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Error bound characterizations of Guignard's constraint qualification in convex programming*

A. Barbara and A. Jourani †

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Abstract

This paper deals with error bound characterizations of Guignard's qualification condition for a convex inequality system in a Banach space X . We establish necessary and sufficient conditions for a closed convex set S defined by a convex function g to have Guignard's condition. These conditions are expressed in terms of the notion of error bound. Our results show that these characterizations hold in the following special cases:

1. g is the maximum of a finite number of differentiable convex functions.
2. S is closed convex and polyhedral.
3. The dimension of the subspace $\text{lin}(S)$ is less than 2 and g is positively homogeneous.

We construct technical examples showing that these characterizations are limited to the three situations above. We introduce a new condition in terms of the gauge function which allows us to give an error bound characterization of convex nondifferentiable systems and to obtain as a direct consequence different characterizations of the concept of strong conical hull intersection property (CHIP) for a finite collection of convex sets.

Keywords: Subdifferential Normal cone Guignard's condition SCHIP property. **AMS**

Classification : 49J52, 49J53, 90C31, 90C25

1 Introduction

Necessary optimality conditions are known to be very important in optimization in the computation of (possible) local or global minima. To be more concrete, consider the following optimization problem

$$\begin{cases} \min f(x) \\ g(x) \leq 0 \end{cases} \quad (1.1)$$

Here $f, g : X \mapsto \mathbb{R} \cup \{+\infty\}$ are extended lower semicontinuous convex functions and X is a Banach space. We denote by

$$S := \{x \in X : g(x) \leq 0\}$$

the feasible set of problem (1.1), $\partial f(\bar{x})$ the Fenchel subdifferential of f at \bar{x} and $N(S, \bar{x})$ the normal cone in the sense of convex analysis to S at \bar{x} . We know that for a feasible point \bar{x} of problem (1.1), the following assertions are equivalent :

1. \bar{x} is a solution of (1.1),

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$$2. 0 \in \partial f(\bar{x}) + N(S, \bar{x})$$

provided that f is Lipschitz continuous around \bar{x} . The problem is

How to compute $N(S, \bar{x})$ in terms of the data g ?

Without additional constraint qualification, there is no way to obtain this computation (take $g(x) = x^2$). So we are looking for conditions which allow us to obtain this computation. One of them is known to be Abadie's constraint qualification expressed as follows :

$$T(S, \bar{x}) = \{h \in X : g'(\bar{x}, h) \leq 0\} \quad (1.2)$$

where $T(S, \bar{x})$ denotes the tangent cone of S at \bar{x} , that is, the negative polar of the normal cone $N(S, \bar{x})$, and where $g'(\bar{x}, h)$ is the directional derivative of g at \bar{x} in the direction h , that is, $g'(\bar{x}, h) = \lim_{s \rightarrow 0^+} \frac{g(\bar{x} + sh) - g(\bar{x})}{s}$.

Using a separation theorem, we may easily obtain the equivalence between the two following assertions:

1. Abadie's constraint qualification holds at \bar{x} , with $g(\bar{x}) = 0$,
2. The following property holds at \bar{x} , with $g(\bar{x}) = 0$,

$$N(S, \bar{x}) = \text{cl}^*[\mathbb{R}_+ \partial g(\bar{x})]. \quad (1.3)$$

So that one of these conditions, guarantees the following characterization for a feasible point \bar{x} for (1.1):

1. \bar{x} is a solution of (1.1),
2. $0 \in \partial f(\bar{x}) + \text{cl}^*[\mathbb{R}_+ \partial g(\bar{x})]$,

provided that f is Lipschitz continuous around \bar{x} . Unfortunately, these two conditions are not sufficient to establish the existence of Karush-Kuhn-Tucker (KKT) multipliers, that there exists $\lambda \geq 0$, such that

$$0 \in \partial f(\bar{x}) + \lambda \partial g(\bar{x})$$

provided that f is Lipschitz continuous around \bar{x} (see Example 1). The real number λ is a KKT-Lagrange multiplier for problem (1.1).

Example 1 [13] Endow \mathbb{R}^2 with the usual scalar product $\langle \cdot, \cdot \rangle$ and the associated euclidean norm $\| \cdot \|$. Consider the convex functions f and g defined on \mathbb{R}^2 by

$$f(x, y) = x, \quad g(x, y) = \|(x, y)\| - y.$$

Thus we have

$$T(S, (0, 0)) = \{0\} \times \mathbb{R}_+ \text{ and } N(S, (0, 0)) = \mathbb{R} \times \mathbb{R}^-$$

while

$$\partial g(0, 0) = B((0, -1), 1) \text{ and } \mathbb{R}_+ \partial g(0, 0) = \mathbb{R} \times]-\infty, 0[\cup \{(0, 0)\}.$$

The feasible point $\bar{x} = (0, 0)$ is a solution of (1.1) but

$$0 \notin \partial f(\bar{x}) + \mathbb{R}_+ \partial g(\bar{x}).$$

The situation is quite different when g is the maximum of finite number of convex differentiable functions. Indeed, Abadie's constraint qualification can be expressed at \bar{x} as

$$T(S, \bar{x}) = \{h \in X : \langle \nabla g_i(\bar{x}), h \rangle \leq 0 \text{ } i \in I(\bar{x})\} \quad (1.4)$$

where g_1, \dots, g_m are convex real-valued functions on X which are differentiable at $\bar{x} \in S$, $I(\bar{x}) = \{i : g_i(\bar{x}) = 0\}$ and $g(x) = \max_{i=1, \dots, m} g_i(x)$. Using the Farkas lemma, we obtain that condition (1.4) is equivalent to

$$N(S, \bar{x}) = \mathbb{R}_+ \text{co}\{\nabla g_i(\bar{x}) : i \in I(\bar{x})\} \quad (1.5)$$

where “co” stands for the convex hull. As $\partial g(\bar{x}) = \text{co}\{\nabla g_i(\bar{x}) : i \in I(\bar{x})\}$, then condition (1.5) can be expressed as

$$N(S, \bar{x}) = \mathbb{R}_+ \partial g(\bar{x}). \quad (1.6)$$

This one is called the *Guignard constraint qualification* and ensures the existence of KKT-Lagrange multipliers. Example 1 is very instructive because it shows that Guignard’s and Abadie’s constraint qualifications are clearly distinct, even though (with only the affirmation that) Guignard’s condition leads to that of Abadie.

Using the subdifferential calculus $\partial g_+(\bar{x}) = \text{co}\{0, \partial g(\bar{x})\}$, with $g(\bar{x}) = 0$, where $a_+ = \max(0, a)$, we easily show that condition (1.6) is equivalent to

$$N(S, \bar{x}) = \mathbb{R}_+ \partial g_+(\bar{x}). \quad (1.7)$$

Note that all classical constraint qualifications (Slater condition, Mangasarian-Fromovitz condition, ...) imply Guignard’s constraint qualification (1.6). One of them is the so-called error bound.

Definition 1.1 (Local error bound) *We say that the following system*

$$g(x) \leq 0 \quad (1.8)$$

satisfies error bound at \bar{x} , with $g(\bar{x}) = 0$, if there exist two real numbers $\alpha > 0$ and $r > 0$ such that

$$d(x, S) \leq \alpha g_+(x) \quad \forall x \in B(\bar{x}, r). \quad (1.9)$$

Where

$$d(x, S) = \inf_{u \in S} \|u - x\|$$

is the distance function of S to x . If the error bound property holds at every \bar{x} , with $g(\bar{x}) = 0$, we say that the system satisfies error bound.

This concept is equivalent to saying that the set-valued mapping $M : \mathbb{R} \rightrightarrows X$ defined by

$$M(t) = \{x \in X : g(x) \leq t\}$$

is calm at $(0, \bar{x})$. Note that the calmness property should not be confused with the concept of metric regularity.

We recall that following [31], M is calm at $(0, \bar{x})$ of its graph if there exist neighborhoods V and W of 0 and \bar{x} respectively, and some $L > 0$ such that the corresponding distance functions satisfy

$$d(x, M(0)) \leq L|t| \forall x \in M(t) \cap W, \forall t \in V.$$

Obviously, calmness is also weaker than the well-known Aubin property of multifunctions

$$d(x, M(t)) \leq Ld(t, M^{-1}(x)) \forall t \in V, \forall x \in W.$$

The last one is equivalent to saying that M^{-1} is metrically regular at $(\bar{x}, 0)$ as defined by Robinson [29] (see [17, 18, 29, 31] and the references therein for more studies on these concepts including necessary and sufficient conditions).

The study of error bounds has received a lot of attention in the mathematical programming literature in the last decades (see [1, 5, 6, 8, 9, 16, 19, 20, 22–27, 30] and references therein).

Note that a simple condition ensuring error bound for the system (1.8) is Slater's condition, i.e., there exists $u \in X$ such that $g(u) < 0$. Indeed, for all $\bar{x} \in S$ and all $x \notin S$, the convexity of S ensures that $v := x + \frac{g(x)}{g(x)-g(u)}(u-x) \in S$ and

$$d(x, S) \leq \|x - v\| = \frac{g(x)}{g(x) - g(u)} \|x - u\| \quad (1.10)$$

which implies that the local error bound holds for the system (1.8).

In this paper, we are also concerned with the following concepts of error bound which give characterization of Guignard's qualification condition in some special situations.

Definition 1.2 (Bounded error bound) *We say that the system (1.8) satisfies bounded error bound if for all $r > 0$ there exists $\alpha_r > 0$ such that*

$$d(x, S) \leq \alpha_r g_+(x) \quad \forall x \in r\mathbb{B}. \quad (1.11)$$

Definition 1.3 (Global error bound) *We say that the system (1.8) satisfies global error bound if there exists $\alpha > 0$ such that*

$$d(x, S) \leq \alpha g_+(x) \quad \forall x \in X. \quad (1.12)$$

It is easy to see that

Global error bound \implies Bounded error bound \implies Local error bound \implies Guignard condition.

The aim of the present work is to characterize Guignard's condition in terms of these error bound concepts. More precisely, we will show that this characterization holds in the following situations:

1. g is the maximum of a finite number of differentiable convex functions.
2. S is closed convex and polyhedral.
3. The dimension of the subspace $\text{lin}(S)$ is less than 2 and g is positively homogeneous.

The first item has been studied in the paper [23] in finite dimensional spaces by using an euclidean approach. Note that the third item is included in the second one and contains the situations where S is a singleton, a ray or an affine subspace. A similar result has been established in [2], Theorem 3.2) in Banach spaces, but around a reference point.

We will show that there is no way to obtain a characterization outside of the three situations above. We will give examples showing the limit of the cited cases. The first example shows that the third item is no longer true if g is not assumed positively homogeneous. The second example shows the loss of the characterization in spaces when the dimension greater than 3 even if g is positively homogeneous.

We will show that we need more to characterize error bounds. Indeed, we have to introduce a new condition in term of the gauge function. The last one allows us to obtain, as a direct consequence of our results, different characterizations of the concept of strong conical hull intersection property (SCHIP) for a finite collection of convex sets in Banach spaces, which has been extensively studied in the literature(see for example [3, 4, 11, 21] and the references therein).

The paper is organized as follows: Section 2 presents the basic notation and concepts used in this paper, namely tools from convex analysis. Section 3 is devoted to the equivalence between of Guignard's and Abadie's conditions under a closedness assumption as well as to an elementary characterization of Guignard's condition by means of the concept of calmness of the function's value(in the Clarke's sense).

