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Optimal identification experiment design to improve the detection of the topology of a dynamic network

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Abstract

In this paper, we propose a methodology to detect the topology of a dynamic network that is based on the analysis of the uncertainty of an estimate of the static characteristic of the matrix of transfer functions between the external excitations and the node signals. We also show that the reliability of the proposed network topology detection methodology can be improved by an appropriate design of the experiment leading to the estimate of the static characteristic.

Key words: Dynamic network identification, topology detection

1 Introduction

This paper contributes to the efforts of developing techniques for the identification of large-scale or interconnected systems. In these efforts, we can distinguish two main lines of research. The first line consists in determining a model of the network when the interconnection structure (i.e. the topology) of the network is known (see e.g. [16,24,14,12]). The second line consists in determining the topology of the unknown network (see e.g. [15,25,22,21]). This paper belongs to this second line of research.

We consider a dynamical network made up of N_{mod} nodes. A given node w_i of such a network can be written as a function of the other nodes w_k ($k \neq i$) and as a function of known external excitations and unknown noise excitations [24]. In this paper, we are interested in determining the topology of such a network i.e. in determining for each possible pairs of nodes w_i and w_k whether there exists a non-zero causal transfer function $G_{0,ik}(z)$ linking these two nodes. To achieve this objective, one can of course use the node measurements to identify these transfer functions in a direct manner [24] or in an indirect way (i.e. by first estimating the transfer

between the external excitations and the different nodes and then by back-computing the transfer between the nodes) [25,22]. Due to the presence of the unknown noise excitations, even if a given transfer function $G_{0,ik}(z)$ is identically zero, its estimate will not be equal to zero. Consequently, it may be difficult in practice to determine the exact topology of the network if we only look at the estimates of the different transfer functions $G_{0,ik}(z)$. In order to avoid these issues, different sparsity-inducing approaches have been considered. In [10,21], one considers algorithms that favour sparse solutions. In [25] and in another section of [21], one considers algorithms that compare the quality of models identified when supposing that some of the transfer functions $G_{0,ik}(z)$ are indeed equal to zero. In [21], a forward selection approach is used while, in [25], all possible topologies are tested and the quality of the identified models are compared using a criterion penalizing a large number of connections.

In this paper, we develop a topology detection method that is based on the analysis of the uncertainty of the identified network model. For this purpose, we will take inspiration from [25] that shows that the topology of the network can be determined via the inversion of the static characteristic of the transfer matrix between the external excitations and the different nodes signals (the method in [25] is thus an indirect approach). We will however not only consider the estimate of this static characteristic but also its uncertainty. Using this uncer-

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tainty, we will be able to derive an uncertainty interval for the static gain of each transfer function $G_{0,ik}(z)$. If this uncertainty interval contains zero, we will then conclude that $G_{0,ik}(z)$ is equal to zero. We analyze the property of this decision rule and we show that we can increase our confidence in this decision rule by keeping the length of the confidence intervals below a certain threshold. Since the length of these confidence intervals can be linked to the experimental conditions of the identification experiment yielding the network model, we subsequently determine the experimental conditions of the identification experiment with the least powerful excitation signal that nevertheless yields confidence intervals with a length smaller than the chosen threshold. This optimal experiment design procedure is the main contribution of this journal paper with respect to its conference version [5].

Notations: The matrix I_n denotes the identity matrix of dimension n . The matrix

$$\begin{pmatrix} X_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & X_n \end{pmatrix}$$

will be denoted $\text{diag}(X_1, \dots, X_n)$ when the elements X_i ($i = 1, \dots, n$) are scalar quantities, while it will be denoted $\text{bdiag}(X_1, \dots, X_n)$ when the elements X_i ($i = 1, \dots, n$) are matrices.. For a matrix A , A^T denotes the transpose of A . Finally, \otimes denotes the Kronecker product.

2 Network description

We consider a dynamic network made up of N_{mod} nodes that are each characterized by a scalar valued measurable signal $w_i(t)$ ($i = 1, \dots, N_{mod}$). The vector $\bar{w}(t) = (w_1(t), w_2(t), \dots, w_{N_{mod}}(t))^T$ obeys the following equation [24]:

$$\bar{w}(t) = G_0(z) \bar{w}(t) + \bar{r}(t) + \bar{v}(t), \quad (1)$$

with

$$G_0(z) = \begin{pmatrix} 0 & G_{0,12}(z) & \dots & G_{0,1N_{mod}}(z) \\ G_{0,21}(z) & 0 & \dots & G_{0,2N_{mod}}(z) \\ \dots & \dots & \dots & \dots \\ G_{0,N_{mod}1}(z) & G_{0,N_{mod}2}(z) & \dots & 0 \end{pmatrix}. \quad (2)$$

In (1), $\bar{r}(t) = (r_1(t), r_2(t), \dots, r_{N_{mod}}(t))^T$ is a vector of external excitation signals that can be freely chosen by the user e.g., for identification purposes ($\bar{r}(t) = 0$ in normal operations) while the vector $\bar{v}(t) = (v_1(t), v_2(t), \dots, v_{N_{mod}}(t))^T$ represents the process noise acting on the network. In this paper, we will assume that $\bar{v}(t)$ is a wide-sense stationary process with a strictly positive definite power spectrum matrix at all frequencies.

For the sequel, we will need the following *closed-loop* expression of (1):

$$\bar{w}(t) = T_0(z)\bar{r}(t) + \bar{v}_{cl}(t) \quad (3)$$

where $\bar{v}_{cl}(t) = T_0(z)\bar{v}(t)$ and where the transfer matrix $T_0(z) = (I_{N_{mod}} - G_0(z))^{-1}$ is assumed to be stable and to have a frequency response $T_0(e^{j\omega})$ that is full rank at all frequencies ω . Given the (standard) assumptions on $T_0(z)$ and $\bar{v}(t)$ given above, the wide-sense stationary process $\bar{v}_{cl}(t)$ has a strictly positive definite power spectrum matrix at all frequencies and can thus be modeled as [1]:

$$\bar{v}_{cl}(t) = H_0(z)\bar{e}(t) \quad (4)$$

for some matrix of transfer functions $H_0(z)$ that is stable, inversely stable and that is also monic (i.e. $H_0(\infty) = I_{N_{mod}}$) and for some white noise vector $\bar{e}(t) = (e_1(t), e_2(t), \dots, e_{N_{mod}}(t))^T$ such that $E\bar{e}(t)\bar{e}^T(t) = \Sigma_0 > 0$ and $E\bar{e}(t)\bar{e}^T(t - \tau) = 0$ for all $\tau \neq 0$. It is important to note that $T_0(z)$ and $H_0(z)$ can be both very complex matrices of transfer functions.

Remark. The stochastic process $\bar{v}(t)$ in (1) can also be written as $\bar{v}(t) = D_0(z)\bar{e}(t)$ with a transfer matrix $D_0(z)$ which is stable, inversely stable and monic. If $G_0(z)$ is stable and if $G_0(\infty) = 0$, then the transfer matrix $H_0(z)$ describing $v_{cl}(t)$ is given by $T_0(z)D_0(z)$. If $G_0(z)$ does not have these two properties, $H_0(z)$ will have a more complex expression.

3 Topology detection problem

As mentioned in the introduction, in this paper, we wish to determine the topology of the network, i.e. we wish to determine which off-diagonal elements $G_{0,ik}(z)$ are identically equal to zero. In other words, the topology detection problem consists in discriminating between the following hypotheses:

$$\begin{cases} \mathcal{H}_0 : G_{0,ik}(z) = 0 \\ \mathcal{H}_1 : G_{0,ik}(z) \neq 0 \end{cases} \quad (5)$$

for all off-diagonal elements $G_{0,ik}(z)$ of $G_0(z)$ in (1). In this paper, we will develop a topology detection procedure under the following assumption on $G_0(z)$:

Assumption 1 Consider the network configuration described in Section 2. For any arbitrary transfer function $G_{0,ik}(z)$ in $G_0(z)$ (see (2)), if the static gain $G_{0,ik}(1)$ of $G_{0,ik}(z)$ is equal to zero, then the transfer function $G_{0,ik}(z)$ is also equal to zero. ■

There exists physical systems violating this assumption, i.e. they have system zeros corresponding to the zero frequency. A typical example is a system with a piezo-actuator. However, we believe that Assumption 1 covers a wide range of real world systems. We will return to how to relax this assumption in the conclusions.

