1}{\tan(\beta-\vartheta)} > 0 \ \text{ as } t\to+\infty, \text{ locally uniformly in } x_1\in\mathbb{R}.$$

In particular, properties (1.31)-(1.32) of Theorem 1.7 do not hold.

(ii) Consider again a function f of the bistable type (1.7) and (4.21) (hence, Hypothesis 1.3 is fulfilled), assume that $N=2$, fix $\beta\in(0,\pi/2)$ and let ϕ and γ_λ be as in (4.22). Then the solution u of (1.1) with initial condition (1.28) defined with, say, $\gamma=\gamma_{1/2}$ (hence, (1.30) is not fulfilled) is such that $u(t,x_1,x_2)-\phi(x_1,x_2-(c^*/\sin\beta)t+a)\to0$ as $t\to+\infty$ in $C^2(\mathbb{R}^2)$, for some $a\in\mathbb{R}$. As a consequence, $\partial_{x_1}X_\lambda(t,x_1)\to\gamma_\lambda'(x_1)$ as $t\to+\infty$, locally uniformly in $x_1\in\mathbb{R}$, for every $\lambda\in(0,1)$. Since $\gamma_\lambda'(x_1)\to\pm1/\tan\beta\neq0$ as $x_1\to\pm\infty$, property (1.34) of Conjecture 1.8 does not hold for all $x'=x_1\in\mathbb{R}$ (although of course it holds at $x_1=0$, and even $\partial_{x_1}X_\lambda(t,0)=0$ for all $t>0$, by even symmetry in x_1).

(iii)-(iv) Assuming Hypothesis 1.3, consider first equation (1.1) in dimension $N=2$, and let $\gamma:\mathbb{R}\to\mathbb{R}$ be a $C^1(\mathbb{R})$ nonpositive function (hence, (1.30) is satisfied) such that $\gamma(x_1)=-ax_1<0$ for all $x_1\ge1$, for some $a>0$. Let u be the solution of (1.1) with initial condition u_0 given by (1.28). From standard parabolic estimates, the functions

$$(t,x)\mapsto u(t,x_1+r,x_2-ar)$$

converge, as $r\to+\infty$, in $C^{1;2}_{loc}((0,+\infty)\times\mathbb{R}^2)$ to the unique solution u_∞ of (1.1) such that $u_\infty(0,x_1,x_2)=0$ if $x_2>-ax_1-b$ and $u_\infty(0,x_1,x_2)=1$ otherwise. By uniqueness, u_∞ is then a function of the variables t and x_2+ax_1 only, that is,

$$(1,-a)\cdot\nabla u_\infty(t,x_1,x_2)=0 \ \text{ for all } (t,x_1,x_2)\in(0,+\infty)\times\mathbb{R}^2.$$

Furthermore, u_∞ is decreasing with respect to the variable x_2+ax_1 in $(0,+\infty)\times\mathbb{R}^2$ (more precisely, $(a,1)\cdot\nabla u_\infty(t,x_1,x_2)<0$ in $(0,+\infty)\times\mathbb{R}^2$), and, for each $t\ge0$, $u_\infty(t,x)\to1$ as $x_2+ax_1\to-\infty$ and $u_\infty(t,x_1,x_2)\to0$ as $x_2+ax_1\to+\infty$. Therefore, for every $t>0$ and every $\lambda\in(0,1)$, the function $X_\lambda(t,\cdot)$ defined by (1.29) is such that $X_\lambda(t,x_1)+ax_1$ has a finite limit as $x_1\to+\infty$, and also

$$\partial_{x_1}X_\lambda(t,x_1)\to-a \ \text{ as } x_1\to+\infty.$$

Finally, for any $\lambda\in(0,1)$, $\partial_{x_1}X_\lambda(t,x')$ cannot converge to 0 as $t\to+\infty$ uniformly with respect to $x'\in\mathbb{R}$. The conclusion is the same if one just assumes that $\gamma'(x_1)\to-a<0$ as $x_1\to+\infty$, and it also holds in higher dimensions $N\ge2$ under similar assumptions on γ. In particular, if $\ell>0$ in condition (1.35), then the conclusion (1.36) of Theorem 1.9 is not uniform with respect to $x'\in\mathbb{R}^{N-1}$. □

4.3 Proof of Theorem 1.9

We start with the proof of (1.36) firstly under condition (1.35) if $N = 2$, secondly under condition (1.35) if $N \geq 3$, thirdly under the condition $\gamma(x')/|x'| \to -\infty$ as $|x'| \to +\infty$ in any dimension $N \geq 2$, fourthly if γ is nonincreasing with respect to $|x' - x'_0|$ for large $|x'|$ and for some $x'_0 \in \mathbb{R}^{N-1}$ in any dimension $N \geq 2$, and fifthly if γ has small derivatives with respect to $|x' - x'_0|$ as $|x' - x'_0| \to +\infty$. The main idea is to argue by way of contradiction and to compare the solution with its reflection with respect to a suitable hyperplane at time 0 and then at all positive times from the maximum principle. This will eventually contradict the Hopf lemma at a suitable point of this hyperplane. We finally derive (1.37) in any dimension $N \geq 2$, from (1.36). Throughout the proof, one assumes Hypothesis 1.1.

Step 1: property (1.36) in dimension $N = 2$ under condition (1.35). Assume by way of contradiction that (1.36) does not hold. Then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0, 1)$, a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers diverging to $+\infty$, and a bounded sequence $(x'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, such that $\sup_{n\in\mathbb{N}} \lambda_n < 1$ and $\inf_{n\in\mathbb{N}} |\partial_{x'} X_{\lambda_n}(t_n, x'_n)| > 0$. Up to extraction of a subsequence and changing the variable x' into $-x'$, it is not restrictive to assume that

$$\sup_{n\in\mathbb{N}} \partial_{x'} X_{\lambda_n}(t_n, x'_n) \leq -2\varepsilon$$

for some $\varepsilon > 0$. In the sequel, we denote y the variable x_2 and set $y_n := X_{\lambda_n}(t_n, x'_n)$ and $\sigma_n := \partial_{x'} X_{\lambda_n}(t_n, x'_n) < 0$. Since $u(t_n, x'_n, y_n) = \lambda_n$ is away from 1 and since $t_n \to +\infty$ as $n \to +\infty$, it follows from Hypothesis 1.1 and from the boundedness of $(x'_n)_{n\in\mathbb{N}}$ that $y_n \to +\infty$ as $n \to +\infty$. Notice that

$$(1, \sigma_n) \cdot \nabla u(t_n, x'_n, y_n) = (1, \partial_{x'} X_{\lambda_n}(t_n, x'_n)) \cdot \nabla u(t_n, x'_n, y_n) = 0$$

by definition of X_{λ_n}, and denote

$$(\alpha_n, \beta_n) := \frac{\nabla u(t_n, x'_n, y_n)}{|\nabla u(t_n, x'_n, y_n)|} = \frac{(\sigma_n, -1)}{\sqrt{1 + \sigma_n^2}}.$$

(β_n is negative since $\partial_{x_2} u < 0$ in $(0, +\infty) \times \mathbb{R}^2$, and then α_n is negative too since so is σ_n). One then has $\sigma_n = -\alpha_n/\beta_n$ and

$$0 < \varepsilon \leq -\frac{1}{2} \sup_{n\in\mathbb{N}} \sigma_n = \frac{1}{2} \inf_{n\in\mathbb{N}} \frac{\alpha_n}{\beta_n}. \tag{4.23}$$

We use now a reflection argument inspired by Jones [27]. For $n \in \mathbb{N}$, consider the line L_n passing through the point (x'_n, y_n) and directed as $\nabla u(t_n, x'_n, y_n)$. It is the graph of the function

$$x' \mapsto \rho_n(x') := \frac{\beta_n}{\alpha_n}(x' - x'_n) + y_n = -\frac{1}{\sigma_n}(x' - x'_n) + y_n.$$

Then, consider the half-plane given by its open subgraph:

$$\Omega_n := \big\{(x', y) \in \mathbb{R}^2 : y < \rho_n(x')\big\}.$$

The vector $(1, \sigma_n)$ is then an inward normal to Ω_n. Finally, let $\mathcal{R}_n$ denote the affine orthogonal reflection with respect to L_n, that is,

$$\mathcal{R}_n(x', y) = (x', y) - 2\Big[(x' - x'_n, y - y_n) \cdot (-\beta_n, \alpha_n)\Big](-\beta_n, \alpha_n).$$

We then define the function v_n in $[0,+\infty)\times\overline{\Omega_n}$ by

$$v_n(t,x',y) := u(t,\mathcal{R}_n(x',y)).$$

We claim that, for n large enough,

$$v_n(0,\cdot,\cdot)\le u_0 \ \text{ in } \overline{\Omega_n}.$$

To prove this, we need to check that if $(x',y)\in\overline{\Omega_n}$ is such that $\mathcal{R}_n(x',y)\in\operatorname{supp} u_0$, then necessarily $(x',y)\in\operatorname{supp} u_0$, which is equivalent to show that

$$\mathcal{R}_n(\operatorname{supp} u_0\setminus\Omega_n)\subset\operatorname{supp} u_0. \tag{4.24}$$

Since $(x_n')_{n\in\mathbb{N}}$ is bounded and $(y_n)_{n\in\mathbb{N}}$ diverges to $+\infty$, and since γ is locally bounded, we can assume without loss of generality that, for all $n\in\mathbb{N}$, $(x_n',y_n)\notin\operatorname{supp} u_0$. We set

$$\xi_n := \sup\big\{x'<x_n' : \gamma(x')\ge\rho_n(x')\big\} \quad\text{and}\quad \zeta_n := \inf\big\{x'>x_n' : \gamma(x')\ge\rho_n(x')\big\}.$$

If the above sets are empty we define $\xi_n=-\infty$, and $\zeta_n=+\infty$, respectively. Observe that the sequence of functions $(\rho_n)_{n\in\mathbb{N}}$ tends locally uniformly to $+\infty$, because $y_n\to+\infty$ and the sequences $(x_n')_{n\in\mathbb{N}}$ and $(\beta_n/\alpha_n)_{n\in\mathbb{N}}=(-1/\sigma_n)_{n\in\mathbb{N}}$ are bounded. Furthermore, γ is locally bounded, and at least continuous outside a compact interval. It follows that

$$\xi_n\to-\infty \ \text{ and } \zeta_n\to+\infty \quad\text{as}\quad n\to+\infty. \tag{4.25}$$

We have that $(\operatorname{supp} u_0\setminus\Omega_n)\cap\big((\xi_n,\zeta_n)\times\mathbb{R}\big)=\emptyset$ for all n large enough, hence for all n without loss of generality. By hypothesis (1.35), there exists $k>\sup_{n\in\mathbb{N}}|x_n'|+1$ such that γ is of class C^1 in $(-\infty,-k]\cup[k,+\infty)$, and

$$\gamma'\ge\ell-\varepsilon \text{ in } (-\infty,-k] \ \text{ and } \ \gamma'\ge-\ell-\varepsilon \text{ in } [k,+\infty). \tag{4.26}$$

Without loss of generality, we can assume that

$$\xi_n<-k<k<\zeta_n$$

for all n. We finally define

$$\begin{cases} K_n^1 := \mathcal{R}_n(\operatorname{supp} u_0\setminus\Omega_n)\cap\big((-\infty,-k)\times\mathbb{R}\big),\\ K_n^2 := \mathcal{R}_n(\operatorname{supp} u_0\setminus\Omega_n)\cap\big([-k,k]\times\mathbb{R}\big),\\ K_n^3 := \mathcal{R}_n(\operatorname{supp} u_0\setminus\Omega_n)\cap\big((k,+\infty)\times\mathbb{R}\big),\end{cases}$$

These sets are depicted in Figure 1. We show separately that they are contained in $\operatorname{supp} u_0$, for all n large enough. That will provide the desired property (4.24) for n large.

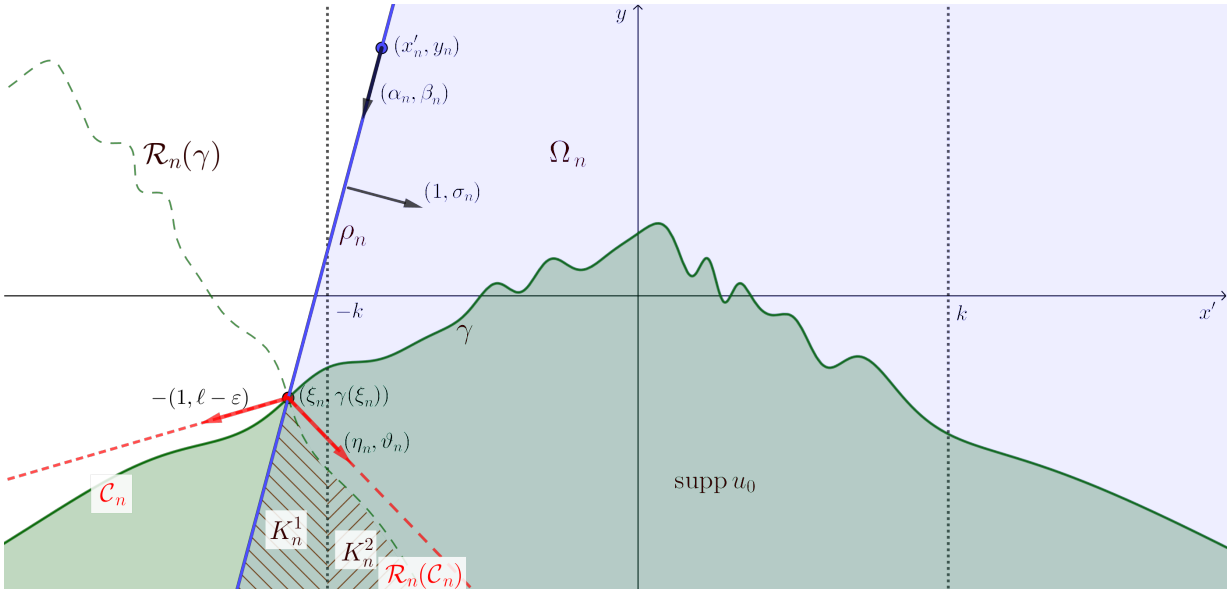

Figure 1: The reflection argument, with the sets K_n^1 and K_n^2.

The inclusion $K_n^1 \subset \operatorname{supp} u_0$. Consider a point in K_n^1. It can be written as $(x', y) + \tau(1, \sigma_n)$ with $(x', y) \in \operatorname{supp} u_0 \setminus \Omega_n$ and $0 \leq \tau < -x' - k$. Notice that $x' < -\tau - k \leq -k$, hence $y \leq \gamma(x')$. We write

$$\gamma(x' + \tau) = \gamma(x') + \int_0^\tau \gamma'(x' + s) ds.$$

Conditions (4.23) and (4.26) yield $\gamma'(x' + s) \geq \ell - \varepsilon > \sigma_n$ for $x' + s \leq -k$. We eventually deduce that $\gamma(x' + \tau) \geq \gamma(x') + \sigma_n \tau \geq y + \sigma_n \tau$. Since $x' + \tau < -k$, this implies that $(x', y) + \tau(1, \sigma_n) \in \operatorname{supp} u_0$.

The inclusion $K_n^2 \subset \operatorname{supp} u_0$ for n sufficiently large. In this case we consider a point of the type $(x', y) + \tau(1, \sigma_n)$ with $(x', y) \in \operatorname{supp} u_0 \setminus \Omega_n$ and $\tau \geq 0$ such that $-k \leq x' + \tau \leq k$. Since $x' \leq k - \tau \leq k$ and $(\operatorname{supp} u_0 \setminus \Omega_n) \cap \big((\xi_n, \zeta_n) \times \mathbb{R}\big) = \emptyset$ with $\xi_n < -k < k < \zeta_n$, we get that $x' \leq \xi_n < -k$. Moreover, by hypothesis, there exists $M > 0$ (independent of n, x', y and τ) such that $\gamma(s) \leq M + \varepsilon |s|$ for all $s \in \mathbb{R}$. As a consequence, using (4.23), we infer that

$$y + \tau\sigma_n \leq M - (\varepsilon + \sigma_n) x' - \sigma_n k \leq M - \sigma_n \left(\frac{x'}{2} + k \right) \leq M - \sigma_n \left(\frac{\xi_n}{2} + k \right).$$

The latter term tends to $-\infty$ as $n \to +\infty$ by (4.23) and (4.25). It follows that for n large enough (independent of x', y, τ) there holds that

$$y + \tau\sigma_n < \inf_{[-k,k]} \gamma - 1,$$

whence $(x', y) + \tau(1, \sigma_n) \in \operatorname{supp} u_0$. Therefore, $K_n^2 \subset \operatorname{supp} u_0$ for all n large enough, and even $K_n^2 \subsetneq \operatorname{supp} u_0$ (by that, we mean that the difference $\operatorname{supp} u_0 \setminus K_n^2$ contains a non-trivial ball).

The inclusion $K_n^3 \subset \operatorname{supp} u_0$ for n sufficiently large. We recall that $\xi_n < -k < k < \zeta_n$ and $(\operatorname{supp} u_0 \setminus \Omega_n) \cap \big((\xi_n, \zeta_n) \times \mathbb{R}\big) = \emptyset$ for all n. We can then divide this case in the following two subcases.

Subcase 1: the points in K_n^3 of the type $\mathcal{R}_n(x',y)$ with $(x',y) \in \operatorname{supp} u_0 \setminus \Omega_n$ and $x' \geq \zeta_n$ $(> k)$. If $\ell > 0$ then γ is bounded from above and such points do not exist for n sufficiently large, since they would satisfy $\rho_n(x') \leq y \leq \gamma(x')$, whereas the sequence of functions $(\rho_n)_{n\in\mathbb{N}}$ tends to $+\infty$ uniformly in any half-line $[A,+\infty)$. In the case $\ell = 0$, we write $\mathcal{R}_n(x',y) = (x',y) + \tau(1,\sigma_n)$ for some $\tau \geq 0$. Then, because $x' \geq \zeta_n > k$, we can argue as in the case of K_n^1 and, by virtue of (4.23) and (4.26), derive

$$\gamma(x'+\tau) \geq \gamma(x') - \varepsilon\tau \geq \gamma(x') + \sigma_n\tau \geq y + \sigma_n\tau,$$

that is, $(x',y) + \tau(1,\sigma_n) \subset \operatorname{supp} u_0$.