Different error bound characterizations of the Guignard's condition are established in Section 4 for differentiable convex inequality systems, namely the equivalence between this

condition and bounded error bound and local error bound. Section 5 contains different characterizations of Guignard's condition in special cases for nondifferentiable convex systems. Section 6 presents three technical examples showing that the equivalence of Guignard's condition and error bound is limited to the situations above. This allows us to introduce in Section 7 a new condition in terms of the gauge function implying Guignard's condition in order to characterize error bounds. Finally, Section 8 provides an illustration of this new condition in the characterization of the SCHIP property.

2 Preliminaries

Unless otherwise stated, space X will be a Banach space equipped with a norm $\|\cdot\|$, X^* is its topological dual with a pairing $\langle \cdot, \cdot \rangle$. The closed and the open unit ball of X (resp. X^*) are identified by \mathbb{B} and $\text{int}(\mathbb{B})$ (resp. \mathbb{B}^* and $\text{int}(\mathbb{B}^*)$), respectively. The closure (resp. w^* -closure) and the convex hull of a set $A \subset X$ (resp. $A^* \subset X^*$) are denoted by $\text{cl}A$ and $\text{co}A$ (resp. cl^*A^*), respectively. Let $\text{int}(C)$ be the interior of a set $C \subset X$ and $\text{lin}(C)$ be the smallest subspace of X containing C .

For an extended-real valued function $f : X \mapsto X \cup \{+\infty\}$, the Fenchel subdifferential is defined by

$$\partial f(x) = \{x^* \in X^* : \langle x^*, u - x \rangle \leq f(u) - f(x) \forall u \in X\}$$

if $f(x) < +\infty$ and \emptyset if $f(x) = +\infty$.

The directional derivative of f at x , with $f(x) < +\infty$, is given by

$$f'(x, h) = \lim_{t \rightarrow 0^+} \frac{f(x + th) - f(x)}{t}$$

and so

$$\partial f(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq f'(x, h) \forall h \in X\}.$$

When f is locally Lipschitz continuous around x , then

$$f'(x, h) = \max_{x^* \in \partial f(x)} \langle x^*, h \rangle \forall h \in X.$$

The tangent cone $T(C, x)$ to a closed convex set $C \subset X$ at $x \in C$ is defined by

$$T(C, x) = \text{cl}(\mathbb{R}_+(C - x))$$

or equivalently

$$T(C, x) = \{h \in X : d'(\cdot, C)(x, h) = 0\}.$$

The normal cone $N(C, x)$ to C at $x \in C$ is given by

$$N(C, x) = \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \forall h \in T(C, x)\}.$$

We also have the following characterization of the normal cone

$$N(C, x) = \mathbb{R}_+ \partial d(x, C).$$

Lemma 2.1 *Let $C \subset X$ be a closed convex set and let $x \notin C$. Then for all $\varepsilon > 0$ there exist $u_\varepsilon \in C$, $x_\varepsilon^* \in X^*$ and $b_\varepsilon^* \in \mathbb{B}^*$ such that*

1. $\|u_\varepsilon - x\| \leq d(x, C) + \varepsilon^2$,
2. $x_\varepsilon^* + \varepsilon b_\varepsilon^* \in (1 + \varepsilon) \partial d(u_\varepsilon, C)$,
3. $\langle x_\varepsilon^*, x - u_\varepsilon \rangle = \|u_\varepsilon - x\|$.

Moreover, if either S is included in a finite dimensional subspace of X or if X is a Hilbert space, then there exist $u \in S$ and $x^ \in X^*$ such that*

1. $d(x, C) = \|x - u\|$,

2. $x^* \in \partial d(u, C)$,
3. $\langle x^*, x - u \rangle = \|u - x\|$.

Proof. Let $v_\varepsilon \in C$ such that $\|x - v_\varepsilon\| \leq d(x, C) + \varepsilon^2$. Define the function f on X by $f(u) = \|u - x\|$. Then

$$f(v_\varepsilon) \leq \inf_{u \in C} f(u) + \varepsilon^2.$$

By Ekeland's variational principle [10], there exists $u_\varepsilon \in C$ such that

$$f(u_\varepsilon) \leq f(v_\varepsilon), \quad \|u_\varepsilon - v_\varepsilon\| \leq \varepsilon, \quad f(u_\varepsilon) \leq f(u) + \varepsilon\|u - u_\varepsilon\| \quad \forall u \in C.$$

This last inequality is equivalent to saying that u_ε minimizes the function $u \mapsto f(u) + \varepsilon\|u - u_\varepsilon\| + (1 + \varepsilon)d(u, C)$ or equivalently

$$0 \in \partial f(u_\varepsilon) + \varepsilon\mathbb{B}^* + (1 + \varepsilon)\partial d(u_\varepsilon, C).$$

So that there exist $-x_\varepsilon^* \in \partial f(u_\varepsilon)$ and $b_\varepsilon^* \in \mathbb{B}^*$ such that $x_\varepsilon^* + \varepsilon b_\varepsilon^* \in (1 + \varepsilon)\partial d(u_\varepsilon, C)$. To conclude, it only remains to see that $\partial f(u_\varepsilon) = \{x^* \in X^* : \langle x^*, u_\varepsilon - x \rangle = \|u_\varepsilon - x\|\}$.

□

Lemma 2.2 *Let $C \subset X$ be a closed convex set and let $x \in C$. Then*

$$\partial d(x, C) = N(C, x) \cap \mathbb{B}^* = \partial d(0, T(C, x)).$$

Lemma 2.3 *Let $K \subset X$ be a closed convex cone with negative polar K^0 ($:= \{x^* \in X^* : \langle x^*, h \rangle \leq 0 \forall h \in K\}$). Then*

$$d(x, K) = \sup_{x^* \in K^0 \cap \mathbb{B}^*} \langle x^*, x \rangle \quad \forall x \in X.$$

The following lemma establishes a subdifferential formula of homogeneous and supremum functions.

Lemma 2.4 (Subdifferential of the supremum of homogeneous functions) *Let $h : \mathbb{R}^m \rightarrow \mathbb{R}$ and $h_k : \mathbb{R}^m \rightarrow \mathbb{R}$, $k \in \mathbb{N}$, be homogeneous convex functions. Then*

1. *For all $x \in \mathbb{R}^m$, we have*

$$x^* \in \partial h(x) \iff \langle x^*, x \rangle = h(x), \quad x^* \in \partial h(0).$$

2. *If $h = \sup_{k \in \mathbb{N}} h_k$, then*

$$\partial h(0) = \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right).$$

Proof. Item 1 is obvious. Let us establish the second one. Using the definition of h , we obtain that for all $k \in \mathbb{N}$, $\partial h_k(0) \subset \partial h(0)$ and hence $\text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right) \subset \partial h(0)$.

Proposition 5.2 in [12] asserts that

$$\partial h(0) \subset \bigcap_{\varepsilon > 0} \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial_\varepsilon h_k(0) \right)$$

where $\partial_\varepsilon h_k(0) = \{x^* \in \mathbb{R}^m : \langle x^*, x \rangle \leq h_k(x) + \varepsilon \forall x \in \mathbb{R}^m\}$ is the ε -subdifferential of h_k at 0. Since h_k is homogeneous, we have $\partial_\varepsilon h_k(0) \subset \partial h_k(0) + \varepsilon\mathbb{B}_{\mathbb{R}^m}^*$. So that

$$\partial h(0) \subset \bigcap_{\varepsilon > 0} \text{clco} \left(\bigcup_{k \in \mathbb{N}} (\partial h_k(0) + \varepsilon\mathbb{B}_{\mathbb{R}^m}^*) \right) \subset \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right).$$

The last inclusion results from the following equality

$$\bigcup_{k \in \mathbb{N}} (\partial h_k(0) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^*) = \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^*$$

and

$$\begin{aligned} \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^* \right) &= \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^* \right) \\ &\subset \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^* \end{aligned}$$

as well as the equality

$$\bigcap_{\varepsilon > 0} \left[\text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right) + \varepsilon \mathbb{B}_{\mathbb{R}^m}^* \right] = \text{clco} \left(\bigcup_{k \in \mathbb{N}} \partial h_k(0) \right).$$

□

3 Some elementary characterizations of Guignard's constraint qualification for nondifferentiable convex systems

In this section, we give two elementary characterisations of Guignard's condition. The first one concerns its equivalence with that of Abadie and the second one with the concept of calmness in the Clarke's sense of the value function. We state them without proof.

As we saw in the introduction, (see Example 1) Guignard's and Abadie's constraint qualifications are not equivalent for nondifferentiable convex systems. The following result shows that both Guignard's and Abadie's constraint qualifications for nondifferentiable convex systems are equivalent under an additional closedness hypothesis.

Proposition 3.1 *The following assertions are equivalent for $x \in S$, with $g(x) = 0$:*

1. *Guignard's constraint qualification (1.6) holds at x ;*
2. *Abadie's constraint qualification (1.2) holds at x and the set $\mathbb{R}_+ \partial g(\bar{x})$ is weak-star closed.*

For the second characterization, consider convex continuous functions $f, g_i : X \rightarrow \mathbb{R}$, $i = 1, \dots, m$ and the optimization problem

$$(P_f) \quad \begin{cases} \min f(x) \\ g_i(x) \leq 0 \quad i = 1, \dots, m \end{cases}$$

To this problem, we associate the following perturbed one

$$(P_y) \quad \begin{cases} \min f(x) \\ g_i(x) \leq y_i \quad i = 1, \dots, m \end{cases}$$

where $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ is the perturbation parameter. The value function associated to (P) is given by

$$v_f(y) = \inf \{ f(x) : g_i(x) \leq y_i \quad i = 1, \dots, m \}.$$

It is easy to see that v_f is convex. Following Clarke [7], v_f is calm at 0, where $v_f(0) \in \mathbb{R}$, if

$$\liminf_{y \rightarrow 0} \frac{v_f(y) - v_f(0)}{\|y\|} > -\infty.$$

In the convex setting, this definition is equivalent to saying that

$$\partial v_f(0) \neq \emptyset.$$

Then, we have:

Proposition 3.2 *Suppose that the solution set S_f of the problem P_f is nonempty. Then*

1. $-\lambda \in \partial v_f(0)$ IFF λ is a KKT multiplier for P_f associated to all $\bar{x} \in S_f$.
2. Guignard's condition holds for the system (1.8), with $g = \max_{i=1, \dots, m} g_i$, IFF for any convex continuous function $f : X \rightarrow \mathbb{R}$ for which $S_f \neq \emptyset$, v_f is calm at 0.

4 Error bound characterization of Guignard's constraint qualification for differentiable convex inequality systems

Recall that a constraint $\{x \in X : g(x) \leq 0\}$, or simply g , satisfies Slater's condition if there exists $u \in X$ such that

$$g(u) < 0$$

and that $g(x) = \max_{i=1, \dots, p} g_i(x)$.

Consider the set $\mathcal{I} := \{J \subset I : g_J := \max_{i \in J} g_i \text{ satisfies Slater's condition}\}$.

The following result states error bound characterizations of Guignard's constraint qualification under the differentiability of the data, especially the equivalence between the last one and the bounded and local error bounds in Banach spaces.