As shown in [25], under Assumption 1, we can derive the topology of the network by simply inspecting the inverse \mathcal{Q}_0 of the static characteristic $\mathcal{T}_0 = T_0(1)$ of the closed-loop transfer matrix $T_0(z)$. This inverse \mathcal{Q}_0 is indeed given by:

$$\mathcal{Q}_0 = \mathcal{T}_0^{-1} = (I_{N_{mod}} - G_0(1)) \quad (6)$$

Consequently, we see that, under Assumption 1, any arbitrary transfer function $G_{0,ik}(z)$ ($i \neq k$) in $G_0(z)$ (see (2)) is equal to zero if and only if

$$\mathcal{Q}_{0,ik} = 0 \quad (7)$$

where $\mathcal{Q}_{0,ik}$ is the (i, k) -entry of the matrix $\mathcal{Q}_0 \in \mathbf{R}^{N_{mod} \times N_{mod}}$. In other words, under Assumption 1, (5) is equivalent to:

$$\begin{cases} \mathcal{H}_0 : \mathcal{Q}_{0,ik} = 0 \\ \mathcal{H}_1 : \mathcal{Q}_{0,ik} \neq 0 \end{cases} \quad (8)$$

4 Estimate of \mathcal{T}_0 and its uncertainty

To be able to discriminate between \mathcal{H}_0 and \mathcal{H}_1 , since \mathcal{Q}_0 is unknown, we will use an estimate $\hat{\mathcal{Q}}$ of \mathcal{Q}_0 . This estimate $\hat{\mathcal{Q}}$ of \mathcal{Q}_0 will be given as:

$$\hat{\mathcal{Q}} = \hat{\mathcal{T}}^{-1} \quad (9)$$

where $\hat{\mathcal{T}}$ is an estimate of \mathcal{T}_0 . It is very difficult to derive a reliable decision rule for the hypothesis test (8) based on such an estimate. Indeed, even if the actual off-diagonal entry $\mathcal{Q}_{0,ik}$ of \mathcal{Q}_0 is identically equal to zero, the off-diagonal entry $\hat{\mathcal{Q}}_{ik}$ of $\hat{\mathcal{Q}}$ will not be equal to zero. Consequently, in this paper, we will propose a decision rule for (8) by combining $\hat{\mathcal{Q}}_{ik}$ with its uncertainty. For this purpose, we need to specify an estimator $\hat{\mathcal{T}}$ of \mathcal{T}_0 and its uncertainty. Since we are only interested in an estimate of the static characteristic of $T_0(z)$ (and not an estimate of $T_0(z)$), the identification problem can be strongly simplified.

Since we wish to characterize the uncertainty of $\hat{\mathcal{T}}$, we will nevertheless need to derive a model of $H_0(z)$. Using time series analysis [8], this can be done in advance based on normal operation data (i.e., data $\bar{w}(t)$ collected on (1) when $\bar{r}(t) = 0$). In order to simplify the identification of this model of $H_0(z)$, we will here use an AR structure for this model. This indeed allows to estimate a model¹ $\hat{H}(z) = \hat{A}(z)^{-1}$ of $H_0(z)$ by solving a least-squares optimization problem. Referring to Theorem 3.1 in [20], $\hat{H}(e^{j\omega})$ will tend to $H_0(e^{j\omega})$ if the number of estimation data tends to infinity and if the order of the AR model increase at a suitable rate with this number of estimation data. In practice, the order of the AR model must

¹ In an AR structure, $\hat{H}(z)$ is modeled as the inverse of a matrix of polynomials $\hat{A}(z)$.

be chosen in such a way that the residuals $\hat{H}^{-1}(z)\bar{w}(t)$ are whitened. The obtained AR model will be used to derive the estimate of \mathcal{T}_0 .

The estimate of the static matrix \mathcal{T}_0 will be determined via N_{mod} identification experiments leading each to an estimate of one column of \mathcal{T}_0 . More precisely, the j^{th} experiment ($j = 1, \dots, N_{mod}$) allows to determine an estimate $\hat{\mathcal{T}}_j$ of the j^{th} column $\mathcal{T}_{0,j}$ of \mathcal{T}_0 . For this purpose, we apply to the network (1) an excitation vector $\bar{r}(t)$ where all the elements except $r_j(t)$ are zero and where $r_j(t)$ is equal to the constant α_j for all t :

$$\bar{r}(t) = \alpha_j \bar{m}_j \quad \forall t \quad (10)$$

where \bar{m}_j ($j = 1, \dots, N_{mod}$) denotes a unit (column) vector of dimension N_{mod} for which the j^{th} entry is equal to 1 and the other entries are equal to zero. After the end of the transient, the vector $\bar{w}(t)$ that is collected during such an experiment obeys the following relation:

$$\bar{w}(t) = \mathcal{T}_{0,j} \alpha_j + H_0(z)\bar{e}(t) \quad (11)$$

Moreover, if we filter the data $\bar{w}(t)$ collected in such an experiment with the inverse of the AR model $\hat{H}(z)$ of $H_0(z)$, we have also that, after the end of the transient²,

$$\bar{w}_H(t) = \hat{H}^{-1}(z)\bar{w}(t) = \Phi_j \mathcal{T}_{0,j} + \hat{H}^{-1}(z)H_0(z)\bar{e}(t) \quad (12)$$

where $\Phi_j = \alpha_j \hat{H}^{-1}(1)$, and where the last term is near white since it approximately equals $\bar{e}(t)$. Using (12), it is clear that an estimate $\hat{\mathcal{T}}_j$ of $\mathcal{T}_{0,j}$ (and an estimate $\hat{\Sigma}$ of the covariance matrix Σ_0 of \bar{e}) can be obtained by considering the following simple prediction error criterion [19]:

$$\min_{\mathcal{T}_j, \Sigma} \frac{1}{N} \sum_{t=N_j,ss}^{N_j+N_j,ss} \bar{\epsilon}_j^T(t, \mathcal{T}_j) \Sigma^{-1} \bar{\epsilon}_j(t, \mathcal{T}_j) \quad (13)$$

$$\bar{\epsilon}_j(t, \mathcal{T}_j) = \bar{w}_H(t) - \Phi_j \mathcal{T}_j \quad (14)$$

where \mathcal{T}_j is a column vector of dimension N_{mod} , Σ is a square matrix of dimension N_{mod} , $N_{j,ss}$ is the moment where $\bar{w}_H(t) = \hat{H}^{-1}(z)\bar{w}(t)$ reaches the steady-state (12), and $N_j + N_{j,ss}$ is the duration of the j^{th} experiment. The criterion (13) can be solved using the iterative least-squares procedure in [14, page 485]³.

² The end of the transient for $\bar{w}(t)$ is the moment where $\bar{w}(t)$ starts to oscillate around a fixed offset (which is equal to $\mathcal{T}_{0,j} \alpha_j$ (see (12)). Since $\hat{H}^{-1}(z)$ is a polynomial matrix, the end of the transient for $\bar{w}_H(t)$ is given by $t_w + t_h$ where t_w is the end of the transient for $\bar{w}(t)$ and t_h is the order of the polynomial of highest order in $\hat{H}^{-1}(z)$.

³ Alternatively, $\hat{\Sigma}$ can be first estimated using the residuals from the AR-model $\hat{H}(z)$ and $\hat{\mathcal{T}}_j$ can then be obtained using weighted least-squares.

