



# Convergence Rate of the Euler-Maruyama Scheme Applied to Diffusion Processes with $L$ $Q - L$ $\rho$ Drift Coefficient and Additive Noise

Benjamin Jourdain, Stéphane Menozzi

## ► To cite this version:

Benjamin Jourdain, Stéphane Menozzi. Convergence Rate of the Euler-Maruyama Scheme Applied to Diffusion Processes with  $L$   $Q - L$   $\rho$  Drift Coefficient and Additive Noise. The Annals of Applied Probability, 2024, 34 (1B), 10.1214/23-AAP2006 . hal-03223426v2

**HAL Id: hal-03223426**

**<https://hal.science/hal-03223426v2>**

Submitted on 3 Aug 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CONVERGENCE RATE OF THE EULER-MARUYAMA SCHEME APPLIED TO DIFFUSION PROCESSES WITH $L^q - L^\rho$ DRIFT COEFFICIENT AND ADDITIVE NOISE

BY BENJAMIN JOURDAIN<sup>1</sup> AND STÉPHANE MENOZZI<sup>2,\*</sup>

<sup>1</sup>*Cermics, Ecole des Ponts, INRIA, Marne-la-Vallée, France, benjamin.jourdain@enpc.fr*

<sup>2</sup>*Laboratoire de Modélisation Mathématique d'Evry (LaMME), Université d'Evry Val d'Essonne, Université Paris Saclay, UMR CNRS 8071, 23 Boulevard de France, 91037 Evry, France, \*stephane.menozzi@univ-evry.fr*

We are interested in the time discretization of stochastic differential equations with additive  $d$ -dimensional Brownian noise and  $L^q - L^\rho$  drift coefficient when the condition  $\frac{d}{\rho} + \frac{2}{q} < 1$ , under which Krylov and Röckner [30] proved existence of a unique strong solution, is met. We show weak convergence with order  $\frac{1}{2}(1 - (\frac{d}{\rho} + \frac{2}{q}))$  which corresponds to half the distance to the threshold for the Euler scheme with randomized time variable and cutoffed drift coefficient so that its contribution on each time-step does not dominate the Brownian contribution. More precisely, we prove that both the diffusion and this Euler scheme admit transition densities and that the difference between these densities is bounded from above by the time-step to this order multiplied by some centered Gaussian density.

**1. Introduction.** In the present paper, we are interested in the Euler-Maruyama discretization of the stochastic differential equation

$$(1.1) \quad X_t = x + W_t + \int_0^t b(s, X_s) ds, \quad t \in [0, T],$$

where  $x \in \mathbb{R}^d$ ,  $(W_t)_{t \geq 0}$  is a  $d$ -dimensional Brownian motion on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ ,  $T \in (0, +\infty)$  is a finite time horizon and the drift coefficient  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is measurable and satisfies the integrability condition :  $\|b\|_{L^q([0, T], L^\rho(\mathbb{R}^d))} =: \|b\|_{L^q-L^\rho} < \infty$  for some  $\rho, q \in (0, +\infty]$  such that

$$(1.2) \quad \frac{d}{\rho} + \frac{2}{q} < 1,$$

which clearly implies that  $\rho > d$  and  $q > 2$ . When  $\rho$  and  $q$  are both finite,

$$\|b\|_{L^q-L^\rho} = \left( \int_0^T \left( \left( \int_{\mathbb{R}^d} |b(t, y)|^\rho dy \right)^{1/\rho} \right)^q dt \right)^{1/q}$$

and when  $\rho = +\infty$  then  $(\int_{\mathbb{R}^d} |b(t, y)|^\rho dy)^{1/\rho}$  is replaced by the essential supremum of  $y \mapsto |b(t, y)|$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  while, when  $q = +\infty$ , the power  $1/q$  of the integral of the  $q$ -th power of the function of the time variable over  $[0, T]$  is replaced by the essential supremum of this function with respect to the Lebesgue measure on  $[0, T]$ . This framework was introduced by Krylov and Röckner [30], who established strong existence and uniqueness for the above equation under the integrability condition (1.2) when moreover  $\rho \geq 2$  (which is implied by (1.2) in dimension  $d \geq 2$ ). Existence of a unique strong solution in dimension  $d = 1$  when  $\rho \in (1, 2)$  and (1.2) holds is covered by Theorem 1.1 (i) [58],

---

*MSC2020 subject classifications:* Primary 60H35, 60H10; secondary 65C30, 65C05.

*Keywords and phrases:* diffusion processes, singular drift, Euler scheme, weak error analysis.

the hypothesis  $q < \infty$  in this theorem leading to no restriction since  $L^\infty([0, T], L^\rho(\mathbb{R})) \subset L^{\frac{4\rho}{\rho-1}}([0, T], L^\rho(\mathbb{R}))$ . The critical case has recently been treated by Krylov [31] and Röckner and Zhao [51], [52] who respectively addressed the strong well-posedness of (1.1) in the time-homogeneous case (then  $q = +\infty$ ) when  $\rho = d$ , and the weak and strong well-posedness when  $\frac{d}{\rho} + \frac{2}{q} = 1$  for a time dependent drift coefficient.

Let us emphasize that dynamics of type (1.1) appear in many applicative fields. In [30], the authors discussed the connection with some models arising from statistical mechanics or interacting particle systems, see [1]. Singular kernels appear as well in several domains related to mathematical physics like fluid dynamics or electro-magnetism. We can for instance mention the Biot-Savart kernel behaving in  $y/|y|^d$  near the origin. A similar singularity also appears in the parabolic elliptic Keller-Segel equation. Let us emphasize that for such singularity, the integrability conditions (1.2) are not met. However, in dimension  $d = 2$ , a kernel behaving around 0 as  $|y|^{\varepsilon+1-d}$ ,  $\varepsilon > 0$  could be considered. In this last setting we can refer e.g. to the work by Jabin and Wang [21] for related applications. We can eventually quote the important work of Zhang and Zhao [61] who established existence of a stochastic Lagrangian path for a Leray solution of the 3d Navier-Stokes equation. In that case,  $d = 3$ ,  $\rho = 2$ ,  $q = \infty$  so condition (1.2) is not met but the drift has some additional properties, namely it is divergence free.

It is therefore important to address the question of the approximation of (1.1). To this end, the easiest, and maybe the most natural at first sight, way consists in introducing the Euler-Maruyama scheme with step  $h > 0$ . Anyhow, in the current singular context it needs to be tailored appropriately. Namely, we consider a *cutoff* with order related to the singularity of the drift. The coefficient with cutoff is defined by

(1.3)

$$\forall (t, y) \in [0, T] \times \mathbb{R}^d, b_h(t, y) = \begin{cases} b(t, y) & \text{if } \rho = q = \infty \\ \mathbb{I}_{\{|b(t, y)| > 0\}} \frac{|b(t, y)| \wedge (Bh^{-\frac{1}{q} + \frac{d}{2\rho}})}{|b(t, y)|} b(t, y) & \text{otherwise} \end{cases}.$$

for some constant  $B \in (0, +\infty)$ . Since, according to (1.2),  $\frac{1}{q} + \frac{d}{2\rho} < \frac{1}{2}$ , the contribution of the cutoffed drift on each time step does not dominate the Brownian contribution.

Furthermore, to get rid of any assumption stronger than mere measurability (and integrability) concerning the regularity of the drift coefficient with respect to the time variable, we choose to randomize the time variable.

The time randomization relies on independent random variables  $(U_k)_{k \in \llbracket 0, n-1 \rrbracket}$ , where from now on we will denote by  $\llbracket \cdot, \cdot \rrbracket$  the integer intervals, which are respectively distributed according to the uniform law on  $[kh, (k+1)h]$  and independent from  $(W_t)_{t \geq 0}$ . Notice that this sequence is of course not needed when the drift coefficient is time-homogeneous. The resulting scheme is initialized by  $X_0^h = x$  and evolves inductively on the regular time-grid  $(t_k = kh)_{k \in \llbracket 0, n \rrbracket}$  with  $h = \frac{T}{n}$  by:

$$(1.4) \quad X_{t_{k+1}}^h = X_{t_k}^h + (W_{t_{k+1}} - W_{t_k}) + b_h(U_k, X_{t_k}^h) h.$$

To illustrate the usefulness of the time randomization technique, let us remark that in dimension  $d = 1$ , for the bounded drift coefficient  $b(t, x) = 1 - \mathbb{I}_{\mathbb{Q}}(t/T)$ ,  $X_T^h = X_T = x + W_T + T$ , while the replacement of  $b_h(U_k, X_{t_k}^h)$  by  $b_h(t_k, X_{t_k}^h)$  in the above induction equation would lead to  $X_T^h = x + W_T$ .

We then consider the following continuous time interpolation of the scheme:

$$(1.5) \quad X_t^h = x + W_t + \int_0^t b_h\left(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h\right) ds, \quad t \in [0, T] \text{ where } \tau_s^h = \lfloor s/h \rfloor h.$$

The cutoff threshold in (1.3) permits to get rid of the drift in the Gaussian estimates that we will derive for the transition densities of the scheme : for all  $c > 0$ ,  $s \in (t_k, t_{k+1}]$  and  $y \in \mathbb{R}^d$ ,

$$\exp\left(c \frac{|b_h(t, y)(s - t_k)|^2}{s - t_k}\right) \leq \exp(cB^2 h^{1 - (\frac{2}{q} + \frac{d}{\rho})}) \xrightarrow[h \rightarrow 0]{(1.2)} 1.$$

For this sole purpose, the natural threshold would have been in  $h^{-1/2}$  rather than  $h^{-(\frac{1}{q} + \frac{d}{2\rho})}$ . The interest of the stronger cutoff is that it also permits to control in the proof of Theorem 1.1 the error on the first time-step when the drift coefficient is computed at the deterministic initial position  $x$ , while it is computed at positions with densities satisfying some Gaussian estimates at the subsequent steps. The error bound of Theorem 1.1 remains valid for the  $h^{-\frac{1}{2}}$  cutoff scale provided we set the drift to zero on the first time step. The alternative scheme writes:

$$(1.6) \quad \bar{X}_{t_{k+1}}^h = \bar{X}_{t_k}^h + (W_{t_{k+1}} - W_{t_k}) + \bar{b}_h(U_k, \bar{X}_{t_k}^h) h,$$

with

$$(1.7) \quad \bar{b}_h(t, y) = \mathbb{I}_{\{t \geq h, |b(t, y)| > 0\}} \frac{|b(t, y)| \wedge (Bh^{-\frac{1}{2}})}{|b(t, y)|} b(t, y), \quad (t, y) \in [0, T] \times \mathbb{R}^d.$$

It can be convenient to choose one scheme or the other. One can wonder in view of (1.7) why there is no cutoff in the definition of the drift coefficient (1.3) of the previous scheme when  $\rho = q = \infty$ . We point out that, in that case,  $Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} = B$ , and choosing the alternative definition in (1.3) with  $B < \|b\|_{L^\infty - L^\infty}$  would lead to a non vanishing error for the difference between the *true* drift and the corresponding truncation. Such a problem does not appear for the truncation in (1.7) which, is such that, when  $\rho = q = \infty$ , for any fixed  $B > 0$ ,  $\bar{b}_h = b$  for  $h$  small enough. Let us stress that, apart from the contributions of the cutoff error and the first time step (see in particular the analysis of the terms  $\Delta_t^2$  and  $\Delta_t^5$  in Section 2.2.2), the choice of the scheme has a minimal impact on the proof of the error estimation since both dynamics (1.4) and (1.6) satisfy the Gaussian estimates of Proposition 2.1. Note that our choice to cutoff the drift rather than to tame it into  $\frac{b(t, x)}{1 + h^\beta |b(t, x)|}$  with  $\beta \in \{\frac{1}{q} + \frac{d}{2\rho}, \frac{1}{2}\}$  avoids some contribution to the error when the norm of the drift is smaller than the cutoff. Taming has been considered in the literature to remedy the possible lack of strong convergence of the standard Euler scheme for stochastic differential equations with non globally Lipschitz coefficients [20, 53, 18]. In particular, when the drift coefficient is locally Lipschitz and one-sided Lipschitz (and the diffusion coefficient Lipschitz), the standard strong order of convergence  $1/2$  is recovered for the tamed Euler scheme with  $\beta = 1$  in the time-homogeneous case [20] and with  $\beta \in (0, 1/2]$  [53].

While the convergence properties of the Euler-Maruyama scheme are well understood for SDEs with smooth coefficients, the case of irregular coefficients is still an active field of research. Concerning the strong error, the additive noise case is investigated in [17] where Halidias and Kloeden only prove convergence and in [8, 45] where rates are derived. Dareiotis and Gerencsér [8] obtain root mean square convergence with order  $1/2 -$  (meaning  $1/2 - \varepsilon$  for arbitrarily small  $\varepsilon > 0$ ) in the time-step for bounded and Dini-continuous time-homogeneous drift coefficients and check that this order is preserved in dimension  $d = 1$  when the Dini-continuity assumption is relaxed to mere measurability. In the scalar  $d = 1$  case, Neuenkirch and Szölgyenyi [45] assume that the drift coefficient is the sum of a  $\mathcal{C}_b^2$  part and a bounded integrable irregular part with a finite Sobolev-Slobodeckij semi-norm of index  $\kappa \in (0, 1)$ . They prove root mean square convergence with order  $\frac{3}{4} \wedge \frac{1+\kappa}{2} -$  for the equidistant Euler-Maruyama scheme, the cutoff of this order at  $\frac{3}{4}$  disappearing for a suitable non-equidistant time-grid. Note that an exact simulation algorithm has been proposed by Étoré

and Martinez [9] for one-dimensional SDEs with additive noise and time-homogeneous and smooth except at one discontinuity point drift coefficient.

More papers have been devoted to the strong error of the Euler scheme for SDEs with a non constant diffusion coefficient : [14, 15, 62, 16, 46, 47, 4]. Recent attention has also been paid to the Euler-Maruyama discretization of SDEs with a piecewise Lipschitz drift coefficient and a globally Lipschitz diffusion coefficient which satisfies some non-degeneracy condition on the discontinuity hypersurface of the drift coefficient : [34, 35, 36, 41, 44].

We will here focus on the so-called *weak error* between the diffusion and the Euler scheme (1.4), namely the quantity

$$\mathcal{E}(x, T, \varphi, h) := \mathbb{E}_x[\varphi(X_T^h)] - \mathbb{E}_x[\varphi(X_T)],$$

for a suitable class of test functions  $\varphi$  which can even be a Dirac mass. In the additive noise case considered in the present paper, Kohatsu-Higa, Lejay and Yasuda [24], prove that for  $\varphi$  thrice continuously differentiable with polynomially growing derivatives, the convergence holds with order  $1/2-$  when  $d \geq 2$  (resp.  $1/3-$  when  $d = 1$ ) and the drift coefficient is time homogeneous, bounded and Lipschitz except on a set  $G$  such that  $\varepsilon^{-d}$  times the Lebesgue measure of  $\{x \in \mathbb{R}^d : \inf_{y \in G} |x - y| \leq \varepsilon\}$  is bounded. Suo, Yuan and Zhang [54] prove convergence in total variation with order  $\frac{\alpha}{2}$  for time-homogeneous drift coefficients with at most linear growth and satisfying an integrated against some Gaussian measure  $\alpha$ -Hölder type regularity condition.

In the much more general multiplicative noise setting, when the diffusion and drift coefficients are smooth, from the seminal work of Talay and Tubaro [55] to the extensions to the hypoelliptic setting, see e.g. the works by Bally and Talay [2], [3], it has been established that the above weak error (with  $b(t_k, X_{t_k}^h)$  replacing  $b_h(U_k, X_{t_k}^h)$  in the right-hand side of (1.4)) has order one w.r.t. the discretization parameter  $h$ . When  $\varphi$  is a Dirac mass we can also refer to [26] or to [28] where non-degenerate bounded Hölder coefficients are considered (see also [10] in the framework of skew diffusions). The common point in all these results is the key role played by the Feynman Kac partial differential equation (PDE) associated with (1.1) which permits to write the error as the expectation of a time integral of the sum of terms with derivatives of the solution to this PDE multiplied by the difference between the drift and squared diffusion coefficients at the current time and position of the Euler scheme and at the last discretization time and corresponding position. This permits to exploit the regularity of these coefficients to derive the order of convergence.

It is however clear that for *rough* coefficients, like in the current  $L^q - L^p$  framework, another strategy is needed. For a bounded measurable drift ( $\rho = q = \infty$ ), a new idea was proposed in [5] consisting in comparing the expansions of the densities at time  $T$  of the diffusion and its Euler scheme with randomized time variable along the solution of the heat equation (in place of the Feynman-Kac PDE) with terminal condition  $\varphi$  equal to  $\delta_y(dz)$ . This solution is  $(s, z) \mapsto g_1(T - s, y - z)$  where  $g_1(t, \cdot)$  denotes the Brownian density at time  $t > 0$ .

We will check in Propositions 2.3 and 2.1 that both the SDE (1.1) and the scheme (1.5) admit transition densities which can be expanded around this Gaussian density as expected from a formal application of Itô's formula. More precisely, for  $s \in [0, T)$  and  $x \in \mathbb{R}^d$ , the solution to

$$(1.8) \quad dX_t = dW_t + b(t, X_t)dt$$

started from  $x$  at time  $s$  admits at time  $t \in (s, T]$  a density with respect to the Lebesgue measure on  $\mathbb{R}^d$  denoted by  $y \mapsto \Gamma(s, x, t, y)$  and, as expected formally by computing  $dg_1(T -$

$s, y - X_s)$  by Itô's formula and taking expectations,

$$\forall y \in \mathbb{R}^d, \Gamma(0, x, T, y) = g_1(T, y - x) - \int_0^T \mathbb{E} [b(s, X_s) \cdot \nabla_y g_1(T - s, y - X_s)] ds.$$

In a similar way, for  $k \in \llbracket 0, n-1 \rrbracket$  and  $x \in \mathbb{R}^d$ , the solution to

$$(1.9) \quad dX_t^h = dW_t + b_h(U_{\lfloor \frac{t}{h} \rfloor}, X_{\tau_t^h}^h) dt$$

(resp. the same dynamics with  $b_h$  replaced by  $\bar{b}_h$ ) started from  $x$  at time  $t_k$  admits at time  $t \in (t_k, T]$  a density with respect to the Lebesgue measure on  $\mathbb{R}^d$  denoted by  $y \mapsto \Gamma^h(t_k, x, t, y)$  (resp.  $y \mapsto \bar{\Gamma}^h(t_k, x, t, y)$ ) and

(1.10)

$$\forall y \in \mathbb{R}^d, \Gamma^h(0, x, T, y) = g_1(T, y - x) - \int_0^T \mathbb{E} \left[ b_h(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h) \cdot \nabla_y g_1(T - s, y - X_s^h) \right] ds$$

(resp. the same equation holds with  $\Gamma^h$  and  $b_h$  replaced by  $\bar{\Gamma}^h$  and  $\bar{b}_h$ ).

Taking the difference of the two expansions, we obtain

$$\begin{aligned} & \Gamma^h(0, x, T, y) - \Gamma(0, x, T, y) \\ &= \int_0^T ds [\Gamma(0, x, s, z) - \Gamma^h(0, x, s, z)] b(s, z) \cdot \nabla_y g_1(T - s, y - z) dz \\ &+ \int_0^T ds \int_{\mathbb{R}^d} \Gamma^h(0, x, s, z) (b(s, z) - b_h(s, z)) \cdot \nabla_y g_1(T - s, y - z) dz \\ &+ \int_0^T ds \int_{\mathbb{R}^d} [\Gamma^h(0, x, s, z) - \Gamma^h(0, x, \tau_s^h, z)] b_h(s, z) \cdot \nabla_y g_1(T - s, y - z) dz \\ &+ \mathbb{E} \left[ \int_0^T ds b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) \cdot (\nabla_y g_1(T - U_{\lfloor s/h \rfloor}, y - X_{\tau_s^h}^h) \right. \\ &\quad \left. - \nabla_y g_1(T - s, y - X_s^h)) \right]. \end{aligned} \quad (1.11)$$

This formula actually emphasizes that, in order to give a convergence rate for the Euler approximation, two preliminary results are needed:

- estimations on the heat kernel  $\Gamma^h$  of the Euler scheme in order to deal with the second (cutoff error) and fourth terms in the right-hand side,
- estimations of its Hölder modulus w.r.t. to the forward time variable to deal with the third term in the right-hand side.