Subcase 2: the points in K_n^3 of the type $\mathcal{R}_n(x',y)$ with $(x',y) \in \operatorname{supp} u_0 \setminus \Omega_n$ and $x' \leq \xi_n$ $(< -k)$. Of course, these points exist only if $\xi_n > -\infty$. By definition of ξ_n, we see that $\rho_n'(\xi_n) \geq \gamma'(\xi_n)$. Then, it follows from (4.26) that

$$\frac{\beta_n}{\alpha_n} = \rho_n'(\xi_n) \geq \gamma'(\xi_n) \geq \inf_{(-\infty,\xi_n]} \gamma' \geq \ell - \varepsilon,$$

whence (x',y) is contained in the cone

$$\mathcal{C}_n := \big\{(\xi_n,\gamma(\xi_n)) + s(-1,-(\ell-\varepsilon)) + t(\alpha_n,\beta_n) : s,t \geq 0\big\},$$

see Figure 1. The point $\mathcal{R}_n(x',y)$ is contained in the reflected cone

$$\mathcal{R}_n(\mathcal{C}_n) = \big\{(\xi_n,\gamma(\xi_n)) + s(\eta_n,\vartheta_n) + t(\alpha_n,\beta_n) : s,t \geq 0\big\},$$

where

$$(\eta_n,\vartheta_n) = \widetilde{\mathcal{R}}_n(-1,-(\ell-\varepsilon)) = (-1,-(\ell-\varepsilon)) - 2\Big[(-1,-(\ell-\varepsilon))\cdot(-\beta_n,\alpha_n)\Big](-\beta_n,\alpha_n),$$

and $\widetilde{\mathcal{R}}_n$ denotes the linear orthogonal reflection with respect to the one-dimensional subspace $\mathbb{R}\,(\alpha_n,\beta_n)$. We see that

$$\eta_n = -1 + 2\beta_n\Big[(-1,-(\ell-\varepsilon))\cdot(-\beta_n,\alpha_n)\Big] = 1 - 2\alpha_n^2 - 2(\ell-\varepsilon)\alpha_n\beta_n \leq 1 - 2\alpha_n(\alpha_n - \varepsilon\beta_n),$$

which is not larger than 1 by (4.23) and the negativity of α_n and β_n. If $\eta_n \leq 0$ then $\mathcal{R}_n(\mathcal{C}_n) \subset (-\infty,\xi_n] \times \mathbb{R} \subset (-\infty,-k) \times \mathbb{R} \subset (-\infty,k] \times \mathbb{R}$, and therefore in this case $K_n^3 = \emptyset$ and we are done. Suppose that $\eta_n > 0$, i.e., that

$$(-1,-(\ell-\varepsilon))\cdot(-\beta_n,\alpha_n) < \frac{1}{2\beta_n}.$$

We deduce that

$$\vartheta_n = -\ell + \varepsilon - 2\alpha_n\Big[(-1,-(\ell-\varepsilon))\cdot(-\beta_n,\alpha_n)\Big] < -\ell + \varepsilon - \frac{\alpha_n}{\beta_n} \leq -\ell - \varepsilon,$$

always by (4.23). This means that $\vartheta_n/\eta_n \leq -\ell - \varepsilon$, whence

$$\mathcal{R}_n(\mathcal{C}_n) \subset \big\{(\xi_n,\gamma(\xi_n)) + s(1,-\ell-\varepsilon) + t(\alpha_n,\beta_n) : s,t \geq 0\big\}.$$

It eventually follows from (4.26) and from the fact that $\xi_n \to -\infty$ and $\gamma(\xi_n)/\xi_n \to -\ell \leq 0$ as $n \to +\infty$, that $\mathcal{R}_n(\mathcal{C}_n) \cap \big((k,+\infty)\times\mathbb{R}\big) \subset \operatorname{supp} u_0$ for n large enough, that is, $\mathcal{R}_n(x',y) \in \operatorname{supp} u_0$ in this last case too.

Conclusion. We have shown that $\mathcal{R}_n(\mathrm{supp}\, u_0 \setminus \Omega_n) \subset \mathrm{supp}\, u_0$, hence $v_n(0,\cdot,\cdot) \le u_0$ in $\overline{\Omega}_n$ for n sufficiently large, and actually that $u_0 - v_n(0,\cdot,\cdot) = 1$ in a non-trivial ball included in Ω_n, because $K_n^2 \subsetneq \mathrm{supp}\, u_0$. The functions v_n and u match on $\partial\Omega_n$. It then follows from the parabolic strong maximum principle that $v_n < u$ in $(0,+\infty)\times\Omega_n$, and from the Hopf lemma that, in particular,

$$\nabla u(t_n, x_n', y_n)\cdot(1,\sigma_n) > \nabla v_n(t_n, x_n', y_n)\cdot(1,\sigma_n) = \widetilde{\mathcal{R}}_n(\nabla u(t_n, x_n', y_n))\cdot(1,\sigma_n).$$

We have reached a contradiction because $\nabla u(t_n, x_n', y_n) = \widetilde{\mathcal{R}}_n(\nabla u(t_n, x_n', y_n))$ (the vector $\nabla u(t_n, x_n', y_n)$ is indeed parallel to (α_n, β_n)). As a consequence, (1.36) has been proved under condition (1.35) in dimension $N = 2$.

Step 2: common notations for the proof of (1.36) under conditions (i)-(iv) with $N \ge 2$. Assume any of the conditions (i)-(iv) of Theorem 1.9 and assume by way of contradiction that (1.36) does not hold. Then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,1)$, a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers diverging to $+\infty$, a bounded sequence $(x_n')_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$, and a sequence $(e_n')_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-2}$ (if $N = 2$, this means that $e_n' \in \{-1,1\}$), such that $\sup_{n\in\mathbb{N}} \lambda_n < 1$ and

$$\sup_{n\in\mathbb{N}} \underbrace{\nabla_{x'} X_{\lambda_n}(t_n, x_n')\cdot e_n'}_{=:\sigma_n} < 0. \tag{4.27}$$

Call

$$y_n = X_{\lambda_n}(t_n, x_n').$$

As in Step 1, one has $y_n \to +\infty$ as $n \to +\infty$. Notice that, for each $n \in \mathbb{N}$,

$$(e_n', \sigma_n)\cdot\nabla u(t_n, x_n', y_n) = 0 \tag{4.28}$$

by definition of X_{λ_n}, and denote H_n the affine hyperplane passing through the point (x_n', y_n) and orthogonal to (e_n', σ_n). This hyperplane is the graph of the function

$$x' \mapsto \rho_n(x') := -\frac{1}{\sigma_n}(x' - x_n')\cdot e_n' + y_n. \tag{4.29}$$

Then, consider the half-space given by its open subgraph:

$$\Omega_n := \big\{(x', x_N) \in \mathbb{R}^N : x_N < \rho_n(x')\big\}. \tag{4.30}$$

The vector (e_n', σ_n) is then an inward normal to Ω_n. Finally, let $\mathcal{R}_n$ denote the affine orthogonal reflection with respect to H_n, that is,

$$\mathcal{R}_n(x', x_N) = (x', x_N) - 2\Big[(x' - x_n', x_N - y_n)\cdot(e_n', \sigma_n)\Big]\,\frac{(e_n', \sigma_n)}{1+\sigma_n^2}. \tag{4.31}$$

We then define the function v_n in $[0,+\infty)\times\overline{\Omega_n}$ by

$$v_n(t, x', x_N) := u(t, \mathcal{R}_n(x', x_N)), \tag{4.32}$$

and we claim that, for n large enough, $v_n(0,\cdot,\cdot) \le u_0$ in $\overline{\Omega_n}$ and that $u_0 - v_n(0,\cdot,\cdot) = 1$ in a non-trivial ball. As in Step 1, this will then lead to a contradiction and complete the proof.

So, we just need to show that

$$\mathcal{R}_n(\operatorname{supp} u_0 \setminus \Omega_n) \subset \operatorname{supp} u_0 \tag{4.33}$$

and that $\operatorname{supp} u_0 \setminus \mathcal{R}_n(\operatorname{supp} u_0 \setminus \Omega_n)$ contains a non-trivial ball, for all n large enough.

To prove the latter, observe firstly that for any non-empty compact set $K \subset \mathbb{R}^N$, one has

$$\min_{(x',x_N)\in K} \left(x_N - 2\big[(x'-x'_n, x_N - y_n)\cdot(e'_n,\sigma_n)\big]\,\frac{\sigma_n}{1+\sigma_n^2}\right) \to +\infty \text{ as } n \to +\infty,$$

and

$$\liminf_{n\to+\infty} \min_{(x',x_N)\in K} \left(\frac{x_N - 2\big[(x'-x'_n, x_N - y_n)\cdot(e'_n,\sigma_n)\big]\,\frac{\sigma_n}{1+\sigma_n^2}}{\left|x' - 2\big[(x'-x'_n, x_N - y_n)\cdot(e'_n,\sigma_n)\big]\,\frac{e'_n}{1+\sigma_n^2}\right|}\right) \geq \liminf_{n\to+\infty} |\sigma_n| > 0,$$

since $y_n \to +\infty$, $\sup_{n\in\mathbb{N}} \sigma_n < 0$ and since the sequence $(x'_n)_{n\in\mathbb{N}}$ is bounded and $(e'_n)_{n\in\mathbb{N}}$ is unitary. But γ in (1.28) is always assumed to be locally bounded, and it is easy to see that

$$\limsup_{|x'|\to+\infty} \frac{\gamma(x')}{|x'|} \leq 0 \tag{4.34}$$

in all cases (i)-(iv) of Theorem 1.9. Therefore, owing to the definition (4.31) of $\mathcal{R}_n$, one gets that $\mathcal{R}_n(K) \cap \operatorname{supp} u_0 = \emptyset$ for all n large enough, that is,

$$K \cap \mathcal{R}_n(\operatorname{supp} u_0) = \emptyset, \quad \text{for all } n \text{ large enough.} \tag{4.35}$$

In particular, $\operatorname{supp} u_0 \setminus \mathcal{R}_n(\operatorname{supp} u_0 \setminus \Omega_n)$ contains a non-trivial ball for any n large enough.

Assume now by way of contradiction that (4.33) does not hold (for all n large enough). Then, up to extraction of a subsequence, there is a sequence of points $z_n = (z'_n, \varpi_n)$ in $\mathbb{R}^N$ such that

$$z_n \in \operatorname{supp} u_0 \setminus \Omega_n \text{ and } \mathcal{R}_n(z_n) \notin \operatorname{supp} u_0, \text{ for all } n \in \mathbb{N}.$$

Denote

$$\delta_n := \frac{(z'_n - x'_n, \varpi_n - y_n)\cdot(e'_n,\sigma_n)}{1+\sigma_n^2}, \tag{4.36}$$

that is,

$$\mathcal{R}_n(z_n) = (z'_n - 2\delta_n e'_n, \varpi_n - 2\delta_n\sigma_n). \tag{4.37}$$

Since $z_n \notin \Omega_n$, one has $\delta_n \leq 0$, and even

$$\delta_n < 0$$

(since otherwise z_n would lie on H_n and $\mathcal{R}_n(z_n)$, which does not belong to $\operatorname{supp} u_0$, would be equal to $z_n \in \operatorname{supp} u_0$). Since $y_n \to +\infty$ as $n \to +\infty$ and $\sup_{n\in\mathbb{N}} \sigma_n < 0$, together with the boundedness of the sequences $(x'_n)_{n\in\mathbb{N}}$ and $(e'_n)_{n\in\mathbb{N}}$, one infers that $\rho_n(x') \to +\infty$ as $n \to +\infty$ locally uniformly in $x' \in \mathbb{R}^{N-1}$. Since $z_n = (z'_n, \varpi_n) \in \operatorname{supp} u_0 \setminus \Omega_n$, it then follows from the local boundedness of γ and the definition (4.30) of Ω_n, that

$$|z'_n| \to +\infty \text{ as } n \to +\infty, \tag{4.38}$$

and, together with (4.34), that

$$\limsup_{n\to+\infty} \frac{\varpi_n}{|z'_n|} \le 0. \tag{4.39}$$

We also claim that

$$|z'_n - 2\delta_n e'_n| \to +\infty \ \text{ as } n \to +\infty. \tag{4.40}$$

Indeed, otherwise, up to extraction of a subsequence, the sequence $(|z'_n - 2\delta_n e'_n|)_{n\to+\infty}$ would be bounded, hence $\delta_n \to -\infty$ and $-2\delta_n \sim |z'_n|$ as $n \to +\infty$, since $|z'_n| \to +\infty$, $\delta_n < 0$ and $|e'_n| = 1$. Furthermore, since the points $\mathcal{R}(z_n)$ given in (4.37) do not belong to $\mathrm{supp}\, u_0$ and since γ is locally bounded, the sequence $(\varpi_n - 2\delta_n\sigma_n)_{n\in\mathbb{N}}$ would then be bounded from below, that is, there would exist $A \in \mathbb{R}$ such that $\varpi_n \ge 2\delta_n\sigma_n + A$ for all $n \in \mathbb{N}$. Finally, together with (4.27) and (4.38), one would have

$$\liminf_{n\to+\infty} \frac{\varpi_n}{|z'_n|} \ge \liminf_{n\to+\infty} \frac{2\delta_n\sigma_n}{|z'_n|} = -\limsup_{n\to+\infty} \sigma_n > 0,$$

a contradiction with (4.39). As a consequence, (4.40) has been proved.

Lastly, since $\mathcal{R}_n(z_n) = (z'_n - 2\delta_n e'_n, \varpi_n - 2\delta_n\sigma_n) \notin \mathrm{supp}\, u_0$ and γ is at least continuous outside a compact set in all cases (i)-(iv) of Theorem 1.9, one gets from (4.40) that

$$\varpi_n - 2\delta_n\sigma_n > \gamma(z'_n - 2\delta_n e'_n) \tag{4.41}$$

for all n large enough, and then for all n without loss of generality. Moreover, since $z_n = (z'_n, \varpi_n) \in \mathrm{supp}\, u_0 \backslash \Omega_n$, it follows from (4.30) and (4.38) that $\rho_n(z'_n) \le \varpi_n \le \gamma(z'_n)$ for all n large enough, and then for all n without loss of generality. Therefore,

$$\gamma(z'_n - 2\delta_n e'_n) - \gamma(z'_n) < -2\delta_n\sigma_n. \tag{4.42}$$

Step 3: property (1.36) in any dimension $N \ge 3$ under condition (1.35). Since the function γ is always locally bounded, the assumption (1.35) and the nonnegativity of ℓ then imply that γ is here globally bounded from above. With the notations of Step 2, define, for each $n \in \mathbb{N}$,

$$\xi_n := \sup\big\{x' \cdot e'_n : \gamma(x') \ge \rho_n(x')\big\}$$

(with the value $-\infty$ if the above set is empty). Since $y_n \to +\infty$ as $n \to +\infty$ and $\sup_{n\in\mathbb{N}} \sigma_n < 0$, together with the boundedness of the sequences $(x'_n)_{n\in\mathbb{N}}$ and $(e'_n)_{n\in\mathbb{N}}$, one infers that $\inf_{x'\cdot e'_n \ge A} \rho_n(x') \to +\infty$ for every $A \in \mathbb{R}$. Together with the boundedness from above of γ and the fact that it is at least continuous (and even C^1) in $\mathbb{R}^{N-1}\backslash B'_R$ for some $R > 0$, one gets that $\xi_n \to -\infty$ as $n \to +\infty$, and then that

$$\xi_n \le -R \ \text{ and } \ \mathrm{supp}\, u_0 \backslash \Omega_n \subset \big\{(x', x_N) \in \mathbb{R}^N : x' \cdot e'_n \le \xi_n\big\}$$

for all n, without loss of generality. In particular, since $z_n = (z'_n, \varpi_n) \in \mathrm{supp}\, u_0 \backslash \Omega_n$, one has $z'_n \cdot e'_n \le \xi_n \le -R$, and

$$z'_n \cdot e'_n \to -\infty \ \text{ as } n \to +\infty. \tag{4.43}$$

Furthermore, owing to the definition (4.36) of δ_n, one has

$$|z'_n|^2 - |z'_n - 2\delta_n e'_n|^2 = -4\delta_n(\delta_n - z'_n \cdot e'_n) = \frac{-4\delta_n}{1+\sigma_n^2}\Big(-\sigma_n^2(z'_n \cdot e'_n) - x'_n \cdot e'_n + \sigma_n(\varpi_n - y_n)\Big).$$

Since $\sup_{n\in\mathbb{N}}\sigma_n<0$, since $z'_n\cdot e'_n\to-\infty$, since the sequences $(x'_n)_{n\in\mathbb{N}}$ and $(e'_n)_{n\in\mathbb{N}}$ are bounded, since the sequence $(\varpi_n)_{n\in\mathbb{N}}$ is bounded from above (because γ is globally bounded from above and $z_n=(z'_n,\varpi_n)\in\mathrm{supp}\,u_0$), and since $y_n\to+\infty$, one infers that

$$-\sigma_n^2(z'_n\cdot e'_n)-x'_n\cdot e'_n+\sigma_n(\varpi_n-y_n)\to+\infty\ \text{ as } n\to+\infty.$$

Together with the negativity of δ_n, one gets that $|z'_n|^2-|z'_n-2\delta_n e'_n|^2>0$ for all n large enough, while $\lim_{n\to+\infty}|z'_n-2\delta_n e'_n|=+\infty$ by (4.40), hence

$$|z'_n|>|z'_n-2\delta_n e'_n|\geq R \tag{4.44}$$

for all n, without loss of generality.