Using the framework of [20] described above, $\hat{H}(e^{j\omega})$ can be made to converge to $H_0(e^{j\omega})$ at a sufficiently fast rate so that the estimate $\hat{\mathcal{T}}_j$ obtained via (13) is asymptotically normally distributed around $\mathcal{T}_{0,j}$ with covariance matrix $P_{\mathcal{T}_j} = (1/N_j) (\Phi_j^T \Sigma_0^{-1} \Phi_j)^{-1}$ [19]. Using the estimate $\hat{\Sigma}$ of Σ_0 , $P_{\mathcal{T}_j}$ can thus be estimated as:

$$P_{\mathcal{T}_j} \approx \frac{1}{\alpha_j^2 N_j} \left(\hat{H}(1) \hat{\Sigma} \hat{H}^T(1) \right) \quad (15)$$

Combining the estimates of the columns of \mathcal{T}_0 obtained in the N_{mod} experiments, we can form the following estimate $\hat{\mathcal{T}}$ of \mathcal{T}_0 :

$$\hat{\mathcal{T}} = \left(\hat{\mathcal{T}}_1, \hat{\mathcal{T}}_2, \dots, \hat{\mathcal{T}}_{N_{mod}} \right) \quad (16)$$

Let us also analyze the variance/uncertainty of this estimate. For this purpose, let us denote by θ_0 (resp. $\hat{\theta}$) the vector of dimension N_{mod}^2 made up of the vectorization of the columns of \mathcal{T}_0 (resp. $\hat{\mathcal{T}}$) i.e.

$$\theta_0 = \left(\mathcal{T}_{0,1}^T, \mathcal{T}_{0,2}^T, \dots, \mathcal{T}_{0,N_{mod}}^T \right)^T \quad (17)$$

$$\hat{\theta} = \left(\hat{\mathcal{T}}_1^T, \hat{\mathcal{T}}_2^T, \dots, \hat{\mathcal{T}}_{N_{mod}}^T \right)^T \quad (18)$$

As the estimates of the different columns of \mathcal{T}_0 are assumed to be obtained from independent experiments, $\hat{\theta}$ is (asymptotically) normally distributed around θ_0 with a covariance matrix P_θ given by:

$$P_\theta = bdiag \left(P_{\mathcal{T}_1}, P_{\mathcal{T}_2}, \dots, P_{\mathcal{T}_{N_{mod}}} \right) \quad (19)$$

with $P_{\mathcal{T}_j}$ ($j = 1, \dots, N_{mod}$) defined in (15).

Remark. The identification procedure described above is in two steps: normal operation data are used to identify $\hat{H}(z)$ and then data collected by exciting the system with $\bar{r}(t)$ are used to identify $\hat{\mathcal{T}}$. If you do not wish to use normal operation data to identify $\hat{H}(z)$, this AR model can also be identified with the data collected under a nonzero excitation (see (11)) by computing the average μ of the steady-state data $\bar{w}(t)$ over the available samples. We can then perform the AR identification based on $\bar{w}(t) - \mu$. We have indeed $\bar{w}(t) - \mu \approx H_0(z)\bar{e}(t)$. Another possible least-squares approach is to use the semi-parametric weighted null-space fitting approach of [13].

5 Decision rule for (8)

As we will show in the next section, using $\hat{\theta}$ and its covariance matrix P_θ , we will be able to derive, for each off-diagonal entry $\mathcal{Q}_{0,ik}$ of \mathcal{Q}_0 , an uncertainty interval $\mathcal{I}_{ik} = [Q_{ik}^{min}, Q_{ik}^{max}]$ that contains $\mathcal{Q}_{0,ik}$ (modulo a given probability level). Based on this uncertainty interval \mathcal{I}_{ik} , we propose the following decision rule for the hypothesis test (8) (equivalent to (5)):

$$\begin{cases} 0 \in \mathcal{I}_{ik} \implies \mathcal{H}_0 \\ 0 \notin \mathcal{I}_{ik} \implies \mathcal{H}_1 \end{cases} \quad (20)$$

When we assume that $\mathcal{Q}_{0,ik} \in \mathcal{I}_{ik}$, this decision rule has the following properties for an uncertainty interval \mathcal{I}_{ik} of length $L_{ik} = Q_{ik}^{max} - Q_{ik}^{min}$. If $\mathcal{Q}_{0,ik} = 0$ (i.e., under \mathcal{H}_0), this decision rule will always yield the right decision i.e., to decide \mathcal{H}_0 . If $\mathcal{Q}_{0,ik} \neq 0$ (i.e., under \mathcal{H}_1), the properties of (20) will depend on the absolute value $|\mathcal{Q}_{0,ik}|$ of the unknown $\mathcal{Q}_{0,ik}$. If $|\mathcal{Q}_{0,ik}|$ is larger than the length L_{ik} of the interval \mathcal{I}_{ik} , the decision rule (20) will always yield the right decision i.e., to decide \mathcal{H}_1 . If $\mathcal{Q}_{0,ik} \neq 0$ and $|\mathcal{Q}_{0,ik}| < L_{ik}$, the decision rule (20) may yield, depending on the actual value of \mathcal{I}_{ik} , to a correct or an incorrect decision.

We can also formulate these properties from another point of view. When we assume that $\mathcal{Q}_{0,ik} \in \mathcal{I}_{ik}$, the decision rule (20) will always yield to the right decision (i.e., \mathcal{H}_1) when $0 \notin \mathcal{I}_{ik}$ while the decision (i.e., \mathcal{H}_0) may be wrong when $0 \in \mathcal{I}_{ik}$ since (20) cannot distinguish between an element $\mathcal{Q}_{0,ik}$ which is exactly equal to zero and an element $\mathcal{Q}_{0,ik} \neq 0$ such that $|\mathcal{Q}_{0,ik}| < L_{ik}$.

From the above analysis, it is thus clear that a small L_{ik} will increase the reliability of the decision rule (20). We will see in the sequel that the smaller P_θ , the smaller the lengths L_{ik} of the intervals \mathcal{I}_{ik} will be. Consequently, the reliability of the decision rule (20) can be improved by an appropriate design of the identification experiments described in Section 4.

To be able to use the decision rule (20), we need a manner to determine the intervals \mathcal{I}_{ik} for each pair (i, k) . This is done in the next section.

Remark. One could wonder whether we could not also derive a decision rule based on the inverse of the AR model $\hat{H}(z)$ of $H_0(z)$ and its uncertainty. This is indeed the case, but only under more restrictive assumptions on the network described in Section 2. For this let us refer to the remark at the end of Section 2. If $G_0(z)$ is stable and satisfies $G_0(\infty) = 0$, we have $H_0^{-1}(1) = D_0^{-1}(1)\mathcal{Q}_0$. Consequently, if, in addition, $D_0(z)$ is assumed diagonal, the topology of the network can be determined by inspecting $H_0^{-1}(1)$ and an alternative decision rule can thus be determined by considering the uncertainty of the identified model $\hat{H}(z)$ of $H_0(z)$.

6 Determination of the uncertainty intervals

6.1 Introduction

Using the statistical properties of $\hat{\theta}$ (see Section 4), it is clear that the following ellipsoid U is a $\beta\%$ -confidence region for the modeling error $\delta_0 = \theta_0 - \hat{\theta}$:

$$U := \left\{ \delta \in \mathbf{R}^{N_{mod}^2} \mid \delta^T P_\theta^{-1} \delta < \chi \right\} \quad (21)$$

where χ is defined by $Pr(\chi^2(N_{mod}^2) < \chi) = \beta$ where $\chi^2(N_{mod}^2)$ is the χ^2 -distribution with N_{mod}^2 degrees of freedom. This ellipsoid U can be considered as an uncertainty set for the unknown modeling error δ_0 . From now onwards, we will therefore assume that $\delta_0 = \theta_0 - \hat{\theta} \in U$ or, in other words, that $\theta_0 \in \{\theta \mid \theta = \hat{\theta} + \delta \text{ and } \delta \in U\}$. If, for a given vector $\delta \in U$, we denote by $T(\delta)$, the matrix obtained from the de-vectorization of the vector $\theta := \hat{\theta} + \delta$, we have that $\hat{\mathcal{T}} = T(0)$ and $\mathcal{T}_0 = T(\delta_0)$. Moreover, since $\mathcal{Q}_0 = T(\delta_0)^{-1}$ and $\delta_0 \in U$, we have also that the unknown (i, k) -entry $\mathcal{Q}_{0,ik}$ of \mathcal{Q}_0 lies in the following set $\mathcal{I}_{ik}^{orig} = \{Q_{ik}(\delta) \in \mathbf{R} \mid Q(\delta) = T(\delta)^{-1} \text{ and } \delta \in U\}$ where $Q_{ik}(\delta)$ is the (i, k) -entry of $Q(\delta)$.