These properties are established in Proposition 2.1 below using an approach inspired from [38]. The first term in the right-hand side will be treated through a Gronwall type argument.

In [5], for  $q = \rho = +\infty$ , starting from a similar decomposition (actually the term involving the modulus of the heat kernel with respect to the forward time variable is written with the transition density of the diffusion instead of that of the Euler scheme), the authors derived a convergence rate of order  $1/2$  w.r.t.  $h$  for the total variation distance between the law of the diffusion and its Euler scheme for a bounded drift.

In our main result, we extend this estimation in a specified way to the case  $b \in L^q - L^\rho$  with  $\frac{d}{\rho} + \frac{2}{q} \in (0, 1)$  by showing that the difference between the densities is bounded from above by  $C h^{\frac{1}{2} \left( 1 - \left( \frac{d}{\rho} + \frac{2}{q} \right) \right)}$  multiplied by some centered Gaussian density. In fact, in the case  $(d, q) = (1, \infty)$ ,  $\rho < \infty$ , for the scheme (1.5), the estimation is perturbed by an extra logarithmic factor.

**THEOREM 1.1** (Convergence Rate for the Euler-Maruyama approximation with  $L^q - L^p$  drift). *Assume that (1.2) holds. Set:*

$$(1.12) \quad \alpha := 1 - \left( \frac{d}{\rho} + \frac{2}{q} \right).$$

*Then, for all  $c > 1$  there exists a constant  $C_c < \infty$  s.t. for all  $h = T/n$  with  $n \in \mathbb{N}^*$ , and all  $t \in (0, T]$ ,  $x, y \in \mathbb{R}^d$*

$$\begin{aligned} |\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)| &\leq C_c h^{\frac{\alpha}{2}} (1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln n) g_c(t, y - x), \\ |\bar{\Gamma}^h(0, x, t, y) - \Gamma(0, x, t, y)| &\leq C_c h^{\frac{\alpha}{2}} g_c(t, y - x). \end{aligned}$$

*where  $g_c(u, \cdot)$  stands for the density of the centered Gaussian vector in dimension  $d$  with covariance matrix  $cuI_d$ ,  $u > 0$ .*

**REMARK 1.2.** *Denoting by  $\mathcal{L}(Y)$  the law of the random vector  $Y$ , we easily deduce convergence rates of  $\mathcal{L}(X_t^h)$  and  $\mathcal{L}(\bar{X}_t^h)$  to  $\mathcal{L}(X_t)$  in total variation distance TV and Wasserstein distance  $\mathcal{W}_1$  with index 1. Indeed,*

$$\begin{aligned} \text{TV}(\mathcal{L}(X_t^h), \mathcal{L}(X_t)) &= \frac{1}{2} \int_{\mathbb{R}^d} |\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)| dy \\ &\leq C_c h^{\frac{\alpha}{2}} (1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln n), \end{aligned}$$

*and, in the same way,  $\text{TV}(\mathcal{L}(\bar{X}_t^h), \mathcal{L}(X_t)) \leq C_c h^{\frac{\alpha}{2}}$ . On the other hand, since  $\mathcal{W}_1(\mathcal{L}(X_t^h), \mathcal{L}(X_t))$  is equal to the supremum of  $|\mathbb{E}[f(X_t^h)] - \mathbb{E}[f(X_t)]|$  over 1-Lipschitz functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and, for such a function,*

$$\begin{aligned} |\mathbb{E}[f(X_t^h)] - \mathbb{E}[f(X_t)]| &= \left| \int_{\mathbb{R}^d} (f(y) - f(0)) (\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)) dy \right| \\ &\leq \int_{\mathbb{R}^d} |y| |\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)| dy, \end{aligned}$$

*we deduce that  $\mathcal{W}_1(\mathcal{L}(X_t^h), \mathcal{L}(X_t)) \leq C_c h^{\frac{\alpha}{2}} t^{1/2} (1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln n)$ . In the same way,  $\mathcal{W}_1(\mathcal{L}(\bar{X}_t^h), \mathcal{L}(X_t)) \leq C_c h^{\frac{\alpha}{2}} t^{1/2}$ .*

**REMARK 1.3** (About the convergence rate). *One can wonder whether the convergence rate obtained in Theorem 1.1 is sharp. It seems rather natural to investigate the weak error through the Duhamel or PDE approach as we do and we actually do not see any alternative. With the Duhamel representation for the density which makes a gradient of the Gaussian density appear, we believe that this is the best achievable rate for a general drift satisfying condition (1.2). Numerical experiments could give some hints and will concern further research.*

*Let us also mention that in specific cases the rate for the weak error can be improved. In [5], the convergence rate in total variation distance improves to  $\frac{h}{\sqrt{t}} \ln(1/h)$  when the bounded drift coefficient has a spatial divergence in the sense of distributions with  $p$ -th power integrable with respect to the Lebesgue measure in space uniformly in time for some  $p \geq d$ . In dimension  $d = 1$ , uniform in time boundedness of the total variation in space of the spatial*

derivative is enough and for the drift  $b(s, x) = a1_{\{x \leq x_0\}} + b1_{\{x > x_0\}}$  (which satisfies this condition and is such that an explicit formula is available for the transition density of the SDE), the convergence rate in  $h$  for fixed  $t > 0$  is confirmed by numerical experiments. For the discretization of a skew one-dimensional diffusion with additive Brownian noise and bounded time-homogeneous drift coefficient, a weak rate of order  $h^{\frac{1}{2}}$  is derived in [10] for an Euler scheme with exact local time contribution.

REMARK 1.4 (About the positive exponent for the time in the error). *We point out that, since we are handling rough drifts, and therefore cannot proceed with the expansions of the heat-kernels beyond orders greater than 2, there is no time singularity appearing in the final bound for the error. We can refer to [26], [13] or more recently [11] for a specific description of the time-singularity for the error expansion when the coefficients are smooth. In that case, the convergence rate is  $h$  and the best upper bound for the time singularity comes from [11] and has order  $t^{-1/2}$ .*

REMARK 1.5 (About the convergence rate for smoother coefficients). *Let us mention that the proof suggests that for drifts Hölder continuous with exponent  $\beta$  with respect to the spatial variable and with exponent  $\frac{\beta}{2}$  with respect to the time variable, the convergence rate of the Euler approximation for an additive noise should be in  $h^{\frac{1}{2} + \frac{\beta}{2}}$  if  $\beta \in (0, 1)$  and not in  $h^{\frac{\beta}{2}}$  as established in e.g. [40], [28] in which a multiplicative noise was anyhow also taken into consideration. We plan to investigate this question in a future work. We can mention the recent work by Dareiotis et al. [7], which investigates the strong error for the Euler scheme with rough drifts, namely quantities of the form  $\mathbb{E}[\sup_{s \in [0, T]} |X_s - X_s^h|^p]^{\frac{1}{p}}$ , and in which are also obtained error bounds of this order (see Theorem 1.5 therein which precisely gives this convergence rate for a bounded drift in the homogeneous Sobolev-Slobodeckij space  $W_m^\alpha$ ,  $m \geq \max(d, 2)$ ,  $\alpha \in (0, 1)$  and an additive Brownian noise).*

REMARK 1.6 (About possible multiplicative Brownian noises). *A natural question concerns the possible extension of our main result to dynamics involving a multiplicative noise, i.e.*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \text{ where } \sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}.$$

*Non degeneracy is needed according to Theorem 1.3 [19] which exhibits an example with constant degenerate diffusion coefficient  $\sigma$  and bounded, time-homogeneous and infinitely differentiable drift coefficient  $b$ , such that the Euler scheme converges weakly at a rate slower than any power law. We expect that provided  $\sigma$  is s.t.  $\sigma\sigma^*$  is bounded and uniformly elliptic and  $\sigma$  is Lipschitz continuous in space uniformly in time, then the convergence rate of Theorem 1.1 should remain valid. Note that existence of a unique strong solution to the stochastic differential equation in the indicated setting is ensured by Theorem 1.1 in [58], see also Lemma 2.3 in [50] for related Gaussian estimates. Roughly speaking the Duhamel expression for the transition density of the diffusion would write in this setting*

$$\begin{aligned} \Gamma(0, x, t, y) = & g_{\sigma, y}(0, t, y - x) - \int_0^t \mathbb{E} [b(s, X_s) \cdot \nabla_z g_{\sigma, y}(s, t, z)|_{z=y-X_s}] ds \\ & + \frac{1}{2} \int_0^t \mathbb{E} \left[ \text{Tr} \left( \left( \sigma\sigma^*(s, X_s) - \sigma\sigma^*(s, y) \right) D_z^2 g_{\sigma, y}(s, t, z)|_{z=y-X_s} \right) \right] ds, \end{aligned}$$

*where, for  $0 \leq s < t \leq T$ ,  $y, z \in \mathbb{R}^d$ ,  $g_{\sigma, y}(s, t, z)$  stands for the density at point  $z$  of a centred  $d$ -dimensional Gaussian vector with covariance matrix  $\int_s^t \sigma\sigma^*(u, y)du$ , see e.g. [26] in the*



*smooth setting. If the diffusion coefficient is Lipschitz continuous in space uniformly in time and such that  $\sigma\sigma^*$  is bounded and uniformly elliptic, then the last term can easily be bounded from above by*

$$C \int_0^t \frac{\mathbb{E}[g_c(t-s, y - X_s)]}{\sqrt{t-s}} ds.$$

*In other words, it gives the same singular behavior for the analysis as a bounded drift. A similar behavior will as well occur for the scheme. In terms of heat kernel estimates, this should be enough to directly adapt the techniques described below. The same should remain valid for the error as well.*

**REMARK 1.7** (About other driving noises). *Another natural extension would concern the class of noises considered. One might wonder e.g. if we could consider dynamics of the form*

$$(1.13) \quad dX_t = b(t, X_t)dt + dZ_t,$$

*with  $Z$  being a rotationally invariant stable process of index  $\gamma \in (1, 2)$ . We think that the techniques used to prove Theorem 2.3 below, adapting somehow the strategy of [38], should be sufficiently robust to obtain an error estimation with order  $h^{\frac{\alpha}{\gamma}}$  for  $\alpha = 1 - \frac{d}{(\gamma-1)\rho} - \frac{\gamma}{(\gamma-1)q}$  when  $b \in L^q - L^\rho$  with  $\frac{d}{(\gamma-1)\rho} + \frac{\gamma}{(\gamma-1)q} < 1$ . Let us mention that the well-posedness in a strong sense and in a weak sense of (1.13) with singular drift coefficient has respectively been addressed in [57] and [6] and that heat-kernel bounds for multiplicative stable noise and unbounded drift have been obtained in [39] even in the super-critical case  $\gamma \in (0, 1)$  through the approach of [38]. The indicated thresholds can be derived following the procedure below adapting Lemma 2.4 to the pure jump stable case.*

The article is organized as follows. We will first prove our main result in Section 2. To this end, we also state therein two propositions giving Gaussian heat kernel estimates and Duhamel representations for the transition densities of the Euler scheme and the SDE (1.1). Section 3 is dedicated to the proof of the estimates for the approximation scheme. In section 4, we deduce the estimates for the diffusion by letting the time-step  $h \rightarrow 0$ . Technical results are gathered in the Appendix.

We denote from now on by  $C$  a generic constant that may change from line to line and might depend on  $b, q, \rho, d, T$ . Other possible dependencies will be explicitly specified. We reserve the notation  $c > 1$  for the concentration constant, or *variance*, in the Gaussian kernels  $g_c$ . For a multi-index  $\zeta \in \mathbb{N}^d$ ,  $x \in \mathbb{R}^d$ , we denote  $\nabla_x^\zeta := \partial_{x_1}^{\zeta_1} \cdots \partial_{x_d}^{\zeta_d}$ . If  $|\zeta| := \sum_{i=1}^d \zeta_i = 0$ ,  $\nabla_x^\zeta$  then simply means that there is no differentiation. Also, for  $a, b > 0$ ,  $B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} du$  stands for the  $\beta$ -function.

Eventually, we will consider from now on that the condition (1.2) is met.

**2. Proof of the convergence rate for the error.** We prove in this section our main result, Theorem 1.1. To this end, we first give two auxiliary results about density/heat kernel estimates for both the scheme and the diffusion.

**2.1. Key results for the proof of Theorem 1.1.** Using an approach inspired from [38], we obtain the following estimations for the scheme.

**PROPOSITION 2.1** (Density estimates for the Euler scheme). *Assume (1.2). Set  $h = \frac{T}{n}$ ,  $n \in \mathbb{N}^*$ . Then the Euler scheme  $X^h$  with dynamics (1.9) (resp.  $\bar{X}^h$  with dynamics*

(1.6)) admits for all  $0 \leq t_k := kh < t \leq T$ ,  $k \in \llbracket 0, n \rrbracket$ ,  $(x, y) \in (\mathbb{R}^d)^2$  a transition density  $\Gamma^h(t_k, x, t, y)$  (resp.  $\bar{\Gamma}^h(t_k, x, t, y)$ ) which enjoys the following Duhamel representation :

$$(2.1) \quad \Gamma^h(t_k, x, t, y) = g_1(t - t_k, y - x) - \int_{t_k}^t \mathbb{E} \left[ b_h(U_{\lfloor \frac{r}{h} \rfloor}, X_{\tau_r}^h) \cdot \nabla_y g_1(t - r, y - X_r^h) \right] dr,$$

(2.2)

$$\text{resp. } \bar{\Gamma}^h(t_k, x, t, y) = g_1(t - t_k, y - x) - \int_{t_k}^t \mathbb{E} \left[ \bar{b}_h(U_{\lfloor \frac{r}{h} \rfloor}, \bar{X}_{\tau_r}^h) \cdot \nabla_y g_1(t - r, y - \bar{X}_r^h) \right] dr.$$

Furthermore, for each  $c > 1$ , there exists a finite constant  $C$  not depending on  $h = \frac{T}{n}$  such that for all  $k \in \llbracket 0, n - 1 \rrbracket$ ,  $t \in (t_k, T]$ ,  $x, y, y' \in \mathbb{R}^d$ ,

$$(2.3) \quad \Gamma^h(t_k, x, t, y) \leq C g_c(t - t_k, y - x)$$

and if  $\alpha$  defined in (1.12) satisfies  $\alpha < 1$ ,

$$(2.4) \quad |\Gamma^h(t_k, x, t, y') - \Gamma^h(t_k, x, t, y)| \leq C \frac{|y - y'|^\alpha \wedge (t - t_k)^{\frac{\alpha}{2}}}{(t - t_k)^{\frac{\alpha}{2}}} (g_c(t - t_k, y - x) + g_c(t - t_k, y' - x)).$$

Also, for all  $0 \leq k < \ell < n$ ,  $t \in [t_\ell, t_{\ell+1}]$ ,  $x, y \in \mathbb{R}^d$ ,

$$(2.5) \quad |\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t_\ell, y)| \leq C \frac{(t - t_\ell)^{\frac{\alpha}{2}}}{(t_\ell - t_k)^{\frac{\alpha}{2}}} g_c(t - t_k, y - x),$$

and the same estimations hold with  $\bar{\Gamma}^h$  replacing  $\Gamma^h$ .

REMARK 2.2. Suppose that  $b \in L^\infty - L^\infty$  which corresponds to  $\alpha = 1$ . Then  $b$  also belongs to  $L^{\tilde{q}} - L^\infty$  for each  $\tilde{q} \in (2, +\infty)$  and the cutoff in (1.3) for  $(q, \rho) = (\tilde{q}, \infty)$  does not play any role if  $B \geq \|b\|_{L^\infty - L^\infty} T^{1/\tilde{q}}$ . As a consequence, (2.4) holds with  $\alpha$  replaced by  $\tilde{\alpha} \in (0, 1)$  in the right-hand side with a multiplicative constant  $C$  possibly depending on  $\tilde{\alpha}$ . The same is of course true for  $\bar{\Gamma}^h$ .

In the limit  $h \rightarrow 0$ , we will deduce the following proposition.

PROPOSITION 2.3. Assume (1.2). Let  $(X_t)_{t \in [0, T]}$  denote the solution to the SDE (1.8). Then for each  $t \in (0, T]$ ,  $X_t$  admits a density  $y \rightarrow \Gamma(0, x, t, y)$  with respect to the Lebesgue measure such that for each  $c > 1$ , there exists  $C < +\infty$  such that for all  $t \in (0, T]$ ,  $x, y, y' \in \mathbb{R}^d$ ,

$$(2.6) \quad \Gamma(0, x, t, y) \leq C g_c(t, y - x)$$

and if  $\alpha$  defined in (1.12) satisfies  $\alpha < 1$ ,

$$(2.7) \quad |\Gamma(0, x, t, y) - \Gamma(0, x, t, y')| \leq C \frac{|y - y'|^\alpha \wedge t^{\frac{\alpha}{2}}}{t^{\frac{\alpha}{2}}} (g_c(t, y - x) + g_c(t, y' - x)).$$

This density enjoys the following Duhamel representation : for all  $t \in (0, T]$ ,  $(x, y) \in (\mathbb{R}^d)^2$ :

$$(2.8) \quad \Gamma(0, x, t, y) = g_1(t, y - x) - \int_0^t \mathbb{E} [b(r, X_r) \cdot \nabla_y g_1(t - r, y - X_r)] dr.$$

Let us indicate the work [50] where estimates are derived for the gradient w.r.t. the initial variable  $x$  of the transition density of an SDE with bounded non-degenerate and Hölder continuous diffusion tensor and time inhomogeneous drift coefficient satisfying an isotropic Krylov and Röckner type condition, i.e.  $q = \rho$  with  $\frac{d+2}{\rho} < 1$  (see Lemma 2.3 therein). When the drift is an Itô process enjoying suitable integrability properties, the representation (2.8) is stated in [37] and two-sided Gaussian estimates are derived. But in this work, no time-singularities are allowed for the drift and even in the time homogeneous Krylov and Röckner setting, it seems delicate to recover our estimates from the approach therein.

In a more singular setting, let us mention [23] who derived two-sided Gaussian bounds and a gradient bound w.r.t. the starting variable, for the transition density of Brownian motions drifted by functions in the Kato class  $\mathbf{K}_{d-1}$ . We recall that an  $\mathbb{R}^d$ -valued function  $f$  belongs to the Kato class  $\mathbf{K}_{d-1}$  if  $\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|f(y)|}{|x-y|^{d-1}} dy \xrightarrow{r \rightarrow 0} 0$ , which allows e.g. to handle singularities of the form  $x \mapsto |x|^{-\beta}$ ,  $\beta < 1$  (possibly shifted in space). It can be noted, as indicated in the introduction, that for time homogeneous drifts, the Krylov and Röckner criterion also leads to the same kind of constraint on the exponent of the singularity.

Also, similar Gaussian estimates and gradient bounds were obtained for even rougher (homogeneous) drifts in Besov spaces with negative regularity index by Perkowski and Van Zuijlen in [49] using Littlewood-Paley decompositions for the drift. For time-homogeneous drifts, heat kernel estimates of the same type were obtained by Zhang and Zhao [60], see Theorem 5.1 therein, under the condition  $\|(I - \Delta)^{-\frac{\alpha}{2}} b\|_\rho < +\infty$ ,  $\alpha \in (0, \frac{1}{2})$ ,  $\rho \in (\frac{d}{1-\alpha}, +\infty)$  through change of probability techniques. However, it seems that the  $L^q - L^\rho$  case had not been considered so far. Hence, to the best of our knowledge, the above heat-kernel estimates are new and can be of interest independently of the approximation procedure.

**2.2. Proof of the main Theorem for the discretization error.** From the results of the previous subsection we are almost in position to prove our main result: the error bound of Theorem 1.1 for the densities. We state in the next paragraph the additional technical lemmas also needed.

**2.2.1. Preliminary results.** We state here three technical lemmas that turn out to be useful for the error analysis. The first one is related to integrability properties of Gaussian kernels integrating a function in  $L^q - L^\rho$ . Such kind of integrals appear from the decomposition of the error (1.11). The second one gives standard quantitative bounds for Gaussian kernels. The last one is a Gronwall-Volterra lemma. Before stating our *Gaussian lemmas*, we recall that for  $c, u > 0$ ,  $g_c(u, \cdot)$  denotes the centered Gaussian density with covariance matrix  $cuI_d$ .