Let us now complete the argument. Since γ is here assumed to be of class C^1 outside B'_R and since it satisfies (1.35) (use here the condition on the radial gradients at large $|x'|$ and the positivity of η), there is $M>0$ such that $\big|\gamma(x')+\ell|x'|\big|\leq M$ for all $|x'|\geq R$. Together with (4.42)-(4.44) and the nonnegativity of ℓ, it follows that

$$-2\delta_n\sigma_n>-\ell|z'_n-2\delta_n e'_n|-M+\ell|z'_n|-M\geq-2M$$

for all n. But $\delta_n<0$ and $\sup_{n\in\mathbb{N}}\sigma_n<0$. Thus, the sequence $(\delta_n)_{n\in\mathbb{N}}$ is bounded. Together with (4.38), that implies that $|z'_n-2s\delta_n e'_n|\geq R$ for all $s\in[0,1]$ and for all $n\in\mathbb{N}$, without loss of generality. Dividing (4.42) by $-2\delta_n>0$ and using the C^1 smoothness of γ outside B'_R, one then gets the existence of a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ in $(0,1)$ such that

$$\nabla\gamma(z'_n-2\vartheta_n\delta_n e'_n)\cdot e'_n<\sigma_n \tag{4.45}$$

for all $n\in\mathbb{N}$. Since the sequences $(\vartheta_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$ and $(e'_n)_{n\in\mathbb{N}}$ are bounded, one then infers from (1.35) and (4.38) that

$$\nabla\gamma(z'_n-2\vartheta_n\delta_n e'_n)\cdot e'_n=-\ell\,\frac{z'_n\cdot e'_n}{|z'_n-2\vartheta_n\delta_n e'_n|}+o(1)\ \text{ as } n\to+\infty,$$

hence $\liminf_{n\to+\infty}\nabla\gamma(z'_n-2\vartheta_n\delta_n e'_n)\cdot e'_n\geq0$ from (4.43) and the nonnegativity of ℓ. But this last formula contradicts (4.27) and (4.45).

One has then reached a contradiction, implying that the desired property (4.33) holds for all n large enough, while $\mathrm{supp}\,u_0\setminus\mathcal{R}_n(\mathrm{supp}\,u_0\setminus\Omega_n)$ contains a non-trivial ball, for all n large enough. Therefore, the functions v_n defined by (4.32) satisfy $v_n(0,\cdot,\cdot)\leq u_0$ in $\overline{\Omega_n}$ for n large enough, and $u_0-v_n(0,\cdot,\cdot)=1$ in a non-trivial ball included in Ω_n. The functions v_n and u match on $\partial\Omega_n$. It then follows from the parabolic strong maximum principle that $v_n<u$ in $(0,+\infty)\times\Omega_n$, and from the Hopf lemma that, in particular,

$$\nabla u(t_n,x'_n,y_n)\cdot(e'_n,\sigma_n)>\nabla v_n(t_n,x'_n,y_n)\cdot(e'_n,\sigma_n)=\widetilde{\mathcal{R}}_n(\nabla u(t_n,x'_n,y_n))\cdot(e'_n,\sigma_n), \tag{4.46}$$

where $\widetilde{\mathcal{R}}_n$ denotes the linear orthogonal reflection with respect to the linear hyperplane orthogonal to the vector (e'_n,σ_n). But the vector $\nabla u(t_n,x'_n,y_n)$ is orthogonal to (e'_n,σ_n) by (4.28), hence $\nabla u(t_n,x'_n,y_n)=\widetilde{\mathcal{R}}_n(\nabla u(t_n,x'_n,y_n))$. This finally contradicts (4.46). As a consequence, (1.36) has been proved under condition (1.35) in any dimension $N\geq3$.

Step 4: property (1.36) for any $N\geq2$ if $\gamma(x')/|x'|\to-\infty$ as $|x'|\to+\infty$. Arguing by way of contradiction as in Step 2, there exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $(0,1)$, a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers diverging to $+\infty$, a bounded sequence $(x'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$,

and a sequence $(e'_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^{N-2}$ such that $\sup_{n\in\mathbb{N}}\lambda_n < 1$ and $\sup_{n\in\mathbb{N}}\sigma_n < 0$, with $\sigma_n = \nabla_{x'}X_{\lambda_n}(t_n,x'_n)\cdot e'_n$. One still has $y_n := X_{\lambda_n}(t_n,x'_n)\to+\infty$ as $n\to+\infty$ and one can then define the functions ρ_n as in (4.29), the half-spaces Ω_n as in (4.30) and the affine orthogonal reflections $\mathcal{R}_n$ as in (4.31). Since $\sup_{n\in\mathbb{N}}\sigma_n<0$ and $y_n\to+\infty$, it then easily follows that, for all n large enough,

$$\mathrm{supp}\,u_0\subset\Omega_n,$$

hence (4.33) is automatically satisfied with $\mathcal{R}_n(\mathrm{supp}\,u_0\setminus\Omega_n)=\emptyset\subsetneq\mathrm{supp}\,u_0$ for all n large enough. One then concludes as in the last paragraph of Step 3.

Step 5: property (1.36) for any $N\geq 2$ if γ is nonincreasing in $|x'-x'_0|$. More precisely, let us assume here that there are $x'_0\in\mathbb{R}^{N-1}$ and a continuous nonincreasing function $\Gamma:\mathbb{R}^+\to\mathbb{R}$ such that $\gamma(x')=\Gamma(|x'-x'_0|)$ for all x' outside a compact set. Since the desired conclusion (1.36) is invariant by translation with respect to the first $N-1$ variables of $\mathbb{R}^N$, one can assume without loss of generality that $x'_0=0$ and that γ is continuous and nonincreasing with respect to $|x'|$ for $|x'|$ large enough. Since γ is locally bounded, it is then globally bounded from above. By using the same notations and repeating the same arguments as in Steps 2 and 3 above until (4.44) (as far as γ is concerned, the arguments until (4.44) only use the boundedness of γ from above), one gets (4.42)-(4.44). But both $|z'_n|$ and $|z'_n-2\delta_n e'_n|$ converge to $+\infty$ as $n\to+\infty$ by (4.38) and (4.40), and γ is nonincreasing with respect to $|x'|$ outside a compact set. Therefore, (4.44) implies that $\gamma(z'_n-2\delta_n e'_n)-\gamma(z'_n)\geq 0$ for all n large enough, contradicting (4.42) since both δ_n and σ_n are negative. One then concludes as in the last paragraph of Step 3.

Step 6: property (1.36) for any $N\geq 2$ if $\gamma=\Gamma(|\cdot-x'_0|)$ with $\Gamma'(+\infty)=0$. More precisely, let us assume here that there are $x'_0\in\mathbb{R}^{N-1}$ and a C^1 function $\Gamma:\mathbb{R}^+\to\mathbb{R}$ such that $\Gamma'(r)\to 0$ as $r\to+\infty$ and $\gamma(x')=\Gamma(|x'-x'_0|)$ for all x' outside a compact set. As in Step 5, one can assume without loss of generality that $x'_0=0$. With the same notations as in Step 2, both $|z'_n|$ and $|z'_n-2\delta_n e'_n|$ converge to $+\infty$ as $n\to+\infty$ by (4.38) and (4.40). Therefore, for every $\varepsilon>0$, there holds

$$|\gamma(z'_n-2\delta_n e'_n)-\gamma(z'_n)|=\big|\Gamma(|z'_n-2\delta_n e'_n|)-\Gamma(|z'_n|)\big|\leq\varepsilon\,\big||z'_n-2\delta_n e'_n|-|z'_n|\big|\leq 2\varepsilon|\delta_n|$$

for all n large enough (remember that $|e'_n|=1$). Since $\varepsilon>0$ can be arbitrarily small and since $\delta_n<0$ for all n and $\sup_{n\in\mathbb{N}}\sigma_n<0$ by (4.27), the above formula contradicts (4.42). One then concludes as in the last paragraph of Step 3.

Step 7: proof of property (1.37). We assume in this last step any of the assumptions (i)-(iv) of Theorem 1.9. Consider any bounded sequence $(x'_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^{N-1}$, any sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers diverging to $+\infty$, and any sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Two cases may occur, up to extraction of a subsequence.

On the one hand, if $\limsup_{n\to+\infty}u(t_n,x'_n,y_n)<1$, then $|\nabla_{x'}X_{u(t_n,x'_n,y_n)}(t_n,x'_n)|\to 0$ as $n\to+\infty$ from (1.36), hence

$$|\nabla_{x'}u(t_n,x'_n,y_n)|=|\partial_{x_N}u(t_n,x'_n,y_n)|\,|\nabla_{x'}X_{u(t_n,x'_n,y_n)}(t_n,x'_n)|\to 0\ \text{ as } n\to+\infty$$

from the boundedness of $\partial_{x_N}u$ in $[1,+\infty)\times\mathbb{R}^N$.

On the other hand, if $u(t_n,x'_n,y_n)\to 1$ as $n\to+\infty$, then, up to extraction of a subsequence, the functions $u_n:(t,x',x_N)\mapsto u(t+t_n,x'+x'_n,x_N+y_n)$ converge in $C^{1;2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$

to a classical solution u_∞ of $\partial_t u_\infty = \Delta u_\infty + f(u_\infty)$ in $\mathbb{R}\times\mathbb{R}^N$, with $0 \le u_\infty \le 1$ in $\mathbb{R}\times\mathbb{R}^N$, and $u_\infty(0,0,0) = 1$. The strong parabolic maximum principle and the uniqueness of the bounded solutions of the Cauchy problem (1.1) imply that $u_\infty \equiv 1$ in $\mathbb{R}\times\mathbb{R}^N$. In particular, $|\nabla_{x'} u(t_n, x'_n, y_n)| = |\nabla_{x'} u_n(0,0,0)| \to |\nabla_{x'} u_\infty(0,0,0)| = 0$ as $n \to +\infty$. Since the limit (namely, 0) does not depend on the subsequence, one concludes that the whole sequence $(|\nabla_{x'} u(t_n, x'_n, y_n)|)_{n\in\mathbb{N}}$ converges to 0 as $n \to +\infty$.

The previous paragraphs provide property (1.37) under any of the assumptions (i)-(iv) and the proof of Theorem 1.9 is thereby complete. □

4.4 Proof of Proposition 1.11

Let $N = 2$. Consider a function f such that Hypothesis 1.3 is satisfied (hence, Hypothesis 1.1 as well), and let $\theta \in (0,1)$ be given by Hypothesis 1.1. Let us call for short y the variable x_2. We consider a function γ defined for $|x'| > 1$ by

$$\gamma(x') = \sqrt{|x'|}\,\sin(\sqrt{|x'|}),$$

and extended in a smooth way to the whole $\mathbb{R}$. For $x' > 1$, we compute

$$\gamma'(x') = \frac{1}{2\sqrt{x'}}\sin(\sqrt{x'}) + \frac{1}{2}\cos(\sqrt{x'}).$$

The function γ then fulfills condition (1.41). Furthermore, $\gamma'(x' + 4\pi^2 n^2) \to 1/2$ as $n \to +\infty$, locally uniformly in $x' \in \mathbb{R}$. As a consequence, $u_0(\cdot + 4\pi^2 n^2, \cdot) \to H(2y - x')$ as $n \to +\infty$ in $L^p_{loc}(\mathbb{R}^2)$, for any $p \ge 1$, where H is the Heaviside function:

$$H(s) = \begin{cases} 1 & \text{if } s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Then, by parabolic estimates, $u(t, x'+4\pi^2 n^2, y)$ converges as $n \to +\infty$ (up to subsequences) in $C^{1;2}_{loc}((0,+\infty)\times\mathbb{R}^2)$, to the solution v of (1.1) with initial datum $H(2y - x')$. By uniqueness, the function v is of the form $v(t, x', y) = w(t, 2y - x')$. Moreover, as for the x_N-monotonicity of u with initial conditions satisfying (1.28), the comparison principle shows that $w(t,z)$ is nonincreasing with respect to z, and the strong maximum principle applied to $\partial_z w$ implies that $\partial_z w < 0$ in $(0,+\infty)\times\mathbb{R}^2$.

Fix now any $\lambda \in (\theta, 1)$, and consider an arbitrary $t > 0$. Let $z_t \in \mathbb{R}$ be such that $w(t, z_t) = \lambda$ (as in (1.29), such z_t exists and is unique because the function $w(t,\cdot)$ is continuous and decreasing, and $w(t,-\infty) = 1$ and $w(t,+\infty) = 0$). We see that $v(t, 0, z_t/2) = \lambda$ and

$$\frac{\partial_1 v(t,0,z_t/2)}{\partial_2 v(t,0,z_t/2)} = \frac{-\partial_z w(t,z_t)}{2\,\partial_z w(t,z_t)} = -\frac{1}{2}.$$

As a consequence, there holds from one hand that

$$\lim_{n\to+\infty} u(t, 4\pi^2 n^2, z_t/2) = \lambda,$$

and from the other hand that

$$\lim_{n\to+\infty} \frac{\partial_1 u(t, 4\pi^2 n^2, z_t/2)}{\partial_2 u(t, 4\pi^2 n^2, z_t/2)} = -\frac{1}{2}.$$

Hence, owing to (1.33), one has $\partial_{x'} X_{u(t,4\pi^2n^2,z_t/2)}(t,4\pi^2n^2) \to 1/2$ as $n \to +\infty$. Therefore, for every $\lambda_0 \in (\lambda,1)$ and every $t>0$, one has

$$\sup_{\theta\le\lambda'\le\lambda_0,\, x'\in\mathbb{R}} |\partial_{x'} X_{\lambda'}(t,x')| \ge \frac{1}{2}.$$

This shows that u violates the conclusion (1.39) of Conjecture 1.10.

Moreover, since Hypothesis 1.1 holds, [14, Theorem 1.11] implies that the functions $w(t,z_t+\cdot)$ converge as $t\to+\infty$ in $C^2_{loc}(\mathbb{R})$ to the profile of a decreasing or constant solution connecting some values a to b with $1\ge a\ge\lambda\ge b\ge 0$, and belonging to the minimal propagating terrace solution to (1.1) connecting 1 to 0. But this minimal propagating terrace reduces here to a single decreasing traveling front owing to Hypothesis 1.3. It follows in particular that $\lim_{t\to+\infty} -\partial_z w(t,z_t) > 0$. Since

$$\lim_{n\to+\infty} \partial_1 u(t,4\pi^2n^2,z_t/2) = -\partial_z w(t,z_t)$$

for every $t>0$, conclusion (1.40) fails too. □

5 The asymptotic one-dimensional symmetry: proofs of Theorems 1.13 and 1.14, and Corollary 1.15

Section 5.1 is devoted to the proofs of Theorems 1.13 and 1.14 on the asymptotic one-dimensional symmetry of the solutions in all directions, under some conditions on the initial support in (1.2). In Section 5.2, we are concerned with the asymptotic one-dimensional symmetry in some specific directions, related to a notion of directional Ω-limit set, and we also prove Corollary 1.15. We recall that in the Fisher-KPP case (1.43) dealt with throughout this section, Hypotheses 1.1 and 1.3 are fulfilled, with any $\theta\in(0,1)$ and $\rho>0$ in Hypothesis 1.1, and the minimal speed of traveling fronts connecting 1 to 0 is equal to $c^*=2\sqrt{f'(0)}$.

5.1 Asymptotic one-dimensional symmetry in all directions

We start with the proof of Theorem 1.14. The cornerstone of the proof is the following approximation result.

Lemma 5.1. *Assume that f is of the Fisher-KPP type* (1.43). *Let u be a solution to* (1.1) *with an initial condition $u_0=\mathbb{1}_U$, where $U\subset\mathbb{R}^N$ has nonempty interior and satisfies, for some $\delta,L>0$ and some $\sigma\in(0,c^*/2)$,*

$$U_\delta\cap B_L\neq\emptyset \quad \text{and} \quad U\setminus(B'_L\times\mathbb{R}) \subset \Big\{(x',x_N)\in\mathbb{R}^N : x_N\le\frac{\sigma}{2c^*}|x'|\Big\}. \tag{5.1}$$

Let $(u^R)_{R>0}$ be the solutions to (1.1) *emerging from the initial data $(u^R_0)_{R>0}$ defined by*

$$u^R_0=\mathbb{1}_{U\cap(B'_R\times\mathbb{R})}.$$

Then, for any $\varepsilon>0$, there exists $\tau_\varepsilon>0$, only depending on f,N,δ,L,σ and ε, such that

$$\forall\,\tau\ge\tau_\varepsilon,\quad \big\|u(\tau,\cdot)-u^{3\sigma\tau}(\tau,\cdot)\big\|_{C^1(B'_{\sigma\tau}\times\mathbb{R}^+)}<\varepsilon. \tag{5.2}$$

Proof. For $\varepsilon > 0$, we will show the existence of $\tau_\varepsilon > 0$ depending on $f, N, \delta, L, \sigma, \varepsilon$ such that

$$\forall\, \tau \geq \tau_\varepsilon, \quad \big\|u(\tau+1,\cdot) - u^{3\sigma\tau}(\tau+1,\cdot)\big\|_{L^\infty(\mathcal{C}_\tau)} < \varepsilon, \tag{5.3}$$

where $\mathcal{C}_\tau$ is the half-cylinder

$$\mathcal{C}_\tau := B'_{\sigma\tau} \times \mathbb{R}^+ = B'_{\sigma\tau} \times (0, +\infty).$$

Once (5.3) is proved, observing that $u - u^{3\sigma\tau}$ is nonnegative (by the comparison principle) and it solves a linear parabolic equation, one infers from the parabolic Harnack inequality and interior estimates, given for instance by [29], that (5.2) holds with ε replaced by $C\varepsilon$, where C only depends on f and N. Then, to prove the lemma it is sufficient to derive (5.3) for an arbitrary $\varepsilon > 0$.

