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Laurence Bessis

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# HYPERELLIPTIC HYPERMAPS OF EVEN GENUS AND THEIR APPLICATIONS TO GENUS 2.

LAURENCE BESSIS

ABSTRACT. We show that automorphisms groups on hyperelliptic hypermaps of even genus  $g$  are subgroups of the following groups :  $C_{4g+2}$ ,  $(C_2 \times C_{2g+2}) \rtimes C_2$ ,  $E_{8g}$ ,  $GL_2(F_3)$  or  $SL_2(F_5)$ , where  $E_{8g}$  are *Elliptic* groups that can be viewed as an extension of Quasi-Dihedral groups. We construct most of the maximal groups with respect to order and genus according to the group structure. We show that hypermaps of genus 2 are almost all hyperelliptic and that their automorphisms group is a subgroup of one of the following groups :  $C_{10}$ ,  $(C_2 \times C_6) \rtimes C_2$  or  $GL_2(F_3)$ .

*Résumé.* Nous montrons que les groupes d'automorphismes d'hypercartes hyperelliptiques de genre  $g$  pair sont un sous-groupe de l'un des groupes suivants :  $C_{4g+2}$ ,  $(C_2 \times C_{2g+2}) \rtimes C_2$ ,  $E_{8g}$ ,  $GL_2(F_3)$  or  $SL_2(F_5)$ , les groupes *Elliptiques*  $E_{8g}$  pouvant se comprendre comme une extension des groupes quasi-diedraux; nous construisons la plus part des groupes maximaux selon l'ordre, le genre et la structure du groupe. Nous montrons que les hypercartes de genre 2 sont presque toutes hyperelliptiques et que leur groupe d'automorphismes est un sous-groupe de l'un des groupes suivants :  $C_{10}$ ,  $(C_2 \times C_6) \rtimes C_2$  ou  $GL_2(F_3)$ .

## 1. INTRODUCTION

The combinatorial study of Riemann surfaces topology is a research field where either geometers such as G. Jones and D. Singerman (see [JoSi]) or group and graph theorists such as R. Cori and A. Machi (see [CoMa]) have developed to meet at the joint point of the theory of maps. This study is made effective by the tessellation of the surface into polygons that naturally leads to the consideration of a map: a pair of permutations  $(\alpha, \sigma)$ , one corresponding to the edges and the other to the vertices of these polygons. Then, topological properties have combinatorial translations for instance, connexity on the surface is ensured by the transitivity of  $\langle \alpha, \sigma \rangle$  (see below for a detailed explanation). Some properties of the automorphisms of the surface correspond then to the properties of  $Aut(\alpha, \sigma)$ , centralizer of both  $\alpha$  and  $\sigma$  in the symmetric group. An example is given by a classical result known as the Riemann Hurwitz formula which relates the genus of the surface, the genus of its quotient surface with respect to an automorphism group  $G$  and the number of points fixed by all elements in  $G$  on the surface. Defining the *genus* of the map in function of the number of cycles of  $\alpha$ ,  $\sigma$  and  $\alpha^{-1}\sigma$ , Machi showed that the formula holds for  $Aut(\alpha, \sigma)$  (see [Ma]).

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Maps theory has been generalized in a natural way to the one of hypermaps by Cori and can be related to hypergraph theory (see [Co] and below). Searching for links between a hypermap and its quotient hypermap with respect to an automorphism group, we have proposed in [Be2] the concept of induced automorphism which can be understood as follow. Let  $(\alpha, \sigma)$  be a hypermap,  $Aut(\alpha, \sigma)$ ,  $G$  a subgroup of  $Aut(\alpha, \sigma)$  and let  $(\bar{\alpha}, \bar{\sigma})$  be the quotient hypermap of  $(\alpha, \sigma)$  with respect to  $G$ . When  $\psi$  belongs to the normalizer of  $G$  in  $Aut(\alpha, \sigma)$ , the automorphism  $\bar{\psi}$  induced by  $\psi$  on  $(\bar{\alpha}, \bar{\sigma})$  is well defined.

This paper deals with the hyperelliptic hypermaps introduced in [CoMa] as a natural correspondancy to hyperelliptic surfaces. The idea is to study hypermaps on which all automorphisms are induced on the sphere by operating a quotient with respect to the hyperelliptic involution. These can be viewed as 2-sheeted coverings of the sphere (see below). Since it appears that the genus parity modifies greatly the structure of automorphisms groups, we concentrate on hyperelliptic hypermaps of even genus. The general theorem states that  $Aut(\alpha, \sigma)$  is a subgroup of one of the following groups :  $C_{4g+2}$ ,  $(C_2 \times C_{2g+2}) \times C_2$ ,  $E_{8g}$ ,  $GL_2(F_3)$  or  $SL_2(F_5)$ , where  $E_{8g}$  is the *elliptic group* of order  $8g$  that can be defined in the following way :  $E_{8g} = \langle a, b/a^{4g} = b^2 = 1, ba = a^{2g-1}b \rangle$ ; when  $g$  is even, elliptic groups can be viewed as a generalisation of quasidiedral groups; we give a general technique to construct maps owning these automorphisms groups. The former theorem can be directly applied to hypermaps of genus 2 since we prove that “almost” all hypermaps of genus 2 are hyperelliptic. Therefore, if  $(\alpha, \sigma)$  is a hypermap of genus 2, then  $Aut(\alpha, \sigma)$  is a subgroup of one of the following groups :  $C_{10}$ ,  $(C_2 \times C_6) \times C_2$  or  $GL_2(F_3)$ . In annexe, one can find examples and drawings of most of these groups.

## 2. HYPERMAPS, AUTOMORPHISMS AND INDUCED AUTOMORPHISMS

For a general introduction to the theory of hypermaps see [CoMa]. In this section we recall a few definitions and results that will be needed in the sequel.

**Definition 1.** A *hypermap* is a pair of permutations  $(\alpha, \sigma)$  on  $B$  (the set of *brins*) such that the group they generate is transitive on  $B$ . When  $\alpha$  is a fixed point free involution,  $(\alpha, \sigma)$  is a *map*. The cycles of  $\alpha, \sigma$  and  $\alpha^{-1}\sigma$  are called *edges*, *vertices* and *faces*, respectively; but if their specification in terms of edges, vertices or faces is not needed, we will refer to them as *points*. Note that considering the permutation means doing first  $\alpha^{-1}$  and then  $\sigma$  so that permutations are multiplied from left to right.

Euler’s formula gives the relationship between the numbers of cycles of these three permutations:

$$z(\alpha) + z(\sigma) + z(\alpha^{-1}\sigma) = n + 2 - 2g$$

where  $n = \text{card}(B)$ ,  $g$  is a non-negative integer, called the *genus* of  $(\alpha, \sigma)$  and where for any permutation  $\theta$ ,  $z(\theta)$  denotes the number of its cycles (cycles of length 1 are included) (see [CoMa], p.422). If  $g = 0$ , then  $(\alpha, \sigma)$  is *planar*; if  $g = 1$ , then  $(\alpha, \sigma)$  is *toroidal* (on the torus).

**Definition 2.** An *automorphism*  $\phi$  of a hypermap  $(\alpha, \sigma)$  is a permutation commuting with both  $\alpha$  and  $\sigma$  that is :

$$\phi\alpha = \alpha\phi \text{ and } \phi\sigma = \sigma\phi$$

Thus, the full automorphism group of  $(\alpha, \sigma)$ , denoted by  $Aut(\alpha, \sigma)$ , is the centralizer in  $Sym(n)$  of the group generated by  $\alpha$  and  $\sigma$ . A subgroup  $G$  of  $Aut(\alpha, \sigma)$  is an automorphism group of  $(\alpha, \sigma)$ ; the transitivity of  $(\alpha, \sigma)$  implies that  $Aut(\alpha, \sigma)$  is semi-regular. Recall here that a semi-regular group is defined by the fact that all its orbits are of the same length namely  $|G|$ .

We denote by  $\chi_\theta(\phi)$  the number of cycles of a permutation  $\theta$  fixed by an automorphism  $\phi$  and by  $\chi(\phi)$  the total number of cycles of  $\alpha, \sigma$  and  $\alpha^{-1}\sigma$  fixed by  $\phi$ ;  $o(\phi)$  will be the order of  $\phi$ . If  $(\alpha, \sigma)$  is planar ( $g = 0$ ) then  $\chi(\phi) = 2$  for all non trivial automorphisms  $\phi$ . Moreover,  $Aut(\alpha, \sigma)$  is one of  $C_n$  (cyclic),  $D_n$  (dihedral),  $A_4, S_4$  and  $A_5$  (see [CoMa] p.464). We shall need this result later.

We now define an equivalence relation  $R$  on the set  $B$ .

**Definition 3.** Let  $G$  be an automorphism group of the hypermap  $(\alpha, \sigma)$ . Two brins  $b_1$  and  $b_2$  are *equivalent*,  $b_1 R b_2$ , if they belong to the same orbit.

This leads to the following definition.

**Definition 4.** The *quotient hypermap*  $(\bar{\alpha}, \bar{\sigma})$  of  $(\alpha, \sigma)$ , with respect to an automorphism group  $G$ , is a pair of permutations  $(\bar{\alpha}, \bar{\sigma})$  acting on the set  $\bar{B}$ , where  $\bar{B} = B/R$  and  $\bar{\alpha}, \bar{\sigma}$  are the permutations induced by  $\alpha$  and  $\sigma$  on  $B$ .