**LEMMA 2.4** (Singularities induced by an  $L^{q'} - L^{\rho'}$  drift in a Gaussian convolution). *Let  $\rho', q' \in [1, +\infty]$ ,  $0 \leq s < t \leq T$ ,  $\varphi : [0, t] \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $f : [s, t] \rightarrow \mathbb{R}_+$  be measurable,  $\beta, \gamma \geq 0$  and  $x, y \in \mathbb{R}^d$ . There exists a finite constant  $C_{\rho'}$  such that*

$$(2.9) \quad \left| \int_{\mathbb{R}^d} g_c(u-s, z-x) |\varphi(u, z)| g_c(t-u, y-z) dz \right| \leq C_{\rho'} \frac{\|\varphi(u, \cdot)\|_{L^{\rho'}(t-s)^{\frac{d}{2\rho'}}}}{(u-s)^{\frac{d}{2\rho'}} (t-u)^{\frac{d}{2\rho'}}} g_c(t-s, x-y),$$

and a finite constant  $C := C(\rho', q', d)$  such that

$$I_{\beta, \gamma, f, \varphi}(s, t) := \int_s^t \frac{du f(u)}{(u-s)^\beta (t-u)^\gamma} \int_{\mathbb{R}^d} g_c(u-s, z-x) |\varphi(u, z)| g_c(t-u, y-z) dz$$

$$\leq C(t-s)^{\frac{d}{2\rho'}} g_c(t-s, y-x) \left( \int_s^t du \left( \frac{f(u)}{(u-s)^{\beta+\frac{d}{2\rho'}} (t-u)^{\gamma+\frac{d}{2\rho'}}} \right)^{q''} \right)^{\frac{1}{q''}} \|\varphi\|_{L^{q'}-L^{\rho'}},$$

where  $\frac{1}{q'} + \frac{1}{q''} = 1$ . When  $f$  is bounded, or in particular when  $f$  is constant, the time singularities are integrable provided that  $(\beta + \frac{d}{2\rho'}) \vee (\gamma + \frac{d}{2\rho'}) < \frac{1}{q''} = 1 - \frac{1}{q'}$ . In that case,

$$(2.10) \quad I_{\beta,\gamma,f,\varphi}(s,t) \leq C \|f\|_{L^\infty} \|\varphi\|_{L^{q'}-L^{\rho'}} g_c(t-s, y-x) (t-s)^{1-\frac{1}{q'}-(\beta+\gamma+\frac{d}{2\rho'})} \times \left( B \left( 1 - q''(\beta + \frac{d}{2\rho'}), 1 - q''(\gamma + \frac{d}{2\rho'}) \right) \right)^{\frac{1}{q''}}.$$

PROOF. Set  $G_{s,t}^{x,y}(u) := \int_{\mathbb{R}^d} g_c(u-s, z-x) |\varphi(u, z)| g_c(t-u, y-z) dz$ . When  $\rho' > 1$ , from the Hölder inequality, we get that there exists a finite constant  $C_{\rho'}$  s.t.

$$\begin{aligned} |G_{s,t}^{x,y}(u)| &\leq C_{\rho'} \|\varphi(u, \cdot)\|_{L^{\rho'}} \left( \int_{\mathbb{R}^d} \frac{g_{\frac{c}{\rho''}}(u-s, x-z) g_{\frac{c}{\rho''}}(t-u, y-z)}{(u-s)^{(\rho''-1)\frac{d}{2}} (t-u)^{(\rho''-1)\frac{d}{2}}} dz \right)^{\frac{1}{\rho''}}, \quad \frac{1}{\rho'} + \frac{1}{\rho''} = 1 \\ &\leq C_{\rho'} \frac{\|\varphi(u, \cdot)\|_{L^{\rho'}}}{(u-s)^{\frac{d}{2\rho'}} (t-u)^{\frac{d}{2\rho'}}} \times \frac{1}{(t-s)^{\frac{d}{2\rho''}}} \exp\left(-\frac{|x-y|^2}{2c(t-s)}\right) \\ &\leq C_{\rho'} \frac{\|\varphi(u, \cdot)\|_{L^{\rho'}} (t-s)^{\frac{d}{2\rho'}}}{(u-s)^{\frac{d}{2\rho'}} (t-u)^{\frac{d}{2\rho'}}} g_c(t-s, x-y), \end{aligned}$$

up to a modification of  $C_{\rho'}$  from line to line. Since

$$\frac{|x-z|^2}{u-s} + \frac{|z-y|^2}{t-u} = \frac{|x-y|^2}{t-s} + \frac{1}{t-s} \left| \sqrt{\frac{t-u}{u-s}}(x-z) - \sqrt{\frac{u-s}{t-u}}(z-y) \right|^2 \geq \frac{|x-y|^2}{t-s},$$

(2.9) still holds when  $\rho' = 1$ .

From the definition of  $I_{\beta,\gamma,f,\varphi}(s,t)$  we derive the second statement from (2.9) and the Hölder inequality (in time). Namely,

$$\begin{aligned} I_{\beta,\gamma,f,\varphi}(s,t) &\leq C_{\rho'} (t-s)^{\frac{d}{2\rho'}} g_c(t-s, x-y) \int_s^t du f(u) \frac{\|\varphi(u, \cdot)\|_{L^{\rho'}}}{(u-s)^{\beta+\frac{d}{2\rho'}} (t-u)^{\gamma+\frac{d}{2\rho'}}} \\ &\leq C(t-s)^{\frac{d}{2\rho'}} g_c(t-s, y-x) \left( \int_s^t du \left( \frac{f(u)}{(u-s)^{\beta+\frac{d}{2\rho'}} (t-u)^{\gamma+\frac{d}{2\rho'}}} \right)^{q''} \right)^{\frac{1}{q''}} \|\varphi\|_{L^{q'}-L^{\rho'}}. \end{aligned}$$

The integrability conditions and the explicit control of (2.10) then readily follow when  $f$  is bounded. The proof is complete.  $\square$

For the computations to be performed, we will also often need quantitative bounds for sensitivities of Gaussian kernels. We state the following result the proof of which is *standard* and postponed to Appendix A for the sake of completeness.

LEMMA 2.5 (Gaussian Sensitivities). *For each  $c > 1$  there exists  $C < +\infty$  s.t. for each multi-index  $\zeta$  with length  $|\zeta| \leq 2$ , and for all  $0 < u \leq u' \leq T$ ,  $x, x' \in \mathbb{R}^d$ :*

$$(2.11) \quad |\nabla_x^\zeta g_1(u, x)| \leq \frac{C}{u^{\frac{|\zeta|}{2}}} g_c(u, x) \text{ and } |\partial_u \nabla_x^\zeta g_1(u, x)| \leq \frac{C}{u^{1+\frac{|\zeta|}{2}}} g_c(u, x),$$

$$(2.12) \quad \left| \nabla_x^\zeta g_1(u, x) - \nabla_x^\zeta g_1(u, x') \right| \leq C \frac{|x - x'| \wedge u^{\frac{1}{2}}}{u^{\frac{1+|\zeta|}{2}}} (g_c(u, x) + g_c(u, x')),$$

$$(2.13) \quad \left| \nabla_x^\zeta g_1(u', x) - \nabla_x^\zeta g_1(u, x) \right| \leq C \frac{|u' - u| \wedge u}{u^{1+\frac{|\zeta|}{2}}} (g_c(u, x) + g_c(u', x)).$$

The next lemma, the proof of which is postponed to Appendix B roughly says that the usual Gronwall inequality extends to integral inequalities involving integrable singularities.

**LEMMA 2.6 (Gronwall-Volterra Lemma).** (i) *Let  $\beta_1 < 1$ ,  $\beta > \beta_1 - 1$  and  $\eta, \delta, T > 0$ . There exists some finite constant  $C_{\beta, \beta_1, \delta, T}$  not depending on  $\eta$  such that  $\sup_{t \in [0, T]} f(t) \leq \eta C_{\beta, \beta_1, \delta, T}$  for each measurable and bounded function  $f : [0, T] \rightarrow \mathbb{R}_+$  satisfying*

$$(2.14) \quad \forall t \in [0, T], f(t) \leq \eta + \delta t^\beta \int_0^t \frac{f(s) ds}{s^{\beta_1}}.$$

(ii) *Let  $\beta_1, \beta_2 < 1$ ,  $\beta_3 > \beta_1 + \beta_2 - 1$  and  $a, b, T > 0$ . There exists some finite constant  $C_{\beta_1, \beta_2, \beta_3, b, T}$  not depending on  $a$  such that  $\sup_{t \in [0, T]} f(t) \leq a C_{\beta_1, \beta_2, \beta_3, b, T}$  for each measurable and bounded function  $f : [0, T] \rightarrow \mathbb{R}_+$  satisfying*

$$(2.15) \quad \forall t \in [0, T], f(t) \leq a + b t^{\beta_3} \int_0^t \frac{f(s) ds}{s^{\beta_1} (t-s)^{\beta_2}}.$$

**REMARK 2.7.** Under the assumptions of (i),  $\forall t \in [0, T]$ ,  $f(t) \leq \eta + \frac{\delta t^{\beta+1-\beta_1}}{1-\beta_1} \sup_{s \in [0, t]} f(s)$  so that  $\sup_{s \in [0, t]} f(s) \leq \eta + \frac{\delta t^{\beta+1-\beta_1}}{1-\beta_1} \sup_{s \in [0, t]} f(s)$  and when  $t < \left( \frac{1-\beta_1}{\delta} \right)^{\frac{1}{\beta+1-\beta_1}}$ ,

$$\sup_{s \in [0, t]} f(s) \leq \frac{\eta(1-\beta_1)}{1-\beta_1 - \delta t^{\beta+1-\beta_1}}.$$

Similarly, under the assumptions of (ii), for  $t \in [0, T]$  such that  $t < (bB(1-\beta_1, 1-\beta_2))^{-\frac{1}{\beta_3+1-\beta_1-\beta_2}}$ ,  $\sup_{s \in [0, t]} f(s) \leq \frac{a}{1-bB(1-\beta_1, 1-\beta_2)t^{\beta_3+1-\beta_1-\beta_2}}$ .

**2.2.2. Final derivation of the error bounds.** By (2.1) and (2.8), the discretization error writes

$$\begin{aligned} & \Gamma^h(0, x, t, y) - \Gamma(0, x, t, y) \\ &= \mathbb{E} \left[ \int_0^t \left( b(s, X_s) \cdot \nabla_y g_1(t-s, y - X_s) - b_h(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h) \cdot \nabla_y g_1(t-s, y - X_s^h) \right) ds \right]. \end{aligned}$$

For  $s \in (t_1, T] \setminus \{t_k : k \in \llbracket 2, n-1 \rrbracket\}$ ,  $\varphi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$  measurable and bounded, we have using  $X_s^h = X_{\tau_s^h}^h + W_s - W_{\tau_s^h} + b_h(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h)(s - \tau_s^h)$  and the independence of  $X_{\tau_s^h}^h$ ,  $W_s - W_{\tau_s^h}$  and  $U_{\lfloor \frac{s}{h} \rfloor}$ ,

$$\begin{aligned} & \mathbb{E} \left[ \varphi(X_{\tau_s^h}^h, X_s^h, U_{\lfloor \frac{s}{h} \rfloor}) \right] \\ &= \frac{1}{h} \int_{\tau_s^h}^{\tau_s^h+h} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(w, z, r) \Gamma^h(0, x, \tau_s^h, w) g_1 \left( s - \tau_s^h, z - w - b_h(r, w)(s - \tau_s^h) \right) dz dw dr. \end{aligned}$$

We deduce that the error decomposes as

(2.16)

$\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y) = \Delta_t^1 + \Delta_t^2 + \Delta_t^3 + \Delta_t^4 + \Delta_t^5 + \Delta_t^6$  where

$$\begin{aligned}\Delta_t^1 &= \int_0^t ds \int_{\mathbb{R}^d} [\Gamma(0, x, s, z) - \Gamma^h(0, x, s, z)] b(s, z) \cdot \nabla_y g_1(t-s, y-z) dz, \\ \Delta_t^2 &= \mathbb{I}_{\{t \geq 3h\}} \int_{t_1}^{\tau_t^h - h} ds \int_{\mathbb{R}^d} \Gamma^h(0, x, s, z) (b(s, z) - b_h(s, z)) \cdot \nabla_y g_1(t-s, y-z) dz, \\ \Delta_t^3 &= \mathbb{I}_{\{t \geq 3h\}} \int_{t_1}^{\tau_t^h - h} ds \int_{\mathbb{R}^d} [\Gamma^h(0, x, s, z) - \Gamma^h(0, x, \tau_s^h, z)] b_h(s, z) \cdot \nabla_y g_1(t-s, y-z) dz, \\ \Delta_t^4 &= \mathbb{I}_{\{t \geq 3h\}} \int_{t_1}^{\tau_t^h - h} ds \mathbb{E} \left[ b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) \cdot (\nabla_y g_1(t - U_{\lfloor s/h \rfloor}, y - X_{\tau_s^h}^h) \right. \\ &\quad \left. - \nabla_y g_1(t-s, y - X_s^h)) \right], \\ \Delta_t^5 &= \frac{1}{h} \int_{s=0}^{t_1 \wedge t} \int_{r=0}^h \int_{\mathbb{R}^d} g_1(s, z-x-b_h(r, x)s) (b(s, z) - b_h(r, x)) \cdot \nabla_y g_1(t-s, y-z) dz dr ds, \\ \Delta_t^6 &= \mathbb{I}_{\{t \geq h\}} \frac{1}{h} \int_{s=(\tau_t^h - h) \vee t_1}^t \int_{r=\tau_s^h}^{\tau_s^h + h} \int_{\mathbb{R}^{2d}} \Gamma^h(0, x, \tau_s^h, w) g_1(s - \tau_s^h, z - w - b_h(r, w)(s - \tau_s^h)) \\ &\quad \times (b(s, z) - b_h(r, w)) \cdot \nabla_y g_1(t-s, y-z) dz dw dr ds.\end{aligned}$$

In the same way,

$$\bar{\Gamma}^h(0, x, t, y) - \Gamma(0, x, t, y) = \bar{\Delta}_t^1 + \bar{\Delta}_t^2 + \bar{\Delta}_t^3 + \bar{\Delta}_t^4 + \bar{\Delta}_t^5 + \bar{\Delta}_t^6,$$

where, for  $i \in \{1, \dots, 6\}$ ,  $\bar{\Delta}_t^i$  is defined like  $\Delta_t^i$  with  $\Gamma^h$ ,  $b^h$  and  $X^h$  respectively replaced by  $\bar{\Gamma}^h$ ,  $\bar{b}^h$  and  $\bar{X}^h$ .

Let us first deal with the cutoff error terms  $\Delta_t^2$  and  $\bar{\Delta}_t^2$  when  $t \geq 3h$  (the contributions  $\Delta_t^1$  and  $\bar{\Delta}_t^1$  will be handled at the end of the proof by a Gronwall type argument). When  $\rho = q = \infty$ , we recall, see (1.3), that  $b_h = b$  and  $\Delta_t^2 = 0$ . Let us then suppose that  $\frac{2}{q} + \frac{d}{\rho} > 0$ , choose  $\tilde{\alpha} \in [\alpha, \alpha + \frac{1}{2})$  and set  $\lambda = 1 + \frac{\tilde{\alpha}}{\frac{2}{q} + \frac{d}{\rho}}$ . We have

$$|b - b_h| = \left( |b| - Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} \right)^+ \leq |b| \mathbb{I}_{\{|b| \geq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}\}} \leq \frac{h^{\frac{\tilde{\alpha}}{2}} |b|^\lambda}{B^{\lambda-1}}.$$

Combining this inequality with  $\tau_t^h - h \leq t - h$ , (2.11) and (2.3) then applying (2.9) with  $\rho' = \frac{\rho}{\lambda}$  (assuming that  $\tilde{\alpha}$  is chosen so that  $\rho' \geq 1$ ) and last Hölder's inequality in time with the integrability of  $s \mapsto s^{-\frac{d\lambda}{2\rho} \times \frac{q}{q-\lambda}}$  on  $[0, \frac{t}{2}]$  deduced from  $\tilde{\alpha} < \alpha + \frac{1}{2}$ , we obtain

$$\begin{aligned}& |\Delta_t^2| \\ & \leq Ch^{\frac{\tilde{\alpha}}{2}} \int_0^{t-h} ds \int_{\mathbb{R}^d} g_c(s, z-x) |b(s, z)|^\lambda \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz ds \\ & \leq Ch^{\frac{\tilde{\alpha}}{2}} g_c(t, y-x) t^{\frac{d\lambda}{2\rho}} \int_0^{t-h} \frac{\|b(s, \cdot)\|_{L^\rho}^\lambda ds}{s^{\frac{d\lambda}{2\rho}} (t-s)^{\frac{1}{2} + \frac{d\lambda}{2\rho}}}\end{aligned}$$

$$\begin{aligned}
&\leq Ch^{\frac{\alpha}{2}} g_c(t, y-x) \left( t^{-\frac{1}{2}} \int_0^{\frac{t}{2}} \frac{\|b(s, \cdot)\|_{L^\rho}^\lambda ds}{s^{\frac{d\lambda}{2\rho}}} + \int_{\frac{t}{2}}^{t-h} \frac{\|b(s, \cdot)\|_{L^\rho}^\lambda ds}{(t-s)^{\frac{1}{2} + \frac{d\lambda}{2\rho}}} \right) \\
&\leq Ch^{\frac{\alpha}{2}} g_c(t, y-x) \|b\|_{L^q-L^\rho}^\lambda \left( t^{-\frac{1}{2}} \times t^{\frac{1+\alpha-\tilde{\alpha}}{2}} + \mathbb{I}_{\{\tilde{\alpha} > \alpha\}} h^{\frac{\alpha-\tilde{\alpha}}{2}} + \mathbb{I}_{\{\tilde{\alpha} = \alpha\}} \left( \ln \left( \frac{t}{2h} \right) \right)^{1-\frac{\lambda}{q}} \right).
\end{aligned}$$

For the application of (2.9) (resp. the Hölder inequality in time), we needed that  $\frac{\rho}{\lambda} \geq 1$  (resp.  $\frac{q}{\lambda} > 1$ ) an inequality of course satisfied when  $\rho = \infty$  (resp.  $q = \infty$ ) and otherwise equivalent to  $\frac{\rho(\frac{2}{q} + \frac{d}{\rho})}{\frac{2}{q} + \frac{d}{\rho} + \tilde{\alpha}} \geq 1$  (resp.  $\frac{q(\frac{2}{q} + \frac{d}{\rho})}{\frac{2}{q} + \frac{d}{\rho} + \tilde{\alpha}} > 1$ ) and thus to  $\tilde{\alpha} \leq \alpha + d - 1 + \frac{2\rho}{q}$  (resp.  $\tilde{\alpha} < \alpha + 1 + \frac{dq}{\rho}$  which always holds true). When  $d \geq 2$  or  $q < \infty$ , we may choose  $\tilde{\alpha} \in (\alpha, \alpha + \frac{1}{2})$  so that the first requirement is satisfied as well, while when  $d = 1$  and  $q = \infty$ ,  $\tilde{\alpha} = \alpha$  is the only possible choice. With the inequality  $t \geq 3h$ , we conclude that

$$(2.17) \quad |\Delta_t^2| \leq Ch^{\frac{\alpha}{2}} \left( 1 + \mathbb{I}_{\{d=1, q=\infty, \rho < \infty\}} \ln \left( \frac{T}{h} \right) \right) g_c(t, y-x).$$

For the scheme (1.6), note that the first time step when  $\bar{b}_h = 0$  does not contribute to the term  $\bar{\Delta}_t^2$ . When  $\rho = q = \infty$ , then the cutoff error vanishes as soon as  $h \leq \frac{B^2}{\|b\|_{L^\infty-L^\infty}^2}$ . When either  $\rho < \infty$  or  $q < \infty$ , we may reproduce the above reasoning with  $\lambda = 1 + \tilde{\alpha}$  since  $|b - \bar{b}_h| \leq \frac{h^{\frac{\tilde{\alpha}}{2}} |b|^{1+\tilde{\alpha}}}{B^{\tilde{\alpha}}}$ . The requirement for the application of (2.9) writes  $\tilde{\alpha} \leq \rho - 1$ . We have  $\rho > d$  so that, when  $d \geq 2$ ,  $\rho + \frac{d}{\rho} > 2$ , inequality which remains valid when  $d = 1$  since for  $\rho < \infty$ ,  $\rho + \frac{1}{\rho} - 2 = \frac{(\rho-1)^2}{\rho} > 0$ . We deduce that  $\alpha < \rho - 1$  so that we can take  $\tilde{\alpha} > \alpha$  satisfying  $\tilde{\alpha} \leq \rho - 1$ . Hence the logarithmic term may be removed when  $d = 1$ ,  $q = \infty$  and  $\rho < \infty$ :

$$|\bar{\Delta}_t^2| \leq Ch^{\frac{\alpha}{2}} g_c(t, y-x).$$

We next suppose that  $t \geq 3h$  to estimate the error contribution  $\Delta_t^3$ , since this contribution vanishes otherwise. Using (2.5), (2.11) and  $|b_h| \leq |b|$ ,  $\tau_s^h \geq \frac{s}{2}$  when  $s \geq t_1$ , then applying Lemma 2.4 with  $\rho' = \rho$ ,  $q' = q$ ,  $\varphi = |b|$ ,  $f = 1$ ,  $\beta = \frac{\alpha}{2}$ ,  $\gamma = \frac{1}{2}$ , we obtain that

$$\begin{aligned}
|\Delta_t^3| &\leq C \int_{t_1}^{\tau_t^h-h} ds \frac{(s - \tau_s^h)^{\frac{\alpha}{2}}}{(\tau_s^h)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} g_c(s, z-x) |b(s, z)| \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz \\
&\leq Ch^{\frac{\alpha}{2}} \int_0^t \frac{ds}{s^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} g_c(s, z-x) |b(s, z)| \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz \\
(2.18) \quad &\leq Ch^{\frac{\alpha}{2}} \|b\|_{L^q-L^\rho} g_c(t, y-x).
\end{aligned}$$

Since we only used  $|b_h| \leq |b|$ , the same estimation holds for  $\bar{\Delta}_t^3$ .