The set \mathcal{I}_{ik}^{orig} defined in the previous paragraph is an interval since the matrix inversion is a continuous operation. Consequently, this interval could be used in the decision rule (20) since \mathcal{I}_{ik}^{orig} contains $\mathcal{Q}_{0,ik}$ (modulo a certain probability level). However, this will not be possible since we cannot compute an explicit expression for \mathcal{I}_{ik}^{orig} . To show this, let us observe that \mathcal{I}_{ik}^{orig} is also equal to $[c_{ik}^{opt,orig} - \sqrt{\rho_{ik}^{opt,orig}}, c_{ik}^{opt,orig} + \sqrt{\rho_{ik}^{opt,orig}}]$ where $c_{ik}^{opt,orig}$ and $\rho_{ik}^{opt,orig}$ are the solutions of the following optimization problem:

$$\min_{c_{ik}, \rho_{ik}} \rho_{ik} \quad (22)$$

$$\text{s.t. } (Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik} \quad \forall \delta \in U \quad (23)$$

As usual in robustness analysis [26], we cannot determine exactly the solutions of the above optimization problem. However, we can use convex relaxation to replace the constraint (23) by an alternative constraint that is linear in the decision variables c_{ik} and ρ_{ik} and that implies (23). If we denote by c_{ik}^{opt} and ρ_{ik}^{opt} the solutions of the convex optimization problem consisting in minimizing ρ_{ik} under the convex constraint mentioned above. Then, the computable interval $\mathcal{I}_{ik} = [c_{ik}^{opt} - \sqrt{\rho_{ik}^{opt}}, c_{ik}^{opt} + \sqrt{\rho_{ik}^{opt}}]$ is such that $\mathcal{I}_{ik}^{orig} \subseteq \mathcal{I}_{ik}$ and can thus be used in the decision rule (20). In the next subsections, we show how we can derive a convex constraint that implies (23).

Remark. From the above analysis, it is clear that the smaller the size P_θ of U , the smaller the length L_{ik} of the intervals \mathcal{I}_{ik}^{orig} and \mathcal{I}_{ik} will be.

6.2 LFT description of $Q_{ik}(\delta)$

To be able to determine a convex alternative for (23), a first step is to rewrite the quantity $Q_{ik}(\delta)$ in an Linear Fractional Transform (LFT) in the variable δ . As we will see below, this can be done by first rewriting $T(\delta)$ as an LFT in δ . For an arbitrary $\delta \in U$, the mapping $\bar{y} = T(\delta)\bar{u}$ can be expressed in the LFT framework as follows:

$$\bar{p} = \underbrace{I_{N_{mod}} \otimes \delta}_{=\Delta(\delta)} \bar{q} \quad (24)$$

$$\begin{pmatrix} \bar{q} \\ \bar{y} \end{pmatrix} = \underbrace{\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}}_{=Z} \begin{pmatrix} \bar{p} \\ \bar{u} \end{pmatrix} \quad (25)$$

where Z is given by:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{0} & I_{N_{mod}} \\ \hline (\bar{m}_1^T, \dots, \bar{m}_{N_{mod}}^T) \otimes I_{N_{mod}} & \hat{\mathcal{T}} \end{array} \right)$$

with \bar{m}_j ($j = 1, \dots, N_{mod}$) and $\hat{\mathcal{T}} = T(0)$ as defined in Section 4. In the sequel, we will use $\bar{y} = \mathcal{F}(Z, \Delta(\delta))\bar{u}$ as a shorthand notation for the LFT (24)-(25).

Let us consider the same $\delta \in U$ as well as the inverse mapping $\bar{u} = Q(\delta)\bar{y} = T^{-1}(\delta)\bar{y}$. This inverse mapping can also be expressed in the LFT framework with (24) and:

$$\begin{pmatrix} \bar{q} \\ \bar{u} \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{=M} \begin{pmatrix} \bar{p} \\ \bar{y} \end{pmatrix} \quad (26)$$

where M can be derived from Z :

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} Z_{11} - Z_{12}Z_{22}^{-1}Z_{21} & Z_{12}Z_{22}^{-1} \\ -Z_{22}^{-1}Z_{21} & Z_{22}^{-1} \end{pmatrix}$$

In other words, we have $\bar{u} = Q(\delta)\bar{y} = \mathcal{F}(M, \Delta(\delta))\bar{y}$. This LFT of the inverse mapping can be adapted to each entry of the matrix $Q(\delta)$. For this purpose, let us note that $Q(\delta) = M_{22} + M_{21}\Delta(\delta)(I - M_{11}\Delta(\delta))^{-1}M_{12}$. Consequently, the (i, k) -entry $Q_{ik}(\delta) = \bar{m}_i^T Q(\delta)\bar{m}_k$ of $Q(\delta)$ is equal to $Q_{ik}(\delta) = \bar{m}_i^T Q(\delta)\bar{m}_k = \bar{m}_i^T M_{22}\bar{m}_k + \bar{m}_i^T M_{21}\Delta(\delta)(I - M_{11}\Delta(\delta))^{-1}M_{12}\bar{m}_k$. Consequently, the scalar mapping $u_i = Q_{ik}(\delta)y_k$ can be rewritten as the LFT made up of (24) and:

$$\begin{pmatrix} \bar{q} \\ u_i \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11}^{ik} & M_{12}^{ik} \\ M_{21}^{ik} & M_{22}^{ik} \end{pmatrix}}_{=M^{ik}} \begin{pmatrix} \bar{p} \\ y_k \end{pmatrix} \quad (27)$$

where M^{ik} is the following function of M :

$$M^{ik} = \begin{pmatrix} M_{11}^{ik} & M_{12}^{ik} \\ M_{21}^{ik} & M_{22}^{ik} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12}\bar{m}_k \\ \bar{m}_i^T M_{21} & \bar{m}_i^T M_{22}\bar{m}_k \end{pmatrix}$$

In other words, we have that $Q_{ik}(\delta) = \mathcal{F}(M^{ik}, \Delta(\delta))$ for any $\delta \in U$.

6.3 Set of multipliers related to the uncertainty set U

Another crucial ingredient to determine a convex alternative for (23) is to associate, with the set U , a so-called set of multipliers [23] that takes into account how δ appears in the LFT representation $\mathcal{F}(M^{ik}, \Delta(\delta))$ of $Q_{ik}(\delta)$ (see (24)).

Definition 1 Consider the set U defined in (21). We define the set of multipliers \mathcal{A}_U as a set of affinely parametrized symmetric matrices A (of dimension $N_{mod}(N_{mod}^2 + 1) \times N_{mod}(N_{mod}^2 + 1)$) that all have the following property:

$$\begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix}^T A \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix} \geq 0 \quad \forall \delta \in U \quad (28)$$

In other words, $A \in \mathcal{A}_U \implies (28)$. \blacksquare

We have derived an extensive parametrization of the set of multipliers \mathcal{A}_U corresponding to U in our previous contribution [2, Proposition 2]. That the parametrization of \mathcal{A}_U is extensive is important since the more extensive the parametrization of the set of multipliers, the tighter the embedding \mathcal{I}_{ik} of \mathcal{I}_{ik}^{orig} will be [23]. In the next proposition, we give the parametrization of the set of multipliers \mathcal{A}_U given in [2, Proposition 2]. Note that, unlike in [2, Proposition 2], this parametrization is here restricted to real multipliers $A = A^T$ since the LFT's introduced in Section 6.2 are entirely real.