The following Riemann-Hurwitz formula (*RH*) relates the genus  $\gamma$  of  $(\bar{\alpha}, \bar{\sigma})$  to the genus  $g$  of  $(\alpha, \sigma)$  (see [Ma]):

$$2g - 2 = |G|(2\gamma - 2) + \sum_{\phi \in G - \{id\}} \chi(\phi)$$

It follows that  $\gamma \leq g$ . In case  $G$  is a cyclic group,  $G = \langle \phi \rangle$ , (*RH*) becomes (*RH2*):

$$2g - 2 = |G|(2\gamma - 2) + \sum_{\phi \in G - \{id\}} \chi(\phi)$$

As a consequence, if  $g \geq 2$ ,  $|Aut(\alpha, \sigma)| \leq 84(g - 1)$ .

We reproduce here a part of the proof found in [CoMa]. When the quotient is the torus, it can be proved that  $|Aut(\alpha, \sigma)| \leq 4(g - 1)$ . When the quotient is planar, (*RH*) can be changed into (*RH\**) the following formula :

$$\frac{2g-2}{|G|} = (r - 2) - \sum_{i=1}^r \frac{1}{n_i}$$

where  $r$  is the number of conjugacy classes of fixing automorphisms and  $n_i$  the cardinality of each classe of stabilizer.

If  $r < 3$  then  $g < 2$  ; If  $r \geq 4$  then  $|Aut(\alpha, \sigma)| \leq 12(g - 1)$  ; If  $r = 3$  then we have the following array finishing the proof :

$n_1$	$n_2$	$n$	$1 - \sum_{i=1}^3 \frac{1}{n_i} \geq$	$ G  \leq$	$ G  =$
$n_1 > 3$	$n_2 > 3$	$n > 3$	$\frac{1}{4}$	$8(g - 1)$	
3	$n_2 > 3$	$n > 3$	$\frac{1}{6}$	$12(g - 1)$	
3	3	$n > 3$	$\frac{1}{12}$	$24(g - 1)$	$\frac{6n}{n-3}(g - 1)$
2	$n_2 > 4$	$n > 4$	$\frac{1}{10}$	$20(g - 1)$	
2	4	$n > 4$	$\frac{1}{20}$	$40(g - 1)$	$\frac{8n}{n-4}(g - 1)$
2	3	$n > 6$	$\frac{1}{42}$	$84(g - 1)$	$\frac{12n}{n-6}(g - 1)$

This leads to a new definition :

**Definition 5.** The *signature* of an automorphism group is given by  $(n_1, n_2, \dots, n_r)$  where the  $n_i$  are written in cressant order.

Now, some other propreties :

If  $\phi$  is an automorphism of order  $m$ , then, for all integers  $i$ ,  $\chi(\phi) \leq \chi(\phi^i)$ , and when  $m$  and  $i$  are coprime  $\chi(\phi) = \chi(\phi^i)$ .

Let  $(\alpha, \sigma)$  be a hypermap,  $G$  an automorphism group of  $(\alpha, \sigma)$  and let  $(\bar{\alpha}, \bar{\sigma})$  be the quotient hypermap of  $(\alpha, \sigma)$  with respect to  $G$ . The proof of the following results can be found in [Be2]. For any element  $\psi$  in the normalizer of  $G$  in  $Aut(\alpha, \sigma)$ , the permutation  $\bar{\psi}$ , defined as  $\psi/G$ , is an automorphism of  $(\bar{\alpha}, \bar{\sigma})$ . The two following operations on  $(\alpha, \sigma)$  are equivalent:

- (i) taking the quotient of  $(\alpha, \sigma)$  first by  $G$  and then taking the quotient of  $(\bar{\alpha}, \bar{\sigma})$  by  $\langle \bar{\psi} \rangle$ ,
- (ii) taking the quotient of  $(\alpha, \sigma)$  by  $\langle G, \psi \rangle$ .

**Definition 6.** The permutation  $\bar{\psi}$  is called the induced automorphism of  $\psi$  on  $(\bar{\alpha}, \bar{\sigma})$ . We also say  $\psi$  induces  $\bar{\psi}$  on  $(\bar{\alpha}, \bar{\sigma})$ .

We now give the theorem that counts the fixed cycles of an induced automorphism. (see [Be3])

**Theorem 7.** Let  $(\alpha, \sigma)$  be a hypermap admitting an automorphism  $\psi$  and an automorphism group  $G$  such that  $G$  is normal in  $\langle G, \psi \rangle$ . Then:

$$|G|\chi(\bar{\psi}) = \sum_{\phi \in G} \chi(\phi\psi)$$

where  $\bar{\psi}$  is the permutation induced by  $\psi$  on the quotient hypermap of  $(\alpha, \sigma)$  with respect to  $G$ .

When  $G = C_p$ , we have simpler formulas (see [Be2]).  $G$  is now generated by an automorphism  $\phi$  of prime order  $p$  and  $\psi$  is an element of  $Aut(\alpha, \sigma)$ . Then there are two cases described by the next two propositions.

**Proposition 8.** Let  $\psi$  commute with  $\phi$ .

$\psi$  is of order  $m$ , where  $p$  and  $m$  are coprime, then

$$p\chi(\bar{\psi}) = \chi(\psi) + (p-1)\chi(\phi\psi) \quad (IAF1)$$

$\psi$  is of order  $pn$ ,  $p$  and  $n$  coprime, and  $\phi$  belong to  $\langle \psi \rangle$ , then

$$p\chi(\bar{\psi}) = \chi(\psi^p) + (p-1)\chi(\psi) \quad (IAF2)$$

$\psi$  is of order  $p^m n$ , where  $m > 1$ ,  $p$  and  $n$  coprime, and  $\phi$  belong to  $\langle \psi \rangle$ , then

$$\chi(\bar{\psi}) = \chi(\psi) \quad (IAF3)$$

$\psi$  is of order  $pm$ ,  $m$  being any integer, and  $\phi$  does not belong to  $\langle \psi \rangle$ , then

$$\chi(\psi\phi^i) \equiv 0 \pmod{p} \text{ and } p\chi(\bar{\psi}) = \sum_{i=0}^{p-1} \chi(\psi\phi^i) \quad (IAF4)$$

Now, the other case:

**Proposition 9.** If  $\psi$  does not commute with  $\phi$ , then

$$\chi(\bar{\psi}) = \chi(\psi). \quad (IAF5)$$

In the classical theory of Riemann surfaces, a *hyperelliptic* surface  $S$  is a surface admitting an involution which is central in  $Aut(S)$  and fixes  $2g + 2$  points. This notion applies to hypermaps [CoMa].

**Definition 10.** A hypermap  $(\alpha, \sigma)$  of genus  $g \geq 2$  is said to be *hyperelliptic* if it admits an automorphism  $\phi$  of order 2 such that the quotient hypermap  $(\bar{\alpha}, \bar{\sigma})$  with respect to  $\langle \phi \rangle$  is planar.

*Remark 11.* Definition 10 is equivalent to the classical definition since by (RH) if the quotient hypermap  $(\bar{\alpha}, \bar{\sigma})$  with respect to  $\langle \phi \rangle$  is of genus 0 then  $\phi$  fixes  $2g + 2$  points and it can be proved that only one involution can fix exactly  $2g + 2$  points, therefore it is central in the group. (See [CoMa])

Since an automorphism on the sphere fixes exactly 2 points, it has been proved in [Be2] that:

**Proposition 12.** *Let  $\psi$  be an automorphism on a hyperelliptic hypermap of genus  $g \geq 2$ , then*

- $\psi$  is of odd order, then  $\chi(\psi) + \chi(\phi\psi) = 4$
  - $\psi$  is of order  $2n$  with  $n$  odd, and  $\phi$  belong to  $\langle \psi \rangle$ , then  $\chi(\psi^2) + \chi(\psi) = 4$
  - $\psi$  is of order  $2^m n$ , where  $m > 1$ , with  $n$  odd, and  $\phi$  belong to  $\langle \psi \rangle$ , then  $\chi(\psi) = 2$
  - $\psi$  is of order  $2m$ ,  $m$  being any integer, and  $\phi$  does not belong to  $\langle \psi \rangle$ , then  $\chi(\psi) = 0, 2, 4$  and  $\chi(\psi) + \chi(\phi\psi) = 4$
- It can also be described in the following way :

*Remark 13.* An automorphism on a hyperelliptic hypermap of genus  $g \geq 2$  has one of the following behaviors:

- it is the hyperelliptic involution and therefore fixes  $2g + 2$  points
- it fixes 2 points in common with the hyperelliptic involution and therefore fixes only these 2 points;
- it fixes 1 point in common with the hyperelliptic involution and therefore fixes either 1 or 3 points;
- it fixes no point in common with the hyperelliptic involution and therefore fixes either 0, 2 or 4 points.

Note that a "non hyperelliptic" automorphism fixes at most 2 points in common with the hyperelliptic involution.

**Proposition 14.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap. Then  $Aut(\alpha, \sigma)$  is either  $C_{2n}$  (cyclic) where  $n$  is a divisor of  $2g + 1$ ;  $C_{2n}$  or lifting of  $D_n$  by  $C_2$  admitting  $C_{2n}$  as a subgroup where  $n$  is a divisor of  $2g$ ;  $C_2 \times C_n$  or a lifting of  $D_n$  by  $C_2$  admitting  $C_2 \times C_n$  as a subgroup, where  $n$  is a divisor of  $2g + 2$ ; or  $Aut(\alpha, \sigma)$  is of order 24, 48, 120 (liftings of  $A_4, S_4, A_5$  respectively).*

We shall also need the following result proved in [Be1] based on a result in [BiCo].