Concerning the estimations of  $(\Delta_t^i, \bar{\Delta}_t^i)$ ,  $i \in \{4, 5, 6\}$ , we take advantage of the choices of the cutoff which ensure that for small time transitions of the scheme, the drift is negligible with respect to the diffusive behavior of the Brownian motion. Indeed, the inequality

$$(2.19) \quad \forall c' > 1, \forall x, y, z \in \mathbb{R}^d, |z - x - y|^2 \geq \frac{1}{c'} |z - x|^2 - \frac{1}{c'-1} |y|^2$$

applied with  $c' = c$  and  $y = sb_h(r, x)$  such that  $|y| \leq \frac{Bs}{h^{\frac{1}{q} + \frac{d}{2\rho}}}$  by the definition (1.3) of  $b_h$ , implies that,

$$(2.20) \quad \forall (s, r, x, z) \in (0, h] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, g_1(s, z - x - b_h(r, x)s) \leq c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} g_c(s, z - x).$$

In the same way,

(2.21)

$$\forall (s, r, x, z) \in (0, h] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, g_1(s, z - x - \bar{b}_h(r, x)s) \leq c^{\frac{d}{2}} e^{\frac{B^2}{2(c-1)}} g_c(s, z - x).$$

We suppose that  $t \geq 3h$  to estimate the error contribution  $\Delta_t^4$ , since this contribution vanishes otherwise. Note that for  $\tilde{\alpha} \geq 0$ ,

(2.22)

$$|z - w|^{\tilde{\alpha}} g_{\frac{1+c}{2}}(s - t_j, w - z) \leq \left( \frac{2c}{1+c} \right)^{\frac{d}{2}} \sup_{a \geq 0} \left\{ a^{\tilde{\alpha}} e^{-\frac{(c-1)a^2}{2c(1+c)}} \right\} (s - t_j)^{\frac{\tilde{\alpha}}{2}} g_c(s - t_j, w - z).$$

Let assume first that  $\alpha < 1$  (i.e. either  $\rho$  or  $q < +\infty$ ) and choose  $\tilde{\alpha} \in (\alpha, 1]$ . Using  $|b_h| \leq |b|$  for the first inequality, (2.3), (2.13), (2.12), (2.20) with  $(c, s, x, z)$  replaced by  $(\frac{1+c}{2}, s - t_j, z, w)$  and (2.12) for the second inequality, (2.22) and  $g_{\frac{1+c}{2}} \leq \left( \frac{2c}{1+c} \right)^{\frac{d}{2}} g_c$  combined with Gaussian convolution and  $|r - s| \vee (s - t_j) \leq h$  for the third inequality, (2.9) for the fourth inequality, we obtain :

$$\begin{aligned} & |\Delta_t^4| \\ & \leq \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \frac{1}{h} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^{t_{j+1}} dr \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma^h(0, x, t_j, z) g_1(s - t_j, w - z - b_h(r, z)(s - t_j)) |b(r, z)| \\ & \quad \times (|\nabla_y g_1(t - r, y - z) - \nabla_y g_1(t - s, y - z)| + |\nabla_y g_1(t - s, y - z) - \nabla_y g_1(t - s, y - w)|) dz dw \\ & \leq \frac{C}{h} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^{t_{j+1}} dr \int_{\mathbb{R}^d \times \mathbb{R}^d} g_c(t_j, z - x) g_{\frac{1+c}{2}}(s - t_j, w - z) |b(r, z)| \\ & \quad \left( \frac{|r - s|^{\frac{\tilde{\alpha}}{2}}}{(t - r \vee s)^{\frac{1+\tilde{\alpha}}{2}}} g_c(t - r \vee s, y - z) + \frac{|z - w|^{\tilde{\alpha}}}{(t - s)^{\frac{1+\tilde{\alpha}}{2}}} (g_c(t - s, y - z) + g_c(t - s, y - w)) \right) dz dw \\ & \leq Ch^{\frac{\tilde{\alpha}}{2} - 1} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^{t_{j+1}} dr \int_{\mathbb{R}^d} \frac{g_c(t_j, z - x) |b(r, z)|}{(t - r \vee s)^{\frac{1+\tilde{\alpha}}{2}}} \left( g_c(t - r \vee s, y - z) \right. \\ & \quad \left. + g_c(t - s, y - z) + g_c(t - t_j, y - z) \right) dz \\ & \leq Ch^{\frac{\tilde{\alpha}}{2} - 1} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^{t_{j+1}} dr \frac{t_j^{\frac{d}{2\rho}} \|b(r, \cdot)\|_{L^\rho}}{t_j^{\frac{d}{2\rho}} (t - r \vee s)^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho}}} \left( g_c(t + t_j - r \vee s, y - x) \right. \\ & \quad \left. + g_c(t + t_j - s, y - x) + g_c(t, y - x) \right). \end{aligned}$$

Since  $t \geq 3h$ , for  $r \in [t_j, t_{j+1}]$  with  $j \leq \lfloor \frac{t}{h} \rfloor - 2$ ,  $t \geq t + t_j - r \geq t - h \geq \frac{2t}{3}$  so that

$$(2.24) \quad g_c(t + t_j - r, y - x) \leq \frac{3^{\frac{d}{2}}}{2^{\frac{d}{2}}} g_c(t, y - x).$$



Since, for  $r, s$  chosen in the same time-step and not greater than  $\tau_t^h - h$ ,  $t - (r \vee s) \geq \frac{t-r}{2}$  and  $t_j \geq \frac{r}{2}$  for  $r \in [t_j, t_{j+1}]$  when  $j \geq 1$ , we deduce that

$$\begin{aligned} \frac{|\Delta_t^4|}{g_c(t, y-x)} &\leq Ch^{\frac{\tilde{\alpha}}{2}} t^{\frac{d}{2\rho}} \int_0^{\tau_t^h-h} \frac{\|b(r, \cdot)\|_{L^\rho} dr}{r^{\frac{d}{2\rho}} (t-r)^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho}}} \\ &\leq Ch^{\frac{\tilde{\alpha}}{2}} \left( t^{-\frac{1+\tilde{\alpha}}{2}} \int_0^{\frac{t}{2}} \frac{\|b(r, \cdot)\|_{L^\rho} dr}{r^{\frac{d}{2\rho}}} + \int_{\frac{t}{2}}^{t-h} \frac{\|b(r, \cdot)\|_{L^\rho} dr}{(t-r)^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho}}} \right) \\ &\leq Ch^{\frac{\tilde{\alpha}}{2}} \left( t^{-\frac{1+\tilde{\alpha}}{2}} \|b\|_{L^q-L^\rho} t^{\frac{1+\alpha}{2}} + \|b\|_{L^q-L^\rho} h^{\frac{\alpha-\tilde{\alpha}}{2}} \right), \end{aligned}$$

where we used Hölder's inequality and  $\frac{q}{q-1} \left( \frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho} \right) > \frac{q}{q-1} \left( \frac{1+\alpha}{2} + \frac{d}{2\rho} \right) = 1$  since  $\tilde{\alpha} > \alpha$  for the last inequality. Using that  $t \geq 3h$ , we conclude that, when  $\alpha < 1$ ,

$$(2.25) \quad |\Delta_t^4| \leq Ch^{\frac{\alpha}{2}} g_c(t, y-x).$$

Let us now check that this estimation still holds when  $\alpha = 1$ , i.e.  $q = \rho = \infty$ . The point in this setting is that (2.23) cannot be obtained directly with  $\tilde{\alpha} > \alpha$  and some additional centering argument is needed. Write  $\Delta_t^4 = \Delta_t^{41} + \Delta_t^{42}$  where

$$\Delta_t^{42} = \int_{t_1}^{\tau_t^h-h} ds \mathbb{E} \left[ b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) \cdot (\nabla_y g_1(t-s, y - X_{\tau_s^h}^h) - \nabla_y g_1(t-s, y - X_s^h)) \right]$$

and the term  $\Delta_t^{41}$  with the second factor in the expectation replaced by  $(\nabla_y g_1(t - U_{\lfloor s/h \rfloor}, y - X_{\tau_s^h}^h) - \nabla_y g_1(t-s, y - X_{\tau_s^h}^h))$  can be handled as above with  $\tilde{\alpha} \in (1, 2]$ , i.e. the control of the first term in parenthesis in (2.23) still holds and  $|\Delta_t^{41}| \leq Ch^{\frac{\alpha}{2}} g_c(t, y-x)$ . Let us focus on  $\Delta_t^{42}$  for which we subtract the centred (conditionally to  $(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h)$ ) random variable

$$b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) \cdot \nabla_y^2 g_1(t-s, y - X_{\tau_s^h}^h)(X_s^h - X_{\tau_s^h}^h - b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h)(s - \tau_s^h))$$

to the expectation and obtain

$$\begin{aligned} |\Delta_t^{42}| &\leq \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \frac{1}{h} \int_{t_j}^{t_{j+1}} ds \int_{t_j}^{t_{j+1}} dr \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma^h(0, x, t_j, z) g_1(s - t_j, w - z - b_h(r, z)(s - t_j)) \\ &\quad \times |b_h(r, z)| \left( \left| \nabla_y g_1(t-s, y-z) - \nabla_y g_1(t-s, y-w) - \nabla_y^2 g_1(t-s, y-z)(w-z) \right| \right. \\ &\quad \left. + \left| \nabla_y^2 g_1(t-s, y-z) b_h(r, z)(s - t_j) \right| \right) dz dw =: \Delta_t^{421} + \Delta_t^{422}. \end{aligned}$$

Using  $|b_h| \leq |b|$  where  $b$  is bounded, (2.3), (2.20) with  $(c, s, x, z)$  replaced by  $(c, s - t_j, z, w)$  and (2.11) for the first inequality then  $s - t_j \leq h \leq t - s$  when  $s \in [t_j, t_{j+1}]$  and  $j \leq \lfloor \frac{t}{h} \rfloor - 2$  for the second inequality and (2.24) for the third, we obtain

$$\begin{aligned} |\Delta_t^{422}| &\leq C \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} (s - t_j) ds \int_{\mathbb{R}^d \times \mathbb{R}^d} g_c(t_j, z - x) g_c(s - t_j, w - z) \frac{g_c(t-s, y-z)}{t-s} dz dw \\ &\leq Ch^{\frac{1}{2}} \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \frac{ds}{(t-s)^{\frac{1}{2}}} g_c(t + t_j - s, y-x) \leq Ch^{\frac{\alpha}{2}} g_c(t, y-x) \int_0^t \frac{ds}{(t-s)^{\frac{1}{2}}}. \end{aligned}$$

To handle  $\Delta_t^{421}$ , we first write

$$\nabla_y g_1(t-s, y-z) - \nabla_y g_1(t-s, y-w) = \int_0^1 d\theta \nabla_y^2 g_1(t-s, y-(z+\theta(w-z)))(w-z)$$

and then exploit the Lipschitz continuity in space of the second order derivatives of the Gaussian kernel stated in (2.12). Also using  $|b_h| \leq |b|$  with  $b$  bounded, (2.3) and (2.20) with  $(c, s, x, z)$  replaced by  $(\frac{1+c}{2}, s-t_j, z, w)$  for the first inequality, (2.22) with  $\tilde{\alpha} = 2$  for the second inequality and (2.24) for the third inequality, we obtain

$$\begin{aligned} |\Delta_t^{421}| &\leq C \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} ds \int_{\mathbb{R}^d \times \mathbb{R}^d} g_c(t_j, z-x) g_{\frac{1+c}{2}}(s-t_j, w-z) \\ &\quad \left( \int_0^1 d\theta \left( g_c(t-s, y-z) + g_c(t-s, y-(z+\theta(w-z))) \right) \frac{\theta |w-z|^2}{(t-s)^{\frac{3}{2}}} dz dw \right) \\ &\leq Ch \sum_{j=1}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \frac{ds}{(t-s)^{\frac{3}{2}}} \left( g_c(t+t_j-s, y-x) + \int_0^1 d\theta g_c(t+(1-\theta^2)(t_j-s), y-x) \right) \\ &\leq Ch g_c(t, y-x) \int_0^{t-h} \frac{ds}{(t-s)^{\frac{3}{2}}} = Ch^{\frac{\alpha}{2}} g_c(t, y-x). \end{aligned}$$

Hence (2.25) holds whatever  $\alpha \in (0, 1]$ . Replacing (2.20) by (2.21), we obtain the same estimation for  $\bar{\Delta}_t^4$ .

By (2.20), (2.11) and the definition (1.3) of  $b_h$ , then (2.9), that  $t-s \geq \frac{t}{2}$  for  $s \in [0, t_1]$  when  $t \geq 2h$ , Hölder's inequality in time, we obtain that

$$\begin{aligned} |\Delta_t^5| &\leq \frac{C}{h} \int_{s=0}^{t_1 \wedge t} \int_{r=0}^h \int_{\mathbb{R}^d} g_c(s, z-x) \left( |b(s, z)| + |b(r, x)| \wedge Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} \right) \\ &\quad \times \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz dr ds \\ &\leq C g_c(t, y-x) \left( t^{\frac{d}{2\rho}} \int_0^{t_1 \wedge t} \frac{\|b(s, \cdot)\|_{L^\rho} ds}{s^{\frac{d}{2\rho}} (t-s)^{\frac{1}{2} + \frac{d}{2\rho}}} + Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} \int_0^{t_1 \wedge t} \frac{ds}{\sqrt{t-s}} \right) \\ &\leq C g_c(t, y-x) \left( \mathbb{I}_{\{t < 2h\}} \left( t^{\frac{d}{2\rho}} \int_0^t \frac{\|b(s, \cdot)\|_{L^\rho} ds}{s^{\frac{d}{2\rho}} (t-s)^{\frac{1}{2} + \frac{d}{2\rho}}} + Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} \int_0^t \frac{ds}{\sqrt{t-s}} \right) \right. \\ &\quad \left. + \mathbb{I}_{\{t \geq 2h\}} t^{-\frac{1}{2}} \left( \int_0^{t_1} \frac{\|b(s, \cdot)\|_{L^\rho} ds}{s^{\frac{d}{2\rho}}} + Bh^{-(\frac{1}{q} + \frac{d}{2\rho})} t_1 \right) \right) \\ &\leq C g_c(t, y-x) (\|b\|_{L^q - L^\rho} + B) (\mathbb{I}_{\{t < 2h\}} [t^{\frac{\alpha}{2}} + h^{\frac{\alpha}{2}}] + \mathbb{I}_{\{t \geq 2h\}} h^{\frac{\alpha}{2}}) \\ (2.26) \quad &\leq C g_c(t, y-x) h^{\frac{\alpha}{2}}. \end{aligned}$$

Note that (1.2) ensures that  $(\frac{1}{2} + \frac{d}{2\rho}) \frac{q}{q-1} < 1$  so that the factor  $\left( \int_0^t \frac{ds}{s^{\frac{dq}{2\rho(q-1)}} (t-s)^{(\frac{1}{2} + \frac{d}{2\rho}) \frac{q}{q-1}}} \right)^{1-\frac{1}{q}}$

which appears when applying Hölder's inequality in time is equal to some finite constant multiplied by  $t^{\frac{\alpha}{2} - \frac{d}{2\rho}}$ . To derive the same bound for  $\bar{\Delta}_t^5$ , in addition to replacing (2.20) by (2.21), we crucially use that  $\bar{b}_h(r, \cdot) = 0$  for  $r \in [0, h]$ . If we had not made this choice, then  $Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$  would have been replaced in the previous analysis by  $Bh^{-\frac{1}{2}}$ , leading to a final bound in  $(t \wedge h)^{\frac{1}{2}} h^{-\frac{1}{2}} g_c(t, y-x)$  which does not go to 0 with  $h$ .

Let us now suppose that  $t \geq h$  to estimate  $\Delta_t^6$ . Using (2.20) with  $(s, x)$  replaced by  $(s - \tau_s^h, w)$ , (2.11) and  $|b_h| \leq |b|$  then Gaussian convolution, (2.9), last that  $s \geq \tau_s^h \geq \frac{t}{3}$  for  $s \geq (\tau_t^h - h) \vee t_1$  with  $t \geq h$  ( $\tau_s^h \geq h \geq \frac{t}{3}$  when  $t \in [h, 3h]$  while when  $t > 3h$ ,  $\tau_t^h - h > \frac{t}{2}$ ),  $t - \tau_s^h \geq t - s$  and Hölder's inequality in time, we obtain that

$$\begin{aligned}
|\Delta_t^6| &\leq \frac{C}{h} \int_{s=(\tau_t^h-h)\vee t_1}^t \int_{r=\tau_s^h}^{\tau_s^h+h} \int_{\mathbb{R}^d \times \mathbb{R}^d} g_c(\tau_s^h, w-x) g_c(s-\tau_s^h, z-w) \\
&\quad \times (|b(s, z)| + |b(r, w)|) \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz dw dr ds \\
&\leq C \int_{(\tau_t^h-h)\vee t_1}^t \int_{\mathbb{R}^d} g_c(s, z-x) |b(s, z)| \frac{g_c(t-s, y-z)}{\sqrt{t-s}} dz ds \\
&\quad + \frac{C}{h} \int_{s=(\tau_t^h-h)\vee t_1}^t \int_{r=\tau_s^h}^{\tau_s^h+h} \int_{\mathbb{R}^d} |b(r, w)| g_c(\tau_s^h, w-x) \frac{g_c(t-\tau_s^h, y-w)}{\sqrt{t-s}} dw ds dr \\
&\leq C g_c(t, y-x) \left( t^{\frac{d}{2\rho}} \int_{(\tau_t^h-h)\vee t_1}^t \frac{\|b(s, \cdot)\|_{L^\rho} ds}{s^{\frac{d}{2\rho}} (t-s)^{\frac{1}{2}+\frac{d}{2\rho}}} \right. \\
&\quad \left. + \frac{t^{\frac{d}{2\rho}}}{h} \int_{s=(\tau_t^h-h)\vee t_1}^t \int_{r=\tau_s^h}^{\tau_s^h+h} \frac{\|b(r, \cdot)\|_{L^\rho}}{(\tau_s^h)^{\frac{d}{2\rho}} (t-\tau_s^h)^{\frac{d}{2\rho}}} dr \frac{ds}{\sqrt{t-s}} \right) \\
&\leq C \|b\|_{L^q-L^\rho} g_c(t, y-x) \left( \left( t - (\tau_t^h - h) \vee t_1 \right)^{\frac{\alpha}{2}} + h^{-\frac{1}{q}} \int_{(\tau_t^h-h)\vee t_1}^t \frac{ds}{(t-s)^{\frac{1}{2}+\frac{d}{2\rho}}} \right) \\
&\leq C g_c(t, y-x) \left( \left( t - (\tau_t^h - h) \vee t_1 \right)^{\frac{\alpha}{2}} + h^{-\frac{1}{q}} \left( t - (\tau_t^h - h) \vee t_1 \right)^{\frac{1}{2}-\frac{d}{2\rho}} \right) \\
(2.27) \quad &\leq C g_c(t, y-x) h^{\frac{\alpha}{2}}.
\end{aligned}$$

Replacing (2.20) by (2.21), we obtain the same estimation for  $\bar{\Delta}_t^6$ .