Proposition 1 Consider Definition 1 and the set U defined in (21). The matrices $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}$ belonging to the set \mathcal{A}_U corresponding to U are parametrized as follows:

$$A_{11} = \tilde{W} \quad A_{22} = -\tilde{W} \otimes \frac{P_\theta^{-1}}{\chi} - \tilde{B} \quad (29)$$

$$A_{12} = \begin{pmatrix} 0 & \tilde{v}_{12}^T & \dots & \tilde{v}_{1N_{mod}}^T \\ -\tilde{v}_{12}^T & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & \tilde{v}_{(N_{mod}-1)N_{mod}}^T \\ -\tilde{v}_{1N_{mod}}^T & \dots & -\tilde{v}_{(N_{mod}-1)N_{mod}}^T & 0 \end{pmatrix}$$

where \tilde{W} is a square matrix of dimension N_{mod} that can take any value as long as \tilde{W} is a positive definite symmetric matrix, where \tilde{v}_{lr} are column vectors of dimension N_{mod}^2 that can take any value ($l = 1, \dots, N_{mod}$ and $r = 1, \dots, N_{mod}$) and finally where \tilde{B} is a square matrix of dimension N_{mod}^3 that can take any values as long as:

$$\tilde{B} = \begin{pmatrix} 0 & K_{12} & \dots & K_{1N_{mod}} \\ -K_{12} & 0 & \dots & \vdots \\ \vdots & \dots & \ddots & K_{(N_{mod}-1)N_{mod}} \\ -K_{1N_{mod}} & \dots & -K_{(N_{mod}-1)N_{mod}}(\omega) & 0 \end{pmatrix}$$

with the constraint that $K_{i,j} = -K_{i,j}^T \in \mathbf{R}^{N_{mod}^2 \times N_{mod}^2}$.

The set of multipliers \mathcal{A}_U consists thus in the set of all matrices A that can be parametrized with the fixed variables P_θ and χ defining U and with the free variables \tilde{W} , \tilde{B} , \tilde{v}_{lr} ($l = 1, \dots, N_{mod}$ and $r = 1, \dots, N_{mod}$) satisfying the above constraints. \blacksquare

6.4 LMI optimization to determine the uncertainty interval

We have now all the elements to derive the convex alternative for (23). We will for this purpose adapt a result in [11] to our situation.

Proposition 2 Consider the LFT $\mathcal{F}(M^{ik}, \Delta(\delta))$ for $Q_{ik}(\delta)$ (see (24)-(27)) and the set of multipliers \mathcal{A}_U associated to U (see Definition 1). Then, the constraint (23) holds for given c_{ik} and ρ_{ik} if we can find a matrix $A \in \mathcal{A}_U$ such that the following LMI constraint holds:

$$\left(\begin{array}{c|c} -1 & \mathcal{X}(c_{ik}) \\ \hline \mathcal{X}(c_{ik})^T & \mathcal{K}^T A \mathcal{K} + \begin{pmatrix} 0 & 0 \\ 0 & -\rho_{ik} \end{pmatrix} \end{array} \right) < 0 \quad (30)$$

with $\mathcal{X}(c_{ik}) = \begin{pmatrix} M_{21}^{ik} & M_{22}^{ik} - c_{ik} \end{pmatrix}$ and $\mathcal{K} = \begin{pmatrix} M_{11}^{ik} & M_{12}^{ik} \\ I & 0 \end{pmatrix}$. \blacksquare

Proof. First, let us observe that the matrix inequality (30) is linear in the decision variables A , c_{ik} and ρ_{ik} . Consequently (30) is indeed an LMI. Subsequently, using the Schur complements, (30) is equivalent to:

$$\mathcal{K}^T A \mathcal{K} + \begin{pmatrix} 0 & 0 \\ 0 & -\rho_{ik} \end{pmatrix} + \mathcal{X}(c_{ik})^T \mathcal{X}(c_{ik}) < 0 \quad (31)$$

Let us now consider the LFT (24)-(27) for a given $\delta \in U$ and for $y_k = 1$ and let us consider the corresponding signals \bar{p} , \bar{q} and $u_i = Q_{ik}(\delta)$. Let us then pre- and post-multiply the constraint (31) with $(\bar{p}^T, 1)$ and $(\bar{p}^T, 1)^T$, respectively. Using (27), this yields:

$$\begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix}^T A \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} + (u_i - c_{ik})^2 < \rho_{ik} \quad (32)$$

Since $\bar{p} = (I_{N_{mod}} \otimes \delta)\bar{q}$ and $u_i = Q_{ik}(\delta)$, we can

rewrite (32) as follows:

$$\bar{q}^T \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix}^T A \begin{pmatrix} I_{N_{mod}} \\ I_{N_{mod}} \otimes \delta \end{pmatrix} \bar{q} + (Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik} \quad (33)$$

The above reasoning can be done for any value of $\delta \in U$. In other words, for the matrix $A \in \mathcal{A}_U$ for which (30) holds, (33) holds true for all $\delta \in U$. Consequently, using Definition 1, we have therefore that the existence of $A \in \mathcal{A}_U$ such that (30) holds implies that, for each $\delta \in U$, $(Q_{ik}(\delta) - c_{ik})^2 < \rho_{ik}$; which is the desired result. ■

Using Proposition 1, we can now compute the uncertainty interval \mathcal{I}_{ik} containing $\mathcal{Q}_{0,ik}$ for each pair (i, k) such that $i \neq k$. Indeed, this interval \mathcal{I}_{ik} is given by $\mathcal{I}_{ik} = [c_{ik}^{opt} - \sqrt{\rho_{ik}^{opt}}, c_{ik}^{opt} + \sqrt{\rho_{ik}^{opt}}]$ where c_{ik}^{opt} and ρ_{ik}^{opt} are the solutions of the LMI optimization problem consisting in finding the smallest value of ρ_{ik} for which we can find a scalar c_{ik} and a matrix $A \in \mathcal{A}_U$ such that the LMI constraint (30) holds. In this LMI optimization problem, when we speak of finding a matrix $A \in \mathcal{A}_U$, we more precisely mean finding the free parameters in the affine structure of the matrix A .

7 Numerical illustration

Let us consider a network (1) with $N_{mod} = 3$ nodes and

$$G_0(z) \triangleq \begin{pmatrix} 0 & 0 & G_{0,13}(z) \\ G_{0,21}(z) & 0 & 0 \\ G_{0,31}(z) & G_{0,32}(z) & 0 \end{pmatrix} \quad (34)$$

with $G_{0,21}(z) = \frac{0.095z^{-1}}{1-0.905z^{-1}}$, $G_{0,31}(z) = \frac{0.181z^{-1}}{1-0.819z^{-1}}$ and $G_{0,32}(z) = \frac{0.259z^{-1}}{1-0.741z^{-1}}$ and $G_{0,13}(z) = 0.3 G_{0,32}(z)$. Moreover, $\bar{v}(t) = (I_3 \otimes \frac{1}{1-0.7z^{-1}})\bar{e}(t)$ ($i = 1, 2, 3$) with $\bar{e}(t)$ a white noise vector of covariance matrix $\Sigma_0 = 0.1I_3$. Since $G_0(z)$ is stable and $G_0(\infty) = 0$, the transfer matrix $H_0(z)$ in (4) is given by $T_0(z)(I_3 \otimes \frac{1}{1-0.7z^{-1}})$ (see the remark at the end of Section 2).

In this network, we see that the transfer functions $G_{0,ik}$ ($i \neq k$) that are equal to zero are the transfer function $G_{0,12}(z)$ and $G_{0,23}(z)$. Let us compute the inverse \mathcal{Q}_0 of the static gain matrix \mathcal{T}_0 of the transfer matrix $T_0(z) = (I_{N_{mod}} - G_0(z))^{-1}$:

$$\mathcal{Q}_0 = \begin{pmatrix} 1 & 0 & -0.3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}. \quad (35)$$

We indeed observe that $\mathcal{Q}_{0,12}$ and $\mathcal{Q}_{0,23}$ are the sole entries $\mathcal{Q}_{0,ik}$ in \mathcal{Q}_0 that are equal to zero.