Theorem 4.1 *Suppose that the function g is a maximum of finite number p of convex differentiable functions $g_i : X \rightarrow \mathbb{R}$. Then the following assertions are equivalent:*

- i) *Guignard's constraint qualification holds for system (1.8), that is, for all $\bar{x} \in S$,*

$$N(S, \bar{x}) = \mathbb{R}_+ \text{co}\{\nabla g_i(\bar{x}) : i \in I(\bar{x})\}$$

where $I(\bar{x}) := \{i \in \{1, \dots, p\} : g_i(\bar{x}) = 0\}$ is the index set of active constraints at \bar{x} .

- ii) *The system (1.8) satisfies error bound. More precisely, $\mathcal{I} \neq \emptyset$ and there exists $(x_J)_{J \in \mathcal{I}} \subset X$ such that*

$$g_J(x_J) < 0 \text{ and } d(x, S) \leq g_+(x) \max_{J \in \mathcal{I}} \left(\frac{\|x - x_J\|}{g_{J+}(x) - g_J(x_J)} \right) \quad \forall x \in X.$$

- iii) *The system (1.8) satisfies bounded error bound. More precisely, there exists $c > 0$ such that for all $r > 0$*

$$d(x, S) \leq c(r+1)g_+(x) \forall x \in X, \text{ with } \|x\| \leq r.$$

- iv) *The system (1.8) satisfies local error bound.*

Proof. *iv) \implies i) :* This implication is obvious and is based on the formula $\partial d(x, S) = N(S, x) \cap \mathbb{B}^*$, and the subdifferential calculus of the maximum of convex functions.

ii) \implies iii) : It is enough to take $c = \max_{J \in \mathcal{I}} \frac{1}{-g_J(x_J)} \max(1, \|x_J\|)$.

i) \implies ii) : This implication will be established in three steps.

Step 1: We start by the following lemma whose proof can be deduced from that of Theorem 4.1 in [16]. We give a proof to make the paper self-contained.

Lemma 4.1 Let $x^*, x_1^*, \dots, x_m^* \in X^* \setminus \{0\}$. Suppose there exist $\mu_1, \dots, \mu_m \in \mathbb{R}_+$ such that

$$x^* = \sum_{i=1}^m \mu_i x_i^*.$$

Then there exist $J \subset \{1, \dots, m\}$ and $(\beta_i)_{i \in J}$, with $\beta_i \geq 0$ for all $i \in J$ and not all equal to zero, such that $(x_i^*)_{i \in J}$ are linearly independent and

$$x^* = \sum_{i \in J} \beta_i x_i^*.$$

Proof. It is included for completeness. Set $I^0 = \{1, \dots, m\}$. If $(x_i^*)_{i \in I^0}$ are not linearly independent, then there is nothing to prove. To suppose the contrary would mean there exist $\gamma_i \in \mathbb{R}$, $i \in I^0$, not all equal to zero such that

$$\sum_{i \in I^0} \gamma_i x_i^* = 0.$$

Without loss of generality, we can assume that there is at least $i \in I^0$ such that $\gamma_i < 0$. Hence for all $t \in \mathbb{R}$

$$x^* = \sum_{i \in I^0} (\mu_i + t\gamma_i) x_i^*.$$

Set $t_{max} = \max\{t : \mu_i + t\gamma_i \geq 0, \forall i \in I^0\}$. Then $t_{max} = \min_{i \in I^0} \{-\frac{\mu_i}{\gamma_i} : \gamma_i < 0\}$. Let then $i_0 \in I^0$ be such that $-\frac{\mu_{i_0}}{\gamma_{i_0}} = t_{max}$, that is, $\mu_{i_0} + t_{max}\gamma_{i_0} = 0$. Hence setting $I^1 = I^0 \setminus \{i_0\}$ and $\mu_i^{(1)} = \mu_i + t_{max}\gamma_i$, $\forall i \in I^1$, we have

$$x^* = \sum_{i \in I^1} \mu_i^{(1)} x_i^*, \text{ with } \mu_i^{(1)} \geq 0, \forall i \in I^1$$

By induction we show that there exist $I \subset \{1, \dots, m\}$ and $(\beta_i \geq 0)_{i \in I}$ such that $(x_i^*)_{i \in I}$ is linearly independent and

$$x^* = \sum_{i \in I} \beta_i x_i^*.$$

□

Step 2: Suppose first that $\mathcal{I} \neq \emptyset$. Then for all $J \in \mathcal{I}$ there exists $x_J \in X$ such that

$$g_J(x_J) < 0 \text{ and (by (1.10)) } d(x, S_J) \leq \frac{g_{J^+}(x)}{g_{J^+}(x) - g_J(x_J)} \|x - x_J\| \forall x \in X \quad (4.1)$$

where $S_J := \{x \in X : g_J(x) \leq 0\}$. Hence

$$d(x, S_J) \leq \frac{g_+(x)}{g_{J^+}(x) - g_J(x_J)} \|x - x_J\| \forall x \in X. \quad (4.2)$$

We will prove that the set \mathcal{I} is in fact not empty.

Lemma 4.2 Suppose *i*) holds. Let $x \notin S$ and $x^* \in \partial d(x, S)$. Then $\|x^*\| = 1$ and there exist sequences $(v_n)_{n \in \mathbb{N}} \subset S$, $v_n^* \in N(S, v_n)$, for all $n \in \mathbb{N}$, $(u_n)_{n \in \mathbb{N}} \subset X$ and $(J_n)_{n \in \mathbb{N}} \subset \mathcal{I}$ such that

1. $\|x^* - v_n^*\| \rightarrow 0$,
2. $\|u_n - x\| \rightarrow 0$,
3. $g_{J_n} := \max_{i \in J_n} g_i$ satisfies Slater's condition and
4. $d(u_n, S) \leq \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} d(u_n, S_{J_n})$.

Proof. Fix $x \notin S$ and $x^* \in \partial d(x, S)$. It is easy to see that $\|x^*\| = 1$. For each integer $n > 0$, there exists $w_n \in S$ such that

$$\|x - w_n\| \leq d(x, S) + \frac{1}{n^2}.$$

Therefore

$$\langle x^*, u - x \rangle \leq d(u, S) - d(x, S) \leq \|u - v\| - \|x - w_n\| + \frac{1}{n^2} \quad \forall u \in X, \forall v \in S.$$

Thus the Lipschitz function $g : X \times X \mapsto \mathbb{R}$ defined by

$$g(u, v) = \|u - v\| - \langle x^*, u \rangle$$

satisfies

$$g(x, w_n) \leq \inf_{(u, v) \in X \times S} g(u, v) + \frac{1}{n^2}.$$

So, endowing $X \times X$ with the norm $\|(x, y)\| = \|x\| + \|y\|$, Ekeland's variational principle [10] ensures the existence of $u_n \in X$ and $v_n \in S$ such that

$$\|x - u_n\| + \|w_n - v_n\| \leq \frac{1}{n}, \quad g(u_n, v_n) \leq g(u, v) + \frac{1}{n}[\|u - u_n\| + \|v - v_n\|] \quad \forall u \in X, \forall v \in S$$

or equivalently

$$(x^*, 0) \in \partial \|\cdot - \cdot\|(u_n, v_n) + \{0\} \times N(S, v_n) + \frac{1}{n} B_{X^*} \times B_{X^*}.$$

Due to the fact that $x \notin S$, $u_n \neq v_n$ for n large enough, there exist $u_n^* \in \partial \|\cdot - \cdot\|(u_n, v_n)$, with $\|u_n^*\| = 1$, and $b_n^* \in \frac{1}{n} B_{X^*}$ such that

$$\|x^* - u_n^*\| \leq \frac{1}{n}, \quad v_n^* := u_n^* + b_n^* \in N(S, v_n).$$

By our hypothesis i) there are $\mu_1^n, \dots, \mu_m^n \in \mathbb{R}_+$, not all equal to zero such that

$$v_n^* = \sum_{i=1}^m \mu_i^n \nabla g_i(v_n).$$

Lemma 4.2 ensures the existence of $J_n \subset \{1, \dots, p\}$ such that $(\nabla g_i)_{i \in J_n}$ are linearly independent and

$$v_n^* = \sum_{i \in J_n} \mu_i^n \nabla g_i(v_n).$$

So that $J_n \in \mathcal{I}$ and affirms that $\mathcal{I} \neq \emptyset$ and $v_n^* \in (1 + \frac{1}{n}) \partial d(v_n, S_{J_n})$ (because $\|v_n^*\| \leq 1 + \frac{1}{n}$ and $\partial d(v_n, S_{J_n}) = N(S_{J_n}) \cap B_{X^*}$). Since $u_n^* \in \partial \|\cdot - \cdot\|(u_n, v_n)$, we have

$$\begin{aligned} d(u_n, S) \leq \|u_n - v_n\| &= \langle u_n^*, u_n - v_n \rangle = \langle v_n^*, u_n - v_n \rangle - \frac{1}{n} \langle b_n^*, u_n - v_n \rangle \\ &\leq \langle v_n^*, u_n - v_n \rangle + \frac{1}{n} \|u_n - v_n\| \end{aligned}$$

Then

$$(1 - \frac{1}{n}) \|u_n - v_n\| \leq \langle v_n^*, u_n - v_n \rangle \leq (1 + \frac{1}{n}) d(u_n, S_{J_n})$$

and the result follows. \(\square\)

Step 3: Now, using the previous steps and relation (4.2), we get

$$\begin{aligned} d(u_n, S) &\leq \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} d(u_n, S_{J_n}) \\ &\leq \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} \frac{g_{J_n+}(u_n)}{g_{J_n+}(u_n) - g_{J_n}(x_{J_n})} \|u_n - x_{J_n}\| \\ &\leq \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} g_+(u_n) \max_{J \in \mathcal{I}} \left(\frac{\|u_n - x_J\|}{g_{J+}(u_n) - g_J(x_J)} \right). \end{aligned}$$

Now passing to the limit on n , we obtain

$$d(x, S) \leq g_+(x) \max_{J \in \mathcal{I}} \left(\frac{\|x - x_J\|}{g_+(x) - g_J(x_J)} \right).$$

□

5 Error bound characterization of Guignard's constraint qualification for nondifferentiable convex systems : Special cases

The situation of nondifferentiable systems is quite different and involves an additional hypothesis except in the following special situations:

1. g is a polyhedral function and X is a Banach space. In this case both Guignard's condition and error bound are satisfied.
2. S is a closed convex polyhedral set and X is a Banach space.
3. The dimension of the subspace $\text{lin}(S)$ is less than 2 and g is positively homogeneous.