Let us conclude with the term  $\Delta_t^1$  which can be used in a Gronwall type argument. Namely, set for  $u \in (0, T]$ :

$$f(u) := \sup_{(x,z) \in (\mathbb{R}^d)^2} \frac{|\Gamma^h(0, x, u, z) - \Gamma(0, x, u, z)|}{g_c(u, x-z)}.$$

We know from (2.3) and (2.6) that  $\sup_{s \in (0, T]} f(s) < +\infty$ . By (2.11) then Lemma 2.4 applied with  $(\rho', q', s)$  equal to  $(\rho, q, 0)$ , we obtain that with  $q'$  defined by  $\frac{1}{q'} = 1 - \frac{1}{q}$ :

$$\begin{aligned}
|\Delta_t^1| &\leq \int_0^t ds f(s) \int_{\mathbb{R}^d} g_c(s, z-x) |b(s, z)| \frac{g_c(t-s, y-z)}{(t-s)^{\frac{1}{2}}} dz \\
&\leq C t^{\frac{d}{2\rho}} \left( \int_0^t ds \left( \frac{f(s)}{s^{\frac{d}{2\rho}} (t-s)^{\frac{1}{2}+\frac{d}{2\rho}}} \right)^{q'} \right)^{\frac{1}{q'}} \|b\|_{L^q-L^\rho} g_c(t, y-x).
\end{aligned}$$

The same estimation holds for  $\bar{\Delta}_t^1$  with  $f(s)$  in the right-hand side replaced by  $\bar{f}(s) := \sup_{(x,z) \in (\mathbb{R}^d)^2} \frac{|\bar{\Gamma}^h(0, x, s, z) - \bar{\Gamma}(0, x, s, z)|}{g_c(s, x-z)}$  which satisfies  $\sup_{s \in (0, T]} \bar{f}(s) < +\infty$  by Proposition 2.1. With (2.16), (2.17), (2.26), (2.18), (2.25) and (2.27), we derive:

$$f(t) \leq C \left( h^{\frac{\alpha}{2}} \left( 1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln \left( \frac{T}{h} \right) \right) + t^{\frac{d}{2\rho}} \left( \int_0^t \frac{[f(s)]^{q'} ds}{s^{q' \frac{d}{2\rho}} (t-s)^{q'(\frac{1}{2}+\frac{d}{2\rho})}} \right)^{\frac{1}{q'}} \right).$$

Thus, up to an additional convexity inequality if  $q < +\infty \iff q' > 1$ , we get:

$$[f(t)]^{q'} \leq C^{q'} 2^{q'-1} \left( h^{\frac{\alpha q'}{2}} \left( 1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln \left( \frac{T}{h} \right) \right)^{q'} + t^{\frac{dq'}{2\rho}} \int_0^t \frac{[f(s)]^{q'} ds}{s^{\frac{dq'}{2\rho}} (t-s)^{q'(\frac{1}{2} + \frac{d}{2\rho})}} \right).$$

It eventually follows from Lemma 2.6 applied with  $\beta_3 = \beta_1 = \frac{dq'}{2\rho}$  and  $\beta_2 = q' \left( \frac{1}{2} + \frac{d}{2\rho} \right)$  (by (1.2), since  $\frac{d}{2\rho} + \frac{1}{q} < \frac{1}{2} \iff q' \left( \frac{1}{2} + \frac{d}{2\rho} \right) < 1$ , one has  $\beta_2 < 1$  and  $\beta_1 + \beta_2 - 1 < \beta_3$ ) that

$$\sup_{t \in (0, T]} f(t) \leq C h^{\frac{\alpha}{2}} \left( 1 + \mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}} \ln \left( \frac{T}{h} \right) \right),$$

which concludes the proof of Theorem 1.1 for the scheme (1.5). For the scheme (1.6), we estimate in the same way  $\sup_{t \in (0, T]} \bar{f}(t)$  without  $\mathbb{I}_{\{d=1, q=\infty, \rho<\infty\}}$  in the right-hand side in reason of the improved estimation of the cutoff error  $\bar{\Delta}_t^2$ .

**REMARK 2.8.** *A careful look at the proof shows that the error estimation for  $\bar{X}^h$  generalizes to the scheme with cutoffed drift  $\mathbb{I}_{\{t \geq h, |b(t, y)| > 0\}} \frac{|b(t, y)| \wedge (Bh^{-\mu})}{|b(t, y)|} b(t, y)$  when  $\mu \in (\frac{1}{q} + \frac{d}{2\rho}, \frac{1}{2}]$ . The indicator function saying that no drift is applied on the first time step permits to avoid worsening the estimation of the term with superscript 5. The inequality  $\mu > \frac{1}{q} + \frac{d}{2\rho}$  permits to deal with the cutoff error term with superscript 2. The inequality  $\mu \leq \frac{1}{2}$  prevents the explosion as  $h \rightarrow 0$  of the constant  $c^{\frac{d}{2}} e^{\frac{B^2 h^{1-2\mu}}{2(c-1)}}$  multiplying  $g_c(s, z - x)$  in the estimation that generalizes (2.20) and (2.21).*

**3. Density estimates for the Euler scheme.** This Section is dedicated to the proof of Proposition 2.1.

**3.1. Existence of a transition density satisfying the Duhamel formula (2.1) and the Gaussian estimation (2.3).** For  $k \in \llbracket 0, n \rrbracket$  and  $x \in \mathbb{R}^d$  let

$$X_t^h = x + (W_t - W_{t_k}) + \int_{t_k}^t b_h \left( U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h \right) ds, \quad t \in [t_k, T]$$

denote the Euler scheme started from  $x$  at the discretization time  $t_k = kh = \frac{kT}{n}$ . We emphasize that the cutoffed drift coefficient  $b_h$  defined in (1.3) coincides with  $b$  as long as  $|b| \leq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$  and is bounded from above by the threshold  $Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$ . For  $t \in (t_k, t_{k+1}]$ ,  $X_t^h$  admits the density

$$\begin{aligned} \Gamma^h(t_k, x, t, y) &= \mathbb{E}[g_1(t - t_k, y - x - (t - t_k)b_h(U_k, x))] \\ &= \int_0^h \frac{ds}{h} g_1(t - t_k, y - x - (t - t_k)b_h(t_k + s, x)) \end{aligned}$$

with respect to the Lebesgue measure on  $\mathbb{R}^d$ . Since  $z \mapsto g_1(t - t_k, z)$  is continuous and bounded by  $(2\pi(t - t_k))^{-\frac{d}{2}}$ , Lebesgue's theorem implies that  $y \mapsto \Gamma^h(t_k, x, t, y)$  is continuous. Moreover, (2.20) implies that

$$(3.1) \quad \forall t \in (t_k, t_{k+1}], \forall x, y \in \mathbb{R}^d, \Gamma^h(t_k, x, t, y) \leq c^{\frac{d}{2}} e^{\frac{B^2 h^\alpha}{2(c-1)}} g_c(t - t_k, y - x).$$

By the Markov structure of the Euler scheme, for  $t \in (t_{k+1}, T]$ ,

$$\begin{aligned} & \Gamma^h(t_k, x, t, y) \\ &= \int_{(\mathbb{R}^d)^{\lceil \frac{t}{h} \rceil - k - 1}} \Gamma^h(t_k, x, t_{k+1}, z_1) \prod_{j=k+1}^{\lceil \frac{t}{h} \rceil - 2} \Gamma^h(t_j, z_{j-k}, t_{j+1}, z_{j+1-k}) \\ & \quad \times \Gamma^h(t_{\lceil \frac{t}{h} \rceil - 1}, z_{\lceil \frac{t}{h} \rceil - k - 1}, t, y) dz_1 \cdots dz_{\lceil \frac{t}{h} \rceil - k - 1} \end{aligned}$$

where, since  $y \mapsto \Gamma^h(t_{\lceil \frac{t}{h} \rceil - 1}, z_{\lceil \frac{t}{h} \rceil - k - 1}, t, y)$  is continuous and bounded by  $(2\pi(t - t_{\lceil \frac{t}{h} \rceil - 1}))^{-\frac{d}{2}}$ , Lebesgue's theorem implies that the left-hand side is a continuous function of  $y$ . Moreover, the last equality combined with (3.1) then Gaussian convolution imply that

$$\begin{aligned} & \Gamma^h(t_k, x, t, y) \\ & \leq \left( c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} \right)^{\lceil \frac{t}{h} \rceil - k} \int_{(\mathbb{R}^d)^{\lceil \frac{t}{h} \rceil - k - 1}} g_c(t_{k+1} - t_k, z_1 - x) \\ & \quad \times \prod_{j=k+1}^{\lceil \frac{t}{h} \rceil - 2} g_c(t_{j+1} - t_j, z_{j+1-k} - z_{j-k}) g_c(t - t_{\lceil \frac{t}{h} \rceil - 1}, y - z_{\lceil \frac{t}{h} \rceil - k - 1}) dz_1 \cdots dz_{\lceil \frac{t}{h} \rceil - k - 1} \\ & = \left( c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} \right)^{\lceil \frac{t}{h} \rceil - k} g_c(t - t_k, y - x). \end{aligned}$$

The estimation

$$(3.2) \quad \Gamma^h(t_k, x, t, y) \leq c^{\frac{dT}{2h}} e^{\frac{B^2 T h \alpha - 1}{2(c-1)}} g_c(t - t_k, y - x)$$

is therefore valid for all  $(x, y) \in (\mathbb{R}^d)^2$ ,  $k \in \llbracket 0, n \rrbracket$  and  $t \in (t_k, T]$ . Let us now check that the factor  $c^{\frac{dT}{2h}} e^{\frac{B^2 T h \alpha - 1}{2(c-1)}}$  which goes to  $+\infty$  when  $h \rightarrow 0$  can be replaced by some finite constant not depending on the time-step  $h$  and study the regularity of  $\Gamma^h(t_k, x, t, y)$  in its forward variables  $t$  and  $y$ .

Let  $t \in (t_k, T]$ ,  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a  $C^2$  function with compact support and  $v(s, y) = \mathbb{I}_{\{s < t\}} g_1(t - s, \cdot) \star \varphi(y) + \mathbb{I}_{\{s = t\}} \varphi(y)$ . The function  $v$  is bounded together with its spatial derivatives up to the order 2 and its first order time derivative on the domain  $[0, t] \times \mathbb{R}^d$  where it solves the heat equation

$$\begin{cases} \partial_s v(s, y) + \frac{1}{2} \Delta v(s, y) = 0, & (s, y) \in [0, t] \times \mathbb{R}^d, \\ v(t, y) = \varphi(y), & y \in \mathbb{R}^d. \end{cases}$$

By Itô's formula,

$$\varphi(X_t^h) = v(t_k, x) + \int_{t_k}^t \nabla v(s, X_s^h) \cdot dW_s + \int_{t_k}^t \nabla v(s, X_s^h) \cdot b_h(U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h) ds.$$

Since  $\nabla v$  and  $b_h$  are bounded and, by (2.11), (3.2) and Gaussian convolution,  $\mathbb{E}[|\nabla g_1(t - s, X_s^h - y)|] \leq C \frac{g_c(t - t_k, y - x)}{\sqrt{t - s}}$ , taking the expectation and using Fubini's theorem, we deduce that

$$\int_{\mathbb{R}^d} \varphi(y) \Gamma^h(t_k, x, t, y) dy = \int_{\mathbb{R}^d} \varphi(y) g_1(t - t_k, x - y) dy$$

$$+ \int_{\mathbb{R}^d} \varphi(y) \int_{t_k}^t \mathbb{E} \left[ b_h \left( U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h \right) \cdot \nabla g_1(t-s, X_s^h - y) \right] ds dy.$$

Since  $\varphi$  is arbitrary and  $g_1$  is even in its spatial variable, we deduce that  $dy$  a.e.,

$$\Gamma^h(t_k, x, t, y) = g_1(t - t_k, y - x) - \int_{t_k}^t \mathbb{E} \left[ b_h \left( U_{\lfloor s/h \rfloor}, X_{\tau_s^h}^h \right) \cdot \nabla_y g_1(t-s, y - X_s^h) \right] ds.$$

This equality even holds for each  $y \in \mathbb{R}^d$  since the left-hand side and the first term in the right-hand side are continuous functions of  $y$  and in the derivation of (2.4) below we will check that the second term in the right-hand side satisfies the Hölder estimate in this inequality and is therefore also continuous in  $y$ .

The proof of Proposition 2.1 relies on this Duhamel formula where we expand  $\Gamma^h$  around the Brownian semi-group. We could as well have considered the full parametrix expansion of the density of the scheme, used for instance in [26] or [27], but the *one-step* Duhamel formulation is more consistent with the approach we also used to estimate the error of the Euler scheme. We have, using that for  $r \in [t_j, t_{j+1}]$ ,  $X_r^h = X_{t_j}^h + W_r - W_{t_j} + b_h(U_j, X_{t_j}^h)(r - t_j)$ , the independence between  $X_{t_j}^h$ ,  $W_r - W_{t_j}$  and  $U_j$  and the Gaussian semi-group property for the second equality,

$$\begin{aligned} & \Gamma^h(t_k, x, t, y) \\ &= g_1(t - t_k, y - x) - \sum_{j=k}^{\lceil \frac{t}{h} \rceil - 1} \int_{t_j}^{t_{j+1} \wedge t} \mathbb{E} \left[ b_h(U_j, X_{t_j}^h) \cdot \nabla_y g_1(t-r, y - X_r^h) \right] dr \\ &= g_1(t - t_k, y - x) \\ & \quad - \sum_{j=k}^{\lceil \frac{t}{h} \rceil - 1} \frac{1}{h} \int_{r=t_j}^{t_{j+1} \wedge t} \int_{s=t_j}^{t_{j+1}} \mathbb{E} \left[ b_h(s, X_{t_j}^h) \cdot \nabla_y g_1(t-t_j, y - X_{t_j}^h - b_h(s, X_{t_j}^h)(r-t_j)) \right] ds dr \\ &= g_1(t - t_k, y - x) - \frac{1}{h} \int_{r=t_k}^{t_{k+1} \wedge t} \int_{s=t_k}^{t_{k+1}} b_h(s, x) \cdot \nabla_y g_1(t-t_k, y - x - b_h(s, x)(r-t_k)) ds dr \\ & \quad - \sum_{j=k+1}^{\lceil \frac{t}{h} \rceil - 1} \frac{1}{h} \int_{r=t_j}^{t_{j+1} \wedge t} \int_{s=t_j}^{t_{j+1}} \int_{\mathbb{R}^d} \Gamma^h(t_k, x, t_j, z) \\ & \quad \times b_h(s, z) \cdot \nabla_y g_1(t-t_j, y - z - b_h(s, z)(r-t_j)) dz ds dr. \end{aligned} \tag{3.3}$$

Since, by (2.11),  $\forall c > 1$ ,  $\exists C < \infty$ ,  $\forall u \in (0, T]$ ,  $\forall x \in \mathbb{R}^d$ ,  $|\nabla g_1(u, x)| \leq \frac{C}{\sqrt{u}} g_{\frac{1+c}{2}}(u, x)$  applying (2.19) with  $c' = \frac{2c}{1+c}$ , we obtain that

$$\begin{aligned} & \exists C < \infty, \forall u \in (0, T], \forall u' \in [0, u \wedge h], \forall s \in [0, T], \forall (x, y) \in (\mathbb{R}^d)^2, \\ & |\nabla g_1(u, y - x - b_h(s, x)u')| \leq C \frac{g_c(u, y - x)}{\sqrt{u}}. \end{aligned} \tag{3.4}$$

Set  $m_{k,j} = \sup_{(x,z) \in (\mathbb{R}^d)^2} \frac{\Gamma^h(t_k, x, t_j, z)}{g_c(t_j - t_k, z - x)}$ . It is clear from (3.2) that  $m_{k,j} < +\infty$ . Using that  $\|b_h\|_\infty \leq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$ , (2.9) for the second inequality, and then that  $t_\ell - t_k \geq h$  and  $(t_j - t_k) \geq \frac{1}{2}(s - t_k)$  for  $s \in [t_j, t_{j+1}]$  with  $j \geq k+1$  for the third one, and eventually Hölder's inequality

in time for the fourth inequality, we deduce that for  $\ell \in \llbracket k+1, n \rrbracket$ ,

$$\begin{aligned}
& \frac{|\Gamma^h(t_k, x, t_\ell, y)|}{g_c(t_\ell - t_k, y - x)} \\
& \leq c^{\frac{d}{2}} + \frac{CBh^{1-(\frac{1}{q}+\frac{d}{2\rho})}}{\sqrt{t_\ell - t_k}} \\
& \quad + \sum_{j=k+1}^{\ell-1} \frac{m_{k,j}}{g_c(t_\ell - t_k, y - x)} \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}^d} g_c(t_j - t_k, z - x) |b_h(s, z)| \frac{g_c(t_\ell - t_j, y - z)}{\sqrt{t_\ell - t_j}} dz ds \\
& \leq c^{\frac{d}{2}} + \frac{CBh^{\frac{1}{2}+\frac{\alpha}{2}}}{\sqrt{t_\ell - t_k}} + C \sum_{j=k+1}^{\ell-1} \frac{m_{k,j}(t_\ell - t_k)^{\frac{d}{2\rho}}}{(t_j - t_k)^{\frac{d}{2\rho}}(t_\ell - t_j)^{\frac{1}{2}+\frac{d}{2\rho}}} \int_{t_j}^{t_{j+1}} \|b(s, \cdot)\|_{L^\rho} ds \\
& \leq C + C \max_{j=k+1}^{\ell-1} m_{k,j} (t_\ell - t_k)^{\frac{d}{2\rho}} \int_{t_k}^{t_\ell} \frac{\|b(s, \cdot)\|_{L^\rho} ds}{(s - t_k)^{\frac{d}{2\rho}}(t_\ell - s)^{\frac{1}{2}+\frac{d}{2\rho}}} \\
& \leq C + C \max_{j=k+1}^{\ell-1} m_{k,j} \|b\|_{L^q-L^\rho}(t_\ell - t_k)^{\frac{\alpha}{2}}.
\end{aligned}$$

Taking the supremum over  $(x, y) \in (\mathbb{R}^d)^2$  and remarking that the right-hand side is non-decreasing with  $\ell$ , we deduce that

$$\max_{j=k+1}^{\ell} m_{k,j} \leq C + C \|b\|_{L^q-L^\rho}(t_\ell - t_k)^{\frac{\alpha}{2}} \max_{j=k+1}^{\ell} m_{k,j}.$$

Hence when  $t_\ell - t_k \leq \theta := (2C\|b\|_{L^q-L^\rho})^{-\frac{2}{\alpha}}$ , then  $\max_{j=k+1}^{\ell} m_{k,j} \leq 2C$ . With (3.1) and Gaussian convolution, we deduce that when  $t - t_k \leq \theta$ , then  $\Gamma^h(t_k, x, t, y) \leq 2C c^{\frac{d}{2}} e^{\frac{B^2 T \alpha}{2(c-1)}} g_c(t - t_k, y - x)$ . In view of (3.2), to prove the independent of  $h$  Gaussian estimate for the density of the scheme, it is enough to check that the estimate is independent of  $h \leq \theta$  and we suppose from now on that this inequality is satisfied so that  $\max_{\kappa=1}^{n-1} \max_{j=\kappa+1}^{(\kappa+\lfloor \frac{\theta}{h} \rfloor) \wedge n} m_{\kappa,j} \leq 2C$ . This gives the Gaussian estimate provided that the associated time interval is small enough but at a *macro* scale. For an arbitrary *macro* time interval the idea is now to chain the previous estimates. Assuming that  $t - t_k \geq \theta$  and setting  $J = \lceil \frac{t-t_k}{\tau_\theta^h} \rceil - 1$ , we have, under the convention  $y_0 = x$ ,

$$\begin{aligned}
& \Gamma^h(t_k, x, t, y) \\
& = \int_{(\mathbb{R}^d)^J} \prod_{j=1}^J \Gamma^h(t_k + (j-1)\tau_\theta^h, y_{j-1}, t_k + j\tau_\theta^h, y_j) \Gamma^h(t_k + J\tau_\theta^h, y_J, t, y) dy_1 \dots dy_J.
\end{aligned}$$

Since  $t - (t_k + J\tau_\theta^h) \leq \tau_\theta^h$ , when  $t$  does not belong to the discretization grid  $\{t_j = jh : j \in \llbracket 0, n \rrbracket\}$ , combining the just derived bound and (3.1), we get

$$\begin{aligned}
\Gamma^h(t_k + J\tau_\theta^h, y_J, t, y) & = \int_{\mathbb{R}^d} \Gamma^h(t_k + J\tau_\theta^h, y_J, \tau_t^h, z) \Gamma^h(\tau_t^h, z, t, y) dz \\
& \leq 2C c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} g_c(t - (t_k + J\tau_\theta^h), y_J - y),
\end{aligned}$$

and the same estimation holds without the factor  $c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}}$  when  $t$  belongs to the discretization grid. Hence, when  $h \leq \theta$  which implies  $\tau_\theta^h > \frac{\theta}{2}$ , proceeding similarly to the proof of

(3.2) (Gaussian chaining argument) with  $h$  replaced by  $\tau_\theta^h$ , we derive:

$$\begin{aligned} \forall 0 \leq k < n, \forall t \in (t_k, T], \Gamma^h(t_k, x, t, y) &\leq (2C)^{\lceil \frac{t-t_k}{\tau_\theta^h} \rceil} c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} g_c(t - t_k, y - x) \\ &\leq (2C)^{1+2\frac{t-t_k}{\theta}} c^{\frac{d}{2}} e^{\frac{B^2 h \alpha}{2(c-1)}} g_c(t - t_k, y - x). \end{aligned}$$

This gives the first estimation (2.3) in the proposition.