**Proposition 15.** *An automorphism of prime power order can not fix only one point.*

These two last results that can be found in [Be6]

**Proposition 16.** *Let  $(\alpha, \sigma)$  be a hypermap of genus  $g \geq 2$  such that there exists an automorphism  $\phi$  of prime order  $2g + 1$ . Then the  $2g + 1$ -Sylow subgroups are of order  $2g + 1$  and :*

- (1) Except for one case when  $g = 3$  and  $Aut(\alpha, \sigma) = PSL_2(7)$  (the simple group of order 168),  $(\alpha, \sigma)$  is a  $(2g + 1)$ -elliptic hypermap.
- (2) When  $(\alpha, \sigma)$  is a  $(2g + 1)$ -elliptic hypermap,  $Aut(\alpha, \sigma)$  is either  $C_{2g+1}$  or  $C_{4g+2}$  (in which case it is also hyperelliptic) and if 3 divides  $g$  we also have  $C_{2g+1} \rtimes_j C_3$  (the unique non commutative group of order  $6g + 3$ ) where  $j$  is a cubic root of unity mod  $2g + 1$ .

The equivalent result with  $g + 1$ .

**Proposition 17.** *Let  $(\alpha, \sigma)$  be a hypermap of genus  $g \geq 2$  such that there exists an automorphism  $\phi$  of prime order  $g + 1$ . Then the  $g + 1$ -Sylow subgroups are of order  $g + 1$  and :*

- (1)  $(\alpha, \sigma)$  is a  $(g + 1)$ -elliptic hypermap except for the following cases:  $g = 2$  and  $|Aut(\alpha, \sigma)|$  is equal to 24 or 48 ;  $g = 4$  and  $|Aut(\alpha, \sigma)|$  is equal to 60, 90 or 120; Or  $g = 6$  and  $|Aut(\alpha, \sigma)| = 420$ .
- (2) When  $(\alpha, \sigma)$  is a  $(g + 1)$ -elliptic hypermap, then  $Aut(\alpha, \sigma)$  is one of the following groups:  $C_{g+1}$ ,  $C_{2g+2}$ ,  $C_{3g+3}$ ,  $D_{g+1}$ ,  $D_{2g+2}$  (in which case it is also hyperelliptic),  $C_2 \times C_{2g+2}$  (also hyperelliptic),  $C_{g+1} \rtimes_{-1} C_4$  (also hyperelliptic), if 4 divides  $g$ ,  $C_{g+1} \rtimes_i C_4$  or  $(C_2 \times C_{2g+2}) \rtimes C_2$  reaching the maximum order of  $8(g + 1)$  (also hyperelliptic).

### 3. AUTOMORPHISM GROUPS ON HYPERELLIPTIC HYPERMAPS OF EVEN GENUS

*Notation 18.* Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap ; in the rest of the article, we will denote by  $\phi$  be the hyperelliptic involution fixing  $2g + 2$  points. Since  $\langle \phi \rangle$  is central in  $Aut(\alpha, \sigma)$  any automorphism permutes these fixed points among themselves, and we can define a homomorphism  $h$  from  $Aut(\alpha, \sigma)$  to  $S_{2g+2}$  that will be regularly referred to . We indicate by  $C_p \rtimes_k C_m$  the group such that there exists  $\theta$ , generator of  $C_p$ , and  $\psi$ , generator of  $C_m$ , and  $\psi\theta = \theta^k\psi$ . We also indicate by  $(C_p \times C_n) \rtimes_{(k,l)} C_m$  the group such that there exists  $\theta$ , generator of  $C_p$ , there exists  $\omega$ , generator of  $C_n$ , and  $\psi$ , generator of  $C_m$ , such that  $\psi\theta = \theta^k\psi$  and  $\psi\omega = \omega^l\psi$ .

**Proposition 19.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$ . Let  $\psi$  be an automorphism of  $(\alpha, \sigma)$  of order  $2^n$  such that  $\psi \neq \phi$ . Then,  $\psi$  fixes exactly two points. Moreover, if  $n > 1$ , then  $\phi$  is a power of  $\psi$ .*

*Proof.* Let us suppose first that the automorphism  $\psi$  is of order 2. We know that on a hyperelliptic hypermap any automorphism which is not  $\phi$  fixes at most four points as a consequence of Proposition 13. Now, by (RH) we have

$$2g - 2 = 2(2\gamma - 2) + \chi(\psi) \text{ that is } \chi(\psi) = 2g - 2(2\gamma - 2) \text{ therefore}$$

$\chi(\psi) = 4(\frac{g}{2} - \gamma + 1) - 2$  (since  $g$  is even).  $\chi(\phi)$  is an even number, less than 4 and not dividable by 4 i.e.  $\chi(\psi) = 2$ .

Let now  $\psi$  be of order  $2^n$  with  $n > 1$ . Note that  $\chi(\psi)$  can only be 0 or 2 because its power of order 2 is either  $\phi$  and by Proposition 12  $\chi(\psi) = 2$  (case 3); or its power of order 2 is another automorphism and we have just proved that it fixes two points.

There exists an homomorphism  $h$  from  $Aut(\alpha, \sigma)$  to  $S_{2g+2} = S_{4\frac{g}{2}+2}$ . Since  $\chi(\psi)$  is 0 or 2,  $h(\psi)$  must be written as  $k$  cycles of length  $2^n$  with either a cycle of length 2 (if  $\chi(\psi) = 0$ ) or two cycles of length 1 (if  $\chi(\psi) = 2$ ). But in anycase,  $\chi(\psi^2) = 2$  which means that  $\phi$  belong to  $\langle \psi^2 \rangle$  thus  $\phi$  belong to  $\langle \psi \rangle$ .

By Proposition 12 (case 3)  $\chi(\psi) = 2$ . □

**Corollary 20.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$ . Let  $\psi$  be an automorphism of  $(\alpha, \sigma)$  of order  $2^n m$  such that  $m$  is odd and  $n > 1$ , then  $\phi$  is a power of  $\psi$  and  $\psi$  fixes exactly two points.*

*Proof.* The automorphism  $\psi^m$  of order  $2^n$  satisfies the conditions of Proposition 19 ; thus,  $\phi$  belong to  $\langle \psi^m \rangle$  thus  $\phi$  belong to  $\langle \psi \rangle$ . By Proposition 12 (case 3)  $\chi(\psi) = 2$ . □

**Theorem 21.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$ . Let  $G$  be a 2-Sylow subgroup of  $Aut(\alpha, \sigma)$ , then  $G$  is isomorphic to  $C_{2^n}$  (cyclic) of maximum order  $4g$ ,  $D_{2^n}$  (Dihedral) of maximum order  $4g$ ,  $Q_{2^n}$  (generalized quaternions) of*

maximum order  $4g$  or  $QD_{2^n}$  (QuasiDiedral) where it can reach  $8g$ , its maximum order.

*Proof.* Since  $G$  be a 2-Sylow subgroup, it is the lifting of either  $C_{2^{n-1}}$  (cyclic),  $D_{2^{n-1}}$  (Dihedral) on the plane by the hyperelliptic involution. Let  $\bar{\psi}$  be a generator of the normal subgroup of order  $2^{n-1}$ . Let  $\psi$  be a lifting of  $\bar{\psi}$  in  $G$ .

If  $\bar{G} = id$ , then  $G = C_2$ ; If  $\bar{G} = C_2$ , then  $G = C_2 \times C_2$  or  $G = C_4$ ; If  $\bar{G} = C_{2^{n-1}}$ , with  $n \geq 3$  then  $G = C_{2^n}$  because  $\bar{\psi}$  a generator of the group of order  $2^{n-1}$  (of minimum order 4) must be lifted in an element  $\psi$  of minimum order 4 and we know by Proposition 19 that  $\phi$  is a power of  $\psi$ , which means that all the orders are “pushed up” of a coefficient 2 by the lifting.

If  $\bar{G} = D_2$ , then  $G$  can be  $D_4$  (Dihedral),  $Q_8$  (quaternions),  $C_2 \times C_4$  or  $C_2 \times C_2 \times C_2$ ; Let us show that the two last groups are impossible. Since both groups are abelian, there own  $\theta$  a central involution which is not the hyperelliptic one. By Proposition 19,  $\theta$  fixes two points and non in common with the other elements of the group. Thus, there exists a homomorphism  $k$  from the group onto  $S_2$  and  $ker(k) = \theta$  which means that the group is of maximum order 4: hence, the contradiction.

If  $\bar{G} = D_{2^{n-1}}$ , with  $n \geq 3$  then  $G$  is of order  $2^{n+1}$  owning  $C_{2^n}$  as a subgroup. General group theory tells us that there are exactly four different non abelian groups of order  $2^{n+1}$  owning a cyclic subgroup of index 2:

$$D_{2^n} = \langle \psi, \theta/\psi^{2^n} = \theta^2 = 1, \theta\psi = \psi^{-1}\theta \rangle \text{ (Dihedral);}$$

$Q_{2^{n+1}} = \langle \psi, \theta/\psi^{2^n} = \theta^4 = 1, \theta\psi = \psi^{-1}\theta, \psi^{2^{n-1}} = \theta^2 \rangle$  (Generalized Quaternions);

$$QD_{2^{n+1}} = \langle \psi, \theta/\psi^{2^n} = \theta^2 = 1, \theta\psi = \psi^{2^{n-1}-1}\theta \rangle \text{ (QuasiDiedral);}$$

$$AD_{2^{n+1}} = \langle \psi, \theta/\psi^{2^n} = \theta^2 = 1, \theta\psi = \psi^{2^{n-1}+1}\theta \rangle \text{ (“Almost” Dihedral)}$$

Now, the quotient of the three first groups by  $\langle \psi^{2^{n-1}} = \phi \rangle$  is  $D_{2^{n-1}}$ . But the quotient of the last group by  $\langle \psi^{2^{n-1}} = \phi \rangle$  is  $C_2 \times C_{2^{n-1}}$  which is a contradiction.