Remark 5.1 *Unfortunately, when "g is not positively homogeneous and $\dim \text{lin}(S) \geq 2$ " or "g is positively homogeneous but $\dim \text{lin}(S) \geq 3$ ", the condition of Guignard is not sufficient to guarantee the existence of an error bound concept. In this respect, we shall give counterexamples in Section 6 showing the limit of this characterization.*

Theorem 5.1 (S is a closed polyhedron) *Suppose that S is a closed polyhedral set of a Banach space X . Then the following assertions are equivalent:*

1. *Guignard's constraint qualification holds at all $\bar{x} \in S$, with $g(\bar{x}) = 0$,*
2. *The system (1.8) satisfies global error bound.*

Proof. It is enough to establish the implication 1. \implies 2.. Write S as

$$S := \{x \in X : \langle a_i, x \rangle \leq b_i \forall i = 1, \dots, m\}$$

where $a_i \in X$, with $\|a_i\| = 1$, and $b_i \in \mathbb{R}$, $i = 1, \dots, m$. For each $x \in S$, set $I(x) = \{i \in \{1, \dots, m\} : \langle a_i, x \rangle = b_i\}$ and $\mathcal{J} := \{I(x) : x \in S\}$. Since \mathcal{J} is a finite set, there exist $x_1, \dots, x_p \in S$ such that $\mathcal{J} = \{I(x_i) : i = 1, \dots, p\}$. For all $i = 1, \dots, p$, set

$$S_i = \{x \in X : \langle a_j, x \rangle \leq b_j \forall j \in I(x_i)\}.$$

Then, we have the following lemma which is interesting in its self.

Lemma 5.1 *For all $x \notin S$ and all $\varepsilon \in]0, 1[$ there exists $i \in \{1, \dots, p\}$ such that*

$$d(x, S) \leq \frac{1 + \varepsilon}{1 - \varepsilon} d(x, S_i)$$

and hence $d(x, S) = \max_{i=1, \dots, p} d(x, S_i)$. Consequently, there exists $\alpha > 0$ such that

$$d(x, S) \leq \alpha \max_{i=1, \dots, p} \max_{j \in I(x_i)} (\langle a_j, x \rangle - b_j)^+ \quad \forall x \in X.$$

Proof. Let $x \notin S$ and $\varepsilon \in]0, 1[$. Lemma 2.1 there exist $u_\varepsilon \in S$, $x_\varepsilon^* \in X^*$ and $b_\varepsilon^* \in \mathbb{B}^*$ such that

- (a) $\|u_\varepsilon - x\| \leq d(x, S) + \varepsilon^2$,
- (b) $x_\varepsilon^* + \varepsilon b_\varepsilon^* \in (1 + \varepsilon)\partial d(u_\varepsilon, S)$,
- (c) $\langle x_\varepsilon^*, x - u_\varepsilon \rangle = \|u_\varepsilon - x\|$.

Assertion (b) is equivalent to saying that $\frac{x_\varepsilon^* + \varepsilon b_\varepsilon^*}{1 + \varepsilon} \in \partial d(u_\varepsilon, S_{I(u_\varepsilon)})$. Since $I(u_\varepsilon) \in \mathcal{J}$, there exists $i \in \{1, \dots, p\}$ such that $I(u_\varepsilon) = I(x_i)$ and hence

$$\frac{x_\varepsilon^* + \varepsilon b_\varepsilon^*}{1 + \varepsilon} \in \partial d(u_\varepsilon, S_i). \quad (5.1)$$

Using relation (5.1) and (c), one obtains

$$\frac{1 - \varepsilon}{1 + \varepsilon} \|x - u_\varepsilon\| \leq d(x, S_i).$$

Hence

$$\frac{1 - \varepsilon}{1 + \varepsilon} d(x, S) \leq d(x, S_i) \leq \max_{j=1, \dots, p} d(x, S_j).$$

As ε is arbitrary and for all $j = 1, \dots, p$, $S \subset S_i$, one gets the desired equality

$$d(x, S) = \max_{j=1, \dots, p} d(x, S_j).$$

The last inequality results from the well-known Hoffmann error bound which asserts that for all $j = 1, \dots, p$, there exists $\alpha_j > 0$ such that

$$d(x, S_j) \leq \alpha_j \max_{i \in I(x_j)} (\langle a_i, x \rangle - b_i)^+ \quad \forall x \in X.$$

To complete the proof, it suffices to set $\alpha = \max_{j=1, \dots, p} \alpha_j$. \(\square\)

Proof of Theorem 5.1 (continued). Note that for all $i = 1, \dots, p$, $N(S, x_i) = \mathbb{R}_+ \text{co}\{a_j : j \in I(x_i)\}$. By Guignard's constraint qualification condition, we have for all $i = 1, \dots, p$, $\mathbb{R}_+ \text{co}\{a_j : j \in I(x_i)\} = \mathbb{R}_+ \partial g(x_i)$. This asserts that for all $i = 1, \dots, p$ and all $j \in I(x_i)$, there exists $\alpha_{ij} > 0$ such that

$$\alpha_{ij} a_j \in \partial g(x_i).$$

Therefore

$$\alpha_{ij} \langle a_j, x - x_i \rangle \leq g(x)$$

or equivalently

$$\alpha_{ij} (\langle a_j, x \rangle - b_j) \leq g(x), \text{ that is, } \alpha_{ij} (\langle a_j, x \rangle - b_j)^+ \leq g(x).$$

Set $\beta = \min_{\substack{i=1, \dots, p \\ j \in I(x_i)}} \alpha_{ij}$. Then

$$\beta \max_{\substack{i=1, \dots, p \\ j \in I(x_i)}} (\langle a_j, x \rangle - b_j)^+ \leq g(x).$$

The proof is then terminated by using Lemma 5.1. \(\square\)

Corollary 5.1 (S is a singleton) Suppose that $S = \{w\}$ and X is a Banach space. Then the following assertions are equivalent:

1. Guignard's constraint qualification holds at w ,
2. $0 \in \text{int}(\partial g(w))$,
3. The system (1.8) satisfies global error bound.

Proof. Since S is polyhedral, the equivalence 1. \iff 3. is a direct consequence of Theorem 5.1. We establish only the implication 1. \implies 2. because the implication 2. \implies 3. is easy to obtain by using the convexity of g and the definition of the Fenchel subdifferential. But this implication is a direct consequence of Baire's theorem. Indeed, since $S = \{w\}$, Guignard's constraint qualification is equivalent to say that

$$X^* = N(S, w) = \mathbb{R}_+ \partial g(w) = \bigcup_{n \in \mathbb{N}} n \partial g(w). \quad (5.2)$$

Since the set $\partial g(w)$ is norm-closed in X^* and, because of relation (5.2), Baire's theorem asserts that $\text{int}(\partial g(w)) \neq \emptyset$. Now, it remains to show that $0 \in \text{int}(\partial g(w))$. Let us suppose the contrary, then by the Hahn-Banach separation theorem there exists $h \in X$, with $\|h\| = 1$, such that

$$\langle u^*, h \rangle \leq 0 \forall u^* \in \partial g(w).$$

Thus, using relation (5.2), we obtain $h = 0$ and this contradiction completes the proof. \square

Corollary 5.2 ($\dim \text{lin}(S) \leq 2$) *Assume that the function g is positively homogeneous and $\dim \text{lin}(S) \leq 2$. If the Guignard condition holds for the system (1.8) then system (1.8) satisfies global error bound.*

Proof. It is enough to prove that S is in fact a closed convex polyhedron and to apply Theorem 5.1. This is based on the following lemma.

Lemma 5.2 *Let $S \subset X$ be a closed convex cone with $\dim \text{lin}(S) \leq 2$. Then S is a polyhedron.*

Proof. Suppose that $S \neq \{0\}$. Let

$$(\bar{x}, \bar{y}) \in \arg \min \{ \langle x, y \rangle : (x, y) \in S^2, \|x\| = \|y\| = \gamma \},$$

where $\gamma = \max_{x \in S \cap \mathbb{B}} \|x\|$. Three cases can arise, namely, $\langle \bar{x}, \bar{y} \rangle = \gamma^2$, $\langle \bar{x}, \bar{y} \rangle = -\gamma^2$ and $-\gamma^2 < \langle \bar{x}, \bar{y} \rangle < \gamma^2$.

Case 1: If $\langle \bar{x}, \bar{y} \rangle = \gamma^2 = \|\bar{x}\| \|\bar{y}\|$, then necessarily $\bar{x} = \bar{y}$. Let us show that $S = \mathbb{R}_+ \bar{x}$. Let $z \in S \setminus \{0\}$. Then $\gamma^2 = \langle \bar{x}, \bar{y} \rangle \leq \langle \bar{x}, \gamma \frac{z}{\|z\|} \rangle \leq \gamma^2$. Hence $\langle \bar{x}, \gamma \frac{z}{\|z\|} \rangle = \|\bar{x}\| \left\| \gamma \frac{z}{\|z\|} \right\|$ and then $\gamma \frac{z}{\|z\|} = \bar{x}$. Thus $z \in \mathbb{R}_+ \bar{x}$ and $S = \mathbb{R}_+ \bar{x}$.

Case 2: If $\langle \bar{x}, \bar{y} \rangle = -\gamma^2 = -\|\bar{x}\| \|\bar{y}\|$, then $\bar{y} = -\bar{x}$. It follows that S contains $\mathbb{R}\bar{x}$. Then either $S = \mathbb{R}\bar{x}$ or $S = \mathbb{R}\bar{x} + \mathbb{R}_+ \bar{u}$, for some $u \in \{\bar{x}\}^\perp \cap \text{lin}(S)$, or $S = \text{lin}(S)$.

Case 3: If $-\gamma^2 < \langle \bar{x}, \bar{y} \rangle < \gamma^2$. Then \bar{x} and \bar{y} are linearly independent and then $\text{lin}(S) = \mathbb{R}\bar{x} + \mathbb{R}\bar{y}$ necessarily. Let us show that in fact $S = \mathbb{R}_+ \bar{x} + \mathbb{R}_+ \bar{y}$. The inclusion $\mathbb{R}_+ \bar{x} + \mathbb{R}_+ \bar{y} \subset S$ holds, since S is a cone and $(\bar{x}, \bar{y}) \in S^2$. Assume for contradiction that $S \not\subset \mathbb{R}_+ \bar{x} + \mathbb{R}_+ \bar{y}$. Let then $z \in S \setminus (\mathbb{R}_+ \bar{x} + \mathbb{R}_+ \bar{y})$. We said that $\text{lin}(S) = \mathbb{R}\bar{x} + \mathbb{R}\bar{y}$. Let then $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $\gamma \frac{z}{\|z\|} = \alpha \bar{x} + \beta \bar{y}$. Then either, $\alpha\beta < 0$ or ($\alpha < 0$ and $\beta < 0$).

If $\alpha\beta < 0$. Without loss of generality we can assume for example that $\alpha < 0$ and $\beta > 0$. We have $\frac{\gamma}{\beta} \frac{z}{\|z\|} = \frac{\alpha}{\beta} \bar{x} + \bar{y} \in S$. It follows that

$$\langle \bar{x}, \bar{y} \rangle \leq \left\langle \frac{\gamma}{\beta} \frac{z}{\|z\|}, \bar{x} \right\rangle = \frac{\alpha}{\beta} \|\bar{x}\|^2 + \langle \bar{x}, \bar{y} \rangle$$

and hence $\alpha \geq 0$ which is absurd. So $S = \mathbb{R}_+\bar{x} + \mathbb{R}_+\bar{y}$.

If $\alpha < 0$ and $\beta < 0$. Then $(-\alpha\bar{x}, -\beta\bar{y}) \in S^2$ and then $-\gamma \frac{z}{\|z\|} \in S$. It follows that

$$-\gamma^2 = \left\langle \gamma \frac{z}{\|z\|}, -\gamma \frac{z}{\|z\|} \right\rangle \geq \langle \bar{x}, \bar{y} \rangle > -\gamma^2,$$

which is impossible. Thus $S = \mathbb{R}_+\bar{x} + \mathbb{R}_+\bar{y}$. \(\square\)

Corollary 5.3 *Let K_1 and K_2 be closed convex cones in X . Suppose that $\dim(X) = 3$. Then the following assertions are equivalent:*

1. $N(K_1 \cap K_2, 0) = N(K_1, 0) + N(K_2, 0)$.
2. The system (1.8) satisfies global error bound with the function $g(x) = d(x, K_1) + d(x, K_2)$.