Fix $\varepsilon > 0$. Consider the solutions $(w^R)_{R>0}$ to (1.1) emerging from the initial data $(w_0^R)_{R>0}$ given by

$$w_0^R = \mathbb{1}_{W^R}, \quad W^R := \Big\{(x', x_N) \in \mathbb{R}^N : |x'| \geq R,\ x_N \leq \frac{\sigma}{2c^*}|x'|\Big\}.$$

Note that $u_0 \leq \min(u_0^R + w_0^R, 1)$ for all $R \geq L$, hence, since under the KPP condition (1.43) the minimum between 1 and the sum of two solutions ranging in $[0,1]$ is a supersolution (because $f(1) = 0$ and $f(a+b) \leq f(a) + f(b)$ for all $a, b \in [0,1]$ with $a + b \leq 1$), we infer by comparison that, for $R \geq L$,

$$0 \leq u^R \leq u \leq \min(u^R + w^R, 1).$$

Thus, property (5.3) holds for some $\tau_\varepsilon \geq L/(3\sigma)$ if we show that

$$\forall\, \tau \geq \tau_\varepsilon, \quad \sup_{\mathcal{C}_\tau} \big(\min(u^{3\sigma\tau}(\tau+1,\cdot) + w^{3\sigma\tau}(\tau+1,\cdot), 1) - u^{3\sigma\tau}(\tau+1,\cdot)\big) < \varepsilon. \tag{5.4}$$

For this, with a value $c \in (2\sigma, c^*)$ that will be chosen later, we divide the half-cylinder $\mathcal{C}_\tau$ into the subsets

$$\mathcal{C}_\tau^i := \big(B'_{\sigma\tau} \times \mathbb{R}^+\big) \cap B_{c\tau}, \qquad \mathcal{C}_\tau^e := \big(B'_{\sigma\tau} \times \mathbb{R}^+\big) \setminus B_{c\tau}.$$

Let us first deal with the set $\mathcal{C}_\tau^i$, with any $c \in (2\sigma, c^*)$. By hypothesis, there exists a ball $B_\delta(x_0) \subset U$ with $|x_0| < L$, hence $u_0^{3\sigma\tau}(x + x_0) = 1$ for $|x| < \delta$, provided that $3\sigma\tau \geq L + \delta$. Applying the spreading result of Proposition 1.4 to the solution of (1.1) with initial datum $\mathbb{1}_{B_\delta}$, and then using the comparison principle, we find a value $\tau_1 \geq (L+\delta)/(3\sigma)$ depending on $f, N, c, \delta, L, \sigma, \varepsilon$ such that

$$\forall\, \tau \geq \tau_1, \quad \inf_{x \in B_{(c^*+c)\tau/2}} u^{3\sigma\tau}(\tau+1, x + x_0) > 1 - \varepsilon.$$

Since $\mathcal{C}_\tau^i \subset B_{c\tau} \subset B_{c\tau+L}(x_0)$ for all $\tau > 0$, and $B_{c\tau+L}(x_0) \subset B_{(c^*+c)\tau/2}(x_0)$ if $(c^*-c)\tau/2 \geq L$, there exists $\tau_2 > 0$ depending on $f, N, c, c^*, \delta, L, \sigma, \varepsilon$ (hence on $f, N, c, \delta, L, \sigma, \varepsilon$ since c^* only depends on f) for which

$$\inf_{\mathcal{C}_\tau^i} u^{3\sigma\tau}(\tau+1, x) \geq \inf_{B_{(c^*+c)\tau/2}(x_0)} u^{3\sigma}(\tau+1, x) > 1 - \varepsilon.$$

This shows that (5.4) holds when $\mathcal{C}_\tau$ is replaced by $\mathcal{C}_\tau^i$, for all $\tau \geq \tau_2$ and for any choice of $c \in (2\sigma, c^*)$.

As for the set $\mathcal{C}^e_\tau$, we estimate its distance from $W^{3\sigma\tau}$ in order to derive an upper bound for $w^{3\sigma\tau}$. In this paragraph, $\tau > 0$ is arbitrary and $c \in (2\sigma, c^*)$ will be fixed at the end of the paragraph. Take two arbitrary points $x = (x', x_N) \in \mathcal{C}^e_\tau$ and $y = (y', y_N) \in W^{3\sigma\tau}$. There holds

$$|x'| < \sigma\tau \le \frac{1}{3}|y'| \quad \text{and} \quad x_N > \sqrt{c^2 - \sigma^2}\,\tau, \quad y_N \le \frac{\sigma}{2c^*}|y'|.$$

We compute

$$|x - y|^2 = |x' - y'|^2 + (x_N - y_N)^2 \ge \frac{4}{9}|y'|^2 + (x_N - y_N)^2.$$

If $\sigma|y'|/(2c^*) \ge \sqrt{c^2 - \sigma^2}\,\tau$, we find that

$$|x - y|^2 \ge \frac{16}{9}(c^*)^2\Big(\frac{c^2}{\sigma^2} - 1\Big)\tau^2 \ge \frac{16}{3}(c^*)^2\tau^2$$

since $c > 2\sigma > 0$. Instead, in the opposite case $\sigma|y'|/(2c^*) < \sqrt{c^2 - \sigma^2}\,\tau$, one has $y_N \le \sigma|y'|/(2c^*) < \sqrt{c^2 - \sigma^2}\,\tau < x_N$, whence

$$\begin{aligned} |x - y|^2 &\ge \frac{4}{9}|y'|^2 + \Big(\sqrt{c^2 - \sigma^2}\,\tau - \frac{\sigma}{2c^*}|y'|\Big)^2 \\ &= \frac{4}{9}|y'|^2 + (c^2 - \sigma^2)\tau^2 + \frac{\sigma^2}{4(c^*)^2}|y'|^2 - \frac{\sigma}{c^*}\sqrt{c^2 - \sigma^2}\,\tau|y'|, \end{aligned}$$

and we estimate the negative terms by observing that

$$\frac{4}{9}|y'|^2 - \sigma^2\tau^2 - \frac{\sigma}{c^*}\sqrt{c^2 - \sigma^2}\,\tau|y'| \ge |y'|\Big(\frac{1}{3}|y'| - \frac{\sigma}{c^*}\sqrt{c^2 - \sigma^2}\,\tau\Big) \ge 0$$

since $|y'|/3 \ge \sigma\tau \ge \sigma\sqrt{c^2 - \sigma^2}\tau/c^*$. Thus, in such case one has

$$|x - y|^2 \ge c^2\tau^2 + \frac{\sigma^2}{4(c^*)^2}|y'|^2 \ge c^2\tau^2 + \frac{9\sigma^4\tau^2}{4(c^*)^2},$$

which is larger than $(c^*)^2\tau^2$ for $c \in (2\sigma, c^*)$ close enough to c^*, depending on $c^* = 2\sqrt{f'(0)}$ and σ only. Summing up, we have shown the existence of some $c \in (2\sigma, c^*)$ and $c' > c^*$, depending on $c^* = 2\sqrt{f'(0)}$ and σ, such that

$$\mathrm{dist}(x, W^{3\sigma\tau}) \ge c'\tau \ \text{ for all } \tau > 0 \text{ and } x \in \mathcal{C}^e_\tau. \tag{5.5}$$

Finally, we invoke the supersolutions $(v^T)_{T>0}$ provided by Proposition 3.2, associated with a fixed $\widetilde{c} \in (c^*, c')$ and $\lambda = \varepsilon$; they satisfy (3.1) with $\widetilde{c}$ instead of c and a quantity R depending on $f, N, \widetilde{c}, c^*$ and ε (hence, R depends on f, N, σ and ε, since c' and $\widetilde{c}$ only depend on $c^* = 2\sqrt{f'(0)}$ and σ). Take $\tau_3 > 0$ large enough (depending on $R, \widetilde{c}, c'$, hence on $f, N, \sigma, \varepsilon$) so that $R + \widetilde{c}(T+1) \le c'T$ for all $T \ge \tau_3$, whence $v^{T+1}(0, x) \ge 1$ for $|x| \ge c'T$. On the other hand, for all $\tau > 0$ and $x_0 \in \mathcal{C}^e_\tau$, we know from (5.5) that $B_{c'\tau}(x_0) \cap W^{3\sigma\tau} = \emptyset$, which implies that $w_0^{3\sigma\tau}(x + x_0) = 0$ for $|x| < c'\tau$. This means that, for $\tau \ge \tau_3$, $w^{3\sigma\tau}(0, \cdot + x_0) \le v^{\tau+1}(0, \cdot)$ in $\mathbb{R}^N$, and thus $w^{3\sigma\tau}(t, \cdot + x_0) \le v^{\tau+1}(t, \cdot)$ in $\mathbb{R}^N$ for all $t \ge 0$ by comparison. We conclude by (3.1) that

$$\forall\, \tau \ge \tau_3,\ \forall\, x_0 \in \mathcal{C}^e_\tau, \quad w^{3\sigma\tau}(\tau + 1, x_0) \le v^{\tau+1}(\tau + 1, 0) < \varepsilon.$$

This yields that (5.4) holds in the set $\mathcal{C}^e_\tau$ too, for a suitable choice of c depending on $c^* = 2\sqrt{f'(0)}$ and σ, and for all $\tau \ge \tau_3 > 0$ with τ_3 depending on $f, N, \sigma, \varepsilon$. Therefore (5.4) holds true in the whole $\mathcal{C}_\tau$, for some $\tau_\varepsilon \ge \max(\tau_2, \tau_3) > 0$ depending on $f, N, \delta, L, \sigma, \varepsilon$. The proof of the lemma is complete. □

We now derive Theorem 1.14 combining Lemma 5.1 with some of our previous results about the speed of propagation, and finally with a reflection argument "à la Jones" [27].

Proof of Theorem 1.14. Consider a function $\psi \in \Omega(u)$, and let $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ be the associated sequences given in the definition (1.3) of $\Omega(u)$, with $(t_n)_{n\in\mathbb{N}}$ diverging to $+\infty$ and $u(t_n, x_n + \cdot) \to \psi$ as $n \to +\infty$ in $L^\infty_{loc}(\mathbb{R}^N)$. In order to show that ψ is one-dimensional, we proceed in several steps: we first derive some properties of $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ and use them to define some new coordinate systems; next, assuming by contradiction that ψ is not one-dimensional, we show that a line orthogonal to a level set of u at time t_n is far from a suitable half-cylinder with radius of order t_n, which in the following step is used for the truncation of the initial support U to which we apply the convergence result of Lemma 5.1 (this is where the geometric assumption (1.47) is used); finally, we get a contradiction by applying Jones' reflection argument to the truncated solution.

First of all, if $U = \emptyset$, then $u(t, x) = 0$ for all $t \geq 0$ and $x \in \mathbb{R}^N$, hence $\Omega(u)$ is reduced to the zero function and the desired conclusion trivially holds in this case. Therefore, we can assume in the sequel without loss of generality that U is not empty, and then its positive-distance-interior U_δ is not empty either for some $\delta > 0$, thanks to (1.44).

Step 1: properties of the sequences and coordinates transformations. For $n \in \mathbb{N}$, we call for short $h_n := \mathrm{dist}(x_n, \overline{U})$ and we take a point ξ_n in the set π_{x_n} of the projections of x_n onto $\overline{U}$ (i.e., $\xi_n \in \overline{U}$ and $|x_n - \xi_n| = h_n$). Using some of our previous results about the spreading speed, we claim that (1.44) yields

$$\forall\, c \in (0, c^*), \quad h_n \geq ct_n \ \text{ for all } n \text{ sufficiently large}, \tag{5.6}$$

unless $\psi \equiv 1$, in which cases ψ is trivially one-dimensional. Suppose indeed that $h_n < ct_n$ for infinitely many $n \in \mathbb{N}$, for some $c \in (0, c^*)$. By assumption (1.44), for given $c < c_1 < c_2 < c^*$, the inclusion $U + B_{c_1 t} \subset U_\delta + B_{c_2 t}$ holds for $t > 0$ sufficiently large. Hence, Lemma 3.1 yields $\inf_{x\in U+B_{c_1 t}} u(t, x) \to 1$ as $t \to +\infty$, from which one infers $\psi \equiv 1$.

Thus, in the rest of the proof, we assume that $\psi \not\equiv 1$ hence (5.6) holds and then, in particular, $h_n > 0$ for all $n \in \mathbb{N}$ (up to extraction of a subsequence). We set

$$e_n := \frac{x_n - \xi_n}{h_n}.$$

Next, we consider a family of $N \times N$ orthogonal matrices $(M_n)_{n\in\mathbb{N}}$ such that $M_n(\mathrm{e}_N) = e_n$, and we define, for $t \geq 0$ and $x \in \mathbb{R}^N$,

$$u_n(t, x) := u(t, \xi_n + M_n(x)).$$

These are still solutions to (1.1), because the equation is invariant under isometry. We have that, up to subsequences, $(e_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ converge respectively to some direction $e \in \mathbb{S}^{N-1}$ and some orthogonal matrix M, with $M(\mathrm{e}_N) = e$. It follows that

$$u_n(t_n, h_n \mathrm{e}_N + x) = u(t_n, x_n + M_n(x)) \longrightarrow \psi(M(x)) =: \widetilde{\psi}(x) \quad \text{as} \quad n \to +\infty, \tag{5.7}$$

locally uniformly in $x \in \mathbb{R}^N$. Moreover, $u_n(0, \cdot) = \mathbb{1}_{U_n}$ with

$$U_n := M_n^{-1}(U) - M_n^{-1}(\xi_n)$$

(a rigid transformation of U). The set U_n is constructed in a way that $0 \in \overline{U_n}$ is an orthogonal projection of $h_n \mathrm{e}_N$ onto $\overline{U_n}$ and thus, since $\mathrm{dist}(h_n \mathrm{e}_N, \overline{U_n}) = h_n \to +\infty$ as $n \to +\infty$ by (5.6), the geometric assumption (1.47) (which is invariant by isometries) yields

$$U_n \subset \big\{ y \in \mathbb{R}^N : y \cdot \mathrm{e}_N \le \alpha_n |y'| \big\} \quad \text{with } \alpha_n \to 0 \text{ as } n \to +\infty. \tag{5.8}$$

Step 2: the choice of the truncation. We claim that

$$\nabla_{x'} \widetilde{\psi} \equiv 0 \ \text{ in } \mathbb{R}^N, \tag{5.9}$$

that is,

$$\nabla \psi \cdot e' \equiv (M(\nabla \widetilde{\psi})) \cdot e' \equiv \nabla \widetilde{\psi} \cdot (M^t(e')) \equiv 0 \ \text{ in } \mathbb{R}^N$$

for any direction $e' \in \mathbb{S}^{N-1}$ such that $M^t(e') \perp \mathrm{e}_N$, i.e., $e' \perp M(\mathrm{e}_N) = e$. Hence this would imply that $\psi = \psi(x \cdot e)$.

Assume by contradiction that the above claim (5.9) fails, that is, that $\nabla_{x'} \widetilde{\psi}(\bar{x}) \neq 0$ for some $\bar{x} \in \mathbb{R}^N$. By interior parabolic estimates, up to extraction of a subsequence, the $L^\infty_{loc}(\mathbb{R}^N)$ convergence (5.7) holds true in $C^1_{loc}(\mathbb{R}^N)$, hence in particular

$$\nabla u_n(t_n, h_n \mathrm{e}_N + \bar{x}) \to \nabla \widetilde{\psi}(\bar{x}) \quad \text{as } \ n \to +\infty. \tag{5.10}$$

Take a real number $\vartheta > 0$, that will be fixed at the end of this paragraph. Let $(H_n)_{n \in \mathbb{N}}$ be the family of closed half-cylinders in $\mathbb{R}^N$ defined by

$$H_n := \overline{B'_{\vartheta t_n}} \times (-\infty, \vartheta t_n].$$

Consider also the conical sets $(V_n)_{n \in \mathbb{N}}$ given by

$$V_n := \big\{ h_n \mathrm{e}_N + s(\nabla \widetilde{\psi}(\bar{x}) + \zeta) : s \in \mathbb{R},\, \zeta \in B_\vartheta \big\}. \tag{5.11}$$

We look for ϑ small enough so that

$$H_n \cap V_n = \emptyset \ \text{ for all } n \text{ sufficiently large.} \tag{5.12}$$

To do so, consider a generic point $P \in V_n$, written as $P = h_n \mathrm{e}_N + \bar{x} + s\big(\nabla \widetilde{\psi}(\bar{x}) + \zeta\big)$ for some $s \in \mathbb{R}$ and $\zeta \in B_\vartheta$, and suppose that $P \in H_n$, whence

$$\vartheta t_n \ge \big(|\nabla_{x'} \widetilde{\psi}(\bar{x})| - \vartheta \big) |s| - |\bar{x}|.$$

We then impose that $\vartheta < |\nabla_{x'} \widetilde{\psi}(\bar{x})|$ to infer that, when $P \in H_n$,

$$|s| \le \frac{\vartheta t_n + |\bar{x}|}{|\nabla_{x'} \widetilde{\psi}(\bar{x})| - \vartheta},$$

that we use to estimate

$$P \cdot \mathrm{e}_N \ge h_n - |\bar{x}| - \big(|\partial_{x_N} \widetilde{\psi}(\bar{x})| + \vartheta \big) |s| \ge h_n - |\bar{x}| - \frac{|\partial_{x_N} \widetilde{\psi}(\bar{x})| + \vartheta}{|\nabla_{x'} \widetilde{\psi}(\bar{x})| - \vartheta} \, (\vartheta t_n + |\bar{x}|).$$

Using (5.6), one finds that the above right-hand side is larger than ϑt_n for n large, provided that ϑ is sufficiently small, only depending on $|\partial_{x_N} \widetilde{\psi}(\bar{x})|$, $|\nabla_{x'} \widetilde{\psi}(\bar{x})|$ and the quantity $c > 0$

chosen in (5.6). This means that, for such values of ϑ, condition (5.12) holds. We then choose ϑ small enough in such a way that (5.12) holds and, in addition,

$$0 < \vartheta < \frac{3c^*}{2}. \tag{5.13}$$

This fixes the choice of the half-cylinders $(H_n)_{n\in\mathbb{N}}$.