A 2-Sylow subgroup of  $Aut(\alpha, \sigma)$  is of maximum order  $\frac{8g}{m}$  if  $g = 2^k \times m$  where  $m$  is odd because it is the lifting of  $D_{\frac{2g}{m}}$  by the hyperelliptic involution. If  $m = 1$ , then  $|G| = |Aut(\alpha, \sigma)| = 8g$ . In this case, its signature is necessarily  $(2, 4, 4g)$  because the quotient by  $\phi$  is planar, it owns a cyclic subgroup of order  $4g$  fixing two points, no odd number can appear in the signature. Since  $Aut(\alpha, \sigma)$  is the lifting of  $D_{2g}$  by  $\phi$ , the element of order 4 appearing in the signature does not belong to  $C_{4g}$  because all elements in this group fix the same points. Let  $\tau_i$  be the generators of the groups contained in this class, then  $\tau_i^2 = \phi$ . How many groups of order 4 do we have in  $Aut(\alpha, \sigma)$ ? At least two: the one in  $C_{4g}$  and  $\tau_i$ , so that  $Aut(\alpha, \sigma)$  can't be Dihedral. At maximum:  $g + 1$  because they all fix two points in common with  $\phi$  (never the same ones). Since  $8g = 2^{n+1}$ ,  $g = 2^{n-2}$  and therefore  $g + 1 = 2^{n-2} + 1$ . This implies that the only possible group is QuasiDiedral because a Generalized Quaternions would have  $2^{n-1} + 1$  groups of order 4.

Of course, when  $Aut(\alpha, \sigma) = QD_{8g}$ ,  $D_{2g}$  and  $Q_{4g}$  are natural subgroups of index 2 in  $Aut(\alpha, \sigma)$  and therefore reach their maximum order of  $4g$ .  $\square$

The definition of a new group is now necessary :

**Definition 22.** For all integer  $g \geq 2$ , let us call *elliptic* group of order  $8g$  the group defined by generators and relations as

$$E_{8g} = \langle a, b/a^{4g} = b^2 = 1, ba = a^{2g-1}b \rangle.$$

When  $g$  is a power of 2, the elliptic group is the usual QuasiDiedral group; when  $g$  is even, it is also related to QuasiDiedral group as it will be seen below.

**Proposition 23.** *For all even number  $g \geq 4$ , there exist  $(\alpha, \sigma)$  a hyperelliptic hypermap of genus  $g$  such that*

$$\begin{aligned} \text{Aut}(\alpha, \sigma) &= E_{8g} = \langle \psi, \theta / \psi^{4g} = \theta^2 = 1, \theta\psi = \psi^{2g-1}\theta \rangle \text{ of order } 8g; \\ \text{when } g &= 2^n \text{ then } \text{Aut}(\alpha, \sigma) = QD_{8g}. \end{aligned}$$

*Proof.* This proof is based on the construction of a hyperelliptic map over  $24g$  brins. It is a lifting of  $D_{2g}$  where  $g \geq 4$  is any even number. If one considers  $D_{2g}$  as the representation of the globe, cut by  $2g$  meridians with an arctic pole and an antartic pole (both represented as large faces and not vertices) then this map can be viewed as the double covering of it. The larger  $g$  is, the more refined the cutting by meridians is.

$\alpha = (1, 2)(3, 4) \dots (12g - 1, 12g)(\bar{1}, \bar{2})(\bar{3}, \bar{4}) \dots (\overline{12g - 1}, \overline{12g})$  that is  $12g$  edges of length 2 (the meridians).

$$\begin{aligned} \sigma &= (1, \overline{4g}, 4g + 1)(2, \overline{4g + 3}, 3)(4, \overline{4g + 5}, 5) \dots (4g - 2, \overline{8g - 1}, 4g - 1) \\ &(\bar{1}, \overline{4g}, \overline{4g + 1})(\bar{2}, \overline{4g + 3}, \bar{3})(\bar{4}, \overline{4g + 5}, \bar{5}) \dots (\overline{4g - 2}, \overline{8g - 1}, \overline{4g - 1}) \\ &(\overline{8g + 1}, \overline{4g + 2}, \overline{12g})(\overline{8g + 3}, \overline{4g + 4}, \overline{8g + 2})(\overline{8g + 5}, \overline{4g + 6}, \overline{8g + 4}) \dots (\overline{12g - 1}, \overline{8g}, \overline{12g - 2}) \\ &(\overline{8g + 1}, \overline{4g + 2}, \overline{12g})(\overline{8g + 3}, \overline{4g + 4}, \overline{8g + 2})(\overline{8g + 5}, \overline{4g + 6}, \overline{8g + 4}) \dots \\ &(\overline{12g - 1}, \overline{8g}, \overline{12g - 2}) \text{ that is } 8g \text{ vertices of length 3.} \end{aligned}$$

$$\begin{aligned} \alpha\sigma &= (4g, \overline{4g - 2}, \overline{4g - 4}, \dots, \overline{4}, \overline{2}, \overline{4g}, \overline{4g - 2}, \overline{4g - 4}, \dots, \overline{4}, \overline{2}) \\ &(\overline{8g + 1}, \overline{8g + 3}, \overline{8g + 5}, \overline{8g + 7}, \dots, \overline{12g - 1}, \overline{8g + 1}, \overline{8g + 3}, \overline{8g + 5}, \overline{8g + 7}, \dots, \overline{12g - 1}) \\ &(1, \overline{4g + 3}, \overline{8g + 2}, \overline{4g + 2}, \bar{1}, \overline{4g + 3}, \overline{8g + 2}, \overline{4g + 2}) \\ &(3, \overline{4g + 5}, \overline{8g + 4}, \overline{4g + 4}, \bar{3}, \overline{4g + 5}, \overline{8g + 4}, \overline{4g + 4}) \\ &(5, \overline{4g + 7}, \overline{8g + 6}, \overline{4g + 6}, \bar{5}, \overline{4g + 7}, \overline{8g + 6}, \overline{4g + 6}) \end{aligned}$$

$\vdots$

$$\begin{aligned} &(\overline{4g - 3}, \overline{8g - 1}, \overline{12g - 2}, \overline{8g - 2}, \overline{4g - 3}, \overline{8g - 1}, \overline{12g - 2}, \overline{8g - 2}) \\ &(\overline{4g - 1}, \overline{4g + 1}, \overline{12g}, \overline{8g}, \overline{4g - 1}, \overline{4g + 1}, \overline{12g}, \overline{8g}) \text{ that is two faces of length } 4g \text{ and } \\ &2g \text{ faces of length } 8. \end{aligned}$$

There exists an automorphism  $\psi$  of order  $4g$  whose power of order 2 is the hyperelliptic involution  $\phi$  :

$$\begin{aligned} \psi &= (1, 3, 5, \dots, \overline{4g - 1}, \bar{1}, \bar{3}, \bar{5}, \dots, \overline{4g - 1}) (2, 4, 6, \dots, \overline{4g}, \bar{2}, \bar{4}, \bar{6}, \dots, \overline{4g}) \\ &(\overline{4g + 1}, \overline{4g + 3}, \dots, \overline{8g - 1}, \overline{4g + 1}, \overline{4g + 3}, \dots, \overline{8g - 1}) \\ &(\overline{4g + 2}, \overline{4g + 4}, \dots, \overline{8g}, \overline{4g + 2}, \overline{4g + 4}, \dots, \overline{8g}) \\ &(\overline{8g + 1}, \overline{8g + 3}, \dots, \overline{12g - 1}, \overline{8g + 1}, \overline{8g + 3}, \dots, \overline{12g - 1}) \\ &(\overline{8g + 2}, \overline{8g + 4}, \dots, \overline{12g}, \overline{8g + 2}, \overline{8g + 4}, \dots, \overline{12g}) \end{aligned}$$

$\psi$  fixes the two faces of length  $4g$  and permutes the others. The hyperelliptic involution  $\phi = (1, \bar{1})(2, \bar{2}) \dots (12g - 1, \overline{12g})$  is its power of order 2 and fixes the  $2g + 2$  faces.

There exists an involution  $\theta$  :

$$\begin{aligned} \theta &= (\overline{4g}, \overline{8g + 1})(\overline{4g - 2}, \overline{8g + 3}) \dots (\overline{4g - 4k}, \overline{8g + 4k + 1})(\overline{4g - 4k - 2}, \overline{8g + 4k + 3}) \dots \\ &(\overline{2}, \overline{12g - 1}) \\ &(\overline{4g}, \overline{8g + 1})(\overline{4g - 2}, \overline{8g + 3}) \dots (\overline{4g - 4k}, \overline{8g + 4k + 1})(\overline{4g - 4k - 2}, \overline{8g + 4k + 3}) \dots \\ &(\bar{2}, \overline{12g - 1}) \\ &(1, \overline{12g})(3, \overline{12g - 2}) \dots (\overline{4k + 1}, \overline{12g - 4k})(\overline{4k + 3}, \overline{12g - 4k - 2}) \dots (\overline{4g - 1}, \overline{8g + 2}) \\ &(\bar{1}, \overline{12g})(\bar{3}, \overline{12g - 2}) \dots (\overline{4k + 1}, \overline{12g - 4k})(\overline{4k + 3}, \overline{12g - 4k - 2}) \dots (\overline{4g - 1}, \overline{8g + 2}) \\ &(\overline{4g + 3}, \overline{8g})(\overline{4g + 5}, \overline{8g - 2})(\overline{4g + 7}, \overline{8g - 4}) \dots (\overline{6g - 1}, \overline{6g + 4}) \\ &(\overline{4g + 3}, \overline{8g})(\overline{4g + 5}, \overline{8g - 2})(\overline{4g + 7}, \overline{8g - 4}) \dots (\overline{6g - 1}, \overline{6g + 4}) \end{aligned}$$

$$\begin{aligned} & (4g+4, \overline{8g-1})(4g+6, \overline{8g-3})(4g+8, \overline{8g-5}) \dots (6g, \overline{6g+3}) \\ & (4g+4, \overline{8g-1})(4g+6, \overline{8g-3}) \dots (\overline{6g}, \overline{6g+3}) \\ & (4g+1, \overline{4g+2})(4g+1, \overline{4g+2})(6g+1, \overline{6g+2})(\overline{6g+1}, \overline{6g+2}) \end{aligned}$$

$\theta$  fixes 2 edges (its last two cycles) and  $\theta\phi$  will fix 2 edges too (the two cycles before).