Proof. If $\text{int}(K_1 \cap K_2) \neq \emptyset$, then there exists $u \in K_1 \cap K_2$, such that $0 \in \text{int}(K_1 \cap (u + \mathbb{B}) - K_2)$ and this condition ensures both items (see for example Lemma 4.1 in [14]). The case $\text{int}(K_1 \cap K_2) = \emptyset$ follows from Lemma 5.2 and Theorem 5.1. \(\square\)

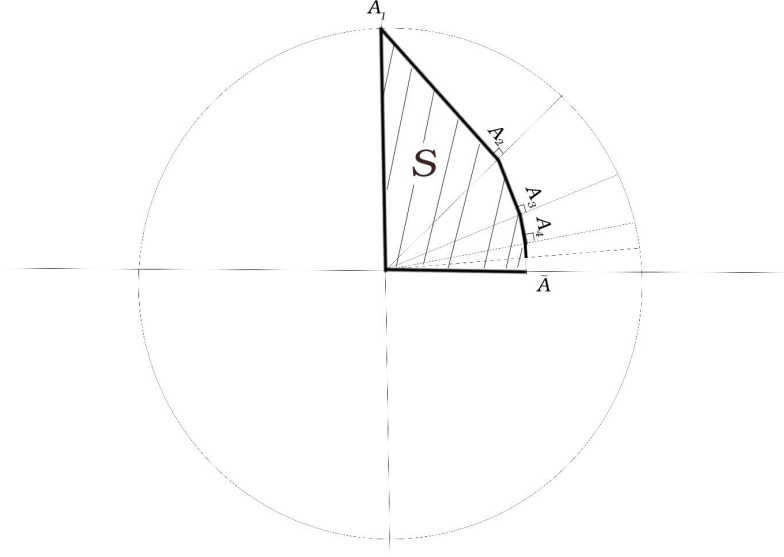
6 Counterexamples

In the previous sections, we have established the equivalence of Guignard's condition and error bound in the situation where $\dim(\text{lin}(S)) \leq 2$ and g is positively homogeneous. Our aim in the first counterexample is to show that if $\text{int}(S) \neq \emptyset$, g is not positively homogeneous and the dimension of X is equal to 2, then Guignard's condition is satisfied but not error bound. In the second Counterexample, we provide in \mathbb{R}^3 an homogeneous function g such that the corresponding set S has an interior for which Guignard's condition holds but is not error bound. Relying on the second example, the last counterexample is given in \mathbb{R}^n for $n \geq 4$.

Example 2 ($\dim X = 2$ and g is not positively homogeneous but $\text{int}(S) \neq \emptyset$) *Set $X = \mathbb{R}^2$ and let $\{e_1, e_2\}$ be its canonical basis. Let $(A_n)_{n \in \mathbb{N}^*}$ the sequence defined by*

$$A_1 = e_2 \text{ and } A_n = \prod_{i=2}^n \cos \frac{\pi}{2^i} \left(\cos \left(\frac{\pi}{2^n} \right) e_1 + \sin \left(\frac{\pi}{2^n} \right) e_2 \right), \text{ for } n \geq 2. \quad (6.1)$$

So A_{n+1} appears as the orthogonal projection of A_n onto $\mathbb{R} \left(\cos \left(\frac{\pi}{2^{n+1}} \right) e_1 + \sin \left(\frac{\pi}{2^{n+1}} \right) e_2 \right)$ as shown in the following figure



It follows that

$$\langle A_n - A_{n+1}, A_{n+1} \rangle = 0, \quad \forall n \in \mathbb{N}^* \quad (6.2)$$

and

$$\|A_1\| = 1 \text{ and } \|A_n\| = \prod_{i=2}^n \cos \frac{\pi}{2^i}, \quad \text{for } n \geq 2 \quad (6.3)$$

Thus $(\|A_j\|)_{j \in \mathbb{N}^*}$ is decreasing, its limit is

$$\bar{\lambda} = \prod_{i \geq 2} \cos \frac{\pi}{2^i} \quad (6.4)$$

and the limit of $(A_j)_{j \in \mathbb{N}^*}$ denoted by \bar{A} , is $\bar{\lambda}e_1$. Furthermore we have

$$\langle A_n, A_k \rangle = \cos \left(\frac{\pi}{2^n} - \frac{\pi}{2^k} \right) \|A_n\| \|A_k\|, \quad \text{for all } (n, k) \in \mathbb{N}^{*2} \quad (6.5)$$

Let us consider now the function g defined as follow

$$g(x) = \sup_{k \geq 1} \frac{1}{4^k} (\langle x - A_k, A_k \rangle^+) + \langle x - \bar{A}, e_1 \rangle^+ + \langle -e_1, x \rangle^+ + \langle -e_2, x \rangle^+, \quad \forall x \in \mathbb{R}^2. \quad (6.6)$$

It is easy to see that g is convex and continuous on \mathbb{R}^2 , and that

$$\begin{aligned} S = \{x \in \mathbb{R}^2 : g(x) \leq 0\} &= \{x \in \mathbb{R}^2 : g(x) = 0\} \\ &= \bigcap_{k \geq 1} \{x \in \mathbb{R}^2 : \langle x - A_k, A_k \rangle \leq 0\} \cap \mathbb{R}_+^2. \end{aligned}$$

Then we have the following result whose proof is very technical.

Proposition 6.1 *With the function g , the Guignard condition holds for the system (1.8) whereas all error bound concepts are violated.*

Proof. *N.B. : The proof is very technical and to simplify we have deliberately omitted the proofs of some statements.*

Let $(\delta_n^k)_{(n,k) \in \mathbb{N}^{*2}}$ and $(\epsilon_k)_{k \in \mathbb{N}^*}$ respectively defined by

$$\delta_n^k = \begin{cases} \cos\left(\frac{\pi}{2^n} - \frac{\pi}{2^k}\right) - \prod_{i=n+1}^k \cos\left(\frac{\pi}{2^i}\right) & \text{if } k > n, \\ \prod_{i=k+1}^n \cos\left(\frac{\pi}{2^i}\right) \cos\left(\frac{\pi}{2^n} - \frac{\pi}{2^k}\right) - 1 & \text{if } k < n \\ 0 & \text{if } k = n. \end{cases} \quad (6.7)$$

and

$$\epsilon_k = -\frac{1}{2}\delta_k^{k+2} \quad (6.8)$$

The following relations hold:

1. $\delta_n^n = \delta_n^{n+1} = 0$, $\forall n \in \mathbb{N}^*$ and $\delta_n^k < 0$ whenever $k < n$ or $k > n + 1$. Furthermore $\delta_n^{k+1} < \delta_n^k \leq 0$, whenever $k > n > 1$.
2. $1 > \bar{\lambda} > \frac{1}{2}$ and $\bar{\lambda} \sin \frac{\pi}{2^n} \geq -\delta_n^{n+2} = 2\epsilon_n$, $\forall n \in \mathbb{N}^*$.
3. For every $(n, k) \in \mathbb{N}^{*2}$ we have

$$\begin{cases} \langle A_n - A_k, A_k \rangle = 0 & \text{if } n = k \text{ or } k = n + 1 \\ \langle A_n - A_k, A_k \rangle \leq \cos \frac{\pi}{2^n} - 1 < 0 & \text{if } k \leq n - 1, \\ \langle A_n - A_k, A_k \rangle \leq \delta_n^{n+2} < 0 & \text{if } k \geq n + 2. \end{cases} \quad (6.9)$$

4. For every $n \in \mathbb{N} \setminus \{0, 1\}$ we have

$$\langle A_n - \bar{\lambda}e_1, e_1 \rangle \leq \delta_n^{n+2} < 0, \quad (6.10)$$

$$\langle A_n, e_1 \rangle \geq -\delta_n^{n+2} = 2\epsilon_n \quad (6.11)$$

and

$$\langle A_n, e_2 \rangle \geq -\delta_n^{n+2} = 2\epsilon_n. \quad (6.12)$$

5. $\text{int}(S) = \bigcap_{n \geq 2} \{x : \langle x - A_n, A_n \rangle < 0\} \cap \{x : \bar{\lambda} > \langle e_1, x \rangle > 0\} \cap \{x : 1 > \langle e_2, x \rangle > 0\} \neq \emptyset$.
6. $\text{bd}S = \bigcup_{n \geq 1} [A_n, A_{n+1}] \cup [0, \bar{\lambda}]e_1 \cup [0, 1]e_2$ and $S = \text{co}\{0, A_k, k \in \mathbb{N}^*\}$.

Computing g : Using the previous relations, we may now compute g around elements of the $\text{bd}(S)$. So let $n \in \mathbb{N}^*$, then

$$1. \ g(x) = \begin{cases} \max\left(\frac{1}{4^n} \langle x - A_n, A_n \rangle^+, \frac{1}{4^{n+1}} \langle x - A_{n+1}, A_{n+1} \rangle^+\right) & \text{if } n \geq 2, \\ \max\left(\frac{1}{4} \langle x - A_1, A_1 \rangle^+, \frac{1}{4^2} \langle x - A_2, A_2 \rangle^+\right) + \langle -e_1, x \rangle^+ & \text{if } n = 1, \end{cases}$$

$\forall x \in B(A_n, \epsilon_n)$, where ϵ_n is given by (6.8).

$$2. \ g(x) = \frac{1}{4^{n+1}} \langle x - A_{n+1}, A_{n+1} \rangle^+, \forall x \in B(A_n^t, \tilde{\epsilon}_n), \text{ where } \tilde{\epsilon}_n^t = \frac{1}{2} \min((1-t)\epsilon_n, t\epsilon_{n+1}^2), \\ A_n^t = A_n + t(A_{n+1} - A_n) \text{ and } t \in (0, 1).$$

3. For $t \in (0, \bar{\lambda})$, we set $\epsilon_1^t = \min\left(\frac{t}{2}, \frac{(\bar{\lambda}-t)\bar{\lambda}}{4}\right)$. Then

$$g(x) = \langle -e_2, x \rangle^+$$

$\forall x \in B(te_1, \epsilon_1^t)$.

$$4. g(x) = \langle -e_1, x \rangle^+ + \langle -e_2, x \rangle^+, \forall x \in B\left(0, \frac{\bar{\lambda}^2}{2}\right).$$

5. For $t \in (0, 1)$, we set $\epsilon_2^t = \frac{1}{4} \min(t, 1-t, -\delta_1^3)$. Then $g(x) = \langle -e_1, x \rangle^+, \forall x \in B(te_2, \epsilon_2^t)$.