Step 3: the approximation procedure. We now apply Lemma 5.1 to the sequence of solutions $(u_n)_{n\in\mathbb{N}}$. Take $\delta > 0$ from hypothesis (1.44),

$$\sigma := \frac{\vartheta}{3} > 0, \qquad L := d_{\mathcal{H}}(U, U_\delta) + 1 > 0, \qquad \varepsilon := \frac{\vartheta}{2} > 0, \tag{5.14}$$

where ϑ is given in the previous step. One has that $\sigma < c^*/2$ by (5.13) and $0 < L < +\infty$ by (1.44). We further have, on the one hand, that $(U_n)_\delta \cap B_L \neq \emptyset$, because $0 \in \overline{U}_n$ and $d_{\mathcal{H}}(U_n, (U_n)_\delta) = d_{\mathcal{H}}(U, U_\delta) < L$. On the other hand, it follows from (5.8) that, for n large,

$$U_n \subset \Big\{(x', x_N) \in \mathbb{R}^N : x_N \leq \frac{\sigma}{2c^*}|x'|\Big\}.$$

This means that the sets U_n fulfill the hypotheses of Lemma 5.1 for n large enough. Therefore, for such values of n, considering the solutions $u_n^{\vartheta t_n}$ of (1.1) whose initial datum is given by the indicator function of the set

$$U_n \cap (B'_{\vartheta t_n} \times \mathbb{R}),^{9}$$

the estimate (5.2) implies in particular that

$$\big\|u_n(t_n, \cdot) - u_n^{\vartheta t_n}(t_n, \cdot)\big\|_{C^1(B'_{\vartheta t_n/3}\times\mathbb{R}^+)} < \frac{\vartheta}{2},$$

provided that $t_n > \tau_\varepsilon$, where τ_ε is independent of n. This means that the above estimate holds true for all n sufficiently large. One infers, using also (5.10),

$$\big|\nabla u_n^{\vartheta t_n}(t_n, h_n \mathrm{e}_N + \bar{x}) - \nabla\widetilde{\psi}(\bar{x})\big| < \vartheta \ \text{ for all } n \text{ sufficiently large.} \tag{5.15}$$

This means that, for such values of n, the line Γ_n passing through the point $h_n\mathrm{e}_N + \bar{x}$ and directed as $\nabla u_n^{\vartheta t_n}(t_n, h_n\mathrm{e}_N + \bar{x})$ is contained in the set V_n defined in (5.11), and therefore, by (5.12),

$$\big(\Gamma_n \cap H_n\big) \subset \big(V_n \cap H_n\big) = \emptyset \ \text{ for all } n \text{ sufficiently large.}$$

Next, owing to (5.8), we also have that $U_n \cap (B'_{\vartheta t_n} \times \mathbb{R}) \subset H_n$ for all n sufficiently large, i.e.,

$$\operatorname{supp} u_n^{\vartheta t_n}(0, \cdot) \subset H_n \ \text{ for all } n \text{ sufficiently large.} \tag{5.16}$$

Step 4: the reflection argument. Let H_n, V_n, Γ_n and $u_n^{\vartheta t_n}$ be as in the previous steps. For n large enough, the half-cylinder H_n and the line Γ_n are convex, closed and disjoint; we can then separate them with an hyperplane, which, up to translation, can be assumed without loss of generality to contain Γ_n. Namely, for n large, there exists an open half-space Ω_n such that

$$\Gamma_n \subset \partial\Omega_n \quad \text{and} \quad H_n \subset \Omega_n. \tag{5.17}$$

[9]Notice that, for every n large enough, this set contains a non-empty open ball, since $(U_n)_\delta \cap B_L \neq \emptyset$.

By (5.16), one has $\operatorname{supp} u_n^{\vartheta t_n}(0,\cdot) \subset \Omega_n$ for n large. Let $\mathcal{R}_n$ denote the affine orthogonal reflection with respect to $\partial\Omega_n$. Then define the function v^n in $[0,+\infty)\times\overline{\Omega_n}$ by

$$v^n(t,x) := u_n^{\vartheta t_n}(t,\mathcal{R}_n(x)).$$

The function v^n coincides with $u_n^{\vartheta t_n}$ on $[0,+\infty)\times\partial\Omega_n$, and furthermore it vanishes identically at $t=0$ in Ω_n, provided n is large enough for (5.16)-(5.17) to hold. Then, for such values of n, it follows from the comparison principle that $v^n \le u_n^{\vartheta t_n}$ in $(0,+\infty)\times\Omega_n$, and moreover, by the Hopf lemma, that $\partial_{\nu_n} v^n > \partial_{\nu_n} u_n^{\vartheta t_n}$ on $(0,+\infty)\times\partial\Omega_n$, where ν_n is the exterior normal to Ω_n. Since clearly $\partial_{\nu_n} v^n = -\partial_{\nu_n} u_n^{\vartheta t_n}$, this means that $\partial_{\nu_n} u_n^{\vartheta t_n} < 0$ on $(0,+\infty)\times\partial\Omega_n$, and thus in particular that $\partial_{\nu_n} u_n^{\vartheta t_n}(t_n, h_n \mathrm{e}_N + \bar{x}) < 0$, because $h_n \mathrm{e}_N + \bar{x} \in \Gamma_n \subset \partial\Omega_n$. This is however impossible because $\nabla u_n^{\vartheta t_n}(t_n, h_n \mathrm{e}_N + \bar{x})$ is parallel to Γ_n and thus orthogonal to ν_n. We have reached a contradiction. This shows that $\psi = \psi(x\cdot e)$ and then concludes the proof of Theorem 1.14. □

Theorem 1.13 will be a consequence of Theorem 1.14 and the following lemma.

Lemma 5.2. *For any $U \subset \mathbb{R}^N$, the map*

$$R \mapsto \sup_{x\in\mathbb{R}^N,\, \operatorname{dist}(x,U)=R} \mathcal{O}(x)$$

is nonincreasing, where $\mathcal{O}(x)$ is defined in (1.46)*. Moreover, for any $U' \subset \mathbb{R}^N$ satisfying $d_{\mathcal{H}}(U,U') < +\infty$, then U fulfills* (1.47) *if and only if U' does (with the corresponding $\mathcal{O}$ defined as in* (1.46) *with U' instead of U).*

Proof. The monotonicity property involving $\mathcal{O}$ is readily derived. Consider indeed any

$$0 < R' < R.$$

If the set $\{x\in\mathbb{R}^N : \operatorname{dist}(x,U) = R\}$ is empty, then $\sup_{x\in\mathbb{R}^N,\,\operatorname{dist}(x,U)=R} \mathcal{O}(x) = -\infty$ and the inequality $\sup_{x\in\mathbb{R}^N,\,\operatorname{dist}(x,U)=R} \mathcal{O}(x) \le \sup_{x\in\mathbb{R}^N,\,\operatorname{dist}(x,U)=R'} \mathcal{O}(x)$ is trivially true. Assume now that the set $\{x\in\mathbb{R}^N : \operatorname{dist}(x,U) = R\}$ is not empty, and consider any x in this set and any $\xi \in \pi_x$, that is, $\xi \in \overline{U}$ and $|x-\xi| = \operatorname{dist}(x,U) = R$. Consider the point $x' := \xi + (R'/R)(x-\xi)$. Its unique projection onto $\overline{U}$ is ξ, that is, $\pi_{x'} = \{\xi\}$. Furthermore, $\operatorname{dist}(x',U) = |x'-\xi| = R'$. One also observes that, for any $y \in U\setminus\{\xi\}$,

$$\frac{x-\xi}{|x-\xi|}\cdot\frac{y-\xi}{|y-\xi|} = \frac{x'-\xi}{|x'-\xi|}\cdot\frac{y-\xi}{|y-\xi|} \le \mathcal{O}(x') \le \sup_{z\in\mathbb{R}^N,\,\operatorname{dist}(z,U)=R'} \mathcal{O}(z).$$

Since x with $\operatorname{dist}(x,U) = R$, together with $\xi\in\pi_x$ and $y\in U\setminus\{\xi\}$, were arbitrary, this shows that

$$\sup_{z\in\mathbb{R}^N,\,\operatorname{dist}(z,U)=R} \mathcal{O}(z) \le \sup_{z\in\mathbb{R}^N,\,\operatorname{dist}(z,U)=R'} \mathcal{O}(z).^{10}$$

Let us turn to the second statement of the lemma. One considers any two subsets U and U' of $\mathbb{R}^N$ satisfying $d_{\mathcal{H}}(U,U') < +\infty$. Denote π'_x and $\mathcal{O}'(x)$ the objects defined as in (1.45)-(1.46) with U' instead of U.

[10] Notice that this property is also satisfied when there is no y in $U\setminus\{\xi\}$, that is, when $U = \{\xi\}$, since in this case $\mathcal{O}(z) = -\infty$ for all $z \notin U$.

Assume by way of contradiction that U fulfills (1.47) and U' does not. Then there are $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N\setminus\overline{U'}$ such that

$$0 < R'_n := \mathrm{dist}(x_n, U') \to +\infty \text{ as } n \to +\infty, \text{ and } \mathcal{O}'(x_n) \geq 2\varepsilon > 0 \text{ for all } n \in \mathbb{N}. \quad (5.18)$$

Calling $d := d_{\mathcal{H}}(U, U') < +\infty$, one then has $R_n := \mathrm{dist}(x_n, U) \to +\infty$ as $n \to +\infty$, and moreover

$$R_n - d \leq R'_n \leq R_n + d \text{ for all } n \in \mathbb{N}. \quad (5.19)$$

Without loss of generality, one has $R_n > 0$ for every $n \in \mathbb{N}$. Since U is assumed to satisfy (1.47), there holds $\limsup_{n\to+\infty} \mathcal{O}(x_n) \leq 0$, that is,

$$\mathcal{O}(x_n)^+ := \max\big(\mathcal{O}(x_n), 0\big) \to 0 \text{ as } n \to +\infty. \quad (5.20)$$

Now, from (5.18), for each $n \in \mathbb{N}$, there are $\xi'_n \in \pi'_{x_n}$, that is, $\xi'_n \in \overline{U'}$ and $|x_n - \xi'_n| = \mathrm{dist}(x_n, U') = R'_n > 0$, and $y'_n \in U'\setminus\{\xi'_n\}$ such that

$$\frac{x_n - \xi'_n}{|x_n - \xi'_n|} \cdot \frac{y'_n - \xi'_n}{|y'_n - \xi'_n|} \geq \varepsilon > 0. \quad (5.21)$$

For each $n \in \mathbb{N}$, consider any $\xi_n \in \pi_{x_n}$, that is, $\xi_n \in \overline{U}$ and $|x_n - \xi_n| = \mathrm{dist}(x_n, U) = R_n > 0$, and then there is a point $y_n \in U$ such that

$$|y_n - y'_n| \leq d_{\mathcal{H}}(U, U') + 1 = d + 1. \quad (5.22)$$

We estimate from above the quantities in (5.21) by writing:

$$\frac{x_n-\xi'_n}{|x_n-\xi'_n|}\cdot\frac{y'_n-\xi'_n}{|y'_n-\xi'_n|} \leq \underbrace{\Big|\frac{x_n-\xi'_n}{|x_n-\xi'_n|}-\frac{x_n-\xi_n}{|x_n-\xi_n|}\Big|}_{=:I_{1,n}} + \underbrace{\Big|\frac{y'_n-\xi'_n}{|y'_n-\xi'_n|}-\frac{y_n-\xi_n}{|y_n-\xi_n|}\Big|}_{=:I_{2,n}} + \underbrace{\frac{x_n-\xi_n}{|x_n-\xi_n|}\cdot\frac{y_n-\xi_n}{|y_n-\xi_n|}}_{=:I_{3,n}}. \quad (5.23)$$

This inequality is understood to hold whenever $y_n \neq \xi_n$, which we will show to occur for n sufficiently large. We will then prove that $I_{1,n}$, $I_{2,n}$, $I_{3,n} \to 0$ as $n \to +\infty$, which will eventually contradict (5.21). In order to estimate $I_{1,n}$, we take $z_n \in \overline{U}$ such that $|z_n - \xi'_n| \leq d$ and we compute

$$\begin{aligned}(x_n - \xi_n)\cdot(x_n - \xi'_n) &= R_n^2 + (x_n - \xi_n)\cdot(\xi_n - z_n) + (x_n - \xi_n)\cdot(z_n - \xi'_n)\\ &\geq R_n^2 - \mathcal{O}(x_n)R_n|z_n - \xi_n| - R_n d\\ &\geq R_n\big(R_n - 2(R_n + d)\mathcal{O}(x_n)^+ - d\big),\end{aligned} \quad (5.24)$$

where the last inequality follows from

$$|z_n - \xi_n| \leq |z_n - \xi'_n| + |\xi'_n - x_n| + |x_n - \xi_n| \leq d + R'_n + R_n \leq 2(R_n + d).$$

One then derives from (5.19) and (5.24) that

$$0 \leq I_{1,n} \leq \sqrt{2 - \frac{2\big(R_n - 2(R_n + d)\mathcal{O}(x_n)^+ - d\big)}{R'_n}} \leq 2\sqrt{\frac{(R_n + d)\mathcal{O}(x_n)^+ + d}{R'_n}}. \quad (5.25)$$

Together with (5.18)-(5.20), one gets that

$$I_{1,n} \to 0 \text{ as } n \to +\infty. \quad (5.26)$$

Next, let us check that $y_n \neq \xi_n$ for n large. We first control $|y'_n - \xi'_n|$ from below. We write

$$|y'_n - x_n|^2 = |y'_n - \xi'_n|^2 + (R'_n)^2 - 2(y'_n - \xi'_n) \cdot (x_n - \xi'_n),$$

which together with (5.21) and the inequality $|y'_n - x_n| \geq \mathrm{dist}(x_n, U') = R'_n$ yields

$$|y'_n - \xi'_n|\big(|y'_n - \xi'_n| - 2\varepsilon R'_n\big) \geq |y'_n - x_n|^2 - (R'_n)^2 \geq 0.$$

Since $y'_n \neq \xi'_n$, this means that

$$|y'_n - \xi'_n| \geq 2\varepsilon R'_n. \tag{5.27}$$

Now, using (5.24) and $R'_n \leq R_n + d$, one infers

$$|\xi_n - \xi'_n|^2 = R_n^2 + (R'_n)^2 - 2(x_n - \xi_n, x_n - \xi'_n) \leq 4R_n d + d^2 + 4R_n(R_n + d)\mathcal{O}(x_n)^+. \tag{5.28}$$

Gathering together the inequalities (5.22), (5.27) and (5.28) shows that

$$\begin{array}{rcl} |y_n - \xi_n| & \geq & |y'_n - \xi'_n| - |y_n - y'_n| - |\xi'_n - \xi_n| \\ & \geq & 2\varepsilon R'_n - (d+1) - \sqrt{4R_n d + d^2 + 4R_n(R_n + d)\mathcal{O}(x_n)^+}. \end{array} \tag{5.29}$$

The right-hand side is positive for all n large enough and is equivalent to $2\varepsilon R'_n$ as $n \to +\infty$, because of (5.18)-(5.20). This means that $y_n \neq \xi_n$ for n large enough. Let us estimate $I_{2,n}$. One has, for n large,

$$\begin{array}{rcl} 0 \leq I_{2,n} & = & \sqrt{2 - 2\dfrac{(y'_n - \xi'_n) \cdot (y_n - \xi_n)}{|y'_n - \xi'_n| \times |y_n - \xi_n|}} \\ & = & \sqrt{\dfrac{|(y'_n - \xi'_n) - (y_n - \xi_n)|^2 - (|y'_n - \xi'_n| - |y_n - \xi_n|)^2}{|y'_n - \xi'_n| \times |y_n - \xi_n|}} \\ & \leq & \dfrac{|(y'_n - \xi'_n) - (y_n - \xi_n)|}{\sqrt{|y'_n - \xi'_n| \times |y_n - \xi_n|}} \leq \dfrac{|y_n - y'_n| + |\xi_n - \xi'_n|}{\sqrt{|y'_n - \xi'_n| \times |y_n - \xi_n|}} \leq \dfrac{d + 1 + |\xi_n - \xi'_n|}{\sqrt{|y'_n - \xi'_n| \times |y_n - \xi_n|}}, \end{array} \tag{5.30}$$

where the last inequality follows from (5.22). Putting together (5.27)-(5.30) leads to

$$0 \leq I_{2,n} \leq \frac{d + 1 + \sqrt{4R_n d + d^2 + 4R_n(R_n + d)\mathcal{O}(x_n)^+}}{\sqrt{2\varepsilon R'_n} \times \sqrt{2\varepsilon R'_n - (d+1) - \sqrt{4R_n d + d^2 + 4R_n(R_n + d)\mathcal{O}(x_n)^+}}}$$

for all n large enough. Using again (5.18)-(5.20), it follows that

$$I_{2,n} \to 0 \quad \text{as } n \to +\infty. \tag{5.31}$$

Finally, one has that $0 \leq I_{3,n} \leq \mathcal{O}(x_n) \leq \mathcal{O}(x_n)^+$ for all n, hence $I_{3,n} \to 0$ as $n \to +\infty$, by (5.20). Together with (5.23), (5.26) and (5.31), one gets that

$$\limsup_{n \to +\infty} \frac{x_n - \xi'_n}{|x_n - \xi'_n|} \cdot \frac{y'_n - \xi'_n}{|y'_n - \xi'_n|} \leq 0,$$

a contradiction with (5.21). The conclusion of the lemma then follows by changing the roles of U and U'. □

Proof of Theorem 1.13. If the set U is convex, then the quantity $\mathcal{O}(x)$ defined by (1.46) satisfies $\mathcal{O}(x) \leq 0$ for all $x \notin \overline{U}$, hence condition (1.47) holds in this case. Condition (1.47) holds true as well when U is at bounded Hausdorff distance from a convex set U', thanks to Lemma 5.2. The conclusion then follows from that of Theorem 1.14. □

Remark 5.3. The conclusions of Theorems 1.13 and 1.14 still hold for the solutions to (1.1) with measurable initial conditions $u_0 : \mathbb{R}^N \to [0,1]$ more general than characteristic functions. To be more precise, if there are $h \in (0,1]$ and $\delta > 0$ such that (1.44) is replaced by

$$d_{\mathcal{H}}\big(\{u_0 \geq h\}, \operatorname{supp} u_0\big) < +\infty \quad \text{and} \quad d_{\mathcal{H}}\big(\{u_0 \geq h\}, \{u_0 \geq h\}_\delta\big) < +\infty, \tag{5.32}$$

and if (1.47) is replaced by

$$\lim_{R \to +\infty} \left(\sup_{x \in \mathbb{R}^N,\, \mathrm{dist}(x, \operatorname{supp} u_0) = R} \mathcal{O}(x) \right) \leq 0, \tag{5.33}$$

then the conclusion of Theorem 1.14 is satisfied. Indeed, first of all, it is straightforward to check that Lemma 5.1 still hlods with $u_0^R := u_0 \, \mathbb{1}_{B'_R \times \mathbb{R}}$ and the assumption

$$\{u_0 \geq h\}_\delta \cap B_L \neq \emptyset \quad \text{and} \quad \operatorname{supp} u_0 \setminus (B'_L \times \mathbb{R}) \subset \left\{ (x', x_N) \in \mathbb{R}^N : x_N \leq \frac{\sigma}{2c^*} |x'| \right\}$$

instead of (5.1) (but now in the conclusion (5.2) the time τ_ε depends on h too). Since Lemma 3.1 is still valid with U replaced by $\{u_0 \geq h\}$, it is easy to see that the proof of Theorem 1.14 works, where now $h_n := \mathrm{dist}(x_n, \operatorname{supp} u_0)$, U_n are rigid transformations of $\operatorname{supp} u_0$ instead of U, and $L := d_{\mathcal{H}}\big(\{u_0 \geq h\}, \{u_0 \geq h\}_\delta\big) + 1$ in (5.14). It then follows that the conclusion of Theorem 1.13 is satisfied when u_0 fulfills (5.32)-(5.33) instead of (1.2) and (1.44), and when the convexity of U is replaced by the convexity of $\operatorname{supp} u_0$ or the convexity of a set at a bounded Hausdorff distance from $\operatorname{supp} u_0$.