Now, by construction (see the figure in annexe),  $\theta\psi = \phi\psi^{-1}\theta = \psi^{2g-1}\theta$ . Therefore, it is  $E_{8g} = \langle \psi, \theta/\psi^{4g} = \theta^2 = 1, \theta\psi = \psi^{2g-1}\theta \rangle$ .  $\square$

*Remark 24.* This map has been constructed for  $g = 4$  and  $g = 6$ ; its is drawn in Annexe A. When  $g = 2$ , this construction exists but leads to a larger group with respect to the genus and with a different structure. It is explicite in Annexe B. When  $g$  is odd, this construction exists but leads to very different groups (object of a for-coming paper).

**Proposition 25.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$  such that  $Aut(\alpha, \sigma)$  is the lifting of  $D_n$  (Dihedral) by the hyperelliptic involution, owning a cyclic subgroup of index 2 then  $Aut(\alpha, \sigma)$  is one of the following groups of order  $4n$ :*

$$\begin{aligned} D_{2n} &= \langle \psi, \theta/\psi^{2n} = \theta^2 = 1, \theta\psi = \psi^{-1}\theta \rangle \text{ (Dihedral);} \\ Dic_n &= \langle \psi, \theta/\psi^{2n} = \theta^4 = 1, \theta\psi = \psi^{-1}\theta, \psi^n = \theta^2 \rangle \text{ (Dicyclic) ;} \\ E_{4n} &= \langle \psi, \theta/\psi^{2n} = \theta^2 = 1, \theta\psi = \psi^{n-1}\theta \rangle \text{ (Elliptic).} \end{aligned}$$

The signature of the maximal groups of each structure is  $(2, 2, 2, 2g)$ ,  $(4, 4, 2g)$ ,  $(2, 4, 4g)$  and of maximal order  $4g, 4g$  and  $8g$  respectively.

*Proof.*  $Aut(\alpha, \sigma)$  is the lifting of  $D_n$  (Dihedral) on the plane by the hyperelliptic involution. Let  $\bar{\psi}$  be a generator of the normal subgroup of order  $n$  and  $\theta$  a reflexion such that  $\theta\bar{\psi} = \bar{\psi}^{-1}\theta$  in  $D_n$ . Let  $\psi$  be a lifting of  $\bar{\psi}$  in  $Aut(\alpha, \sigma)$ , by hypothesis of order  $2n$  (that is  $\phi \in \langle \psi \rangle$ ), then there are four cases.

Case 1 :  $o(\theta) = 2$  and  $\theta\psi = \psi^{-1}\theta$ ; it is the definition of a dihedral group of order  $4n$

Case 2 :  $o(\theta) = 4$  and  $\theta\psi = \psi^{-1}\theta$  and  $\psi^n = \theta^2$ ; it is the definition of a dicyclic group of order  $4n$

Case 3 :  $o(\theta) = 2$  and  $\theta\psi = \phi\psi^{-1}\theta = \psi^{n-1}\theta$ ; it is our definition of an elliptic group of order  $4n$  (if  $n$  is dividable by 4).

Case 4 :  $o(\theta) = 4$  and  $\theta\psi = \phi\psi^{-1}\theta$  and  $\psi^n = \theta^2$ ; In this case,  $\theta\psi = \phi\psi^{-1}\theta = \psi^{-1}\theta\phi = \psi^{-1}\theta^3 = \psi^{-1}\theta^{-1} = (\theta\psi)^{-1}$  which means that  $(\theta\psi)^2 = id$ ; now,  $(\theta\psi)\psi = (\phi\psi^{-1}\theta)\psi = \phi\psi^{-1}(\theta\psi)$ . So that this group can be defined by  $\langle \psi, \theta' \rangle$  where  $\theta' = \theta\psi$  and we are back to the third case.

$D_{2g}$  signature is  $(2, 2, 2, 2g)$  because the element of order  $2g$  fixes two points in common with the hyperelliptic involution that are exchanged by the reflexions so that is one class. the other three classes of order 2 are one for each class of reflexion and one for the hyperelliptic involution

Note that when  $Dic_{4g}$  is of maximum order it is structured by a generalized quaternion (see next corollary and Theorem 21) the signature is  $(4, 4, 2g)$  because the element of order  $2g$  fixes two points in common with the hyperelliptic involution that are exchanged by the elements of order 4 so that is one class. each classe of order 4 is composed of  $g$  conjugated elements such that their power of order 2 is the hyperelliptic involution so that all  $\phi$  fixed points are already fixed by an automorphism of greater order; thus, 2 does not appear in the signature.

$E_{8g}$  signature is  $(2, 4, 4g)$  because the element of order  $4g$  fixes two points in common with the hyperelliptic involution that are exchanged by the elements of order 4 so that is one class. The class of order 4 is composed of  $2g$  conjugated elements such that their power of order 2 is the hyperelliptic involution so that all  $\phi$  fixed points are already fixed by an automorphism of greater order; the 2 which appears in the signature is the reflexion class.  $\square$

**Corollary 26.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$  such that  $Aut(\alpha, \sigma)$  is the lifting of a Dihedral group by the hyperelliptic involution, owning a cyclic subgroup of index 2 then  $Aut(\alpha, \sigma)$  is a subgroup of the following groups ::  $D_{4g}$  (Dihedral);  $C_{2m+1} \rtimes_{-1} C_4$ ;  $C_{2m+1} \rtimes Q_{\frac{4g}{2m+1}}$ ;  $C_{2m+1} \rtimes QD_{\frac{8g}{2m+1}}$  where  $2m+1$  is the odd factor of  $Aut(\alpha, \sigma)$  order.*

*Proof.* First group is immediate; the next two groups come from the rewriting of  $Dic_{4g}$  considering  $C_4$  as the first generalized quaternion group (see [DE] page 202 and seq. also 408 and seq.); the last group is the rewriting of  $E_{8g}$  following exactly the same steps in the demonstration of the former groups. The automorphism which structures the semi-product in each case is the one induced by each group  $Dic_{4g}$  or  $E_{8g}$ .  $\square$

**Proposition 27.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$  such that the induced automorphism group is either  $C_n$  or  $D_n$  where  $o(\psi) = n$  divides  $g+1$ ; then  $Aut(\alpha, \sigma)$  is a subgroup of either  $C_2 \times C_{2g+2}$ ,  $D_{2g}$  or  $(C_2 \times C_{2g+2}) \rtimes_A C_2$  reaching the maximum order of  $8(g+1)$ , where  $A$  is an automorphism described in the proof.*

*Proof.* If the quotient group is  $C_n$ , then by Theorem 10,  $Aut(\alpha, \sigma)$  is  $C_2 \times C_n$  and it is a subgroup of  $C_2 \times C_{2g+2}$ .

If the quotient group is  $D_n$  with  $n$  dividing  $g+1$  (thus odd), then  $Aut(\alpha, \sigma)$  owns a subgroup of type  $C_2 \times C_n = C_{2n} = \langle \phi\psi = \psi' \rangle$ . Let  $\theta$  be a reflexion such that  $\theta\psi = \psi^{-1}\theta$  in  $D_n$ ; then there are two cases.

Case 1 :  $\theta\psi' = \theta(\phi\psi) = \phi\psi^{-1}\theta = (\phi\psi)^{-1}\theta = (\psi')^{-1}\theta$ ; it is either the definition of a dihedral group if  $o(\theta) = 2$  or a dicyclic group if  $o(\theta) = 4$ .

Case 2 :  $\theta(\phi\psi) = \psi^{-1}\theta$ , that is  $\theta^2(\phi\psi) = \theta\psi^{-1}\theta$  (E1). If  $o(\theta) = 2$ , this last equation becomes  $\phi\psi = \theta\psi^{-1}\theta$  which is impossible because the left handside is of even order and the right handside is of odd order as a conjugate of  $\psi^{-1}$ . Thus,  $o(\theta) = 4$  and  $\theta^2 = \phi$ . Then (E1) becomes  $\psi = \theta\psi^{-1}\theta$  that is  $\psi\theta^{-1} = \psi^{-1}\theta$  i.e.  $\psi\theta^{-1} = (\psi\theta^{-1})^{-1}$  thus  $(\psi\theta^{-1})^2 = id$ . Let  $\psi\theta^{-1} = \theta'$ . Then,  $\theta'(\phi\psi) = \psi^{-1}\theta(\phi\psi) = \psi^{-1}(\theta(\phi\psi)) = \psi^{-2}\theta = \psi^{-1}\theta'$  that is  $\psi^{-1} = \theta'\phi\psi\theta'$  which is impossible because the left handside is of odd order and the right handside is of even order as a conjugate of  $\phi\psi$ .