Computing the subdifferential of g and the normal cone to S : Let $\tilde{x} \in S$, with $\tilde{x} \neq \bar{A}$. Then the following items hold:

1. If $\tilde{x} \in \text{int}(S)$ then $\partial g(\tilde{x}) = \{0\}$ and $N(S, \tilde{x}) = \{0\}$.
2. $\partial g(0) = [0, 1](-e_2) + [0, 1](-e_1)$ and $N(S, 0) = \mathbb{R}_+(-e_1) + \mathbb{R}_+(-e_2)$.
3. $\partial g(A_1) = [0, \frac{1}{4}]e_2 + [0, \frac{1}{16}]A_2 + [0, 1](-e_1)$ and $N(S, A_1) = \mathbb{R}_+A_1 + \mathbb{R}_+A_2 + \mathbb{R}_+(-e_1)$.
4. $\partial g(A_{n+1}) = [0, \frac{1}{4^{n+1}}]A_{n+1} + [0, \frac{1}{4^{n+2}}]A_{n+2}$ and $N(S, A_{n+1}) = \mathbb{R}_+A_{n+1} + \mathbb{R}_+A_{n+2}$.
5. $\partial g(A_{n+1}^t) = [0, \frac{1}{4^{n+2}}]A_{n+2}$ and $N(S, A_{n+1}^t) = \mathbb{R}_+A_{n+2}$, for every $n \in \mathbb{N}^*$ and $t \in (0, 1)$.
6. Guignard's condition holds at \tilde{x} .

Computing the normal cone to S at \bar{A} : By using the definition of g and the following formula

$$\lim_{n \rightarrow \infty} \mu^n \left(\cos \frac{\pi}{2^n} - \prod_{i \geq n+1} \cos \frac{\pi}{2^i} \right) = 0$$

where $\mu \in [0, 4]$, we obtain the following relations:

1. $[0, \bar{\lambda}]e_1 + [0, 1](-e_2) \subset \partial g(\bar{A})$.
2. $\mathbb{R}_+ \partial g(\bar{A}) = N(S, \bar{A}) = \mathbb{R}_+e_1 + \mathbb{R}_+(-e_2)$.

So, we have established that Guignard's constraint qualification holds at each element of the boundary of S .

Violation of error bound: Consider the sequence (x_n) defined by

$$x_n = (1 + \epsilon_n)A_n.$$

According to the previous relations, we have

$$g(x_n) = \max\left(\frac{1}{4^n} \langle x_n - A_n, A_n \rangle, \frac{1}{4^{n+1}} \langle x_n - A_{n+1}, A_{n+1} \rangle\right).$$

Now $\langle x_n - A_n, A_n \rangle = \epsilon_n \|A_n\|^2$ and $\langle x_{n+1} - A_{n+1}, A_{n+1} \rangle = \epsilon_n \langle A_n, A_{n+1} \rangle + \langle A_n - A_{n+1}, A_{n+1} \rangle$. But $\langle A_n - A_{n+1}, A_{n+1} \rangle = 0$ and $\langle A_n, A_{n+1} \rangle \leq \|A_n\| \|A_{n+1}\| < \|A_n\|^2$. Hence $g(x_n) = \frac{\epsilon_n}{4^n} \|A_n\|^2$. Since $A_n \in N(S, A_n)$, it follows that the projection of x_n over S is A_n . Then

$$d(x_n, S) = \epsilon_n \|A_n\| > 4^n g(x_n). \quad (6.13)$$

The last one shows that the local error bound does not hold at \bar{A} . Otherwise, there exist $\alpha > 0$ and $r > 0$ such that

$$d(x, S) \leq \alpha g_+(x) \quad \forall x \in \bar{A} + r\mathbb{B}.$$

Since $\lim_{n \rightarrow +\infty} x_n = \bar{A}$, then, for n sufficiently large, we should obtain

$$d(x_n, S) \leq \alpha g(x_n)$$

which combined with the inequality (6.13) gives $4^n < \alpha$, with n large enough, and this contradiction completes the proof. \square

It follows that g is positively homogeneous, convex and continuous on \mathbb{R}^3 , and thus

$$S = \bigcap_{k \geq 1} \{x \in \mathbb{R}^3 : \langle B_k, x \rangle \leq 0\} \cap \{x \in \mathbb{R}^3 : \langle \bar{B}, x \rangle \leq 0\} \cap \mathbb{R}_+^3. \quad (6.19)$$

Proposition 6.2 *With this function g , the Guignard condition holds for the system (1.8) whereas all error bound concepts are violated.*

Proof. *The proof is based on the following lemmas.*

Lemma 6.1 (Normal cone of S at 0) *We have*

$$N(S, 0) = \mathbb{R}_+ \text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\} \cup \{\bar{B}\}) + \mathbb{R}_-^3$$

Proof. *Set $\mathcal{A}_1 = \mathbb{R}_+ \text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\} \cup \{\bar{B}\})$. By (6.19), $\{0\} \cup \{B_k : k \in \mathbb{N}^*\} \cup \{\bar{B}\} \subset N(S, 0)$ and then $\mathcal{A}_1 \subset N(S, 0)$. The second inclusion is established in four steps.*

Step 1 : *Let us first prove that the set $\mathcal{A}_1 := \mathbb{R}_+ \text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\} \cup \{\bar{B}\})$ is closed. Let $(x_n)_{n \in \mathbb{N}^*}$ be a sequence of \mathcal{A}_1 converging to some \bar{x} . Let us show that $\bar{x} \in \mathcal{A}_1$. Set then $x_n = \lambda_n u_n$, $n \in \mathbb{N}^*$, where $u_n \in \text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\} \cup \{\bar{B}\})$ and $\lambda_n \geq 0$.*

Using Caratheodory theorem, u_n can be written as $u_n = \sum_{i=1}^4 \alpha_{k_{n_i}} h_{k_{n_i}}$, with, $\alpha_{k_{n_i}} \geq 0$,

$$\sum_{i=1}^4 \alpha_{k_{n_i}} \leq 1 \text{ and } h_{k_{n_i}} \in \{B_k, \bar{B}, k \in \mathbb{N}^*\}. \text{ Recall that } (B_k)_3 = -\|A_k\|^2 \leq -\bar{\lambda}^2, \forall k \in \mathbb{N}^*.$$

It follows that $(x_n)_3 = -\lambda_n \sum_{i=1}^4 \alpha_{k_{n_i}} \|A_{k_{n_i}}\|^2 \leq -\lambda_n \bar{\lambda}^2$ and thus $(\lambda_n)_{n \in \mathbb{N}^}$ is bounded.*

The result follows since $\{B_k, \bar{B}, k \in \mathbb{N}^\}$ is compact and $(\alpha_{k_{n_i}})_{k_{n_i} \in \mathbb{N}^*}$, $i = 1, \dots, 4$, are bounded.*

Step 2 : *Let us prove that $\mathcal{A}_1 \cap \mathbb{R}_+^3 = \{0\}$. It is enough to prove that $\text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\}) \cap \mathbb{R}_+^3 = \{0\}$. Let $x \in \text{co}(\{0\} \cup \{B_k : k \in \mathbb{N}^*\}) \cap \mathbb{R}_+^3$. Write then $x = \sum_{i=1}^4 \alpha_{k_i} h_{k_i}$, with,*

$$\alpha_{k_i} \geq 0, \sum_{i=1}^4 \alpha_{k_i} \leq 1 \text{ and } h_{k_i} \in \text{co}(\{B_k : k \in \mathbb{N}^*\}). \text{ Then}$$

$$0 \leq x_3 = -\sum_{i=1}^4 \alpha_{k_i} \|A_{k_i}\|^2 \leq -\bar{\lambda}^2 \sum_{i=1}^4 \alpha_{k_i} \leq 0.$$

Hence $\sum_{i=1}^4 \alpha_{k_i} = 0$ and thus $x = 0$.

Step 3 *Let us prove now that $\mathcal{A} = \mathcal{A}_1 + \mathbb{R}_-^3$ is closed. Let $(x_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be a convergent sequence to some \bar{x} . Let then $(u_n)_{n \in \mathbb{N}} \subset \mathcal{A}_1$ and $(v_n)_{n \in \mathbb{N}} \subset \mathbb{R}_-^3$ such that $x_n = u_n + v_n$.*

Claims : $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are bounded.

Suppose the contrary. Without loss of generality we can assume that $\lim_{n \rightarrow +\infty} \|u_n\| = +\infty$.

Then

$$\lim_{n \rightarrow +\infty} \frac{x_n}{\|u_n\|} = \lim_{n \rightarrow +\infty} \frac{u_n}{\|u_n\|} + \frac{v_n}{\|u_n\|} = 0 \quad (6.20)$$

Let then $u^ \neq 0$ be a limit of a convergent subsequence of $\left(\frac{u_n}{\|u_n\|}\right)_{n \in \mathbb{N}}$. By step 1, $u^* \in \mathcal{A}_1$.*

By (6.20), $-u^$ is a limit of a convergent subsequence of $\left(\frac{v_n}{\|u_n\|}\right)_{n \in \mathbb{N}} \subset \mathbb{R}_-^3$. It follows*

that $u^ \in \mathcal{A}_1 \cap \mathbb{R}_+^3 \setminus \{0\}$, which contradicts the step 2's result.*

Step 4 : *We will establish the inclusion $N(S, 0) \subset \mathcal{A}$. So let us suppose, by contradiction, that there exists $x^* \in N(S, 0) \setminus \mathcal{A}$. We said (step 3) that \mathcal{A} is closed. According to the separation theorem, there exist $h \in \mathbb{R}^3$, $\|h\| = 1$ and $\alpha \in \mathbb{R}$ such that*

$$\langle x^*, h \rangle < \alpha \leq \langle u^*, h \rangle, \forall u^* \in \mathcal{A}_1.$$

Then necessarily

$$\langle x^*, h \rangle < 0 \text{ and } \langle u^*, h \rangle \geq 0, \forall u^* \in \mathcal{A}.$$

But

$$\begin{aligned} \langle u^*, h \rangle \geq 0, \forall u^* \in \mathcal{A} &\Leftrightarrow \langle u^*, h \rangle \geq 0, \forall u^* \in \mathcal{A}_1 \text{ and } \langle v^*, h \rangle \geq 0, \forall v^* \in \mathbb{R}_-^3 \\ &\Leftrightarrow h \leq 0 \text{ and } \langle u^*, h \rangle \geq 0, \forall u^* \in \mathcal{A}_1 \\ &\Leftrightarrow -h \in S \text{ (by definition of } S \text{)}. \end{aligned}$$

Thus we have a contradiction with the fact that $\langle x^*, -h \rangle > 0$ and $x^* \in N(S, 0)$. \(\square\)

Using Lemma 2.4, we obtain the following estimation of the subdifferential of g .

Lemma 6.2 (Subdifferential of g)

$$1. \partial g(0) = \text{clco} \left(\bigcup_{k \geq 1} [0, \frac{1}{4^k}] B_k \right) + [0, 1] \bar{B} + [0, 1](-e_1) + [0, 1](-e_2) + [0, 1](-e_3).$$

$$2. \forall x \in \mathbb{R}^3, \partial g(x) = \{x^* \in \mathbb{R}^3 : \langle x^*, x \rangle = g(x)\} \cap \partial g(0).$$

Proof. Define the functions h_k and h on \mathbb{R}^3 by $h_k(x) = \frac{1}{4^k} \langle B_k, x \rangle^+$ and $h := \sup h_k$. Then

$$g = h + \langle \bar{B}, \cdot \rangle^+ + \langle -e_1, \cdot \rangle^+ + \langle -e_2, \cdot \rangle^+ + \langle -e_3, \cdot \rangle^+.$$

Thus

$$\partial g(0) = \partial h(0) + [0, 1] \bar{B} + [0, 1](-e_1) + [0, 1](-e_2) + [0, 1](-e_3).$$

It remains to compute the subdifferential of h at 0. By Lemma 2.4, we obtain

$$\partial h(0) = \text{clco} \left(\bigcup_{k \geq 1} \partial h_k(0) \right) = \text{clco} \left(\bigcup_{k \geq 1} [0, \frac{1}{4^k}] B_k \right)$$

Whence the first item.