Following the procedure in Section 4, we identify an AR model $\hat{H}(z) = \hat{A}^{-1}(z)$ of $H_0(z)$ using 10000 samples of $\bar{w}(t)$ collected on (1) with $\bar{r}(t) = 0$. Using the whiteness test of [19, page 512], we observe that a polynomial matrix $\hat{A}(z)$ having entries of degree one is sufficient to whiten $\hat{H}^{-1}(z)\bar{w}(t) = \hat{A}(z)\bar{w}(t)$. This AR model is then used to subsequently identify the columns of \mathcal{T}_0 via three identification experiments and the identification criterion (13). In these three experiments, we choose $\alpha_j = 0.5$ and $N_j = 800$ ($j = 1, 2, 3$). This allows to deduce an estimate $\hat{\mathcal{T}}$ of \mathcal{T}_0 (see (16)) and we then obtain the following estimate of \mathcal{Q}_0 :

$$\hat{\mathcal{Q}} = \hat{\mathcal{T}}^{-1} = \begin{pmatrix} 0.9905 & 0.0522 & -0.3106 \\ -0.9693 & 0.9847 & -0.0269 \\ -1.2323 & -1.0247 & 1.1004 \end{pmatrix} \quad (36)$$

We observe that $\hat{Q}_{12} = 0.0522$ and $\hat{Q}_{23} = -0.0269$ are smaller than the other \hat{Q}_{ik} ($i \neq k$), but not extremely small. To have an idea of the uncertainty of the estimates \hat{Q}_{ik} ($i \neq k$), the estimate of the covariance matrix $P_{\hat{\theta}}$ of $\hat{\theta}$ (see (18) and (19)) is used to determine the uncertainty ellipsoid U (the confidence level is chosen equal to 95% and χ is thus chosen equal to 16.919). In our case, U indeed contains $\delta_0 = \theta_0 - \hat{\theta}$. We then use the procedure described in Section 6 to derive the uncertainty intervals \mathcal{I}_{ik} corresponding to each non-diagonal entry \hat{Q}_{ik} of $\hat{\mathcal{Q}}$. In Table 1, we give these intervals $\mathcal{I}_{ik} = [Q_{ik}^{min}, Q_{ik}^{max}]$ together with their length L_{ik} , the decisions of the decision rule (20) and the true coefficients $\mathcal{Q}_{0,ik}$. We observe that, for all pairs (i, k) , the true coefficient $\mathcal{Q}_{0,ik}$ lies in its confidence interval \mathcal{I}_{ik} which was to be expected since $\delta_0 \in U$. We also observe that the lengths L_{ik} of these intervals are relatively large with respect to $|\mathcal{Q}_{0,ik}|$ and this has as consequence that the decision rule (20) is not able to yield to the right decision for the non-diagonal entry $\mathcal{Q}_{0,13}$ of \mathcal{Q}_0 . Indeed, 0 not only lies in \mathcal{I}_{12} and \mathcal{I}_{23} , but also in \mathcal{I}_{13} . Note however that, even if L_{ik} is large, the decision rule (20) allows to definitely exclude the hypothesis that $\mathcal{Q}_{0,21}$, $\mathcal{Q}_{0,31}$, $\mathcal{Q}_{0,32}$ are equal to zero. Indeed, zero is not an element of the corresponding (large) intervals.

Let us check, via Monte Carlo simulations, whether the observations of the previous paragraph contain some generalities. For this purpose, we have repeated the above procedure 500 times with different realizations of the noise vector \bar{e} in each step (i.e. the identification of $\hat{H}(z)$ and of $\hat{\mathcal{T}}$) and we have observed that δ_0 lies in U in 90.6% of these Monte Carlo simulations (which is close to the prescribed confidence level of 95%)⁴. We have also observed that the decision rule (20) is only able to determine the correct topology in 51% of the cases

⁴ The difference is certainly to be explained by the low order AR model $\hat{H}(z)$ for the relatively complex $H_0(z)$.

with the chosen experimental conditions. In the sequel, we will show that we can increase the performance of the decision rule (20) by an appropriate design of the amplitudes α_j ($j = 1, \dots, N_{mod}$) of the excitation signals used in the identification procedure of Section 4.

Table 1
Intervals for the experiment in Section 7

(i, k)	$Q_{0,ik}$	Q_{ik}^{min}	Q_{ik}^{max}	L_{ik}	\mathcal{H}
(1, 2)	0	-0.4665	0.5578	1.0242	\mathcal{H}_0
(1, 3)	-0.3	-0.8244	0.0220	0.8463	\mathcal{H}_0
(2, 1)	-1	-2.0406	-0.4102	1.6304	\mathcal{H}_1
(2, 3)	0	-0.4626	0.5247	0.9873	\mathcal{H}_0
(3, 1)	-1	-2.4219	-0.6720	1.7499	\mathcal{H}_1
(3, 2)	-1	-1.8308	-0.5469	1.2839	\mathcal{H}_1

8 Optimal experiment design for topology detection

In the previous sections, we have derived a topology detection procedure that is based on a decision rule involving uncertainty intervals. We have seen that the decision rule can become unreliable if the length L_{ik} of the intervals containing zero is large. In the sequel, we will assume that a nonzero $Q_{0,ik}$ cannot have an absolute value $|Q_{0,ik}|$ smaller than a known small threshold ε . If we make this additional assumption together with the assumption that $Q_{0,ik} \in \mathcal{I}_{ik}$, it is clear from the analysis in Section 5 that the decision rule (20) will then always lead to the right decision if each interval \mathcal{I}_{ik} containing zero has a length $L_{ik} < \varepsilon$. Since the lengths L_{ik} of the intervals are directly related to the covariance matrix P_θ of the identified parameter vector θ and thus directly related to the amplitudes α_j ($j = 1, \dots, N_{mod}$) of the excitation signals used to collect the data yielding $\hat{\theta}$ (see (15) and (19)), we can achieve this accuracy objective by an appropriate design of these amplitudes. In particular, we wish to determine the amplitudes α_j ($j = 1, \dots, N_{mod}$) corresponding to the smallest identification cost $\mathcal{J} = \sum_{j=1}^{N_{mod}} \alpha_j^2$ that nevertheless guarantees that this accuracy objective will be respected after an identification procedure of fixed duration.

Like in all optimal experiment design problems, the optimal experiment design problem described in the previous paragraph will require some initial information on the to-be-identified network. This information will be here gathered by an initial identification procedure with small α_j ($j = 1, \dots, N_{mod}$). This initial identification procedure has led to intervals \mathcal{I}_{ik}^{init} having relatively large lengths L_{ik} (such as the one in Section 7). Let us denote by \mathcal{X} the set of all pairs (i, k) such that the interval \mathcal{I}_{ik}^{init} contains zero and such that its length is larger than ε . As an example, if the initial identification procedure is the one described in Section 7 and if ε is chosen equal to 0.1, the set \mathcal{X} is given by $\mathcal{X} = \{(1, 2), (1, 3), (2, 3)\}$. The set \mathcal{X} describes the intervals whose length has to be

reduced in a to-be-designed second identification procedure in order to increase the confidence in the decision rule (20). Besides \mathcal{X} , this initial identification procedure has also led to an AR model $\hat{H}(z)$ of $H_0(z)$ and initial estimates of \mathcal{T}_0 and Σ_0 that will also be used to optimally design the second identification procedure i.e., to design the amplitudes α_j ($j = 1, \dots, N_{mod}$) of this second identification procedure that corresponds to the smallest value of \mathcal{J} for which we have still the guarantee that the intervals \mathcal{I}_{ik} (that will be computed after this second identification procedure) are such that $L_{ik} < \varepsilon$ for all $(i, k) \in \mathcal{X}$.

If we denote the LMI (30) by $X(A, \rho_{ik}, c_{ik}, M^{ik}) < 0$ and if we denote $\gamma_j = \alpha_j^2$ ($j = 1, \dots, N_{mod}$), then the optimal amplitudes can be determined using the following optimization problem with as decision variables N_{mod} real scalars $\gamma_j > 0$ and, for each $(i, k) \in \mathcal{X}$, a scalar c_{ik} and a matrix $A_{ik} \in \mathcal{A}_U$:

$$\min \sum_{j=1}^{N_{mod}} \gamma_j \quad (37)$$

$$X(A_{ik}, \frac{\varepsilon^2}{4}, c_{ik}, M^{ik}) < 0 \quad \forall (i, k) \in \mathcal{X} \quad (38)$$

The optimal amplitudes α_j^{opt} are given by $\sqrt{\gamma_j^{opt}}$ ($j = 1, \dots, N_{mod}$) where γ_j^{opt} are the solutions of the above optimization problem. These optimal amplitudes correspond well to the desired ones since, via Proposition 2, (38) implies that, for each $(i, k) \in \mathcal{X}$, the interval \mathcal{I}_{ik} that can be obtained via this new identification experiment has a length L_{ik} smaller than $2\sqrt{\frac{\varepsilon^2}{4}} = \varepsilon$. In the above optimization problem, for each $(i, k) \in \mathcal{X}$, the free variables in the matrix A_{ik} can be chosen differently while the fixed variable P_θ^{-1} is replaced by its linear expression in function of $\gamma_j = \alpha_j^2$ ($j = 1, \dots, N_{mod}$) i.e.

$$P_\theta^{-1} = bdiag \left(P_{\mathcal{T}_1}^{-1}, P_{\mathcal{T}_2}^{-1}, \dots, P_{\mathcal{T}_{N_{mod}}}^{-1} \right) \quad (39)$$

with $P_{\mathcal{T}_j}^{-1} = \gamma_j N_j (H_0(1) \Sigma_0 H_0^T(1))^{-1}$.