If the quotient group is  $D_{2n}$  with  $n$  dividing  $g+1$  (thus odd), then  $Aut(\alpha, \sigma)$  2-Sylows subgroups can only be  $D_4$  or  $Q_8$  by Theorem 21. Since  $Aut(\alpha, \sigma)$  owns a subgroup of type  $C_2 \times C_{2n}$  (by Theorem 10) which contains already three involutions,  $Q_8$  is impossible, so that the 2-Sylows subgroups can only be  $D_4$ . Now,  $D_4 = \langle \tau, \theta/\tau^4 = \theta^2 = id, \theta\tau = \tau^{-1}\theta \rangle$  with  $\tau^2 = \phi$  and  $Aut(\alpha, \sigma) = \langle \phi, \psi, \theta \rangle$  because the latter is included in the former and order coincide. We know that  $\bar{\theta}\bar{\psi} = \bar{\psi}^{-1}\bar{\theta}$  because the induced automorphism group is  $D_{2n}$ . This equation becomes in  $Aut(\alpha, \sigma)$  either  $\theta\psi = \psi^{-1}\theta$  or  $\theta\psi = \phi\psi\theta$ . If  $\theta\psi = \psi^{-1}\theta$  were the correct formula, then  $\langle \psi, \theta \rangle = D_{2n}$  and, since  $\phi$  is central,  $Aut(\alpha, \sigma) = \langle \phi, \psi, \theta \rangle$  could

not owns an element of order 4 : a contradiction. This means that the correct formula is  $\theta\psi = \psi\psi^{-1}\theta$  and the group definition is :

$$\langle \psi, \phi, \theta / \psi^{2n} = \phi^2 = \theta^2 = 1, \psi\phi = \phi\psi, \theta\phi = \phi\theta, \theta\psi = \phi\psi^{-1}\theta \rangle$$

So that  $Aut(\alpha, \sigma)$  is a subgroup of  $(C_2 \times C_{2g+2}) \rtimes_A C_2$  reaching the maximum order of  $8(g+1)$ , where A is the automorphism described above.  $\square$

*Remark 28.* Proposition 27 could have been stated in the following way : let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$  such that the induced automorphism group is either  $C_n$  or  $D_n$  where  $o(\psi) = n$  divides  $g+1$  ; then  $Aut(\alpha, \sigma)$  is a subgroup of  $(C_2 \times C_{2g+2}) \rtimes_A C_2$  reaching the maximum order of  $8(g+1)$ . Indeed,  $Aut(\alpha, \sigma)$  subgroups of index 2 are  $C_2 \times C_{2g+2} = \langle \phi, \psi \rangle$ , the subgroup  $\langle \tau, \psi^2 \rangle$  is  $Dic_n = C_n \rtimes_{-1} C_4$  because it owns an element of order 4 and the subgroup  $\langle \theta, \phi\psi^2 \rangle$  is  $D_{2n}$  since it is non abelian and owns no element of order 4. Note that  $C_{2n} = \langle \phi\psi^2 \rangle$  is also a subgroup, but of index 4.

**Proposition 29.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$ . If the quotient of  $Aut(\alpha, \sigma)$  with respect to  $\phi$  is  $A_4$ ,  $S_4$  or  $A_5$  then  $Aut(\alpha, \sigma)$  is  $SL_2(F_3) = Q_8 \rtimes C_3$  of order 24 ,  $GL_2(F_3) = (Q_8 \rtimes C_3) \rtimes C_2$  of order 48, or  $SL_2(F_5)$  of order 120 respectively.*

*Proof.* Let us suppose that the quotient of  $Aut(\alpha, \sigma)$  with respect to  $\phi$  is  $A_4$ . The 2- Sylow subgroups are of order 8 and by Theorem 21, they can only be  $D_4$  or  $Q_8$ . In both cases, there exists an element  $\tau_1$  of order 4, and by Proposition 19,  $\tau_1^2 = \phi$ . Let  $\tau_2$  be a representative of the other automorphism such that  $\langle \tau_1, \tau_2 \rangle = C_2 \times C_2$  and of course  $\tau_3 = \tau_1\tau_2$ . Let  $\theta_1$  be an automorphism of order 3 on the quotient,  $\theta_1$  acts on the common fixed points of  $\phi$  and the  $\tau_i$ s. Hence, two cases. First case :  $\phi$  and the  $\tau_i$ s have strictly less than 6 fixed points in common (minimum the 2 points fixed by  $\tau_1$  up to 4 if another  $\tau_i$  fixes points), then, since at least one representative  $\theta_1$  of  $\theta_1$  is an automorphism of order 3,  $\theta_1$  fixes at least one of these points, which means that there exists an element of order 6 on the quotient. This is impossible because  $A_4$  has no element of order 4. Second case :  $\phi$  and the  $\tau_i$ s have 6 fixed points in common (2 points fixed by each  $\tau_i$ ), in which case all the  $\tau_i$ s are of order 4 and are such that  $\tau_i^2 = \phi$ . This group is  $Q_8$ . Now, there exists a unique group of order 24 such that the quotient by a  $C_2$  is  $A_4$  and owning 2- Sylows equal to  $Q_8$  : this group is  $SL_2(F_3) = Q_8 \rtimes C_3$ .

Let us suppose that the quotient of  $Aut(\alpha, \sigma)$  with respect to  $\phi$  is  $S_4$ . The 2- Sylow subgroups are of order 16 and by Theorem 21, they can only be  $D_8$ ,  $Q_{16}$  or  $QD_{16}$ . Since  $A_4$  is a subgroup of  $S_4$ ,  $SL_2(F_3) = Q_8 \rtimes C_3$  is a subgroup of  $Aut(\alpha, \sigma)$  and  $Q_8$  is a subgroup of  $Aut(\alpha, \sigma)$  two-Sylows. Thus,  $D_8$  is impossible. We are left with  $Q_{16}$  or  $QD_{16}$ . The first one owns 5  $\tau_i$  of order 4 such that  $\tau_i^2 = \phi$  and the second one owns only 3 of them. Now,  $S_4$  contains four  $D_3$  that are not in  $A_4$  (even if the  $C_3$  are in  $A_4$ ). Those groups can only be lifted in  $D_6$  or  $C_3 \times C_4$  ; in the second case, three new  $\tau_i$ s are created by the lifting ; but  $SL_2(F_3)$  contains already three  $\tau_i$ s, so that it is one to many. This means that the four  $D_3$  are lifted in  $D_6$  and that the number of  $\tau_i$ s, remains three.  $Aut(\alpha, \sigma)$  2- Sylow subgroups are therefore  $QD_{16}$ . Now, there exists a unique group of order 48 such that the quotient by a  $C_2$  is  $S_4$ , owning 2- Sylows equal to  $QD_{16}$  and admitting  $SL_2(F_3)$  as a subgroup: this group is  $GL_2(F_3) = (Q_8 \rtimes C_3) \rtimes C_2$ .

Let us suppose that the quotient of  $Aut(\alpha, \sigma)$  with respect to  $\phi$  is  $A_5$ . There are only three non resoluble groups of order 120 :  $S_5$ ,  $A_5 \times C_2$  and  $SL_2(F_5)$  ; but only the last group admits  $SL_2(F_3)$  as a subgroup. Hence the result.  $\square$

*Remark 30.* Those last three results can be tested with the GAP computation and their well presented lattices are searchable in [DE].  $SL_2(F_3)$  and  $GL_2(F_3)$  are constructed in Annexe B on a map of genus 2. The first possible hyperelliptic hypermap of even genus on which one could construct  $SL_2(F_5)$  is  $g = 14$  because the signature must contain the minimum of conjugacy classes (three), the orders must be the lowest possible and reduced to the compulsory ones; thus, the signature is  $(3, 4, 5)$  ; 2 does not appear because all  $\phi$  fixed points are “taken” by automorphisms of order 4. Since there are 15 groups of order 4 (all conjugates) in  $SL_2(F_5)$  fixing each 2 points in common with the hyperelliptic involution,  $2g + 2 = 30$  and  $g = 14$ .

All these results lead to a general theorem which describes automorphisms groups on hyperelliptic hypermaps of even genus:

**Theorem 31.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of even genus  $g \geq 2$ . Then,  $Aut(\alpha, \sigma)$  is a subgroup of one of the following groups :  $C_{4g+2}$ ,  $(C_2 \times C_{2g+2}) \rtimes C_2$ ,  $E_{8g}$ ,  $GL_2(F_3)$  or  $SL_2(F_5)$  .*

*Proof.* It is a simple summ up of the former results;  $E_{8g}$  contains all Dicyclic and Dihedral groups (thus cyclic and quaternions); both  $GL_2(F_3)$  or  $SL_2(F_5)$  contain  $SL_2(F_3)$ .  $\square$

#### 4. HYPERMAPS OF GENUS 2

The general results on hyperelliptic hypermaps of even genus find a natural application on genus 2 because of the following theorem. But, before we need a technical lemma to help the next theorem proof :

**Lemma 32.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of genus 2. Then  $Aut(\alpha, \sigma)$  is either reduced to identity,  $C_2$ ,  $C_4$  ; or  $(\alpha, \sigma)$  is 5-elliptic, 3-elliptic or hyperelliptic.*

*Proof.* We know by (RH) that an automorphism of prime order can only be of order 2,  $3 = g + 1$ ,  $5 = 2g + 1$ . If 5 divides  $|Aut(\alpha, \sigma)|$  then by Proposition 16  $Aut(\alpha, \sigma)$  is either  $C_5$  or  $C_{10}$  therefore  $(\alpha, \sigma)$  is 5-elliptic.

If 3 divides  $|Aut(\alpha, \sigma)|$  and if  $(\alpha, \sigma)$  is not 3-elliptic then by Proposition 17 ,  $|Aut(\alpha, \sigma)|$  is either 24 or 48. Let us consider  $\psi$  a central automorphism in  $G$  a 2-Sylow subgroup of  $Aut(\alpha, \sigma)$ ; it fixes either 2 or 6 points. If it fixes two points, there exists an homomorphism  $h$  from  $G$  to  $S_2$ . If  $Ker(h) \neq \langle \psi \rangle$  there exists an automorphism of order 4 that fixes 2 points and by (RH2)  $2 = 4(2\gamma - 2) + 3 \times 2$  which is impossible. Thus  $|G| \leq 4$ , a contradiction. If  $\psi$  fixes six points, then it is the hyperelliptic involution.