Similar estimates with the factors  $e^{\frac{B^2 h \alpha}{2(c-1)}}$  replaced by  $e^{\frac{B^2}{2(c-1)}}$  and with  $e^{\frac{B^2 T h \alpha - 1}{2(c-1)}}$  replaced by  $e^{\frac{B^2 T}{2(c-1)h}}$  in (3.2) can be derived for the scheme  $\bar{X}^h$  defined in (1.6) (and even for the scheme with the same cutoff when the cutoffed drift is kept on the first time-step).

**3.2. Hölder regularity of the transition density in the forward time variable.** We now prove (2.5). Let  $1 \leq k < \ell < n$ ,  $x, y \in \mathbb{R}^d$  and  $t \in [t_\ell, t_{\ell+1}]$ . We want to estimate  $\Gamma^h(t_k, x, t_\ell, y) - \Gamma^h(t_k, x, t, y)$ , which, according to (3.3), is equal to  $\Delta^1 + \Delta^2 + \Delta^3 + \Delta^4$  with

$$\begin{aligned} \Delta^1 &= g_1(t_\ell - t_k, y - x) - g_1(t - t_k, y - x), \\ \Delta^2 &= \frac{1}{h} \int_{r=t_k}^{t_{k+1}} \int_{s=t_k}^{t_{k+1}} b_h(s, x) \cdot [\nabla g_1(t - t_k, w) - \nabla g_1(t_\ell - t_k, w)]|_{w=y-x-b_h(s,x)(r-t_k)} ds dr, \\ \Delta^3 &= \sum_{j=k+1}^{\ell-1} \frac{1}{h} \int_{r=t_j}^{t_{j+1}} \int_{s=t_j}^{t_{j+1}} \int_{\mathbb{R}^d} \Gamma^h(t_k, x, t_j, z) b_h(s, z) \cdot [\nabla g_1(t - t_j, w) \\ &\quad - \nabla g_1(t_\ell - t_j, w)]|_{w=y-z-b_h(s,z)(r-t_j)} dz ds dr, \\ \Delta^4 &= \frac{1}{h} \int_{r=t_\ell}^t \int_{s=t_\ell}^{t_{\ell+1}} \int_{\mathbb{R}^d} \Gamma^h(t_k, x, t_\ell, z) b_h(s, z) \cdot \nabla_y g_1(t - t_\ell, y - z - b_h(s, z)(r - t_\ell)) dz ds dr. \end{aligned}$$

From (2.3), (3.4), recalling that  $|b_h| \leq |b|$ , applying (2.9), then Hölder's inequality in time and using lastly that  $t - t_k \leq 2(t_\ell - t_k)$  and  $t - t_\ell \leq h$ , we obtain that

$$\begin{aligned} |\Delta^4| &\leq \frac{C(t - t_\ell)}{h} \int_{t_\ell}^{t_{\ell+1}} \int_{\mathbb{R}^d} g_c(t_\ell - t_k, z - x) |b(s, z)| \frac{g_c(t - t_\ell, y - z)}{\sqrt{t - t_\ell}} dz ds \\ &\leq \frac{C(t - t_\ell)(t - t_k)^{\frac{d}{2\rho}}}{h(t_\ell - t_k)^{\frac{d}{2\rho}}(t - t_\ell)^{\frac{1}{2} + \frac{d}{2\rho}}} \int_{t_\ell}^{t_{\ell+1}} \|b(s, \cdot)\|_{L^\rho} ds g_c(t - t_k, y - x) \\ &\leq \frac{C(t - t_\ell)^{\frac{1}{2} - \frac{d}{2\rho}}}{h} h^{1 - \frac{1}{q}} \|b\|_{L^q - L^\rho} g_c(t - t_k, y - x) \\ &\leq C(t - t_\ell)^{\frac{\alpha}{2}} \|b\|_{L^q - L^\rho} g_c(t - t_k, y - x). \end{aligned}$$

Reasoning like in the above derivation of (3.4) with (2.13) replacing (2.11), we obtain the existence of a finite constant  $C$  such that for all  $0 < u < u' \leq T$ , all  $u'' \in [0, u \wedge h]$ , all  $s \in [0, T]$  and all  $(y, z) \in (\mathbb{R}^d)^2$ ,

$$\begin{aligned} &|\nabla g_1(u, y - z - b_h(s, x)u'') - \nabla g_1(u', y - z - b_h(s, z)u'')| \\ &\leq C \frac{|u' - u| \wedge u}{u^{\frac{3}{2}}} (g_c(u, y - z) + g_c(u', y - z)) \\ (3.5) \quad &\leq C \frac{|u' - u| \wedge u}{u^{\frac{3}{2}}} g_c(u', y - z) \text{ if } u' \leq 2u. \end{aligned}$$



This inequality, together with  $|b_h| \leq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$  and  $t_\ell - t_k \geq h$ , implies that ,  
(3.6)

$$|\Delta^2| \leq C \frac{(t - t_\ell)^{\frac{\alpha}{2}}}{(t_\ell - t_k)^{\frac{1+\alpha}{2}}} g_c(t - t_k, y - x) \int_{t_k}^{t_{k+1}} |b_h(s, x)| ds \leq C \frac{(t - t_\ell)^{\frac{\alpha}{2}}}{(t_\ell - t_k)^{\frac{\alpha}{2}}} h^{\frac{\alpha}{2}} g_c(t - t_k, y - x).$$

The estimation of  $\Delta_3$  is a bit more involved. We suppose that  $\ell \geq k + 2$  since  $\Delta_3 = 0$  otherwise. Let  $\alpha_1 \in (\alpha, 2]$ . Using (2.3), (3.5) and  $|b_h| \leq |b|$ , (2.9), then that  $s - t_k \leq 2(t_j - t_k)$  when  $s \in [t_j, t_{j+1}]$  with  $j \geq k + 1$  and  $(t_{\ell-1} - t_k) > \frac{t - t_k}{3}$  and last Hölder's inequality in time, we obtain that

$$\begin{aligned} |\Delta^3| &\leq C \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}^d} g_c(t_j - t_k, z - x) |b(s, z)| \frac{g_c(t - t_j, y - z) (t - t_\ell)^{\frac{\alpha_1}{2}}}{(t_\ell - t_j)^{\frac{1+\alpha_1}{2}}} dz ds \\ &\leq C (t - t_\ell)^{\frac{\alpha_1}{2}} (t - t_k)^{\frac{d}{2\rho}} g_c(t - t_k, y - x) \sum_{j=k+1}^{\ell-1} \frac{\int_{t_j}^{t_{j+1}} \|b(s, \cdot)\|_{L^\rho} ds}{(t_j - t_k)^{\frac{d}{2\rho}} (t_\ell - t_j)^{\frac{1+\alpha_1}{2} + \frac{d}{2\rho}}} \\ &\leq C (t - t_\ell)^{\frac{\alpha_1}{2}} g_c(t - t_k, y - x) \left[ (t - t_k)^{\frac{d}{2\rho}} \int_{t_k}^{t_{\ell-1}} \frac{\|b(s, \cdot)\|_{L^\rho}}{(s - t_k)^{\frac{d}{2\rho}} (t_\ell - s)^{\frac{1+\alpha_1}{2} + \frac{d}{2\rho}}} ds \right. \\ &\quad \left. + h^{-(\frac{1+\alpha_1}{2} + \frac{d}{2\rho})} \int_{t_{\ell-1}}^{t_\ell} \|b(s, \cdot)\|_{L^\rho} ds \right] \\ &\leq C (t - t_\ell)^{\frac{\alpha_1}{2}} g_c(t - t_k, y - x) \left[ (t_\ell - t_k)^{-\frac{1+\alpha_1}{2}} \int_{t_k}^{\frac{t_k+t_\ell}{2}} \frac{\|b(s, \cdot)\|_{L^\rho} ds}{(s - t_k)^{\frac{d}{2\rho}}} \right. \\ &\quad \left. + \int_{\frac{t_k+t_\ell}{2}}^{t_{\ell-1}} \frac{\|b(s, \cdot)\|_{L^\rho} ds}{(t_\ell - s)^{\frac{1+\alpha_1}{2} + \frac{d}{2\rho}}} + h^{-(\frac{1+\alpha_1}{2} + \frac{d}{2\rho})} \int_{t_{\ell-1}}^{t_\ell} \|b(s, \cdot)\|_{L^\rho} ds \right] \\ &\leq C (t - t_\ell)^{\frac{\alpha_1}{2}} g_c(t - t_k, y - x) \|b\|_{L^q-L^\rho} \times \left[ (t_\ell - t_k)^{\frac{\alpha - \alpha_1}{2}} + h^{\frac{\alpha - \alpha_1}{2}} \right] \\ &\leq C (t - t_\ell)^{\frac{\alpha}{2}} g_c(t - t_k, y - x). \end{aligned}$$

Using (2.13) to deal with  $\Delta^1$ , we conclude that (2.5) holds. Similar estimates with  $h^{\frac{\alpha}{2}}$  replaced by 1 in the right-hand side of (3.6) can be derived for the scheme  $\bar{X}^h$  defined in (1.6) (and even for the scheme with the same cutoff when the cutoffed drift is kept on the first time-step).

**3.3. Hölder regularity of the transition density in the forward spatial variable.** Let us now suppose that  $\alpha < 1$  and prove (2.4). First of all, by (2.3),

$$\begin{aligned} |\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t, y')| &\leq \Gamma^h(t_k, x, t, y) + \Gamma^h(t_k, x, t, y') \\ &\leq C \left( g_c(t - t_k, y - x) + g_c(t - t_k, y' - x) \right), \end{aligned}$$

so that (2.4) holds in the *global off-diagonal regime*  $|y - y'|^2 > (t - t_k)/4$  where  $\frac{|y - y'|^\alpha \wedge (t - t_k)^{\frac{\alpha}{2}}}{(t - t_k)^{\frac{\alpha}{2}}} > 4^{-\frac{\alpha}{2}}$ . Therefore, it is enough to focus on the so-called *global diagonal regime*

$$(3.7) \quad |y - y'|^2 \leq (t - t_k)/4.$$

Setting now  $u = t - |y - y'|^2$ , we derive from the Duhamel formula (2.1) that

$$\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t, y') = g_1(t - t_k, y - x) - g_1(t - t_k, y' - x) + T_1 + T_2 + T_3 \text{ where}$$

$$\begin{aligned}
T_1 &= \int_{t_k}^{t_{k+1} \wedge t} \mathcal{E}_r^h dr + \mathbb{I}_{\{t > t_{k+1}\}} \int_{(\tau_t^h - h) \vee t_{k+1}}^t \mathcal{E}_r^h dr, \\
T_2 &= \mathbb{I}_{\{\tau_t^h - h > t_{k+1}, u \geq \tau_t^h - h\}} \int_{t_{k+1}}^{\tau_t^h - h} \mathcal{E}_r^h dr + \mathbb{I}_{\{u < \tau_t^h - h\}} \int_{t_{k+1}}^{\tau_u^h} \mathcal{E}_r^h dr, \\
T_3 &= \mathbb{I}_{\{u < \tau_t^h - h\}} \int_{\tau_u^h}^{\tau_t^h - h} \mathcal{E}_r^h dr,
\end{aligned}$$

$$\text{with } \mathcal{E}_r^h := \mathbb{E} \left[ b_h(U_{\lfloor \frac{r}{h} \rfloor}, X_{\tau_r^h}^h) \cdot \left( \nabla_{y'} g_1(t - r, y' - X_r^h) - \nabla_y g_1(t - r, y - X_r^h) \right) \right] dr.$$

Note that when  $u < \tau_t^h - h$ , then  $|y - y'|^2 > h$  so that, in view of (3.7),  $t > t_{k+4}$  and  $(\tau_t^h - h) \wedge u > t_{k+3}$ . By (2.12) and since  $\alpha \leq 1$ , we first get

$$\begin{aligned}
& |g_1(t - t_k, y - x) - g_1(t - t_k, y' - x)| \\
& \leq C \frac{|y - y'|^\alpha \wedge (t - t_k)^{\frac{\alpha}{2}}}{(t - t_k)^{\frac{\alpha}{2}}} (g_c(t - t_k, y - x) + g_c(t - t_k, y' - x)).
\end{aligned}$$

Using (2.12), (2.3) and  $|b_h| \leq B h^{-\left(\frac{1}{q} + \frac{d}{2\rho}\right)}$ , then Gaussian convolution, we then obtain that

$$\begin{aligned}
\mathcal{E}_r^h &\leq C \int_{\mathbb{R}^d} g_c(r - t_k, z - x) \frac{|y - y'|^\alpha}{h^{\frac{1}{q} + \frac{d}{2\rho}} (t - r)^{\frac{1+\alpha}{2}}} (g_c(t - r, y' - z) + g_c(t - r, y - z)) dz \\
&\leq C (g_c(t - t_k, y - x) + g_c(t - t_k, y' - x)) \frac{|y - y'|^\alpha}{h^{\frac{1}{q} + \frac{d}{2\rho}} (t - r)^{\frac{1+\alpha}{2}}}.
\end{aligned}$$

Therefore, when  $y' \neq y$ , using  $\int_{t_k}^{t_{k+1} \wedge t} \frac{dr}{(t-r)^{\frac{1+\alpha}{2}}} \leq \int_{t_k}^t \frac{dr}{(t-r)^{\frac{1+\alpha}{2}}}$  when  $t - t_k \leq 2h$  and  $(t - r) > (t - t_k)/2$  for  $r \in [t_k, t_{k+1}]$  when  $t - t_k > 2h$  for the second inequality,

$$\begin{aligned}
& \frac{|T_1|}{(g_c(t - t_k, y - x) + g_c(t - t_k, y' - x)) |y - y'|^\alpha} \\
& \leq \frac{C}{h^{\frac{1}{q} + \frac{d}{2\rho}}} \left( \int_{t_k}^{t_{k+1} \wedge t} \frac{dr}{(t-r)^{\frac{1+\alpha}{2}}} + \mathbb{I}_{\{t > t_{k+1}\}} \int_{(\tau_t^h - h) \vee t_{k+1}}^t \frac{dr}{(t-r)^{\frac{1+\alpha}{2}}} \right) \\
& \leq \frac{C}{h^{\frac{1}{q} + \frac{d}{2\rho}}} \left( \mathbb{I}_{\{t - t_k \leq 2h\}} \frac{(t - t_k)^{\frac{1-\alpha}{2}}}{(1 - \alpha)/2} \right. \\
& \quad \left. + \mathbb{I}_{\{t - t_k > 2h\}} \frac{h}{((t - t_k)/2)^{\frac{1+\alpha}{2}}} + \mathbb{I}_{\{t > t_{k+1}\}} \frac{(t - ((\tau_t^h - h) \vee t_{k+1}))^{\frac{1-\alpha}{2}}}{(1 - \alpha)/2} \right) \leq C.
\end{aligned}$$

For the scheme  $\bar{X}^h$  defined in (1.6) (and even for the scheme with the same cutoff when the cutoffed drift is kept on the first time-step), the constant  $C$  in the right-hand side should be replaced by  $C(t - t_k)^{-\frac{\alpha}{2}}$ , where the denominator does not prevent from deriving (2.4). The forthcoming estimations of  $T_2$  and  $T_3$  rely on the bound  $|b_h| \leq |b|$  and are valid for the two schemes.

Let  $\tilde{\alpha} \in (\alpha, 1]$ . Using  $|b_h| \leq |b|$ , (2.3), (2.20) and (2.12), then Gaussian convolution, we obtain that

$$\mathcal{E}_r^h \leq \frac{C}{h} \int_{\tau_r^h}^{\tau_r^h + h} \int_{\mathbb{R}^d \times \mathbb{R}^d} g_c(\tau_r^h - t_k, z - x) g_c(r - \tau_r^h, w - z) |b(s, z)| \frac{|y - y'|^{\tilde{\alpha}}}{(t - r)^{\frac{1+\tilde{\alpha}}{2}}} dz dw ds$$

$$\begin{aligned}
& \times (g_c(t-r, y' - w) + g_c(t-r, y - w)) dz dw ds \\
& \leq \frac{C}{h} \int_{\tau_r^h}^{\tau_r^h+h} \int_{\mathbb{R}^d} g_c(\tau_r^h - t_k, z - x) |b(s, z)| \frac{|y - y'|^{\tilde{\alpha}}}{(t-r)^{\frac{1+\tilde{\alpha}}{2}}} \\
& \quad \times (g_c(t - \tau_r^h, y' - z) + g_c(t - \tau_r^h, y - z)) dz ds.
\end{aligned}$$

Let us assume that  $\tau_t^h - h > t_{k+1}$  (so that  $\tau_t^h - h - t_k > \frac{t-t_k}{2}$ ) and set  $\ell = \mathbb{I}_{\{u \geq \tau_t^h - h\}}(\lfloor \frac{t}{h} \rfloor - 1) + \mathbb{I}_{\{u < \tau_t^h - h\}}(\lfloor \frac{u}{h} \rfloor)$ . Using that  $t - r \geq \frac{t - \tau_r^h}{2}$  for  $r \leq \tau_t^h - h$  then Hölder's inequality in space, we deduce that when  $y' \neq y$

$$\begin{aligned}
& \frac{|T_2|}{|y - y'|^{\tilde{\alpha}}} \\
& \leq \frac{C}{h} \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t_{j+1}} \int_{\mathbb{R}^d} g_c(t_j - t_k, z - x) \frac{|b(s, z)|}{(t - t_j)^{\frac{1+\tilde{\alpha}}{2}}} \\
& \quad \times (g_c(t - t_j, y' - z) + g_c(t - t_j, y - z)) dz ds dr \\
& \leq C (g_c(t - t_k, y' - x) + g_c(t - t_k, y - x)) \sum_{j=k+1}^{\ell-1} \int_{t_j}^{t_{j+1}} \frac{(t - t_k)^{\frac{d}{2\rho}} \|b(s, \cdot)\|_{L^\rho}}{(t_j - t_k)^{\frac{d}{2\rho}} (t - s)^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho}}} ds.
\end{aligned}$$