The second item follows from the first part of Lemma 2.4. \(\square\)

The following result shows that Guignard's condition holds.

Lemma 6.3 (Guignard's condition) For all $x \in \text{bd}(S)$,

$$N(S, x) = \mathbb{R}_+ \partial g(x).$$

Proof. Since $x \in S$ then $g(x) = 0$. Using Lemma 6.2, $\partial g(x) = \{x\}^\perp \cap \partial g(0)$, where $\{x\}^\perp = \{x^* \in \mathbb{R}^3 : \langle x^*, x \rangle = 0\}$. Invoking Lemmas 6.1 and 6.2 we obtain $N(S, 0) = \mathbb{R}_+ \partial g(0)$. As $N(S, x) = \{x\}^\perp \cap N(S, 0)$ then

$$N(S, x) = \mathbb{R}_+ \partial g(x). \(\square\)$$

Violation of error bound: Consider the sequence (x_n) defined by

$$x_n = \tilde{A}_n + \epsilon_n B_n, \quad n \geq 2$$

where $\epsilon_n = \frac{1}{2} \min \left(1 - \cos \frac{\pi}{2^n}, -\delta_n^{n+2} \right)$. Then

$$g(x_n) = \frac{1}{4^n} \epsilon_n \|B_n\|^2 \text{ and } d(x_n, S) = \|x_n - \tilde{A}_n\| = \epsilon_n \|B_n\|$$

which show the violation of the local error bound. \(\square\)

We use the last example to build homogeneous functions for which Guignard's condition holds but not error bound whenever $\dim(X) \geq 4$. Let us recall that a Counterexample has been given in [3] in \mathbb{R}^4 with g positively homogeneous and $\text{int}(S) = \emptyset$.

Example 4 ($\dim X \geq 4$ and g positively homogeneous) Let g be the function considered in Example 3 and the corresponding set $S := \{x \in \mathbb{R}^3 : g(x) \leq 0\}$.

For all $n \in \mathbb{N}$, with $n \geq 4$, consider the two functions g_n and \tilde{g}_n defined on \mathbb{R}^n by

$$g_n(x) = g(x_1, x_2, x_3) \text{ and } \tilde{g}_n(x) = g(x_1, x_2, x_3) + \sum_{i=4}^n |x_i| \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Let $S_n := \{x \in \mathbb{R}^n : g_n(x) \leq 0\}$ and $\tilde{S}_n := \{x \in \mathbb{R}^n : \tilde{g}_n(x) \leq 0\}$. Then g_n and \tilde{g}_n are positively homogeneous and

$$S_n = S \times \mathbb{R}^{n-3} \text{ and } \tilde{S}_n = S \times \{0_{n-3}\}.$$

This shows $\text{int}(S_n) \neq \emptyset$ and $\text{int}(\tilde{S}_n) = \emptyset$. With the help of Example 3, it is not difficult to see that in both situation Guignard's condition holds but is not error bound.

Remark 6.1 Note that in Example 2 (resp. Example 3), we may also consider the function

$$g_2(x) = \sup_{k \geq 1} \frac{1}{4^k} (\langle x - A_k, A_k \rangle^+) + \langle x - \bar{A}, e_1 \rangle^+, \quad \forall x \in \mathbb{R}^2$$

$$\left(\text{resp. } g_3(x) = \sup_{k \geq 1} \frac{1}{4^k} (\langle B_k, x \rangle^+) + \langle \bar{B}, x \rangle^+, \quad \forall x \in \mathbb{R}^3 \right).$$

With this choice, the set S in the two Examples is unbounded.

7 Error bound characterization of Guignard's constraint qualification for nondifferentiable convex systems : X is a Banach space

As we saw in the previous sections, Guignard's constraint qualification for nondifferentiable convex systems is not sufficient to characterize error bound. In this section, we will introduce the following new condition: For $r > 0$ there exists $\alpha_r > 0$ such that

$$\gamma(x^*/\partial g(x)) \leq \alpha_r \|x^*\| \quad \forall x \in B(0, r) \cap S, \quad \forall x^* \in N(S, x). \quad (7.1)$$

Here $\gamma(\cdot/D)$ is the gauge function of the convex set D , that is,

$$\gamma(v/D) = \inf\{\lambda \geq 0 : v \in \lambda D\}.$$

We have the following proposition which is based on this condition.

Proposition 7.1 Let $\alpha_r > 0$ and $x \in S$, with $g(x) = 0$. Then the following assertions are equivalent

1.

$$\partial d(x, S) \subset \alpha_r \partial g_+(x). \quad (7.2)$$

2. For all $u \in X$,

$$d(u, T(S, x)) \leq \alpha_r g'_+(x, u). \quad (7.3)$$

3.

$$\gamma(x^*/\partial g_+(x)) \leq \alpha_r \|x^*\| \quad \forall x^* \in N(S, x). \quad (7.4)$$

Each one of the three conditions ensures that Guignard's constraint qualification holds at x .

Proof. 1. \Leftrightarrow 2. This equivalence follows from Lemma 2.2 and the definition of the Fenchel subdifferential.

2. \Rightarrow 3.: By 2. and Proposition 7.1, we have

$$\partial d(x, S) \subset \alpha_r \partial g_+(x).$$

Let $x^* \in N(S, x)$, with $x^* \neq 0$, then $\frac{x^*}{\|x^*\|} \in \partial d(x, S)$ and hence $\frac{x^*}{\|x^*\|} \in \alpha_r \partial g_+(x)$. Thus $\gamma(x^*/\partial g_+(x)) \leq \alpha_r \|x^*\|$.

3. \Rightarrow 2.: Suppose that there exists $u \in X$ such that $d(u, T(S, x)) > \alpha_r g'_+(x, u)$. Lemma 2.3 ensures the existence of $x^* \in N(S, x) \cap \mathbb{B}^*$ such that

$$d(u, T(S, x)) = \langle x^*, u \rangle > \alpha_r g'_+(x, u).$$

Note that $g'_+(x, u) > 0$, otherwise $u \in T(S, x)$ and one obtains a contradiction with the last inequality. Then for all $\varepsilon \in]0, \frac{\langle x^*, u \rangle - \alpha_r g'_+(x, u)}{\alpha_r g'_+(x, u)}[$ there exists $\lambda > 0$ satisfying $\gamma(x^*/\partial g_+(x)) > \lambda - \alpha_r \varepsilon$ and $x^* \in \lambda \partial g_+(x)$. So that

$$\langle x^*, u \rangle > \varepsilon \alpha_r g'_+(x, u) + \alpha_r g'_+(x, u) \geq \frac{\alpha_r (1 + \varepsilon)}{\lambda} \langle x^*, u \rangle$$

which implies that $\lambda > \alpha_r (1 + \varepsilon)$ and hence $\gamma(x^*/\partial g_+(x)) > \lambda - \alpha_r \varepsilon > \alpha_r$. But item 3. guarantees the inequality

$$\gamma(x^*/\partial g_+(x)) \leq \alpha_r \|u^*\| \leq \alpha_r.$$

This contradiction completes the proof. \(\square\)

Remark 7.1 This equivalence 1. \Leftrightarrow 2. has been observed first by Burke and Ferris [5] in finite dimension.

Now we may state and prove our characterization for nondifferentiable convex inequality systems.

Theorem 7.1 The following assertions are equivalent:

1. For all $r > 0$ there exists $\alpha_r > 0$ such that for all $x \in r\mathbb{B}$, with $g(x) = 0$, relation (7.4) holds.
2. For all $r > 0$ there exists $\alpha_r > 0$ such that

$$d(x, S) \leq \alpha_r g_+(x) \quad \forall x \in r\mathbb{B}. \tag{7.5}$$

Proof. 1. \implies 2.: Let $r > 0$, $x \in r\mathbb{B} \setminus S$ and $\bar{x} \in S$. By Lemma 2.1, for all $\varepsilon \in]0, \min(\frac{1}{2}, \sqrt{r})[$ there exists $u_\varepsilon \in S$, $x_\varepsilon^* \in X^*$ and $b_\varepsilon^* \in \mathbb{B}^*$ such that

- (a) $\|u_\varepsilon - x\| \leq d(x, S) + \varepsilon^2$,
- (b) $x_\varepsilon^* + \varepsilon b_\varepsilon^* \in (1 + \varepsilon) \partial d(u_\varepsilon, S)$,
- (c) $\langle x_\varepsilon^*, x - u_\varepsilon \rangle = \|u_\varepsilon - x\|$.

Items (b) and (c) ensure that $g(u_\varepsilon) = 0$ while item (a) implies that $u_\varepsilon \in (3r + \|\bar{x}\|)\mathbb{B}$. Let $s \geq 3r + \|\bar{x}\|$. Assertion 1. ensures the existence of $\alpha_s > 0$ (depending only on r and \bar{x}) such that

$$\gamma(x^*/\partial g(u_\varepsilon)) \leq \alpha_s \|x^*\| \forall x^* \in N(S, u_\varepsilon)$$

or equivalently via Proposition 7.1,

$$\partial d(u_\varepsilon, S) \subset \alpha_s \partial g_+(u_\varepsilon).$$

Using assertion (b), we get $x_\varepsilon^* + \varepsilon b_\varepsilon^* \in (1 + \varepsilon)\alpha_s \partial g_+(u_\varepsilon)$, which ensures the inequality

$$\langle x_\varepsilon^* + \varepsilon b_\varepsilon^*, x - u_\varepsilon \rangle \leq (1 + \varepsilon)\alpha_s g(x).$$

Using (c), we obtain

$$\|x - u_\varepsilon\| (1 - \varepsilon) \leq (1 + \varepsilon)\alpha_s g(x)$$

and hence

$$(1 - \varepsilon) d(x, S) \leq (1 + \varepsilon)\alpha_s g(x).$$

As ε is arbitrary, we get

$$d(x, S) \leq \alpha_s g(x)$$

whence 2..

2. \implies 1.: It is obvious that 2. ensures that for all $r > 0$ there exists $\alpha_r > 0$ such that for all $x \in r\mathbb{B}$

$$d(x, S) \leq \alpha_r g_+(x) \quad \forall x \in r\mathbb{B}.$$

Now, let $x \in X$ be such that $\|x\| < r$, with $g(x) = 0$, and $x^* \in \partial d(x, S)$. Then

$$\langle x^*, u - x \rangle \leq d(u, S) \quad \forall u \in X$$

and hence

$$\langle x^*, u - x \rangle \leq \alpha_r g_+(u) \quad \forall u \in r\mathbb{B}.$$

As $\|x\| < r$, the later inequality is equivalent to

$$\langle x^*, u - x \rangle \leq \alpha_r g_+(u) \quad \forall u \in X$$

and hence $x^* \in \alpha_r \partial g(x)$. Consequently,

$$\partial d(x, S) \subset \alpha_r \partial g_+(x)$$

which is equivalent to relation (7.4). \(\square\)

As a consequence of this theorem, we obtain the following characterization for global error bound.