5.2 Directional asymptotic one-dimensional symmetry

This section is devoted to the proof of Corollary 1.15 and further asymptotic one-dimensional symmetry results. As a matter of fact, the same arguments as in the proof of Theorem 1.14 can be used to precise the asymptotic one-dimensional symmetry property when one focuses on a fixed direction. For this, it is sufficient to have a "localized" version of the geometric assumption (1.47). Namely, we derive a result for the *directional Ω-limit set*, defined as follows.

Definition 5.4. *For a given function $u : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ and for any direction $e \in \mathbb{S}^{N-1}$, the set*

$$\begin{aligned} \Omega_e(u) := \big\{ \psi \in L^\infty(\mathbb{R}^N) \ : \ & u(t_n, x_n + \cdot) \to \psi \text{ in } L^\infty_{loc}(\mathbb{R}^N) \\ & \text{for some sequences } (t_n)_{n \in \mathbb{N}} \text{ in } \mathbb{R}^+ \text{ diverging to } +\infty \\ & \text{and } (x_n)_{n \in \mathbb{N}} \text{ in } \mathbb{R}^N \setminus \{0\} \text{ such that } x_n/|x_n| \to e \text{ as } n \to +\infty \big\} \end{aligned}$$

is called the Ω-limit set in the direction *e of u. For any bounded solution u of* (1.1) *and any $e \in \mathbb{S}^{N-1}$, the set $\Omega_e(u)$ is not empty and included in $C^2(\mathbb{R}^N)$, from standard parabolic estimates.*

Theorem 5.5. *Assume that f is of the Fisher-KPP type* (1.43). *Let u be a solution of* (1.1) *with an initial condition $u_0 = \mathbb{1}_U$, where $U \subset \mathbb{R}^N$ has nonempty interior and satisfies*

$$U \subset \big\{(x', x_N) \in \mathbb{R}^N : x_N \leq \gamma(x')\big\}, \tag{5.34}$$

for a function $\gamma \in L^\infty_{loc}(\mathbb{R}^{N-1})$ satisfying (1.30). *Then, any function $\psi \in \Omega_{\mathrm{e}_N}(u)$ is of the form $\psi = \psi(x_N)$. In particular, for any $X \in \mathbb{R}$, there holds*

$$\nabla_{x'} u(t, x', x_N) \to 0 \quad \text{as } t \to +\infty, \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } x_N \in [X, +\infty),$$

and if the inclusion is replaced by an equality in (5.34), *then*

$$\nabla_{x'} u(t, x', x_N) \to 0 \quad \text{as } t \to +\infty, \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } x_N \in \mathbb{R}.$$

Proof. The proof consists in showing that, when restricted to the directional Ω-limit set, the arguments of the proof of Theorem 1.14 can be performed when hypotheses (1.44) and (1.47) are replaced by the assumptions that U has non-empty interior and fulfills (5.34) and (1.30). We also need to prove that the functions in $\Omega_{\mathrm{e}_N}(u)$ are one-dimensional precisely in the direction e_N. The only place where (1.44) and (1.47) were used in the proof of Theorem 1.14 is the Step 1. In that step, we also determined the direction e which, in the end, turns out to be the direction of symmetry; we then need to verify that $e = \mathrm{e}_N$ in the present case, in addition to the onther conclusions of that Step 1, and we will be done here with $\Omega_{\mathrm{e}_N}(u)$. In the sequel, we fix a point z_0 in U (U is not empty since its interior itself is not empty).

Consider a function $\psi \in \Omega_{\mathrm{e}_N}(u)$, and let $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N\backslash\{0\}$ be the associated sequences given in the Definition 5.4 of $\Omega_{\mathrm{e}_N}(u)$, with $t_n \to +\infty$ and $x_n/|x_n| \to \mathrm{e}_N$ as $n \to +\infty$. We set $h_n := \mathrm{dist}(x_n, \overline{U})$ and $R_n := |x_n|$. Let us first deal with property (5.6), which was previously deduced from assumption (1.44). Take $c \in (0, c^*)$. If $R_n \leq ct_n$ for infinite values of n, then, remembering that U has a non-empty interior, Proposition 1.4 yields $\psi \equiv 1$ and the result trivially holds. Suppose then without loss of generality that $R_n > ct_n$ for all n large enough. Owing to (5.34), (1.30) and $\lim_{n\to+\infty} x_n/|x_n| = \mathrm{e}_N$, one infers that $h_n \to +\infty$ as $n \to +\infty$, hence $h_n > 0$ for all n without restriction. We further call $e_n := (x_n - \xi_n)/h_n$, where $\xi_n \in \pi_{x_n}$. For any $\varepsilon > 0$, (5.34) and (1.30) yield the existence of $C_\varepsilon > 0$ such that

$$\xi_n \cdot \mathrm{e}_N \leq C_\varepsilon + \varepsilon|\xi_n'| \quad \text{for all } n \in \mathbb{N},$$

where $\xi_n' \in \mathbb{R}^{N-1}$ is the vector of the first $N-1$ components of ξ_n, and moreover

$$|\xi_n'| \leq |\xi_n| \leq h_n + |x_n| = \mathrm{dist}(x_n, \overline{U}) + R_n \leq |x_n - z_0| + R_n \leq 2R_n + |z_0|.$$

We deduce from these properties

$$\begin{aligned}
\frac{h_n^2}{R_n^2} = \frac{|x_n - \xi_n|^2}{R_n^2} &= 1 + \frac{|\xi_n|^2}{R_n^2} - 2\frac{x_n}{R_n}\cdot\frac{\xi_n}{R_n} \geq 1 - 2\Big|\frac{x_n}{R_n} - \mathrm{e}_N\Big| \times \frac{|\xi_n|}{R_n} - 2\mathrm{e}_N \cdot \frac{\xi_n}{R_n} \\
&\geq 1 - 4\Big|\frac{x_n}{R_n} - \mathrm{e}_N\Big|\Big(1 + \frac{|z_0|}{2R_n}\Big) - 2\frac{C_\varepsilon}{R_n} - 2\varepsilon\frac{|\xi_n'|}{R_n} \\
&\geq 1 - 4\Big|\frac{x_n}{R_n} - \mathrm{e}_N\Big|\Big(1 + \frac{|z_0|}{2R_n}\Big) - 2\frac{C_\varepsilon}{R_n} - 4\varepsilon\Big(1 + \frac{|z_0|}{2R_n}\Big),
\end{aligned}$$

which tends to $1-4\varepsilon$ as $n \to +\infty$ because $x_n/R_n \to \mathrm{e}_N$ and $R_n > ct_n \to +\infty$ as $n \to +\infty$. Since $\varepsilon > 0$ is arbitrary and $h_n \le R_n + |z_0|$, we deduce that

$$\frac{h_n}{R_n} \to 1 \quad \text{as } n \to +\infty$$

and therefore h_n fulfills property (5.6), because R_n does without loss of generality, for every $c \in (0, c^*)$. We also infer that, for every $\varepsilon > 0$,

$$e_n \cdot \mathrm{e}_N = \frac{x_n \cdot \mathrm{e}_N}{h_n} - \frac{\xi_n \cdot \mathrm{e}_N}{h_n} \ge \frac{x_n \cdot \mathrm{e}_N}{h_n} - \frac{C_\varepsilon}{h_n} - \frac{\varepsilon |\xi_n'|}{h_n} \ge \frac{x_n \cdot \mathrm{e}_N}{h_n} - \frac{C_\varepsilon}{h_n} - \frac{\varepsilon(2R_n + |z_0|)}{h_n} \underset{n \to +\infty}{\longrightarrow} 1 - 2\varepsilon,$$

whence

$$e = \lim_{n \to +\infty} e_n = \mathrm{e}_N.$$

It only remains to prove property (5.8), that was a consequence of (1.47) in the proof of Theorem 1.14. Here, to get (5.8), it is therefore sufficient to show that

$$\limsup_{n \to +\infty} \Big(\sup_{y \in U \setminus \{\xi_n\}} \frac{y - \xi_n}{|y - \xi_n|} \cdot e_n \Big) \le 0. \tag{5.35}$$

Assume by contradiction that this property fails. Then up to extraction of a subsequence, there exist $\ell > 0$ and a sequence $(y_n)_{n \in \mathbb{N}}$ in U such that

$$y_n \neq \xi_n \quad \text{and} \quad \frac{y_n - \xi_n}{|y_n - \xi_n|} \cdot e_n \ge \ell \quad \text{for all } n. \tag{5.36}$$

Combining (5.36) with the fact that $|y_n - x_n| \ge \mathrm{dist}(x_n, U) = h_n$, one finds

$$1 \le \frac{|y_n - x_n|^2}{h_n^2} = \frac{|y_n - \xi_n|^2}{h_n^2} + 1 - 2\frac{y_n - \xi_n}{h_n} \cdot e_n \le \frac{|y_n - \xi_n|^2}{h_n^2} + 1 - 2\ell \frac{|y_n - \xi_n|}{h_n},$$

that is,

$$|y_n - \xi_n| \ge 2\ell h_n. \tag{5.37}$$

Next, using that $h_n/R_n \to 1$, $x_n/R_n = x_n/|x_n| \to \mathrm{e}_N$ and also $e_n \to \mathrm{e}_N$ as $n \to +\infty$, one gets

$$\frac{\xi_n \cdot e_n}{R_n} = \frac{\xi_n - x_n}{R_n} \cdot e_n + \frac{x_n}{R_n} \cdot e_n = -\frac{h_n}{R_n} + \frac{x_n}{R_n} \cdot e_n \longrightarrow 0 \quad \text{as } \ n \to +\infty, \tag{5.38}$$

which, together with (5.36) and (5.37), yields

$$\frac{|y_n|}{R_n} \ge \frac{y_n \cdot e_n}{R_n} \ge \ell \frac{|y_n - \xi_n|}{R_n} + \frac{\xi_n \cdot e_n}{R_n} \ge 2\ell^2 \, \frac{h_n}{R_n} + \frac{\xi_n \cdot e_n}{R_n} \underset{n \to +\infty}{\longrightarrow} 2\ell^2 \quad \text{as } n \to +\infty,$$

hence in particular $|y_n| \to +\infty$ as $n \to +\infty$. Finally, gathering the above estimate with the contradictory assumption (5.36) one infers, for n large,

$$\frac{y_n}{|y_n|} \cdot e_n = \frac{|y_n - \xi_n|}{|y_n|} \times \frac{y_n - \xi_n}{|y_n - \xi_n|} \cdot e_n + \frac{R_n}{|y_n|} \times \frac{\xi_n}{R_n} \cdot e_n \ge \ell \frac{|y_n - \xi_n|}{|y_n|} - \frac{|\xi_n \cdot e_n|}{\ell^2 R_n}. \tag{5.39}$$

Let us analyze the behavior of the last two terms as $n \to +\infty$. The first term satisfies $\liminf_{n \to +\infty} |y_n - \xi_n|/|y_n| > 0$, since otherwise one would have $|y_n - \xi_n|/|y_n| \to 0$

as $n \to +\infty$ up to extraction of a subsequence, hence $h_n/|y_n| \to 0$ as $n \to +\infty$ by (5.37), but then $|y_n - \xi_n|/|y_n| \geq 1 - |\xi_n|/|y_n| \geq 1 - (2R_n + |z_0|)/|y_n| \to 1$ as $n \to +\infty$ (since $R_n/h_n \to 1$ as $n \to +\infty$), leading to a contradiction. As for the second term of the right-hand side of (5.39), it converges to 0 because, we recall, $\xi_n \cdot e_n/R_n$ does by (5.38). In conclusion,

$$\liminf_{n\to+\infty} \frac{y_n}{|y_n|} \cdot e_n > 0$$

and thus

$$\liminf_{n\to+\infty} \frac{y_n}{|y_n|} \cdot e_N > 0,$$

because $(e_n)_{n\in\mathbb{N}}$ converges to e_N. Recalling that the points $(y_n)_{n\in\mathbb{N}}$ belong to U and that $|y_n| \to +\infty$ as $n \to +\infty$, we have found a contradiction with the hypotheses (5.34) and (1.30), and the local boundedness of γ. This finally shows (5.35) and this concludes the proof of the fact that any function in $\Omega_{\mathrm{e}_N}(u)$ is of the form $\psi(x_N)$.

The last statements of the theorem then immediately follow from parabolic estimates and Proposition 1.4. □

The following result, which is a consequence of Theorem 5.5 and Lemma 4.1, states that, under conditions (1.28) and (1.30), the level curves of u become locally uniformly flat along sequences of times diverging to $+\infty$.

Corollary 5.6. *Assume that f is of the Fisher-KPP type* (1.43). *Let u be a solution of* (1.1) *with an initial condition of the type* (1.28). *If γ satisfies* (1.30), *then*

$$\liminf_{t\to+\infty} \left|\nabla_{x'} X_\lambda(t, x')\right| = 0$$

for every $\lambda \in (0,1)$ and $x' \in \mathbb{R}^{N-1}$, and even

$$\liminf_{t\to+\infty} \Big(\max_{a\leq\lambda\leq b,\, |x'|\leq A} \left|\nabla_{x'} X_\lambda(t, x')\right| \Big) = 0 \tag{5.40}$$

for every $0 < a \leq b < 1$ and $A > 0$.[11]

Proof. Fix $A > 0$, $0 < a \leq b < 1$ and then any a', b' and b'' such that

$$0 < a' < a \leq b < b' < b'' < 1.$$

Let ζ be the solution of the ordinary differential equation $\dot\zeta(t) = f(\zeta(t))$ for $t \in \mathbb{R}$, with $\zeta(0) = a'$. Because of (1.43), there is $\tau > 0$ such that $\zeta(\tau) = b''$. Now, for $\rho > 0$, let v_ρ denote the solution of (1.1) with initial condition

$$v_\rho(0, \cdot) = a' \mathbb{1}_{B_\rho}.$$

Since f is Lipschitz continuous in $[0,1]$, it is easy to see that $v_\rho(\tau, \cdot) \to b''$ as $\rho \to +\infty$, locally uniformly in $\mathbb{R}^N$. In particular, let us fix in the sequel a large enough real number ρ such that

$$\rho > c^*\tau \quad \text{and} \quad v_\rho(\tau, 0) > b', \tag{5.41}$$

where we recall that $c^* = 2\sqrt{f'(0)}$ is the minimal speed of traveling fronts connecting 1 to 0 in the Fisher-KPP case (1.43).

[11] We recall that the function $(\lambda, t, x') \mapsto \nabla_{x'} X_\lambda(t, x')$ is continuous in $(0,1) \times (0, +\infty) \times \mathbb{R}^{N-1}$.

We now claim that there exist $\varepsilon > 0$ and $T > 0$ such that

$$\begin{array}{l} \forall\, t \geq T,\ \forall\, |x'| \leq A,\ \forall\, \lambda \in [a,b], \\ |\partial_{x_N} u(t,x',X_\lambda(t,x'))| \leq \varepsilon \Longrightarrow a' < \displaystyle\min_{\overline{B_{\rho+A}(x',X_\lambda(t,x'))}} u(t,\cdot) \leq \max_{\overline{B_{\rho+A}(x',X_\lambda(t,x'))}} u(t,\cdot) < b'. \end{array} \tag{5.42}$$

Indeed, otherwise, there would exist a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ diverging to $+\infty$, a sequence $(x_n)_{n\in\mathbb{N}}$ in $\overline{B'_A} \times \mathbb{R}$ such that $\partial_{x_N} u(t_n,x_n) \to 0$ as $n \to +\infty$, together with

$$\left\{ \begin{array}{l} a \leq u(t_n,x_n) \leq b, \\ \text{and either } \displaystyle\min_{\overline{B_{\rho+A}(x_n)}} u(t_n,\cdot) \leq a' < a \ \text{ or } \ \max_{\overline{B_{\rho+A}(x_n)}} u(t_n,\cdot) \geq b' > b, \ \text{ for all } n \in \mathbb{N}. \end{array} \right. \tag{5.43}$$

Up to extraction of a subsequence, the functions $(t,x) \mapsto u(t_n+t,x_n+x)$ converge in $C^{1;2}_{loc}(\mathbb{R}\times\mathbb{R}^N)$ to a solution u_∞ of (1.1) such that $0 \leq u_\infty \leq 1$ and $\partial_{x_N} u_\infty \leq 0$ in $\mathbb{R}\times\mathbb{R}^N$ (remember that $\partial_{x_N} u < 0$ in $(0,+\infty)\times\mathbb{R}^N$), while $\partial_{x_N} u_\infty(0,0) = 0$. The strong parabolic maximum principle applied to the function $\partial_{x_N} u_\infty$ then yields $\partial_{x_N} u_\infty \equiv 0$ in $(-\infty,0]\times\mathbb{R}^N$ and then in $\mathbb{R}\times\mathbb{R}^N$. Furthermore, since the sequence $(x'_n)_{n\in\mathbb{N}}$ is bounded (in $\mathbb{R}^{N-1}$), it follows from Theorem 5.5 that $\nabla_{x'} u_\infty \equiv 0$ in $\mathbb{R}\times\mathbb{R}^N$. Finally, $\nabla u_\infty \equiv 0$ in $\mathbb{R}\times\mathbb{R}^N$ and there holds in particular $\max_{\overline{B_{\rho+A}(x_n)}} |u(t_n,\cdot) - u(t_n,x_n)| \to 0$ as $n \to +\infty$, a contradiction with (5.43). Therefore, the claim (5.42) has been proved.