There are a number of issues with the optimization problem (37)-(38). First, like in all optimal experiment design problem, (38) depends on a number of unknown variables (the so-called chicken-and-egg issue). Indeed, the matrices M^{ik} defined in (27) depend on the to-be-identified estimate of \mathcal{T}_0 while the expression (39) of P_θ^{-1} depends on $H_0(z)$ and Σ_0 . This issue can, e.g., be resolved by replacing, in (38), these unknown quantities by their estimates obtained during the initial identification procedure. The second issue is related to the particular parametrization of one term in A_{ik} . We indeed observe, in (29), the Kronecker product of the decision variable \bar{W}_{ik} and of P_θ^{-1} , which is a linear function of the decision variables γ_j ($j = 1, \dots, N_{mod}$). The optimization problem (37)-(38) is thus bilinear. To

tackle this bilinearity, we will use the iterative approach given in Algorithm 1 below which is inspired by the so called D-K iterations [26] (see also [6]). Before presenting Algorithm 1, we note that, if we arbitrarily choose the squared amplitudes γ_j ($j = 1, \dots, N_{mod}$), we can use (39) and the solution to the chicken-and-egg issue given above to compute the corresponding P_θ^{-1} . With this P_θ^{-1} and M^{ik} (that can also be evaluated via the solution to the chicken-and-egg issue given above), we can easily determine \mathcal{I}_{ik} for each $(i, k) \in \mathcal{X}$ using the procedure of Section 6 and verify that these intervals \mathcal{I}_{ik} have all a length smaller than ε . If that is the case, we will say that these squared amplitudes γ_j ($j = 1, \dots, N_{mod}$) are validated.

Algorithm 1. *The algorithm is made up of an initialization step (step S.0) and each iteration consists of the two steps S.1 and S.2.*

- S.0. *We initialize the algorithm by arbitrarily choosing positive values for the squared amplitudes γ_j ($j = 1, \dots, N_{mod}$).*
- S.1. *Using a subdivision algorithm, we determine, using the notion of validation defined above, the minimal positive scalar $\kappa \in \mathbf{R}$ such that the squared amplitudes $\kappa\gamma_j$ ($j = 1, \dots, N_{mod}$) remain validated. Denote this minimal κ by κ_{min} . To validate $\kappa_{min}\gamma_j$, we have used, for each $(i, k) \in \mathcal{X}$, the LMI optimization problem given at the end of Section 6.4. The optimal values of \tilde{W}_{ik} ($(i, k) \in \mathcal{X}$) in these LMI optimization problems are conserved for Step S.2.*
- S.2. *The optimal experiment design problem (37)-(38) is transformed into an LMI optimization problem by fixing the decision variables \tilde{W}_{ik} to the ones determined in Step S.1. The optimal values γ_j^{opt} of γ_j ($j = 1, \dots, N_{mod}$) of this transformed optimization problem can then be used in Step S.1 for a new iteration.*

The algorithm is stopped when the optimal cost $\mathcal{J}_{opt} = \sum_{j=1}^{N_{mod}} \gamma_j^{opt}$ in Step S.2 no longer decreases significantly after each iteration. The optimal squared amplitudes are then the ones corresponding to this last iteration (Step S.1 is used a last time to further refine the solution).

Since the squared amplitudes γ_j ($j = 1, \dots, N_{mod}$) delivered by Algorithm 1 are validated, they are guaranteed to lead to a covariance matrix P_θ and an uncertainty set U for which the intervals \mathcal{I}_{ik} have a length smaller than ε for all $(i, k) \in \mathcal{X}$ (modulo the chicken-and-egg issue). However, there is no guarantee that the corresponding cost \mathcal{J}_{opt} is the smallest possible. In order to approach the optimal cost of the original optimization problem (37)-(38), it is important to choose, in Step S.0 of Algorithm 1, initial values of γ_j that are as close as possible to the optimal solution of (37)-(38). In order to do that, we can use the procedure in Appendix A. This procedure consists in a convex formulation of the opti-

mization problem (37)-(38) based on the linearization of the relationship between δ and $Q(\delta)$.

Remark. In order to further reduce the cost of the second identification procedure, the information obtained during the initial identification procedure can be combined to the information obtained during the second identification procedure to derive the identified parameter vector $\hat{\theta}$ and the corresponding covariance matrix P_θ (see [19, page 464]). The above experiment design approach can be easily adapted to this case (see e.g., [4]).

9 Numerical illustration (cont'ed)

Let us illustrate the results of the previous section by supposing that the threshold ε is chosen equal to 0.1 and that the initial identification procedure is the identification procedure described in Section 7 for which $\mathcal{X} = \{(1, 2), (1, 3), (2, 3)\}$. We wish to design a second identification procedure with a duration $N_j = 800$ ($j = 1, 2, 3$) that is such that $L_{ik} \leq 0.1$ for all $(i, k) \in \mathcal{X}$. To initialize Algorithm 1, we use the procedure in Appendix A which leads to $\gamma_1 = 4.0245$, $\gamma_2 = 20.2858$ and $\gamma_3 = 25.1854$. Using this initialization, Algorithm 1 is then used for one iteration⁵ yielding $\gamma_1^{opt} = 4.0939$, $\gamma_2^{opt} = 20.5735$ and $\gamma_3^{opt} = 25.2335$. With respect to its initialization with Appendix A, Algorithm 1 thus only slightly modify the experimental conditions. This could indicate that the approach in Appendix A allows to closely approach the global optimum of the optimization problem (37)-(38). The obtained experimental conditions correspond to $\alpha_1^{opt} = 2.0233$, $\alpha_2^{opt} = 4.5358$ and $\alpha_3^{opt} = 5.0233$. Using these experimental conditions, we collect data and, using the AR model $\hat{H}(z)$ identified in the initial identification procedure, we identify the following estimate $\hat{Q} = \hat{\mathcal{T}}^{-1}$ of Q_0 :

$$\hat{Q} = \begin{pmatrix} 0.9689 & -0.0081 & -0.2866 \\ -0.9708 & 1.0131 & -0.0138 \\ -1.0009 & -1.0069 & 1.0029 \end{pmatrix} \quad (40)$$

To have an idea of the uncertainty of the estimates \hat{Q}_{ik} ($i \neq k$), we follow the same procedure as in Section 7 and we obtain Table 2 where the intervals $\mathcal{I}_{ik} = [Q_{ik}^{min}, Q_{ik}^{max}]$ are given together with their length L_{ik} , the decisions of the decision rule (20) and the true coefficients $Q_{0,ik}$. From Table 2, it is clear that the second identification procedure leads to intervals \mathcal{I}_{ik} with satisfactory lengths i.e., $L_{ik} \leq 0.1$ for all $(i, k) \in \mathcal{X}$. Moreover, unlike in Section 7, the decision rule (20) allows now to correctly conclude that the true unknown

⁵ Further iterations do indeed not reduce the optimal cost \mathcal{J}_{opt} . Note also that, in the step S.1 of the single iteration of Algorithm 1, $\kappa_{min} = 1.006$ which shows that the squared amplitudes obtained via the procedure of Appendix A are nearly validated.

non-diagonal entries $\mathcal{Q}_{0,13}$, $\mathcal{Q}_{0,21}$, $\mathcal{Q}_{0,31}$, $\mathcal{Q}_{0,32}$ are all non-zero (since 0 is not an element of their confidence interval) and that both $\mathcal{Q}_{0,12}$ and $\mathcal{Q}_{0,23}$ are equal to zero (since 0 lies in both \mathcal{I}_{12} and \mathcal{I}_{23}).