Corollary 7.1 *Let $\alpha > 0$. Then the following assertions are equivalent:*

1. For all $x \in S$, with $g(x) = 0$,

$$\gamma(x^*/\partial g(x)) \leq \alpha \|x^*\| \quad \forall x^* \in N(S, x).$$

2.

$$d(x, S) \leq \alpha g_+(x) \quad \forall x \in X. \tag{7.6}$$

8 Guignard's constraint qualification and SCHIP property

The aim of this section is to give a relationship between Guignard's constraint qualification and SCHIP property.

Before giving this connexion, we recall that the collection C_1, \dots, C_m of closed convex subset of X satisfies the conical hull intersection property (CHIP) if

$$T(C, x) = \cap_{i=1}^m T(C_i, x) \forall x \in C \quad (8.1)$$

where $C := \cap_{i=1}^m C_i \neq \emptyset$. This collection satisfies the strong conical hull intersection property (SCHIP) if

$$N(C, x) = \sum_{i=1}^m N(C_i, x) \forall x \in C. \quad (8.2)$$

Let $1 \leq p, q \leq +\infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Define the function g on X by

$$g(x) = \left(\sum_{i=1}^m d(x, C_i)^p \right)^{\frac{1}{p}} \quad \forall x \in X.$$

Before computing the subdifferential of g , we rewrite it in the following form

$$g(x) = (h \circ w)(x) \forall x \in X$$

where the mappings $h : \mathbb{R}^m \mapsto \mathbb{R}$ and $w : X \mapsto \mathbb{R}^m$ are defined by

$$h(u) = \|u^+\|_p := \left(\sum_{i=1}^m (u_i^+)^p \right)^{\frac{1}{p}} \quad \text{and} \quad w(x) = (d(x, C_1), \dots, d(x, C_m))$$

and $a^+ = \max(a, 0)$. Let

$$\mathbb{B}_q^* = \{u^* = (u_1^*, \dots, u_m^*) \in \mathbb{R}^m : \|u^*\|_q := \left(\sum_{i=1}^m |u_i^*|^q \right)^{\frac{1}{q}} \leq 1\}.$$

Proposition 8.1 *Let $x \in C$, that is, $w(x) = 0$. Then*

1. *h is a convex function and $\partial h(0) = \mathbb{B}_q^* \cap \mathbb{R}_+^m$.*

2. $\partial g(x) = \bigcup_{u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m} \left(\sum_{i=1}^m u_i^* \partial d(x, C_i) \right)$.

Proof. 1. Taking into account that the functions $t \mapsto t^+$ and $u \mapsto \|u\|_p$ are convex, we conclude that h is also convex. Let $u^* \in \partial h(0)$, that is,

$$\langle u^*, u \rangle \leq \left(\sum_{i=1}^m (u_i^+)^p \right)^{\frac{1}{p}} \quad \forall u \in \mathbb{R}^m. \quad (8.3)$$

Now, take all the components of u equal to zero except the i th one $u_i < 0$. Then $u_i^* \geq 0$ and this shows that $u^* \in \mathbb{R}_+^m$. This fact and relation (8.3) ensure that

$$\langle u^*, u \rangle \leq \sum_{i=1}^m u_i^* |u_i| \leq \left(\sum_{i=1}^m |u_i|^p \right)^{\frac{1}{p}} \quad \forall u \in \mathbb{R}^m$$

and hence

$$\|u^*\|_q \leq 1.$$

So that $\partial h(0) \subset \mathbb{B}_q^* \cap \mathbb{R}_+^m$.

Let $u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m$. Then, for all $u \in \mathbb{R}^m$, $\langle u^*, u^+ \rangle \leq \|u^+\|_p$. As $u^* \in \mathbb{R}_+^m$, we have

$$\langle u^*, u \rangle \leq \langle u^*, u^+ \rangle \leq \|u^+\|_p = h(u) \quad \forall u \in \mathbb{R}^m$$

and hence $u^* \in \partial h(0)$. Consequently, $\mathbb{B}_q^* \cap \mathbb{R}_+^m \subset \partial h(0)$.

2. Since h and w are locally Lipschitz, the subdifferential calculus ensures that

$$\begin{aligned} \partial g(x) &\subset \bigcup_{u^* \in \partial h(0)} \partial(u^* \circ w)(x) \\ &= \bigcup_{u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m} \partial \left(\sum_{i=1}^m u_i^* d(\cdot, C_i) \right) (x) \\ &= \bigcup_{u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m} \left(\sum_{i=1}^m u_i^* \partial d(x, C_i) \right). \end{aligned}$$

Now, let $u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m$ and $x^* \in \sum_{i=1}^m u_i^* \partial d(x, C_i) = \partial(\sum_{i=1}^m u_i^* d(\cdot, C_i))(x)$. Then for all $y \in X$

$$\begin{aligned} \langle x^*, y - x \rangle &\leq \sum_{i=1}^m u_i^* d(y, C_i) \\ &\leq \|u^*\|_q \left(\sum_{i=1}^m d(y, C_i)^p \right)^{\frac{1}{p}} \\ &\leq g(y). \end{aligned}$$

Thus $x^* \in \partial g(x)$. Consequently, $\bigcup_{u^* \in \mathbb{B}_q^* \cap \mathbb{R}_+^m} \left(\sum_{i=1}^m u_i^* \partial d(x, C_i) \right) \subset \partial g(x)$. □

Using this proposition, we obtain the following rewriting of the CHIP properties as well as a computation of the gauge function of $\partial g(x)$.

Proposition 8.2 *Let $x \in C$. Then*

1. $C = \{x \in X : g(x) \leq 0\}$.
2. $g'(x, h) = (\sum_{i=1}^m ((d(\cdot, C_i))'(x, h))^p)^{\frac{1}{p}} = (\sum_{i=1}^m (d(h, T(C_i, x))^p)^{\frac{1}{p}}$.
3. Relation (8.1) holds at x IFF

$$T(C, x) = \{u \in X : g'(x, u) \leq 0\}.$$

4. Relation (8.2) holds at x IFF

$$N(C, x) = \mathbb{R}_+ \partial g(x).$$

5. For all $x^* \in X^*$

$$\gamma(x^*/\partial g(x)) = \min \left\{ \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m \right\}. \quad (8.4)$$

Proof. 2.. By Lemmas 2.3 and 2.2, we obtain

$$d(h, T(C_i, x)) = \sup_{x^* \in \partial d(x, C_i)} \langle x^*, h \rangle = (d(\cdot, C_i))'(x, h).$$

So that

$$g'(x, h) = \left(\sum_{i=1}^m ((d(\cdot, C_i))'(x, h))^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^m (d(h, T(C_i, x)))^p \right)^{\frac{1}{p}}.$$

5. Let $x^* \in X^*$. Then the set $\{\lambda \geq 0 : x^* \in \lambda \partial g(x)\}$ is empty IFF the set $\{(x_1^*, \dots, x_m^*) \in N(C_1, x) \times \dots \times N(C_m, x) : x^* = \sum_{i=1}^m x_i^*\}$ is also empty. So that both quantities $\gamma(x^*/\partial g(x))$ and $\min\{(\sum_{i=1}^m \|x_i^*\|^q)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m\}$ are equal to $+\infty$.

Now let $x_i^* \in N(C_i, x), i = 1, \dots, m$ be such that $x^* = \sum_{i=1}^m x_i^*$. Without loss of generality, we may assume that all $x_i^* \neq 0$. Then

$$x^* = \sum_{i=1}^m x_i^* = \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} \sum_{i=1}^m \frac{\|x_i^*\|}{(\sum_{i=1}^m \|x_i^*\|^q)^{\frac{1}{q}}} x_i^*.$$

For $i = 1, \dots, m$, put $u_i^* = \frac{\|x_i^*\|}{(\sum_{i=1}^m \|x_i^*\|^q)^{\frac{1}{q}}}$ and $y_i^* = \frac{x_i^*}{\|x_i^*\|}$. Then $x^* = (\sum_{i=1}^m \|x_i^*\|^q)^{\frac{1}{q}} \sum_{i=1}^m u_i^* y_i^*$, $u^* := (u_1^*, \dots, u_m^*) \in B_q^* \cap \mathbb{R}_+^m$ and $y_i^* \in \partial d(x, C_i)$. Consequently,

$$x^* \in \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} \sum_{i=1}^m u_i^* \partial d(x, C_i) \subset \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} \partial g(x)$$

and hence

$$\gamma(x^*/\partial g(x)) \leq \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}}.$$

Thus

$$\gamma(x^*/\partial g(x)) \leq \min\left\{ \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m \right\}.$$

Now, let $\varepsilon > 0$ and $\beta \geq 0$ be such that $x^* \in \beta \partial g(x)$ and

$$\beta \leq \gamma(x^*/\partial g(x)) + \varepsilon.$$

By Proposition 8.1, there exist $u^* \in B_q^* \cap \mathbb{R}_+^m$ and $z_i^* \in \partial d(x, C_i), i = 1, \dots, m$, such that $x^* = \beta \sum_{i=1}^m u_i^* z_i^*$. Since $\|z_i^*\| \leq 1, i = 1, \dots, m$, and

$$\left(\sum_{i=1}^m \|u_i^* z_i^*\|^q \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^m u_i^{*q} \right)^{\frac{1}{q}} \leq 1$$

then

$$\beta \left(\sum_{i=1}^m \|u_i^* z_i^*\|^q \right)^{\frac{1}{q}} \leq \beta.$$

Consequently, because $\beta u_i^* z_i^* \in N(C_i, x), i = 1, \dots, m$,

$$\min\left\{ \left(\sum_{i=1}^m \|x_i^*\|^q \right)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m \right\} \leq \gamma(x^*/\partial g(x)) + \varepsilon$$

and as $\varepsilon > 0$ is arbitrary we get the desired inequality.

□

By combining the previous result, we obtain the following characterization of the SCHIP property.

Theorem 8.1 *The following assertions are equivalent:*

1. For all $r > 0$ there exists $\alpha_r > 0$ such that for all $x \in \text{rint}(\mathbb{B}) \cap C$

$$\min\left\{\left(\sum_{i=1}^m \|x_i^*\|^q\right)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m\right\} \leq \alpha_r \|x^*\| \forall x^* \in N(C, x).$$

2. For all $r > 0$ there exists $\alpha_r > 0$ such that

$$d(x, S) \leq \alpha_r \left(\sum_{i=1}^m d(x, C_i)^p\right)^{\frac{1}{p}} \quad \forall x \in r\mathbb{B}. \quad (8.5)$$

We may obtain the following result by [22] as a corollary of results.

Corollary 8.1 *Let $\alpha > 0$. Then the following assertions are equivalent:*

1. For all $x \in C$ and $x^* \in N(C, x)$

$$\min\left\{\left(\sum_{i=1}^m \|x_i^*\|^q\right)^{\frac{1}{q}} : x^* = \sum_{i=1}^m x_i^*, x_i^* \in N(C_i, x), i = 1, \dots, m\right\} \leq \alpha \|x^*\|.$$

- 2.

$$d(x, S) \leq \alpha \left(\sum_{i=1}^m d(x, C_i)^p\right)^{\frac{1}{p}} \quad \forall x \in X. \quad (8.6)$$

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