To complete the proof of Corollary 5.6, assume by way of contradiction that the conclusion (5.40) does not hold. Then, using (1.33) and Theorem 5.5, one gets that

$$\min_{a\leq\lambda\leq b,\, |x'|\leq A} |\partial_{x_N} u(t,x',X_\lambda(t,x'))| \to 0 \ \text{ as } t \to +\infty.$$

Together with (5.42), there is then $T' > 0$ such that, for every $t \geq T'$, there are $x'_t \in \overline{B'_A}$ and $\lambda_t \in [a,b]$ such that

$$a' < \min_{\overline{B_{\rho+A}(x'_t,X_{\lambda_t}(t,x'_t))}} u(t,\cdot) \leq \max_{\overline{B_{\rho+A}(x'_t,X_{\lambda_t}(t,x'_t))}} u(t,\cdot) < b'.$$

Since $\overline{B_\rho(0,X_{\lambda_t}(t,x'_t))} \subset \overline{B_{\rho+A}(x'_t,X_{\lambda_t}(t,x'_t))}$, it then follows that

$$a' < \min_{\overline{B_\rho(0,X_{\lambda_t}(t,x'_t))}} u(t,\cdot) \leq \max_{\overline{B_\rho(0,X_{\lambda_t}(t,x'_t))}} u(t,\cdot) < b'.$$

In particular, for every $t \geq T'$, one has on the one hand $X_{b'}(t,0) < X_{\lambda_t}(t,x'_t) - \rho$, and on the other hand $u(t,\cdot + (0,X_{\lambda_t}(t,x'_t))) \geq a' \mathbb{1}_{B_\rho} = v_\rho(0,\cdot)$ in $\mathbb{R}^N$. The maximum principle then yields in particular $u(t+\tau,0,X_{\lambda_t}(t,x'_t)) \geq v_\rho(\tau,0) > b'$ from (5.41), hence $X_{b'}(t+\tau,0) > X_{\lambda_t}(t,x'_t)$. As a consequence, $X_{b'}(t+\tau,0) > X_{b'}(t,0) + \rho$ for every $t \geq T'$, and thus

$$\limsup_{s\to+\infty} \frac{X_{b'}(s,0)}{s} \geq \frac{\rho}{\tau} > c^*$$

owing to (5.41). This last formula is in contradiction with Lemma 4.1. As a conclusion, (5.40) has been proved. □

We complete this section with the proof of Corollary 1.15.

Proof of Corollary 1.15. We claim that, under the assumptions of this corollary, the set $U = \mathrm{supp}\, u_0$ fulfills the hypotheses (1.44) and (1.47) of Theorem 1.14. For the former, observe that, by (1.48), there exists $R > 0$ such that

$$\forall\, \widetilde{x}' \in \mathbb{R}^{N-1},\ \forall\, y' \in \mathbb{R}^{N-1} \backslash B'_R(\widetilde{x}'), \quad \gamma(y') \geq \gamma(\widetilde{x}') - |y' - \widetilde{x}'|. \tag{5.44}$$

Take an arbitrary $\delta > 0$ and any $x \in U$. One has that either $x \in U_\delta$, or there exists $\widetilde{x} \in \partial U$ with $|\widetilde{x} - x| \leq \delta$. In the latter case, writing $\widetilde{x} = (\widetilde{x}', \gamma(\widetilde{x}'))$, one gets from (5.44)

$$U \supset \big\{(y', y_N) \in (\mathbb{R}^{N-1} \backslash B'_R(\widetilde{x}')) \times \mathbb{R} : y_N \leq \gamma(\widetilde{x}') - |y' - \widetilde{x}'|\big\} =: E,$$

and any ball $\overline{B_\delta((\widetilde{x}' + z', \gamma(\widetilde{x}') + z_N))}$ with $|z'| = R + \delta$ and $z_N = -R - 3\delta$ is contained in the latter set, hence in U. This shows that

$$\mathrm{dist}(x, U_\delta) \leq \delta + \mathrm{dist}(\widetilde{x}, U_\delta) \leq \delta + \sqrt{(R+\delta)^2 + (R+3\delta)^2},$$

and, since x was arbitrary in U, one infers that $d_{\mathcal{H}}(U, U_\delta) \leq \delta + \sqrt{(R+\delta)^2 + (R+3\delta)^2}$, i.e., (1.44) holds.

In order to deal with (1.47), we will reduce to the situation of the proof of Theorem 5.5 by suitable translations. Consider $\psi \in \Omega(u)$, and let $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ be the associated sequences given in the definition (1.3). For $n \in \mathbb{N}$, we write $x_n = (x'_n, x_{N,n})$ and then consider the translation $\mathcal{T}_n : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\mathcal{T}_n(x) := x - \big(x'_n, \gamma(x'_n)\big).$$

We then call

$$\widetilde{U}_n := \mathcal{T}_n(U), \qquad \widetilde{x}_n := \mathcal{T}_n(x_n) = \big(0, x_{N,n} - \gamma(x'_n)\big), \qquad \widetilde{\xi}_n := \mathcal{T}_n(\xi_n),$$

where $\xi_n \in \pi_{x_n}$. One sees that $0 \in \partial\widetilde{U}_n$ and $\widetilde{\xi}_n$ is an orthogonal projection of $\widetilde{x}_n$ onto $\widetilde{U}_n$. We want to apply the arguments of the proof of Theorem 5.5 to these new objects. One has

$$R_n := |\widetilde{x}_n| = |\widetilde{x}_n - 0| \geq \mathrm{dist}(\widetilde{x}_n, \widetilde{U}_n) = \mathrm{dist}(x_n, U_n) =: h_n,$$

and, as in the proof of Theorem 1.14, one restricts to the case where $(h_n)_{n\in\mathbb{N}}$ fulfills (5.6), because otherwise $\psi \equiv 1$ due to condition (1.44). Observe further that we now directly have $\widetilde{x}_n/|\widetilde{x}_n| = \mathrm{e}_N$ for all n large enough (since $\mathrm{dist}(\widetilde{x}_n, \widetilde{U}_n) = h_n > 0$ for all n large enough). Finally, the sets $\widetilde{U}_n$ are the subgraphs of the functions $\widetilde{\gamma}_n$ given by $\widetilde{\gamma}_n(x') := \gamma(x' + x'_n) - \gamma(x'_n)$, which, by (1.48) fulfill (1.30) (with equality) uniformly with respect to $n \in \mathbb{N}$. This allows one to repeat exactly the same arguments as in the proof of Theorem 5.5, with $(x_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ and U replaced by $(\widetilde{x}_n)_{n\in\mathbb{N}}$, $(\widetilde{\xi}_n)_{n\in\mathbb{N}}$ and $\widetilde{U}_n$ respectively, and conclude that ψ is one-dimensional in the direction e_N. The corollary is proved. □

6 Lag behind the front in the Fisher-KPP case

This section is devoted to the proofs of Theorem 1.19, Proposition 1.20, and Corollary 1.21 on the lag behind the front in the Fisher-KPP case, and on further asymptotic one-dimensional symmetry results in the direction e_N, when the initial conditions u_0 are of the type (1.28), with $\gamma(x')$ going to $-\infty$ suitably fast as $|x'| \to +\infty$.

Proof of Theorem 1.19. Throughout the proof, we assume that f is of the Fisher-KPP type (1.43) (hence, Hypotheses 1.1 and 1.3 are satisfied and $c^* = 2\sqrt{f'(0)}$ is the minimal speed of traveling fronts connecting 1 to 0), and that u_0 satisfies (1.28) and (1.50). By the monotonicity of the functions X_λ with respect to λ, it is sufficient to derive the result (1.51) for any given $\lambda \in (0,1)$, which is fixed throughout the proof.

Let $\underline{u}$ be the solution to (1.1) emerging from a continuous, compactly supported, radially symmetric and non-trivial initial datum $\underline{u}_0$ such that $0 \le \underline{u}_0 \le u_0 \le 1$ in $\mathbb{R}^N$. We know from [18] or [45, Theorem 1.1] that there exists $\sigma \in \mathbb{R}$ such that, for any $K > 0$, there exists $T_K > 0$ for which there holds

$$\forall\, t \ge T_K,\ \forall\, |x'| \le K, \quad \underline{u}\Big(t, x', c^* t - \frac{N+2}{c^*}\,\log t + \sigma\Big) > \lambda.$$

Since $1 \ge u \ge \underline{u} \ge 0$ in $[0,+\infty) \times \mathbb{R}^N$ by the parabolic comparison principle, we find that

$$X_\lambda(t,x') \ge c^* t - \frac{N+2}{c^*}\log t + \sigma + o(1) \quad \text{as} \quad t \to +\infty, \tag{6.1}$$

locally uniformly in $x' \in \mathbb{R}^{N-1}$.

In order to show the reverse inequality, we will construct a supersolution v larger than u at time 0, for which we are able to explicitly compute the lag. First of all, owing to (1.50), we can take $\beta \in \mathbb{R}$ satisfying

$$\limsup_{|x'|\to+\infty} \frac{\gamma(x')}{\log(|x'|)} < \beta < -\frac{2(N-1)}{c^*}.$$

We then take $M > 0$ large enough so that

$$\forall\, x' \in \mathbb{R}^{N-1}, \quad \gamma(x') \le \beta \log(1+|x'|) + M.$$

Hence, by the parabolic comparison principle, if we show the desired upper bound for X_λ when $\gamma(x')$ is replaced by $\beta\log(1+|x'|)+M$, we are done. Up to a translation of the coordinate system, we can further assume that $M = 0$. We then assume from now on that

$$\gamma(x') = \beta \log(1+|x'|).$$

Next, since γ is globally Lipschitz continuous, we can find a radius $\delta > 0$ large enough, depending on N and the Lipschitz constant of γ, such that

$$\big\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \le \gamma(x')\big\} \subset \bigcup_{k \in \mathbb{Z}^{N-1}} \bigcup_{h \in \mathbb{N}\cup\{0\}} B_\delta(k, \gamma(k) - h). \tag{6.2}$$

We then consider the solution $0 \le w \le 1$ of (1.1) emerging from a C^∞ compactly supported and radially symmetric initial datum w_0 such that $\mathbb{1}_{B_\delta} \le w_0 \le 1$ in $\mathbb{R}^N$, and we define a nonnegative function v in $[0,+\infty) \times \mathbb{R}^N$ by

$$v(t,x) = v(t,x',x_N) := \sum_{k \in \mathbb{Z}^{N-1}} \sum_{h=0}^{\infty} w(t, x'-k, x_N - \gamma(k) + h). \tag{6.3}$$

From Gaussian estimates, for any $T > 0$, there are some positive constants α_T and C_T such that $0 \le w(t,x) \le C_T e^{-\alpha_T |x|^2}$ and $|\partial_t w(t,x)| + |\partial_{x_i} w(t,x)| + |\partial_{x_i x_j} w(t,x)| \le C_T e^{-\alpha_T |x|^2}$ for

all $(t,x) \in [0,T] \times \mathbb{R}^N$ and $1 \le i,j \le N$. Therefore, the function v is well defined in $[0,+\infty) \times \mathbb{R}^N$ and of class $C^{1;2}_{t;x}([0,+\infty) \times \mathbb{R}^N)$. Furthermore, since the function f is at least Lipschitz continuous in $[0,1]$ and satisfies (1.43), then, for any series $\sum a_j$ of nonnegative real numbers such that $\sum_{j\in\mathbb{N}} a_j \le 1$, the series $\sum f(a_j)$ converges too and $f(\sum_{j\in\mathbb{N}} a_j) \le \sum_{j\in\mathbb{N}} f(a_j)$. It then follows that $\min(v,1)$ is a (generalized) supersolution of (1.1) in $[0,+\infty)\times\mathbb{R}^N$. For any $(x',x_N) \in B_\delta(k,\gamma(k)-h)$, with $k \in \mathbb{Z}^{N-1}$ and $h \in \mathbb{N}\cup\{0\}$, there holds that $v(0,x',x_N) \ge w_0(x'-k, x_N-\gamma(k)+h) = 1$, whence $v(0,\cdot) \ge u_0$ due to (6.2). The parabolic comparison principle then implies that

$$0 \le u \le \min(v,1) \ \text{ in } [0,+\infty) \times \mathbb{R}^N. \tag{6.4}$$

Let us estimate the position of the level sets of v. Let φ denote the profile of a traveling front connecting 1 to 0 with minimal speed $c^* = 2\sqrt{f'(0)}$. It is known [13, 18, 45] that

$$w(t,x) - \varphi\big(|x| - \rho(t)\big) \to 0 \ \text{ as } t \to +\infty,$$

uniformly with respect to $x \in \mathbb{R}^N$, where $\rho(t)$ satisfies, for some $\widetilde{t} > 0$ and $\widetilde{C} \in \mathbb{R}$,

$$0 < \rho(t) \le c^* t - \frac{N+2}{c^*} \log t + \widetilde{C} \ \text{ for all } t \ge \widetilde{t}. \tag{6.5}$$

We also know that there exists a constant $A > 0$ such that $\varphi(r) \le A\, r\, e^{-c^* r/2}$ for all $r \ge 1$. Lastly, by [45, Proposition 5], there is a constant $C > 0$ such that

$$w(t,x) \le C\big(|x| - \rho(t)\big) e^{-c^*(|x|-\rho(t))/2} \ \text{ for all } t \ge 1 \text{ and } |x| - \rho(t) \ge 1. \tag{6.6}$$

Using the fact $\gamma(k) \le 0$ for all $k \in \mathbb{Z}^{N-1}$ (since $\beta < -2(N-1)/c^* < 0$), we infer that, for every $t \ge \max(\widetilde{t},1)$ (for which $\rho(t) \ge 0$ by (6.5) and moreover (6.6) holds) and every $y \ge 1$,

$$v(t,0,\rho(t)+y) \le C \sum_{k\in\mathbb{Z}^{N-1}} \sum_{h=0}^{\infty} \big(|(-k,\rho(t)+y-\gamma(k)+h)| - \rho(t)\big) e^{-c^*(|(-k,\rho(t)+y-\gamma(k)+h)|-\rho(t))/2}.$$

Because $r \mapsto re^{-c^* r/2}$ is decreasing for $r \ge 2/c^*$, we then deduce that, for every $t \ge \max(\widetilde{t},1)$ and every $y \ge \max(1, 2/c^*)$,

$$\begin{aligned} v(t,0,\rho(t)+y) \le\ & C \sum_{k\in\mathbb{Z}^{N-1}} \sum_{h=0}^{\infty} (y - \gamma(k) + h) e^{-c^*(y-\gamma(k)+h)/2} \\ =\ & C\Big(\sum_{k\in\mathbb{Z}^{N-1}} e^{c^*\gamma(k)/2} \Big) \Big(\sum_{h=0}^{\infty} (y+h) e^{-c^*(y+h)/2} \Big) \\ & + C\Big(\sum_{k\in\mathbb{Z}^{N-1}} |\gamma(k)| e^{c^*\gamma(k)/2} \Big) \Big(\sum_{h=0}^{\infty} e^{-c^*(y+h)/2} \Big). \end{aligned}$$

As a consequence, calling

$$C_1 := \sum_{h=0}^{\infty} e^{-c^* h/2} = \frac{1}{1-e^{-c^*/2}} \ \text{ and } \ C_2 := \sum_{h=0}^{\infty} h e^{-c^* h/2} = \frac{e^{-c^*/2}}{(1-e^{-c^*/2})^2}, \tag{6.7}$$

we find that, for every $t \geq \max(\widetilde{t}, 1)$ and every $y \geq \max(1, 2/c^*)$,

$$v(t, 0, \rho(t) + y) \leq C(C_1 y + C_2) e^{-c^* y/2} \sum_{k \in \mathbb{Z}^{N-1}} e^{c^* \gamma(k)/2} + C C_1 e^{-c^* y/2} \sum_{k \in \mathbb{Z}^{N-1}} |\gamma(k)| e^{c^* \gamma(k)/2}.$$

Let us study these series in k. Recalling that $\gamma(x') = \beta \log(1 + |x'|)$, we compute

$$\sum_{k \in \mathbb{Z}^{N-1}} |\gamma(k)| e^{c^* \gamma(k)/2} = |\beta| \sum_{k \in \mathbb{Z}^{N-1}} (1 + |k|)^{c^* \beta/2} \log(1 + |k|).$$

We now use the fact that, for any pair of nonnegative functions $p, q : \mathbb{R} \to \mathbb{R}$, with p nonincreasing and q nondecreasing, there holds that

$$\forall\, k \in \mathbb{Z}^{N-1}, \quad p(|k|)\, q(|k|) \leq \int_{k + (0,1)^{N-1}} p(|x'| - \sqrt{N-1})\, q(|x'| + \sqrt{N-1})\, dx',$$

and therefore, for any measurable set $A \subset \mathbb{R}^{N-1}$, we get

$$\sum_{k \in \mathbb{Z}^{N-1} \cap A} p(|k|)\, q(|k|) \leq \int_{A + B'_{\sqrt{N-1}}} p(|x'| - \sqrt{N-1})\, q(|x'| + \sqrt{N-1})\, dx'. \tag{6.8}$$

By using $p(r) = (1 + r^+)^{c^* \beta/2}$ and $q(r) = \log(1 + r^+)$, this allows us to estimate

$$\sum_{k \in \mathbb{Z}^{N-1} \setminus B'_{2\sqrt{N-1}}} (1 + |k|)^{c^* \beta/2} \log(1 + |k|)$$
$$\leq \int_{|x'| \geq \sqrt{N-1}} (1 + |x'| - \sqrt{N-1})^{c^* \beta/2} \log(1 + |x'| + \sqrt{N-1})\, dx',$$

which is finite because $\beta < -2(N-1)/c^*$. This shows that $\sum_{k \in \mathbb{Z}^{N-1}} |\gamma(k)| e^{c^* \gamma(k)/2}$ converges, as well as $\sum_{k \in \mathbb{Z}^{N-1}} e^{c^* \gamma(k)/2}$ (since $|\gamma(k)| \to +\infty$ as $|k| \to +\infty$). It follows that there exists a constant $C' > 0$ such that, for every $t \geq \max(\widetilde{t}, 1)$ and $y \geq \max(1, 2/c^*)$,

$$v(t, 0, \rho(t) + y) \leq C' y\, e^{-c^* y/2}.$$

Because $u \leq v$ by (6.4), we eventually deduce from the definition of $X_\lambda(t, 0)$ that there is a constant $C'' \in \mathbb{R}$ such that $X_\lambda(t, 0) \leq \rho(t) + C''$ for all $t \geq \max(\widetilde{t}, 1)$, that is, by (6.5),

$$X_\lambda(t, 0) \leq c^* t - \frac{N+2}{c^*} \log t + \widetilde{C} + C'' \ \text{ for all } t \geq \max(\widetilde{t}, 1).$$

Finally, since $\gamma(x') = \beta \log(1 + |x'|)$ is radially symmetric and decreasing (remember that β is here negative), it follows from a standard reflection argument with respect to hyperplanes parallel to e_N (similarly as in the proof of Theorem 1.9) that, for every $t > 0$ and $x_N \in \mathbb{R}$, the function $x' \mapsto u(t, x', x_N)$ is radially symmetric and decreasing with respect to $|x'|$. We then deduce from the above estimate that

$$X_\lambda(t, x') \leq c^* t - \frac{N+2}{c^*} \log t + \widetilde{C} + C'' \ \text{ for all } x' \in \mathbb{R}^{N-1} \text{ and } t \geq \max(\widetilde{t}, 1). \tag{6.9}$$

The proof of Theorem 1.19 is thereby complete. □

We continue with the proof of Proposition 1.20 which provides a bound of the lag for functions γ satisfying a logarithmic upper bound at infinity.