If neither 3 or 5 divide  $|Aut(\alpha, \sigma)|$ , then  $|Aut(\alpha, \sigma)| = 2^k$ . Let us consider  $\psi$  a central automorphism in  $G$  a 2-Sylow subgroup of  $Aut(\alpha, \sigma)$ ; it fixes either 2 or 6 points. If it fixes two points, there exists an homomorphism  $h$  from  $G$  to  $S_2$ . If  $Ker(h) \neq \langle \psi \rangle$  there exists an automorphism of order 4 that fixes 2 points and by (RH2)  $2 = 4(2\gamma - 2) + 3 \times 2$  which is impossible. Thus  $Ker(h) = \langle \psi \rangle$  and  $|Aut(\alpha, \sigma)| \leq 4$  which means that  $Aut(\alpha, \sigma)$  is either  $C_2$ ,  $C_4$  or  $C_2 \times C_2$  ; now, by (RH2), the last case leads to the existence of a hyperelliptic involution: if all involutions fix two points then by (RH2)  $2 = 4(2\gamma - 2) + 3 \times 2$  is impossible. If  $\psi$  fixes six points, then it is the hyperelliptic involution.  $\square$

In the classical Riemann Surface Theory, there is a theorem that states that any hypermap of genus 2 is a hyperelliptic. The following theorem gives the combinatorial result equivalent to the geometrical result :

**Theorem 33.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of genus 2. If  $Aut(\alpha, \sigma) > 6$  or if  $Aut(\alpha, \sigma)$  owns a  $C_2 \times C_2$  subgroup then  $(\alpha, \sigma)$  is hyperelliptic.*

*Proof.* We know that if  $(\alpha, \sigma)$  is 5-elliptic then by Proposition 16 since  $Aut(\alpha, \sigma) > 6$ ,  $Aut(\alpha, \sigma)$  is  $C_{10}$  therefore  $(\alpha, \sigma)$  is hyperelliptic. If  $(\alpha, \sigma)$  is 3-elliptic then by Proposition 17 since  $Aut(\alpha, \sigma) > 6$ ,  $Aut(\alpha, \sigma)$  is  $D_6$ ,  $C_2 \times C_6$ ,  $C_3 \rtimes_{-1} C_4$  or  $(C_2 \times C_{2g+2}) \rtimes C_2$  all hyperelliptic hypermaps ( $C_9$  is impossible because the 3-Sylows are of order  $3 = g + 1$ ).

If  $Aut(\alpha, \sigma)$  owns a  $C_2 \times C_2$  subgroup, then it has already been proved in Proposition 32 that  $(\alpha, \sigma)$  is hyperelliptic.  $\square$

This last result can not really be improved because of the following example:

**Example 34.** Let  $(\alpha, \sigma)$  be the following hypermap:  $\alpha = (1, 2, 3, 4, 5)$  and  $\sigma = (5, 4, 3, 2, 1) = \alpha^{-1}$  thus  $\alpha\sigma = \alpha^{-2}$ ;

by (RH) ,  $(\alpha, \sigma)$  of genus  $g = 2$  and  $Aut(\alpha, \sigma) = \langle \alpha, \sigma \rangle = \langle \alpha \rangle$  thus all element of the hypermap is an automorphism of it and none is hyperelliptic since  $Aut(\alpha, \sigma) = C_5$ .

In fact, the hypermap is not hyperelliptic because of a lack in brins, since all the "exceptions" are compatible with hyperellipticity.

In the rest of the paper, we simply adapt results on hypermaps of even genus to hypermaps of genus 2

**Proposition 35.** *Let  $(\alpha, \sigma)$  be a hyperelliptic hypermap of genus 2. Let  $G$  be a 2-Sylow subgroup of  $Aut(\alpha, \sigma)$ , then  $G$  is a subgroup of  $QD_{16}$  (QuasiDiedral).*

*Proof.* Immediate application of Proposition 21.  $\square$

Now, the general description of automorphism groups on hypermaps of genus 2.

**Proposition 36.** *Let  $(\alpha, \sigma)$  be a hypermap of genus 2, then  $Aut(\alpha, \sigma)$  is either  $C_n$  (cyclic) with  $n = 1, 2, 3, 4, 5, 6, 8, 10$  or  $D_n$  (dihedral) with  $n = 1, 2, 3, 4, 6$ ,  $Q_8$ ,  $C_2 \times C_6$ ,  $C_3 \rtimes C_4$ ,  $QD_{16}$  (QuasiDiedral),  $(C_2 \times C_6) \rtimes C_2$ ,  $SL_2(F_3) = Q_8 \rtimes C_3$  of order 24 or  $GL_2(F_3) = (Q_8 \rtimes C_3) \rtimes C_2$  of order 48.*

*Proof.* Immediate application of Theorem 31 and the fact that if  $g = 2$ , then  $|Aut(\alpha, \sigma)| \leq 84$  so that  $SL_2(F_5)$  is impossible.  $\square$

The former proposition can be resumed in the following theorem :

**Theorem 37.** *Let  $(\alpha, \sigma)$  be a hypermap of genus 2, then  $Aut(\alpha, \sigma)$  is a subgroup of one of the following groups :  $C_{10}$ ,  $(C_2 \times C_6) \rtimes C_2$  or  $GL_2(F_3) = (Q_8 \rtimes C_3) \rtimes C_2$ .*

Do all these groups exist ? We have constructed all the groups and subgroups in the annexe except  $(C_2 \times C_6) \rtimes C_2$ .

## 5. Annexe : Examples and drawings

**5.1. The Decagone.** This hyperelliptic map is defined over 10 brins with 5 edges of length 2, one vertex of length 10 and one face of length 10. Euler's formula tells

FIGURE 5.1. The Decagone

us that the genus is 2. It is the cyclic group with maximal order on a hypermap of genus 2.

$$\alpha = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10),$$

$$\sigma = (10, 7, 6, 3, 2)(9, 8, 5, 4, 1)$$

$$\alpha^{-1}\sigma = (10, 8, 6, 4, 2, 9, 7, 5, 3, 1).$$

Since  $\alpha$  and  $\sigma$  commute,  $\text{Aut}(\alpha, \sigma) = \langle \alpha^{-1}\sigma \rangle$ ; thus it is equal to  $C_{10}$ .

See figure 5.1.

**5.2. A : The double hexagon.** This hyperelliptic map is defined over 12 brins with 6 edges of length 2, 2 vertices of length 6 and 2 faces of length 6. Euler's formula tells us that the genus is 2. It is the abelian group with maximal order on a hypermap of genus 2.

$$\alpha = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12),$$

$$\sigma = (11, 10, 7, 6, 3, 2)(12, 9, 8, 5, 4, 1)$$

$$\alpha^{-1}\sigma = (12, 10, 8, 6, 4, 2)(11, 9, 7, 5, 3, 1).$$

Since  $\alpha$  and  $\sigma$  commute,  $\text{Aut}(\alpha, \sigma) = \langle \alpha, \sigma \rangle$ ; thus it is equal to  $C_2 \times C_6$ .

See figure 5.2.

**5.3. The hypercube.** This hyperelliptic map is defined over 48 brins with 24 edges of length 2, 16 vertices of length 3 and 6 faces of length 8. Euler's formula tells us that the genus is 2. Its quotient with respect to the hyperelliptic involution is the cube ( $S_4$ ). It is  $GL_2(F_3) = (Q_8 \rtimes C_3) \rtimes C_2$  of order 48, the group with maximal order on a hypermap of genus 2. Note that its two-Sylows are of order 16, thus of maximal order;

$$\alpha = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$$

$$(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)$$

$$\sigma = (17, 10, 24)(42, 19, 12)(44, 21, 14)(46, 23, 16)(41, 34, 48)(36, 18, 43)(45, 38, 20)$$

$$(47, 40, 22)(1, 32, 9)(25, 8, 33)(2, 11, 3)(26, 35, 27)(4, 13, 5)(28, 37, 29)(6, 15, 7)(30, 39, 31)$$

$$\alpha\sigma = (1, 11, 42, 34, 25, 35, 18, 10)(2, 32, 30, 28, 26, 8, 6, 4)(3, 13, 44, 36, 27, 37, 20, 12)$$

$$(5, 15, 46, 38, 29, 39, 22, 14)(7, 33, 48, 40, 31, 9, 24, 16)(17, 43, 21, 47, 41, 19, 45, 23)$$

GAP computation gives :