Let us check, via Monte Carlo simulations, whether the observations of the previous paragraph contain some generalities. For this purpose, we have repeated 500 times the procedure consisting in first identifying the AR model $\hat{H}(z)$ and then subsequently in identifying \hat{T} with the optimal amplitudes α_j^{opt} ($j = 1, 2, 3$) given above. In each of these Monte-Carlo simulations, different realizations of the noise vector \bar{e} are used in each step. Like in the particular case described above, all these Monte Carlo simulations have also yielded intervals \mathcal{I}_{ik} with which the decision rule (20) can determine the correct topology of the network. The optimal experiment design procedure presented in Section 8 is thus effectively efficient to increase the performance of the decision rule (20).

Table 2
Intervals for the experiment in Section 9

(i, k)	$Q_{0,ik}$	Q_{ik}^{min}	Q_{ik}^{max}	L_{ik}	\mathcal{H}
(1, 2)	0	-0.0568	0.0395	0.0962	\mathcal{H}_0
(1, 3)	-0.3	-0.3265	-0.2502	0.0763	\mathcal{H}_1
(2, 1)	-1	-1.0864	-0.8645	0.2218	\mathcal{H}_1
(2, 3)	0	-0.0601	0.0345	0.0946	\mathcal{H}_0
(3, 1)	-1	-1.1025	-0.9041	0.1983	\mathcal{H}_1
(3, 2)	-1	-1.0609	-0.9550	0.1059	\mathcal{H}_1

10 Conclusion

In this paper, we have presented an optimal identification experiment approach that allows to increase the performance of a topology detection methodology that is based on the identification of the static characteristic of the closed-loop matrix $T_0(z)$. This topology detection methodology is based on Assumption 1. This assumption can however be easily relaxed by deriving uncertainty intervals for the frequency response $T_0(e^{j\omega})$ of $T_0(z)$ at other frequencies than $\omega = 0$. The identification procedure of Section 4 can be easily adapted by considering excitations $\bar{r}(t) = \sin(\omega t) \bar{m}_j$ ($j = 1, \dots, N_{mod}$) and the procedures of Sections 6 and 8 can also be adapted to this setting.

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A Initialization of Algorithm 1

As mentioned at the end of Section 8, it is advantageous to initialize Algorithm 1 using values of γ_j that are as close as possible to the optimal solution of (37)-(38). For this purpose, building on techniques introduced in [18] and later generalized in [7,17], we propose in this section an alternative approach to tackle the experiment design problem. This alternative approach is based on a first-order approximation, but, unlike in Section 8, with this alternative approach, the optimal amplitudes can be determined using convex optimization.

Let us consider the interval \mathcal{I}_{ik}^{orig} defined in Section 6.1 i.e., $\mathcal{I}_{ik}^{orig} = \{ Q_{ik}(\delta) \in \mathbf{R} \mid Q(\delta) = T(\delta)^{-1} \text{ and } \delta \in U \}$ where $Q_{ik}(\delta) = \bar{m}_i^T Q(\delta) \bar{m}_k$ is the (i, k) -entry of $Q(\delta)$. Using a first-order Taylor approximation, $Q_{ik}(\delta)$ can be approximated by $Q_{ik}(\delta) \approx Q_{ik}(0) + \Gamma_{ik}^T \delta$ with, as shown in Appendix B,:

$$\Gamma_{ik} = \left. \frac{dQ_{ik}(\delta)}{d\delta} \right|_{\delta=0} = -(Q(0) \bar{m}_k) \otimes (\bar{m}_i^T Q(0))^T \quad (\text{A.1})$$

where \bar{m}_j ($j = 1, \dots, N_{mod}$) is defined in Section 4 and $Q(0) = Q(\delta = 0)$. Consequently, after an identification experiment yielding a parameter vector $\hat{\theta}$ having a covariance matrix P_θ (recall that $\hat{\theta}$ is the vectorization of the estimate \hat{T} of T_0), the interval \mathcal{I}_{ik}^{orig} can be approximated by the interval \mathcal{I}_{ik}^{fo} having the following analytical expression [3]:

$$\mathcal{I}_{ik}^{fo} = [c_{ik}^{opt,fo} - \sqrt{\rho_{ik}^{opt,fo}}, c_{ik}^{opt,fo} + \sqrt{\rho_{ik}^{opt,fo}}] \quad (\text{A.2})$$

where $c_{ik}^{opt,fo} = Q_{ik}(0) = \hat{Q}_{ik}$ i.e. the (i, k) -entry of $\hat{Q} = \hat{T}^{-1}$ and where $\rho_{ik}^{opt,fo} = \chi \Gamma_{ik}^T P_\theta \Gamma_{ik}$ with $\Gamma_{ik} = -(\hat{Q} \bar{m}_k) \otimes (\bar{m}_i^T \hat{Q})^T$.

Based on this first-order approximation \mathcal{I}_{ik}^{fo} of the interval \mathcal{I}_{ik}^{orig} , the accuracy constraints considered in Section 8 can be reformulated as $2\sqrt{\rho_{ik}^{opt,fo}} < \epsilon$ for all

$(i, k) \in \mathcal{X}$. Using the Schur complements [9] and the expression for $\rho_{ik}^{opt,fo}$ given above, these constraints can be equivalently re-formulated as the following matrix inequalities:

$$\begin{pmatrix} \frac{\epsilon^2}{4\chi} & \Gamma_{ik}^T \\ \Gamma_{ik} & P_\theta^{-1} \end{pmatrix} > 0 \quad \forall (i, k) \in \mathcal{X} \quad (\text{A.3})$$

where P_θ^{-1} is given by (39). Since P_θ^{-1} is linear in the decision variables γ_j ($j = 1, \dots, N_{mod}$), the optimization problem consisting in minimizing $\mathcal{J} = \sum_{j=1}^{N_{mod}} \gamma_j$ subject to (A.3) is an LMI optimization problem (the chicken-and-egg issue can be solved in the same way as in Section 8) and the solutions $\gamma_j^{opt,fo}$ ($j = 1, \dots, N_{mod}$) of this LMI optimization problem can be used to initialize Algorithm 1.

B Derivation of (A.1)

The gradient Γ_{ik} is a column vector of dimension N_{mod}^2 whose j^{th} entry Γ_{ik}^j is given by

$$\Gamma_{ik}^j = \left. \frac{\partial Q_{ik}(\delta)}{\partial \delta_j} \right|_{\delta=0}$$

where δ_j is the j^{th} entry of δ . Since $Q_{ik}(\delta) = \bar{m}_i^T Q(\delta) \bar{m}_k$ and $Q(\delta) = T^{-1}(\delta)$, we have that:

$$\Gamma_{ik}^j = -\bar{m}_i^T Q(0) \left. \frac{\partial T(\delta)}{\partial \delta_j} \right|_{\delta=0} Q(0) \bar{m}_k$$

Recall now that

$$T(\delta) = \begin{pmatrix} \delta_1 & \delta_{N_{mod}+1} & \dots & \delta_{(N_{mod}+1)N_{mod}+1} \\ \delta_2 & \delta_{N_{mod}+2} & \dots & \delta_{(N_{mod}+1)N_{mod}+2} \\ \dots & \dots & \dots & \dots \\ \delta_{N_{mod}} & \delta_{2N_{mod}} & \dots & \delta_{N_{mod}^2} \end{pmatrix}$$

Consequently, if δ_j is for example an element of the first column of $T(\delta)$, we have that:

$$\Gamma_{ik}^j = -\bar{m}_i^T Q(0) \bar{m}_j \bar{m}_1^T Q(0) \bar{m}_k = -Q_{ij}(0) Q_{1k}(0)$$

and consequently the column vector $(\Gamma_{ik}^1, \Gamma_{ik}^2, \dots, \Gamma_{ik}^{N_{mod}})^T$ containing the N_{mod} first elements of Γ_{ik} is given by $-Q_{1k}(0) (\bar{m}_i^T Q(0))^T$. Repeating the above argument for each column of $T(\delta)$ leads to the expression (A.1).