Since  $t_j - t_k \geq \frac{s-t_k}{2}$  for  $s \in [t_j, t_{j+1}]$  with  $j \geq k+1$  and when  $j < \frac{k-1+\lfloor \frac{t}{h} \rfloor}{2}$ ,  $t_{j+1} \leq \frac{t_k + \tau_t^h}{2}$  so that  $t - t_{j+1} \geq \frac{t-t_k}{2}$  while when  $j \geq \frac{k-1+\lfloor \frac{t}{h} \rfloor}{2}$ ,  $t_j - t_k \geq \frac{\tau_t^h - h - t_k}{2} > \frac{t-t_k}{4}$ , the last sum is smaller than

$$\begin{aligned}
& \frac{2^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{\rho}}}{(t - t_k)^{\frac{1+\tilde{\alpha}}{2}}} \sum_{j=k+1}^{\ell-1} \mathbb{I}_{\{j < \frac{k-1+\lfloor \frac{t}{h} \rfloor}{2}\}} \int_{t_j}^{t_{j+1}} \frac{\|b(s, \cdot)\|_{L^\rho}}{(s - t_k)^{\frac{d}{2\rho}}} ds \\
& + 4^{\frac{d}{2\rho}} \sum_{j=k+1}^{\ell-1} \mathbb{I}_{\{j \geq \frac{k-1+\lfloor \frac{t}{h} \rfloor}{2}\}} \int_{t_j}^{t_{j+1}} \frac{\|b(s, \cdot)\|_{L^\rho}}{(t - s)^{\frac{1+\tilde{\alpha}}{2} + \frac{d}{2\rho}}} ds \\
& \leq C \|b\|_{L^q - L^\rho} \frac{(\frac{t_k + \tau_t^h}{2} - t_k)^{\frac{1+\alpha}{2}}}{(t - t_k)^{\frac{1+\tilde{\alpha}}{2}}} + C \|b\|_{L^q - L^\rho} (t - t_\ell)^{\frac{\alpha - \tilde{\alpha}}{2}},
\end{aligned}$$

where we used Hölder's inequality in time for the last inequality. Since  $t - t_\ell \geq t - u = |y - y'|^2$  and  $|y - y'|^2 \leq \frac{t-t_k}{4}$ , we deduce that

$$|T_2| \leq C |y - y'|^\alpha (g_c(t - t_k, y' - x) + g_c(t - t_k, y - x)).$$

Using  $|b_h| \leq |b|$ , (2.3), (2.20) and (2.11), then Gaussian convolution, we obtain that

$$\mathcal{E}_r^h \leq \frac{C}{h} \int_{\tau_r^h}^{\tau_r^h+h} \int_{\mathbb{R}^d} g_c(\tau_r^h - t_k, z - x) \frac{|b(s, z)|}{(t-r)^{\frac{1}{2}}} (g_c(t - \tau_r^h, y' - z) + g_c(t - \tau_r^h, y - z)) dz ds.$$

When  $u < \tau_t^h - h$  which implies that  $t - u > h$ , using  $t - r > \frac{t - \tau_r^h}{2}$  for  $r \leq \tau_t^h - h$ , then (2.9) and last that

$$\begin{aligned}
\tau_u^h - t_k & > u - h - t_k = t - t_k - (t - u) - h > t - t_k - 2(t - u) = t - t_k - 2|y - y'|^2 \\
& \geq \frac{t - t_k}{2},
\end{aligned}$$

and Hölder's inequality in time, we deduce that

$$\begin{aligned} |T_3| &\leq C \left( g_c(t - t_k, y' - x) + g_c(t - t_k, y - x) \right) \sum_{j=\lfloor \frac{u}{h} \rfloor}^{\lfloor \frac{t}{h} \rfloor - 2} \int_{t_j}^{t_{j+1}} \frac{(t - t_k)^{\frac{d}{2\rho}} \|b(s, \cdot)\|_{L^\rho}}{(t_j - t_k)^{\frac{d}{2\rho}} (t - s)^{\frac{1}{2} + \frac{d}{2\rho}}} ds \\ &\leq C \left( g_c(t - t_k, y' - x) + g_c(t - t_k, y - x) \right) \|b\|_{L^q - L^\rho} (t - \tau_u^h)^{\frac{\alpha}{2}}, \end{aligned}$$

where, by definition of  $u$  and since  $u < \tau_t^h - h$ ,  $(t - \tau_u^h)^{\frac{\alpha}{2}} \leq (|y' - y|^2 + h)^{\frac{\alpha}{2}} < (2|y - y'|^2)^{\frac{\alpha}{2}}$ .

**4. Density estimates for the diffusion.** The section is devoted to the proof of Proposition 2.3. We focus without loss of generality on the case  $\alpha < 1$ , which in particular implies that either  $\rho$  or  $q$  is finite. Indeed, when  $b \in L^\infty - L^\infty$  i.e.  $\alpha = 1$ , since we are considering a compact time interval,  $b$  also belongs to  $L^q - L^\infty$  for each  $q \in (2, \infty)$ . Notice that, for a time-space bounded drift, the estimates of the proposition are known, see e.g. [38].

**Step 1: tightness.** Let us first assume that  $\rho \geq 2$ . Using  $|b_h| \leq B h^{-\left(\frac{1}{q} + \frac{d}{2\rho}\right)} \wedge |b|$ , the independence between  $U_{\lfloor \frac{s}{h} \rfloor}$  and  $X_{\tau_s^h}^h$ , then (2.3), Hölder's inequality in space combined with  $\|g_c(t, \cdot)\|_{L^{\rho'}} = \rho'^{-\frac{d}{2\rho'}} (2\pi c t)^{-(1 - \frac{1}{\rho'})\frac{d}{2}}$ , that  $t_k \geq \frac{s}{2}$  for  $s \in [t_k, t_{k+1}]$  with  $k \geq 1$  and last Hölder's inequality in time, we obtain

$$\begin{aligned} &\mathbb{E} \left[ \int_0^T \left| b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) \right|^2 ds \right] \\ &\leq B^2 h^\alpha + \mathbb{E} \left[ \int_h^T \left| b \left( s, X_{\tau_s^h}^h \right) \right|^2 ds \right] \\ &\leq B^2 h^\alpha + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} |b(s, y)|^2 g_c(t_k, y - x) dy ds \\ &\leq B^2 h^\alpha + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \|b(s, \cdot)\|_{L^\rho}^2 \|g_c(t_k, \cdot)\|_{L^{\frac{\rho}{\rho-2}}} ds \\ &\leq B^2 h^\alpha + C \sum_{k=1}^{n-1} t_k^{-\frac{d}{\rho}} \int_{t_k}^{t_{k+1}} \|b(s, \cdot)\|_{L^\rho}^2 ds \leq B^2 h^\alpha + C \int_0^T s^{-\frac{d}{\rho}} \|b(s, \cdot)\|_{L^\rho}^2 ds \\ (4.1) \quad &\leq B^2 T^\alpha + C \|b\|_{L^q - L^\rho}^2 T^\alpha. \end{aligned}$$

When  $\rho \in (1, 2)$ , which in view of (1.2) implies that  $d = 1$  and  $\frac{\rho}{q} < \frac{\rho-1}{2} < \frac{1}{2}$ , we obtain in the same way that

$$(4.2) \quad \mathbb{E} \left[ \int_0^T \left| b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) \right|^\rho ds \right] \leq B^\rho T^{\frac{1}{2} - \frac{\rho}{q}} + C \|b\|_{L^q - L^\rho}^\rho T^{\frac{1}{2} - \frac{\rho}{q}}.$$

Since, by Hölder's inequality,

$$\forall 0 \leq u \leq t \leq T, \left| \int_u^t b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) ds \right| \leq (t - u)^{\frac{2 \wedge \rho - 1}{2 \wedge \rho}} \left( \int_0^T \left| b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) \right|^{2 \wedge \rho} ds \right)^{\frac{1}{2 \wedge \rho}},$$

with the Ascoli-Arzelà theorem, we deduce the tightness of the laws of the continuous processes

$$\left( \int_0^t b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) ds \right)_{t \in [0, T]}$$

indexed by  $h = \frac{T}{n}$  with  $n \in \mathbb{N}^*$ , when the space  $\mathcal{C}$  of continuous functions from  $[0, T]$  to  $\mathbb{R}^d$  is endowed with the supremum norm. With the continuity of the sum on this space, we deduce that the laws  $P^h$  of  $X^h$  are tight. We may extract a subsequence still denoted by  $(P^h)$  for notational simplicity such that  $P^h$  weakly converges to some limit  $P$  as  $h \rightarrow 0$ . For fixed  $t \in (0, T]$ , the weak convergence of  $P_t^h(dy) = \Gamma^h(0, x, t, y)dy$  to  $P_t(dy)$  together with (2.4), (2.3) and the Ascoli-Arzelà theorem, ensure that  $P_t(dy) = \Gamma(0, x, t, y)dy$  for some function  $\Gamma$  satisfying (2.7) and (2.6).

**Step 2:  $P$  solves the martingale problem.** Let  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^2$  function with compact support,  $\psi : (\mathbb{R}^d)^p \rightarrow \mathbb{R}$  be continuous and bounded,  $0 \leq s_1 \leq s_2 \leq \dots \leq s_p < u \leq t \leq T$  with  $u > 0$  and  $F$  denote the functional on  $\mathcal{C}$  defined by

$$F(\xi) = \left( \varphi(\xi_t) - \varphi(\xi_u) - \int_u^t \left( \frac{1}{2} \Delta \varphi(\xi_s) + b(s, \xi_s) \cdot \nabla \varphi(\xi_s) \right) ds \right) \psi(\xi_{s_1}, \dots, \xi_{s_p}).$$

We are going to check in the next step of the proof that  $\lim_{h \rightarrow 0} \mathbb{E}[F(X^h)] = 0$ . Unfortunately, the lack of continuity of the functional  $F$  on  $\mathcal{C}$  prevents from deducing immediately that  $\int_{\mathcal{C}} F(\xi) P(d\xi) = 0$ . That is why we introduce for  $\varepsilon \in (0, 1]$ , a smooth and bounded function  $b_\varepsilon$  approximating the original drift  $b$  in (1.1) such that, setting  $b_\varepsilon^K(t, x) = \mathbb{I}_{[-K, K]^d}(x) b_\varepsilon(t, x)$  and  $b^K(t, x) = \mathbb{I}_{[-K, K]^d}(x) b(t, x)$  for  $K \in \mathbb{N}^*$ ,

$$(4.3) \quad \forall K \in \mathbb{N}^*, \|b_\varepsilon^K - b^K\|_{L^{\tilde{q}} - L^{\tilde{p}}} \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$\text{with } (\tilde{\rho}, \tilde{q}) = \begin{cases} (\rho, q) & \text{if } \rho < \infty \text{ and } q < \infty, \\ (\rho, \frac{2\rho+1}{\rho-d}) & \text{if } \rho < \infty \text{ and } q = \infty, \\ (\frac{dq+1}{q-2}, q) & \text{if } \rho = \infty \text{ and } q < \infty. \end{cases}$$

Note that  $\frac{d}{\tilde{\rho}} + \frac{2}{\tilde{q}} < 1$ . The functional  $F_\varepsilon$  defined like  $F$  but with  $b_\varepsilon$  replacing  $b$  is continuous and bounded and therefore, for fixed  $\varepsilon \in (0, 1]$ ,  $\int_{\mathcal{C}} F_\varepsilon(\xi) P(d\xi) = \lim_{h \rightarrow 0} \mathbb{E}[F_\varepsilon(X^h)] = \lim_{h \rightarrow 0} \mathbb{E}[F_\varepsilon(X^h) - F(X^h)]$ . We deduce that

$$\left| \int_{\mathcal{C}} F(\xi) P(d\xi) \right| \leq \limsup_{\varepsilon \rightarrow 0} \int_{\mathcal{C}} |F(\xi) - F_\varepsilon(\xi)| P(d\xi) + \limsup_{\varepsilon \rightarrow 0} \limsup_{h \rightarrow 0} \mathbb{E}[|F_\varepsilon(X^h) - F(X^h)|].$$

Let  $K \in \mathbb{N}^*$  be such that  $\varphi$  vanishes outside  $[-K, K]^d$ . One has, using (2.3) then Hölder's inequality in space together with  $\|g_c(s, \cdot)\|_{L^{\frac{\tilde{\rho}}{\tilde{\rho}-1}}} = \left( \frac{\tilde{\rho}}{\tilde{\rho}-1} \right)^{-\frac{d(\tilde{\rho}-1)}{2\tilde{\rho}}} (2\pi cs)^{-\frac{d}{2\tilde{\rho}}}$  and last Hölder's inequality in time,

$$\begin{aligned} \mathbb{E}[|F_\varepsilon(X^h) - F(X^h)|] &\leq \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \int_u^t \mathbb{E}[|b_\varepsilon^K(s, X_s^h) - b^K(s, X_s^h)|] ds \\ &\leq C \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \int_u^t \int_{\mathbb{R}^d} |b_\varepsilon^K(s, y) - b^K(s, y)| g_c(s, y-x) dy ds \\ &\leq C \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \int_u^t \frac{\|b_\varepsilon^K(s, \cdot) - b^K(s, \cdot)\|_{L^{\tilde{\rho}}}}{s^{\frac{d}{2\tilde{\rho}}}} ds \\ &\leq C \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \|b_\varepsilon^K - b^K\|_{L^{\tilde{q}} - L^{\tilde{p}}}(t-u)^{1 - \left(\frac{1}{\tilde{q}} + \frac{d}{2\tilde{\rho}}\right)}. \end{aligned}$$

Since the same estimation holds for  $\int_{\mathcal{C}} |F(\xi) - F_\varepsilon(\xi)| P(d\xi)$ , we conclude that  $\int_{\mathcal{C}} F(\xi) P(d\xi) = 0$ . Taking  $\varphi, \psi, u, s_1, \dots, s_p, t$  in countable dense subsets, we deduce that  $P$  solves the martingale problem associated with the stochastic differential equation

$$X_t = x + W_t + \int_0^t b(s, X_s) dr, \quad t \in [0, T].$$

Since by [30] and [58], existence of a pathwise unique strong solution holds for this equation,  $P$  is the distribution of the solution.

**Step 3: proof of  $\lim_{h \rightarrow 0} \mathbb{E}[F(X^h)] = 0$ .** We compute  $\varphi(X_t^h) - \varphi(X_u^h)$  by Itô's formula and take expectations to obtain that

$$\begin{aligned} \mathbb{E}[F(X^h)] &= \mathbb{E} \left[ \left( \int_u^t \left( b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) - b(s, X_s^h) \right) \cdot \nabla \varphi(X_s^h) ds \right) \psi(X_{s_1}^h, \dots, X_{s_p}^h) \right] \\ &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \text{ with} \\ \Delta_1 &= \mathbb{E} \left[ \left( \int_u^t b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) \cdot \left( \nabla \varphi(X_s^h) - \nabla \varphi(X_{\tau_s^h}^h) \right) ds \right) \psi(X_{s_1}^h, \dots, X_{s_p}^h) \right], \\ \Delta_2 &= \mathbb{E} \left[ \left( \int_u^t \left( b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) - b_h(s, X_{\tau_s^h}^h) \right) \cdot \nabla \varphi(X_{\tau_s^h}^h) ds \right) \psi(X_{s_1}^h, \dots, X_{s_p}^h) \right], \\ \Delta_3 &= \mathbb{E} \left[ \left( \int_u^t \left( b_h(s, X_{\tau_s^h}^h) - b(s, X_{\tau_s^h}^h) \right) \cdot \nabla \varphi(X_{\tau_s^h}^h) ds \right) \psi(X_{s_1}^h, \dots, X_{s_p}^h) \right], \\ \Delta_4 &= \mathbb{E} \left[ \left( \int_u^t \left( b(s, X_{\tau_s^h}^h) \cdot \nabla \varphi(X_{\tau_s^h}^h) - b(s, X_s^h) \cdot \nabla \varphi(X_s^h) \right) ds \right) \psi(X_{s_1}^h, \dots, X_{s_p}^h) \right]. \end{aligned}$$

By Hölder's inequality and either (4.1) or (4.2),

$$\begin{aligned} |\Delta_1| &\leq C \|\psi\|_{L^\infty} \left( \int_u^t \mathbb{E} \left| \nabla \varphi(X_s^h) - \nabla \varphi(X_{\tau_s^h}^h) \right|^{\frac{2 \wedge \rho}{2 \wedge \rho - 1}} ds \right)^{\frac{2 \wedge \rho - 1}{2 \wedge \rho}} \\ &\leq C \|\psi\|_{L^\infty} (2 \|\nabla \varphi\|_{L^\infty})^{\frac{1}{2 \wedge \rho}} \|\nabla^2 \varphi\|_{L^\infty}^{\frac{2 \wedge \rho - 1}{2 \wedge \rho}} \left( \int_u^t \mathbb{E} \left[ |W_s - W_{\tau_s^h}| + \int_{\tau_s^h}^s |b_h(U_{\lfloor \frac{r}{h} \rfloor}, X_{\tau_r^h}^h)| dr \right] ds \right)^{\frac{2 \wedge \rho - 1}{2 \wedge \rho}} \\ &\leq Ch^{\left( \frac{2 \wedge \rho - 1}{2 \wedge \rho} \right)^2}. \end{aligned}$$

Since  $\mathbb{E} \left[ \int_{\lfloor \frac{u}{h} \rfloor h}^{\tau_t^h} \left( b_h \left( U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h \right) - b_h(s, X_{\tau_s^h}^h) \right) \cdot \nabla \varphi(X_{\tau_s^h}^h) ds | \sigma(X_r^h, r \in [0, \lfloor \frac{u}{h} \rfloor h]) \right] = 0$ ,  $|b_h| \leq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}$ ,  $\lfloor \frac{u}{h} \rfloor h - u \leq h$  and  $t - \tau_t^h \leq h$ , we have

$$\begin{aligned} |\Delta_2| &\leq \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left[ \int_u^{\lfloor \frac{u}{h} \rfloor h} |b_h(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h)| + |b_h(s, X_{\tau_s^h}^h)| ds \right. \\ &\quad \left. + \int_{\tau_t^h}^t |b_h(U_{\lfloor \frac{s}{h} \rfloor}, X_{\tau_s^h}^h)| + |b_h(s, X_{\tau_s^h}^h)| ds \right] \leq Ch^{\frac{1+\alpha}{2}}. \end{aligned}$$

Since  $|b_h - b| \leq |b| \mathbb{I}_{\{|b| \geq Bh^{-(\frac{1}{q} + \frac{d}{2\rho})}\}} \leq B^{1-2\wedge\rho} h^{(2\wedge\rho-1)(\frac{1}{q} + \frac{d}{2\rho})} |b|^{2\wedge\rho}$ , using (4.1) or (4.2) for the last inequality, we have that when  $h \leq u$ ,

$$|\Delta_3| \leq B^{1-2\wedge\rho} h^{(2\wedge\rho-1)(\frac{1}{q} + \frac{d}{2\rho})} \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \mathbb{E} \left[ \int_h^T |b(s, X_{\tau_s^h}^h)|^{2\wedge\rho} ds \right] \leq Ch^{(2\wedge\rho-1)(\frac{1}{q} + \frac{d}{2\rho})}.$$

We recall that  $s_p < u$  and suppose that  $h$  is small enough so that  $s_p \leq \tau_u^h - h$ . Using (2.5), (2.3) and Gaussian convolution for the third inequality, then  $\tau_s^h + h - \tau_u^h > s - \tau_u^h \geq s - u$ ,  $\|g_c(s, \cdot)\|_{L^{\rho'}} = \rho'^{-\frac{d}{2\rho'}} (2\pi cs)^{-(1-\frac{1}{\rho'})\frac{d}{2}}$  with  $\frac{1}{\rho'} = 1 - \frac{1}{\rho}$ , Hölder's inequality in space and in time, we obtain that

$$\begin{aligned}
|\Delta_4| &\leq \|\psi\|_{L^\infty} \mathbb{E} \left[ \left| \mathbb{E} \left[ \int_u^t \left( b(s, X_{\tau_s^h}^h) \cdot \nabla \varphi(X_{\tau_s^h}^h) - b(s, X_s^h) \cdot \nabla \varphi(X_s^h) \right) ds \middle| (X_r^h)_{0 \leq r \leq \tau_u^h - h} \right] \right| \right] \\
&\leq \|\psi\|_{L^\infty} \|\nabla \varphi\|_{L^\infty} \int_u^t \int_{\mathbb{R}^d \times \mathbb{R}^d} |\Gamma^h(\tau_u^h - h, y, \tau_s^h, z) - \Gamma^h(\tau_u^h - h, y, s, z)| \\
&\quad |b(s, z)| \Gamma^h(0, x, \tau_u^h - h, y) dz dy ds \\
&\leq C \int_u^t \int_{\mathbb{R}^d} \frac{(s - \tau_s^h)^{\frac{\alpha}{2}}}{(\tau_s^h + h - \tau_u^h)^{\frac{\alpha}{2}}} g_c(s, z - x) |b(s, z)| dz ds \leq Ch^{\frac{\alpha}{2}} \int_u^t \frac{\|b(s, \cdot)\|_{L^\rho} ds}{(s - u)^{\frac{\alpha}{2} + \frac{d}{2\rho}}} \\
&\leq Ch^{\frac{\alpha}{2}} \|b\|_{L^q - L^\rho}(t - u)^{\frac{1}{2}}.
\end{aligned}$$

We conclude that  $\lim_{h \rightarrow 0} \mathbb{E}[F(X^h)] = 0$ .