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gap> StructureDescription( Aut ); "GL(2,3)"
```

FIGURE 5.2. The Double Hexagone

Note that this group owns  $D_6$ ,  $Q_8$ ,  $C_3 \times C_4$ ,  $QD_{16}$  and  $SL_2(F_3) = Q_8 \times C_3$  as subgroups.

See Figure 5.3.

**5.4. The hyperglobe on genus 4.** This map is defined over 96 brins with 48 edges of length 2, 32 vertices of length 3 and two faces of length 16 and 8 faces of length 8. Euler's formula tells us that the genus is 4. It is a lifting of  $D_8$ . This map is the less refined double covering of the globe with Proposition 23 technique (see above).

$$\begin{aligned} \alpha &= (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24) \\ &\quad (25, 26)(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48) \\ &\quad (49, 50)(51, 52)(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72) \\ &\quad (73, 74)(75, 76)(77, 78)(79, 80)(81, 82)(83, 84)(85, 86)(87, 88)(89, 90)(91, 92)(93, 94)(95, 96) \\ \sigma &= (1, 64, 17)(2, 19, 3)(4, 21, 5)(6, 23, 7)(8, 25, 9)(10, 27, 11)(12, 29, 13)(14, 31, 15) \\ &\quad (49, 16, 65)(50, 67, 51)(52, 69, 53)(54, 71, 55)(56, 73, 57)(58, 75, 59)(60, 77, 61)(62, 79, 63) \\ &\quad (33, 18, 48)(35, 20, 82)(37, 22, 84)(39, 24, 86)(41, 26, 88)(43, 28, 90)(45, 30, 92)(47, 32, 94) \\ &\quad (81, 66, 96)(83, 68, 34)(85, 70, 36)(87, 72, 38)(89, 74, 40)(91, 76, 42)(93, 78, 44)(95, 80, 46) \\ \alpha\sigma &= (1, 19, 82, 66, 49, 67, 34, 18)(2, 64, 62, 60, 58, 56, 54, 52, 50, 16, 14, 12, 10, 8, 6, 4) \\ &\quad (3, 21, 84, 68, 51, 69, 36, 20)(5, 23, 86, 70, 53, 71, 38, 22)(7, 25, 88, 72, 55, 73, 40, 24) \\ &\quad (9, 27, 90, 74, 57, 75, 42, 26)(11, 29, 92, 76, 59, 77, 44, 28)(13, 31, 94, 78, 61, 79, 46, 30) \\ &\quad (15, 65, 96, 80, 63, 17, 48, 32)(33, 83, 37, 87, 41, 91, 45, 95, 81, 35, 85, 39, 89, 43, 93, 47). \end{aligned}$$

This map owns  $\psi$ , the following automorphism of order 16 :

FIGURE 5.3. The Hypercube

$\psi = (1, 3, 5, 7, 9, 11, 13, 15, 49, 51, 53, 55, 57, 59, 61, 63)$   
 $(2, 4, 6, 8, 10, 12, 14, 16, 50, 52, 54, 56, 58, 60, 62, 64)$   
 $(17, 19, 21, 23, 25, 27, 29, 31, 65, 67, 69, 71, 73, 75, 77, 79)$   
 $(18, 20, 22, 24, 26, 28, 30, 32, 66, 68, 70, 72, 74, 76, 78, 80)$   
 $(33, 35, 37, 39, 41, 43, 45, 47, 81, 83, 85, 87, 89, 91, 93, 95)$   
 $(34, 36, 38, 40, 42, 44, 46, 48, 82, 84, 86, 88, 90, 92, 94, 96)$ ;  
 and  $\theta$ , the following automorphism of order 2 :  
 $\theta := (16, 33)(14, 83)(12, 37)(10, 87)(8, 41)(6, 91)(4, 45)(2, 95)$   
 $(64, 81)(62, 35)(60, 85)(58, 39)(56, 89)(54, 43)(52, 93)(50, 47)$   
 $(1, 96)(3, 46)(5, 92)(7, 42)(9, 88)(11, 38)(13, 84)(15, 34)$

FIGURE 5.4. The Hyperglobe

(49, 48)(51, 94)(53, 44)(55, 90)(57, 40)(59, 86)(61, 36)(63, 82)  
 (19, 80)(21, 30)(23, 76)(67, 32)(69, 78)(71, 28)  
 (20, 79)(22, 29)(24, 75)(68, 31)(70, 77)(72, 27)  
 (17, 66)(65, 18)(25, 26)(73, 74).

Now,  $\theta\psi = \psi^7\theta$ , thus

$$\text{Aut}(\alpha, \sigma) = \langle \psi, \theta / \psi^{4 \times 4} = \theta^2 = 1, \theta\psi = \psi^{2 \times 4 - 1}\theta \rangle = E_{32} = QD_{32}$$

We give here the result of GAP computation:

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gap> StructureDescription( Aut ); "QD32"
```

Note that this group owns  $D_8$ ,  $C_{16}$  and  $Q_{16}$ , as maximal subgroups.

See Figure 5.4.

**5.5. The hyperglobe on genus 6.** This map is defined over 144 brins with 72 edges of length 2, 48 vertices of length 3 and two faces of length 24 and 12 faces of length 8. Euler's formula tells us that the genus is 6. It is a lifting of  $D_{12}$ . This map is the second less refined double covering of the globe with Proposition 23 technique after the one of genus 4 (see above).

$\alpha = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24)$   
 $(25, 26)(27, 28)(29, 30)(31, 32)(33, 34)(35, 36)(37, 38)(39, 40)(41, 42)(43, 44)(45, 46)(47, 48)$   
 $(49, 50)(51, 52)(53, 54)(55, 56)(57, 58)(59, 60)(61, 62)(63, 64)(65, 66)(67, 68)(69, 70)(71, 72)$   
 $(73, 74)(75, 76)(77, 78)(79, 80)(81, 82)(83, 84)(85, 86)(87, 88)(89, 90)(91, 92)(93, 94)(95, 96)$   
 $(97, 98)(99, 100)(101, 102)(103, 104)(105, 106)(107, 108)(109, 110)(111, 112)(113, 114)$   
 $(115, 116)(117, 118)(119, 120)(121, 122)(123, 124)(125, 126)(127, 128)(129, 130)(131, 132)$   
 $(133, 134)(135, 136)(137, 138)(139, 140)(141, 142)(143, 144);$   
 $\sigma = (1, 96, 25)(2, 27, 3)(4, 29, 5)(6, 31, 7)(8, 33, 9)(10, 35, 11)(12, 37, 13)(14, 39, 15)$   
 $(16, 41, 17)(18, 43, 19)(20, 45, 21)(22, 47, 23)(73, 24, 97)(74, 99, 75)(76, 101, 77)(78, 103, 79)$   
 $(80, 105, 81)(82, 107, 83)(84, 109, 85)(86, 111, 87)(88, 113, 89)(90, 115, 91)(92, 117, 93)(94, 119, 95)$   
 $(49, 26, 72)(51, 28, 122)(53, 30, 124)(55, 32, 126)(57, 34, 128)(59, 36, 130)(61, 38, 132)(63, 40, 134)$   
 $(65, 42, 136)(67, 44, 138)(69, 46, 140)(71, 48, 142)(121, 98, 144)(123, 100, 50)(125, 102, 52)$   
 $(127, 104, 54)(129, 106, 56)(131, 108, 58)(133, 110, 60)(135, 112, 62)(137, 114, 64)$   
 $(139, 116, 66)(141, 118, 68)(143, 120, 70);$   
 $\alpha\sigma = (96, 94, 92, 90, 88, 86, 84, 82, 80, 78, 76, 74, 24, 22, 20, 18, 16, 14, 12, 10, 8, 6, 4, 2)$   
 $(49, 123, 53, 127, 57, 131, 61, 135, 65, 139, 69, 143, 121, 51, 125, 55, 129, 59, 133, 63, 137, 67, 141, 71)$   
 $(1, 27, 122, 98, 73, 99, 50, 26)(3, 29, 124, 100, 75, 101, 52, 28)(5, 31, 126, 102, 77, 103, 54, 30)$   
 $(7, 33, 128, 104, 79, 105, 56, 32)(9, 35, 130, 106, 81, 107, 58, 34)(11, 37, 132, 108, 83, 109, 60, 36)$   
 $(13, 39, 134, 110, 85, 111, 62, 38)(15, 41, 136, 112, 87, 113, 64, 40)(17, 43, 138, 114, 89, 115, 66, 42)$   
 $(19, 45, 140, 116, 91, 117, 68, 44)(21, 47, 142, 118, 93, 119, 70, 46)(23, 97, 144, 120, 95, 25, 72, 48).$

This map owns  $\psi$ , the following automorphism of order 24 :

$\psi = (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95)$   
 $(2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96)$   
 $(25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119)$   
 $(26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120)$   
 $(49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 121, 123, 125, 127, 129, 131, 133, 135, 137, 139, 141, 143)$   
 $(50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 144);$

and  $\theta$ , the following automorphism of order 2 :

$\theta = (1, 144)(3, 70)(5, 140)(7, 66)(9, 136)(11, 62)(13, 132)(15, 58)(17, 128)(19, 54)(21, 124)(23, 50)$   
 $(73, 72)(75, 142)(77, 68)(79, 138)(81, 64)(83, 134)(85, 60)(87, 130)(89, 56)(91, 126)(93, 52)(95, 122)$   
 $(2, 143)(4, 69)(6, 139)(8, 65)(10, 135)(12, 61)(14, 131)(16, 57)(18, 127)(20, 53)(22, 123)(24, 49)$   
 $(74, 71)(76, 141)(78, 67)(80, 137)(82, 63)(84, 133)(86, 59)(88, 129)(90, 55)(92, 125)(94, 51)(96, 121)$   
 $(27, 120)(29, 46)(31, 116)(33, 42)(35, 112)$

(99, 48)(101, 118)(103, 44)(105, 114)(107, 40)  
 (28, 119)(30, 45)(32, 115)(34, 41)(36, 111)  
 (100, 47)(102, 117)(104, 43)(106, 113)(108, 39)  
 (25, 98)(26, 97)(37, 38)(109, 110).

Now,  $\theta\psi = \psi^{11}\theta$ , thus

$$\text{Aut}(\alpha, \sigma) = \langle \psi, \theta/\psi^{4 \times 6} = \theta^2 = 1, \theta\psi = \psi^{2 \times 6 - 1}\theta \rangle = E_{48}$$

We give here the result of GAP computation:

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gap> StructureDescription( Aut ); "C24 : C2"
```

Note that this group owns  $D_{12}$ ,  $C_{24}$  and  $QD_{16}$ , as maximal subgroups.

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UNIVERSIT PARIS, 10 RUE PIERRE LAROUSSE, 92245 MALAKOFF CEDEX

*Email address:* [laurence.bessis@parisdescartes.fr](mailto:laurence.bessis@parisdescartes.fr), [laurence.bessis@u-paris.fr](mailto:laurence.bessis@u-paris.fr)