Proof of Proposition 1.20. Throughout the proof, λ is any fixed real number in $(0,1)$. Consider first the case where $\sigma > -(N-1)$ and γ is given by

$$\gamma(x') = \frac{2\sigma}{c^*} \log(1+|x'|) \ \text{ for all } x' \in \mathbb{R}^{N-1}. \tag{6.10}$$

Since γ is globally Lipschitz continuous, as in the proof of Theorem 1.19, we can find $\delta > 0$ sufficiently large so that, for the functions w and v defined as in (6.3), the inequality (6.4) holds true, and the function w still fulfills (6.6) for some positive constant C, with $\rho(t)$ satisfying (6.5). Let us fix

$$\beta > \frac{\sigma + N - 1}{c^*} > 0. \tag{6.11}$$

Our aim is to show that

$$v(t, 0, \rho(t) + \beta \log t) \to 0 \ \text{ as } t \to +\infty. \tag{6.12}$$

Let us postpone for a moment the proof of (6.12) and conclude the argument. Together with (6.4), this will imply that $u(t, 0, \rho(t) + \beta \log t) \to 0$ as $t \to +\infty$, hence $X_\lambda(t,0) \le \rho(t) + \beta \log t$ for all t large enough, and then by (6.5),

$$\limsup_{t \to +\infty} \frac{X_\lambda(t,0) - c^* t}{\log t} \le \beta - \frac{N+2}{c^*}.$$

Since γ is given by (6.10), we infer from Theorem 1.9 with assumption (iv) that the above estimate holds true for $X_\lambda(t,x')$, locally uniformly with respect to $x' \in \mathbb{R}^{N-1}$, and then (1.53) follows from the arbitrariness of β in (6.11). If we now consider a general γ satisfying (1.52) with $\sigma \ge -(N-1)$, we take an arbitrary $\sigma' > \sigma$ and then, since γ satisfies $\gamma(x') < (2\sigma'/c^*)\log(1+|x'|)$ for $x' \in \mathbb{R}^{N-1}$ up to an additive constant, we deduce from what precedes and the comparison principle, that (1.53) holds with σ replaced by σ', locally uniformly with respect to $x' \in \mathbb{R}^{N-1}$. This gives the conclusion of the proposition, owing to the arbitrariness of $\sigma' \in (\sigma, +\infty)$.

So, we are left to prove that (6.12) holds with β and σ as in (6.11), when γ is given by (6.10). We can take $\alpha \in (1/2, 1)$, close enough to $1/2$, in such a way that

$$\frac{2\alpha\sigma}{c^*} \le \frac{2\alpha(\sigma + N - 1)}{c^*} < \beta. \tag{6.13}$$

For every $t > 0$, let us divide the following sum

$$0 \le v(t, 0, \rho(t) + \beta \log t) = \sum_{k \in \mathbb{Z}^{N-1}} \sum_{h=0}^{\infty} w(t, -k, \rho(t) + \beta \log t - \gamma(k) + h)$$

into two subsums over k, namely:

$$\begin{aligned} 0 \ \le \ v(t, 0, \rho(t) + \beta \log t) \ &= \underbrace{\sum_{k \in \mathbb{Z}^{N-1} \cap B'_{t^\alpha}} \sum_{h=0}^{\infty} w(t, -k, \rho(t) + \beta \log t - \gamma(k) + h)}_{=: I_1(t)} \\ &\quad + \underbrace{\sum_{k \in \mathbb{Z}^{N-1} \setminus B'_{t^\alpha}} \sum_{h=0}^{\infty} w(t, -k, \rho(t) + \beta \log t - \gamma(k) + h)}_{=: I_2(t)}. \end{aligned}$$

Let us first deal with the sum $I_1(t)$. Recalling that γ is given in (6.10), one has

$$\forall\, t>0,\ \forall\, k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha},\quad \beta\log t-\gamma(k)\ge\frac{\beta}{\alpha}\log|k|-\frac{2\sigma}{c^*}\log(1+|k|).$$

Hence, by (6.13) and the positivity of β, there is $t_1\ge 1$ large enough such that $\beta\log t-\gamma(k)\ge\max(1,2/c^*)$ for all $t\ge t_1$ and $k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}$. In particular, we can use the estimate (6.6) in the expression of $I_1(t)$ for $t\ge t_1$. Then, owing to the monotonicity of the function $r\mapsto r^{-c^*r/2}$ in $[2/c^*,+\infty)$ and the fact that $\rho(t)\ge 0$ by (6.5) for $t\ge\widetilde{t}$, we infer that, for $t\ge\max(t_1,\widetilde{t})$, there holds

$$\begin{array}{rcl} I_1(t) & \le & C\displaystyle\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}\sum_{h=0}^{\infty}(\beta\log t-\gamma(k)+h)\,e^{-c^*(\beta\log t-\gamma(k)+h)/2}\\ & \le & C\,(C_1\beta\log t+C_2)\,t^{-c^*\beta/2}\displaystyle\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}e^{c^*\gamma(k)/2}\\ & & +\,C\,C_1\,t^{-c^*\beta/2}\displaystyle\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}|\gamma(k)|\,e^{c^*\gamma(k)/2}, \end{array}$$

where C, C_1, C_2 are given in (6.6)-(6.7). We now use the estimate (6.8), which, we recall, holds if p is nonnegative and nonincreasing and q is nonnegative and nondecreasing. We here use it with $p(s)=1$ and $q(s)=(1+s^+)^\sigma$ if $\sigma\ge 0$, and with $p(s)=(1+s^+)^\sigma$ and $q(s)=1$ if $\sigma<0$. We get, for some $C_N>0$ and with "$\pm$" is in accordance with the sign of σ,

$$\begin{array}{rcl} \displaystyle\sum_{k\in\mathbb{Z}^{N-1}\cap(B'_{t^\alpha}\setminus B'_{2\sqrt{N-1}})}e^{c^*\gamma(k)/2} & \le & C_N\displaystyle\int_{\sqrt{N-1}}^{t^\alpha+\sqrt{N-1}}r^{N-2}(1+r\pm\sqrt{N-1})^\sigma\,dr\\ & \le & C_N\displaystyle\int_0^{t^\alpha+2\sqrt{N-1}}(r+\sqrt{N-1})^{N-2}(1+r)^\sigma\,dr\\ & \le & C_N(N-1)^{N/2-1}\displaystyle\int_0^{t^\alpha+2\sqrt{N-1}}(1+r)^{\sigma+N-2}\,dr. \end{array}$$

As a consequence, since $\sigma+N-1>0$, we find

$$\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}e^{c^*\gamma(k)/2}\le\frac{C_N(N-1)^{N/2-1}}{\sigma+N-1}\,(1+t^\alpha+2\sqrt{N-1})^{\sigma+N-1}.$$

Together with (6.13), one concludes that

$$C\,(C_1\beta\log t+C_2)\,t^{-c^*\beta/2}\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}e^{c^*\gamma(k)/2}\ \longrightarrow 0\ \ \text{as } t\to+\infty.$$

Similarly, using again (6.10) and (6.13), one gets that

$$C\,C_1\,t^{-c^*\beta/2}\sum_{k\in\mathbb{Z}^{N-1}\cap B'_{t^\alpha}}|\gamma(k)|\,e^{c^*\gamma(k)/2}\ \longrightarrow 0\ \ \text{as } t\to+\infty.$$

As a consequence, $I_1(t)\to 0$ as $t\to+\infty$.

Let us finally deal with the second sum $I_2(t)$. For each $t \geq \max(\widetilde{t},1)$ (with $\widetilde{t}>0$ given by (6.5)) and each $k \in \mathbb{Z}^{N-1} \setminus B'_{t^\alpha}$ and $h \in \mathbb{N} \cup \{0\}$, one has

$$\begin{aligned} &|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|-\rho(t) \\ &\qquad = \frac{|k|^2+(\beta\log t-\gamma(k)+h)^2+2\rho(t)(\beta\log t-\gamma(k)+h)}{|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|+\rho(t)} \\ &\qquad \geq \frac{|k|^2+h^2-2h\gamma(k)-2\gamma(k)(\beta\log t+\rho(t))}{|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|+\rho(t)}, \end{aligned} \tag{6.14}$$

since $(\beta\log t)^2+\gamma(k)^2+2\beta h\log t+2\beta\rho(t)\log t+2h\rho(t)\geq 0$ (remember that $\beta>0$ by (6.11)). In order to estimate the numerator, we use the facts that $\max(\widetilde{t},1)\leq t\leq |k|^{1/\alpha}$, with $0<1/\alpha<2$, and that $\rho(t)\sim c^*t$ as $t\to+\infty$ and $|\gamma(k)|=O(\log|k|)$ as $|k|\to+\infty$. We infer the existence of some $t_2\geq\max(\widetilde{t},1)$ such that, for every $t\geq t_2$, $k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}$ and $h\in\mathbb{N}\cup\{0\}$,

$$|k|^2+h^2-2h\gamma(k)-2\gamma(k)(\beta\log t+\rho(t)) \geq \frac{|k|^2+h^2}{2} \geq 0. \tag{6.15}$$

Using the same estimates and $1<1/\alpha$, one gets for the denominator,

$$\begin{aligned} 0<|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|+\rho(t) &\leq |k|+\beta\log t+|\gamma(k)|+h+2\rho(t) \\ &\leq 3c^*|k|^{1/\alpha}+h\leq 6c^*|(k,h)|^{1/\alpha}, \end{aligned} \tag{6.16}$$

for all t larger than some $t_3\geq\max(\widetilde{t},1)$ and for all $k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}$. Summing up (6.14)-(6.16) and recalling that $1<1/\alpha<2$, one has that, for $t\geq\max(t_2,t_3)$ and $k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}$,

$$|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|-\rho(t)\geq\frac{|(k,h)|^{2-1/\alpha}}{12c^*},$$

which is larger than $\max(1,2/c^*)$ for t larger than some $t_4\geq\max(t_2,t_3)$, since $|k|\geq t^\alpha$. One eventually gets from (6.6) that, for every $t\geq t_4$,

$$\begin{aligned} I_2(t) &\leq C\sum_{k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}}\sum_{h=0}^{\infty}\Big[\big(|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|-\rho(t)\big) \\ &\qquad\qquad \times e^{-c^*(|(-k,\rho(t)+\beta\log t-\gamma(k)+h)|-\rho(t))/2}\Big] \\ &\leq \frac{C}{12c^*}\sum_{k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}}\sum_{h=0}^{\infty}|(k,h)|^{2-1/\alpha}\,e^{-(1/24)|(k,h)|^{2-1/\alpha}} \\ &\leq \frac{C}{12c^*}\sum_{k\in\mathbb{Z}^{N-1}\setminus B'_{t^\alpha}}\sum_{h\in\mathbb{Z}}|(k,h)|^{2-1/\alpha}\,e^{-(1/24)|(k,h)|^{2-1/\alpha}}. \end{aligned}$$

We then use an estimate of the type (6.8) in dimension N, and the inequality $2-1/\alpha>0$, to finally infer that $I_2(t)\to 0$ as $t\to+\infty$.

As a conclusion, $v(t,0,\rho(t)+\beta\log t)\to 0$ as $t\to+\infty$. The claim (6.12) has been shown, and, as already emphasized, this completes the proof of Proposition 1.20. □

Proof of Corollary 1.21. (i) On the one hand, we know from Proposition 1.4 that

$$X_\lambda(t,x')\to+\infty \quad \text{as } t\to+\infty,$$

locally uniformly in $\lambda \in [0,1)$ and $x' \in \mathbb{R}^{N-1}$. On the other hand, we know from Theorem 5.5 that

$$\nabla_{x'} u(t,x',x_N) \to 0 \ \text{ as } t \to +\infty, \ \text{ locally in } x' \in \mathbb{R}^{N-1} \text{ and uniformly in } x_N \in \mathbb{R}.$$

Then, owing to (1.33), in order to show that (1.34) holds locally uniformly in $\lambda \in (0,1)$, it is sufficient to derive a lower bound on $|\partial_{x_N} u|$ on the level sets. Namely, if we assume that (1.34) does not hold locally uniformly in $\lambda \in (0,1)$, then there necessarily exist a sequence $(\lambda_n)_{n\in\mathbb{N}}$ contained in some interval $[\underline{\lambda},\overline{\lambda}] \subset (0,1)$ (with $\underline{\lambda} < \overline{\lambda}$ without loss of generality), a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,+\infty)$ diverging to $+\infty$ and a bounded sequence $(x'_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$ such that

$$\partial_{x_N} u\big(t_n, x'_n, X_{\lambda_n}(t_n,x'_n)\big) \to 0 \quad \text{as} \ \ n \to +\infty.$$

Since the function $\partial_{x_N} u$ is a negative solution of a linear parabolic equation in $(0,+\infty)\times\mathbb{R}^N$, it readily follows from the strong maximum principle and parabolic estimates, as in the proof of Corollary 5.6, that

$$\partial_{x_N} u\big(t_n, x', x_N + X_{\lambda_n}(t_n,x'_n)\big) \to 0 \ \text{ as } n \to +\infty, \ \text{ locally uniformly in } (x',x_N) \in \mathbb{R}^N.$$

Writing

$$\underline{\lambda} - \overline{\lambda} = \int_{X_{\overline{\lambda}}(t_n,x'_n)}^{X_{\underline{\lambda}}(t_n,x'_n)} \partial_{x_N} u(t_n,x'_n,x_N)\,dx_N$$

and observing that $X_{\overline{\lambda}}(t_n,x'_n) \le X_{\lambda_n}(t_n,x'_n) \le X_{\underline{\lambda}}(t_n,x'_n)$, one deduces from the above convergence that $X_{\underline{\lambda}}(t_n,x'_n) - X_{\overline{\lambda}}(t_n,x'_n) \to +\infty$ as $n \to +\infty$. This is impossible because $X_{\underline{\lambda}}(t,x') - X_{\overline{\lambda}}(t,x')$ is bounded uniformly in t large enough and locally in x' thanks to Theorem 1.19. We have reached a contradiction, and the desired property follows.

(ii) By standard parabolic estimates, for given $\lambda \in (0,1)$ and $x'_0 \in \mathbb{R}^{N-1}$ and any sequence $(s_n)_{n\in\mathbb{N}}$ diverging to $+\infty$, the limit

$$\widetilde{u}(t,x',x_N) := \lim_{n\to+\infty} u(s_n+t, x', X_\lambda(s_n,x'_0)+x_N),$$

exists (up to subsequences) locally uniformly in $(t,x',x_N) \in \mathbb{R}\times\mathbb{R}^N$. We apply the estimates derived in the proof of Theorem 1.19. Namely, by (6.1), (6.9), for any $\eta \in (0,1)$, there exist $C_\eta > 0$ such that, for any $x' \in \mathbb{R}^{N-1}$,

$$\left| X_\eta(t,x') - \left(c^* t - \frac{N+2}{c^*} \log t \right) \right| \le C_\eta + o(1) \quad \text{as} \ \ t \to +\infty.$$

We deduce that, for any $\eta_1,\eta_2 \in (0,1)$, any $t \in \mathbb{R}$, and any $x'_1, x'_2 \in \mathbb{R}^{N-1}$,

$$\begin{aligned} |X_{\eta_1}(s,x'_1) + c^* t - X_{\eta_2}(s+t,x'_2)| &\le \frac{N+2}{c^*} \,|\log s - \log(s+t)| + C_{\eta_1} + C_{\eta_2} + o(1) \\ &\le C_{\eta_1} + C_{\eta_2} + o(1) \quad \text{as} \ \ s \to +\infty. \end{aligned}$$

It follows, for any $\eta \in (0,1)$ and any $t \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$, from the one hand that

$$\begin{aligned} \widetilde{u}(t,x',c^*t + C_\lambda + C_\eta + 1) &= \lim_{n\to+\infty} u(s_n+t, x', X_\lambda(s_n,x'_0) + c^* t + C_\lambda + C_\eta + 1) \\ &\le \lim_{n\to+\infty} u(s_n+t, x', X_\eta(s_n+t,x')) = \eta, \end{aligned}$$

and from the other hand that

$$\widetilde{u}(t, x', c^*t - C_\lambda - C_\eta - 1) \geq \lim_{n\to+\infty} u(s_n + t, x', X_\eta(s_n + t, x')) = \eta.$$

Owing to the arbitrariness of $\eta \in (0, 1)$, and the fact that $\widetilde{u}$ is nonincreasing with respect to x_N (as so is u) and independent of x' (from Theorem 5.5), the proof of Corollary 1.21 is complete. □

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