**Step 4: proof of the Duhamel representation (2.8) for the density of the diffusion.** This can be done by reasoning like in the above derivation of (2.1) in Subsection 3.1 and using that by (2.11), (2.6) and Lemma 2.4,

$$\begin{aligned}
&\int_0^t \mathbb{E} [|b(r, X_r) \cdot \nabla_y g_1(t - r, y - X_r)|] dr \\
&\leq C \int_0^t \int_{\mathbb{R}^d} g_c(r, z - x) |b(r, z)| \frac{g_c(t - r, y - z)}{(t - r)^{\frac{1}{2}}} dz dr \\
&\leq C \|b\|_{L^q - L^\rho} g_c(t, y - x) t^{\frac{\alpha}{2}} < +\infty.
\end{aligned}$$

Using that, by (2.12),

$$|\nabla_z g_1(t - r, z) - \nabla_z g_1(t - r, z')| \leq C \frac{|z - z'|^{\frac{\alpha}{2}}}{(t - r)^{\frac{2+\alpha}{4}}} (g_c(t - r, z) + g_c(t - r, z')),$$

one derives in the same way

$$\begin{aligned}
&\int_0^t \mathbb{E} [|b(r, X_r) \cdot (\nabla_y g_1(t - r, y - X_r) - \nabla_y g_1(t - r, y' - X_r))|] dr \\
&\leq C \|b\|_{L^q - L^\rho} (g_c(t, y - x) + g_c(t, y' - x)) t^{\frac{\alpha}{4}} |y - y'|^{\frac{\alpha}{2}},
\end{aligned}$$

which yields the continuity in the spatial variable  $y$ .

#### APPENDIX A: PROOF OF LEMMA 2.5: USUAL GAUSSIAN ESTIMATES

**PROOF.** The bound (2.11) is standard. If  $|x - x'| \geq u^{\frac{1}{2}}$  then (2.12) precisely follows from the first inequality in (2.11). Namely,

$$\left| \nabla_x^\zeta g_1(u, x) - \nabla_x^\zeta g_1(u, x') \right| \leq \left| \nabla_x^\zeta g_1(u, x) \right| + \left| \nabla_x^\zeta g_1(u, x') \right| \leq \frac{C}{u^{\frac{|\zeta|}{2}}} (g_c(u, x) + g_c(u, x')).$$

Assume now  $|x - x'| \leq u^{1/2}$ . Write

$$\nabla_x^\zeta g_1(u, x') - \nabla_x^\zeta g_1(u, x) = \int_0^1 d\lambda \nabla_x \nabla_x^\zeta g_1(u, x + \lambda(x' - x)) \cdot (x' - x).$$

Observe now that from the first inequality in (2.11) applied with  $c$  and  $\zeta$  replaced by  $\frac{1+c}{2} > 1$  and the sum of  $\zeta$  and a vector in the canonical basis of  $\mathbb{R}^d$  (note that (2.11) remains valid for multi-indices with length bounded from above by 3) :

$$(A.1) \quad |\nabla_x^\zeta g_1(u, x') - \nabla_x^\zeta g_1(u, x)| \leq C \frac{|x - x'|}{u^{\frac{d+1+|\zeta|}{2}}} \int_0^1 d\lambda \exp\left(-\frac{|x + \lambda(x' - x)|^2}{(1+c)u}\right).$$

Recall now that since  $\lambda \in [0, 1]$  and that  $|x - x'| \leq u^{\frac{1}{2}}$ :

$$\begin{aligned} |x + \lambda(x' - x)|^2 &\geq \frac{1+c}{2c}|x|^2 - \frac{c+1}{c-1}\lambda^2|x - x'|^2 \geq \frac{1+c}{2c}|x|^2 - \frac{c+1}{c-1}|x - x'|^2 \\ &\geq \frac{1+c}{2c}|x|^2 - \frac{c+1}{c-1}u, \end{aligned}$$

which plugged into (A.1) yields:

$$|\nabla_x^\zeta g_1(u, x) - \nabla_x^\zeta g_1(u, x')| \leq C \frac{|x - x'|}{u^{\frac{d+1+|\zeta|}{2}}} \exp\left(-\frac{|x|^2}{2cu}\right) \exp\left(\frac{1}{c-1}\right),$$

which, up to a modification of the constant  $C$ , concludes the proof of (2.12). The bound (2.13) is obtained the same way by bounding from above each term of the difference using the first inequality in (2.11) when  $|u' - u| \geq u$  and by integrating the second inequality in (2.11) and using that for  $v \in [u, u']$ ,  $\frac{g_c(v, x)}{v^{1+\frac{|\zeta|}{2}}} \leq \frac{u'^{\frac{d}{2}} g_c(u', x)}{u^{\frac{d}{2}+1+\frac{|\zeta|}{2}}} \leq 2^{\frac{d}{2}} \frac{g_c(u', x)}{u^{1+\frac{|\zeta|}{2}}}$  when  $|u' - u| \leq u$ .  $\square$

## APPENDIX B: PROOF OF LEMMA 2.6

PROOF. (i) Iterating (2.14)  $n \geq 1$  times, we get that for all  $t \in [0, T]$ ,

$$\begin{aligned} f(t) &\leq \eta \left( 1 + \sum_{k=1}^{n-1} \delta^k t^\beta \int_0^t s_1^{\beta-\beta_1} \int_0^{s_1} s_2^{\beta-\beta_1} \dots \int_0^{s_{k-2}} s_{k-1}^{\beta-\beta_1} \int_0^{s_{k-1}} \frac{ds_k}{s_k^{\beta_1}} ds_{k-1} \dots ds_1 \right) \\ &\quad + \delta^n t^\beta \int_0^t s_1^{\beta-\beta_1} \int_0^{s_1} s_2^{\beta-\beta_1} \dots \int_0^{s_{n-2}} s_{n-1}^{\beta-\beta_1} \int_0^{s_{n-1}} \frac{f(s_n) ds_n}{s_n^{\beta_1}} ds_{n-1} \dots ds_1 \end{aligned}$$

The term with index  $k$  in the sum is equal to  $\frac{\delta^k t^{k(1+\beta-\beta_1)}}{\prod_{j=1}^k (j(1+\beta-\beta_1)-\beta)}$  so that the first line in the right-hand side is not greater than the finite constant not depending on  $f$

$$\begin{aligned} C_{\beta, \beta_1, \eta, \delta, T} &= \eta + \eta \sum_{k \geq 1} \frac{\delta^k T^{k(1+\beta-\beta_1)}}{\prod_{j=1}^k (j(1+\beta-\beta_1)-\beta)} \leq \eta + \eta \sum_{k \geq 1} \frac{\delta^k T^{k(1+\beta-\beta_1)}}{\prod_{j=1}^k j(1+\beta \wedge 0 - \beta_1)} \\ &= \eta e^{\frac{\delta T^{1+\beta-\beta_1}}{1+\beta \wedge 0 - \beta_1}}. \end{aligned}$$

On the other hand, the last term in the right-hand side is bounded from above by  $\frac{\delta^n t^{n(1+\beta-\beta_1)}}{\prod_{j=1}^n (j(1+\beta-\beta_1)-\beta)} \sup_{s \in [0, t]} f(s)$  and converges to 0 as  $n \rightarrow \infty$ .

(ii) If  $\beta_2 \leq 0$ , then (2.14) holds with  $\eta = a$ ,  $\delta = b$  and  $\beta = \beta_3 - \beta_2$  which is larger than  $\beta_1 - 1$  so that the conclusion follows by (i). Let us suppose that  $\beta_2 > 0$  and check that by iterating the inequality (2.15), we obtain (2.14) for some finite constants  $\eta, \delta$  and some  $\beta > \beta_1 - 1$ . Iterating (2.15) once, we obtain

(B.1)

$$\forall t \in [0, T], f(t) \leq a + abt^{\beta_3} \int_0^t \frac{ds}{s^{\beta_1}(t-s)^{\beta_2}} + b^2 t^{\beta_3} \int_0^t \frac{1}{s^{\beta_1-\beta_3}(t-s)^{\beta_2}} \int_0^s \frac{f(u) du}{u^{\beta_1}(s-u)^{\beta_2}} ds.$$



We now set  $\gamma := 1 + \beta_3 - \beta_1 - \beta_2 > 0$  and distinguish two situations depending on the sign of  $\beta_1 - \beta_3$ . Let us first suppose that  $\beta_1 - \beta_3 \geq 0$ . Using Fubini's theorem and the inequality  $s^{\beta_1 - \beta_3} \geq (s - u)^{\beta_1 - \beta_3}$ , we obtain

$$\begin{aligned} f(t) &\leq a + abT^\gamma B(1 - \beta_1, 1 - \beta_2) + b^2 t^{\beta_3} \int_0^t \frac{f(u)}{u^{\beta_1}} \int_u^t \frac{ds}{(s - u)^{\beta_2 + \beta_1 - \beta_3} (t - s)^{\beta_2}} du \\ &= a_1 + b_1 t^{\beta_3} \int_0^t \frac{f(u) du}{u^{\beta_1} (t - u)^{\beta_2 - \gamma}} \end{aligned}$$

with  $a_1 = a + abT^\gamma B(1 - \beta_1, 1 - \beta_2)$  and  $b_1 = b^2 B(\gamma, 1 - \beta_2)$ . In comparison with (2.15), after this first step, the power of the last factor in the denominator has decreased from  $\beta_2$  to  $\beta_2 - \gamma$ . We may now at each step iterate the inequality obtained from the previous step and obtain after  $n$  steps

$$\forall t \in [0, T], f(t) \leq a_n + b_n t^{\beta_3} \int_0^t \frac{f(s) ds}{s^{\beta_1} (t - s)^{\beta_2 - (2^n - 1)\gamma}}$$

where  $a_n = a_{n-1} + a_{n-1} b_{n-1} T^{2^{n-1}\gamma} B(1 - \beta_1, 1 + (2^{n-1} - 1)\gamma - \beta_2)$  depends linearly on  $a$  and  $b_n = b_{n-1}^2 B(2^{n-1}\gamma, 1 + (2^{n-1} - 1)\gamma - \beta_2)$ . For  $\hat{n} = \lceil \frac{\ln(1 + \beta_2/\gamma)}{\ln 2} \rceil$ ,  $\beta_2 - (2^{\hat{n}} - 1)\gamma \leq 0$ , so that (2.14) holds for  $\eta = a_{\hat{n}}$  and  $\delta = b_{\hat{n}}$  and

$$\beta = \beta_3 + (2^{\hat{n}} - 1)\gamma - \beta_2 = 2^{\hat{n}}\gamma + \beta_1 - 1 > \beta_1 - 1 \text{ since } \hat{n} \geq 1 \text{ as } \beta_2 > 0.$$

Let us now suppose that  $\beta_1 - \beta_3 < 0$ . Inserting the inequality  $s^{\beta_3 - \beta_1} \leq t^{\beta_3 - \beta_1}$  in (B.1), we get

$$\begin{aligned} f(t) &\leq a + abT^\gamma B(1 - \beta_1, 1 - \beta_2) + b^2 t^{2\beta_3 - \beta_1} \int_0^t \frac{f(u)}{u^{\beta_1}} \int_u^t \frac{ds}{(s - u)^{\beta_2} (t - s)^{\beta_2}} du \\ &= a_1 + b_1 t^{2\beta_3 - \beta_1} \int_0^t \frac{f(u) du}{u^{\beta_1} (t - u)^{2\beta_2 - 1}} \end{aligned}$$

with  $a_1 = a + abT^\gamma B(1 - \beta_1, 1 - \beta_2)$  and  $b_1 = b^2 B(1 - \beta_2, 1 - \beta_2)$ . In comparison with (2.15), after this first step, the power of  $t$  has increased from  $\beta_3$  to  $2\beta_3 - \beta_1$  and the power in the last factor of the denominator has decreased from  $\beta_2$  to  $2\beta_2 - 1$ . We may now at each step iterate the inequality obtained from the previous step and obtain after  $n$  steps

$$\forall t \in [0, T], f(t) \leq a_n + b_n t^{2^n(\beta_3 - \beta_1) + \beta_1} \int_0^t \frac{f(s) ds}{s^{\beta_1} (t - s)^{2^n(\beta_2 - 1) + 1}}$$

where  $a_n = a_{n-1} + a_{n-1} b_{n-1} T^{2^{n-1}\gamma} B(1 - \beta_1, 2^{n-1}(1 - \beta_2))$  depends linearly on  $a$  and  $b_n = b_{n-1}^2 B(2^{n-1}(1 - \beta_2), 2^{n-1}(1 - \beta_2))$ . For  $\hat{n} = \lceil \frac{-\ln(1 - \beta_2)}{\ln 2} \rceil$ ,  $2^{\hat{n}}(\beta_2 - 1) + 1 \leq 0$ , so that (2.14) holds for  $\eta = a_{\hat{n}}$  and  $\delta = b_{\hat{n}}$  and

$$\beta = 2^{\hat{n}}(\beta_3 - \beta_1) + \beta_1 + 2^{\hat{n}}(1 - \beta_2) - 1 = 2^{\hat{n}}\gamma + \beta_1 - 1 > \beta_1 - 1 \text{ since } \hat{n} \geq 1 \text{ as } \beta_2 > 0.$$

□

## REFERENCES

- [1] S. Albeverio, Y.G. Kondratiev, and M. Röckner. *Strong Feller properties for distorted Brownian motion and applications to finite particle systems with singular interactions*. In: Finite and Infinite Dimensional Analysis in Honor of Leonard Gross, H. H. Kuo et al., (eds.), Contemporary Mathematics, Vol. 317, Amer. Math. Soc., 2003.
- [2] V. Bally and D. Talay. *The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function*. Prob. Th. Rel. Fields, 104-1:43–60, 1996.

- [3] V. Bally and D. Talay, *The law of the Euler scheme for stochastic differential equations, II. Convergence rate of the density*. Monte-Carlo methods and Appl., 2:93–128, 1996.
- [4] J. Bao, X. Huang and C. Yuan, *Convergence Rate of Euler-Maruyama Scheme for SDEs with Hölder-Dini Continuous Drifts*. Journal of Theoretical Probability, Vol.32, pp.848–871, 2019.
- [5] O. Bencheikh and B. Jourdain, *Convergence in total variation of the Euler-Maruyama scheme applied to diffusion processes with measurable drift coefficient and additive noise*. SIAM J. Numer. Anal., Vol.60, pp.1701–1740, 2022.
- [6] P.E. Chaudru de Raynal and S. Menozzi, *On Multidimensional stable-driven Stochastic Differential Equations with Besov drift*. Electron. J. Probab. 27, Paper No. 163, 52 pp, 2022.
- [7] K. Dareiotis, M. Gerencsér and K. Lê, *Quantifying a convergence theorem of Gyöngy and Krylov*. arXiv:2101.12185, to appear in *Annals of Applied Probability*.
- [8] K. Dareiotis and M. Gerencsér, *On the regularisation of the noise for the Euler-Maruyama scheme with irregular drift*. Electron. J. Probab. 25, Paper No. 82, 2020.
- [9] P. Étoré and M. Martinez, *Exact simulation for solutions of one-dimensional Stochastic Differential Equations with discontinuous drift*. ESAIM: Probability and Statistics, EDP Sciences, Vol.18, pp.686–702, 2014.
- [10] N. Frikha, *On the weak approximation of a skew diffusion by an Euler-type scheme*. Bernoulli, Vol.24(3), pp.1653–1691, 2018.
- [11] E. Gobet and C. Labart, *Sharp estimates for the convergence of the density of the Euler scheme in small time*. Elect. Comm. in Probab., 1,p 352-363, 2008.
- [12] S. Göttlich, K. Lux and A. Neuenkirch, *The Euler scheme for stochastic differential equations with discontinuous drift coefficient: A numerical study of the convergence rate*. Advances in Difference Equation, Vol.429, 2019.
- [13] J. Guyon, *Euler scheme and tempered distributions*. Stochastic Process. Appl. 116 , no. 6, 877-904, 2006.
- [14] I. Gyöngy and N. Krylov, *Existence of strong solutions for Ito's stochastic equations via approximations*. Probability Theory and Related Fields, Vol.105, pp.143–158, 1996.
- [15] I. Gyöngy, *A note on Euler's approximations*. Potential Analysis, Vol.8, pp.205–216 , 1998.
- [16] I. Gyöngy and M. Rasonyi, *A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients*. Stochastic Processes and their Applications, Vol.121, pp.2189–2200, 2011.
- [17] N. Halidias and P.E. Kloeden, *A note on the Euler-Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient*. BIT Numerical Mathematics, Vol.48, pp.51–59, 2008.
- [18] M. Hutzenthaler and A. Jentzen, *Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients*. Mem. Amer. Math. Soc. 236, no. 1112, 2015.
- [19] M. Hairer, M. Hutzenthaler and A. Jentzen, *Loss of regularity for Kolmogorov equations*. Ann. Probab. 43(2): 468–527, 2015.
- [20] M. Hutzenthaler, A.Jentzen and P.E. Kloeden, *Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients*. Ann. Appl. Probab. 22(4):1611–1641, 2012.
- [21] P.E. Jabin and Z. Whang, *Quantitative estimates of propagation of chaos for stochastic systems with  $W^{-1,\infty}$  kernel*. Invent. Math. Vol. 214, 523-591, 2018.
- [22] A. Jentzen, A. Neuenkirch, *A random Euler scheme for Carathéodory differential equations*. J. Comput. Appl. Math., Vol. 224(1), pp.346–359, 2009.
- [23] P. Kim, Panki and R. Song, *Two-sided estimates on the density of Brownian motion with singular drift*. Illinois J. Math. 50, n° 1-4, 635-688
- [24] A. Kohatsu-Higa, A. Lejay and K. Yasuda, *On Weak Approximation of Stochastic Differential Equations with Discontinuous Drift Coefficient*. Mathematical Economics, Kyoto, Japan, pp.94–106, 2012.
- [25] A. Kohatsu-Higa, A. Lejay and K. Yasuda, *Weak rate of convergence of the Euler-Maruyama scheme for stochastic differential equations with non-regular drift*. Journal of Computational and Applied Mathematics, Vol.326, pp.138–158, 2017.
- [26] V. Konakov and E. Mammen, *Edgeworth type expansions for euler schemes for stochastic differential equations*. Monte Carlo Methods Appl., 8–3:271–285, 2002.
- [27] V. Konakov, A. Kozhina, and S. Menozzi, *Stability of densities for perturbed diffusions and Markov chains*. ESAIM Probab. Stat., 21:88–112, 2017.
- [28] V. Konakov and S. Menozzi, *Weak error for the Euler scheme approximation of diffusions with non-smooth coefficients*. Electronic Journal of Probability, Vol.22(46), pp.1–47, 2017.
- [29] V. Konakov and E. Mammen, *Local limit theorems for transition densities of Markov chains converging to diffusions*. Prob. Th. Rel. Fields, 117:551–587, 2000.

- [30] N. Krylov and M. Röckner. *Strong solutions of stochastic equations with singular time dependent drift*. Prob. Theory Rel. Fields, 131:154–196, 2005.
- [31] N. Krylov. *On strong solutions of Itô's equations with  $a \in W_d^1$  and  $b \in L^d$* . Ann. Probab. 49(6):3142–3167, 2021. arXiv:2007.06040.
- [32] K. Lê and C. Ling. *Taming singular stochastic differential equations: A numerical method*. arXiv preprint arXiv:2110.01343, 2021.
- [33] V. Lemaire and S. Menozzi. *On some non asymptotic bounds for the Euler scheme*. Electronic Journal of Probability, 15:1645–1681, 2010.
- [34] G. Leobacher and M. Szölgényi. *A numerical method for SDEs with discontinuous drift*. BIT Numerical Mathematics, Vol.56, pp.151–162, 2016.
- [35] G. Leobacher and M. Szölgényi. *A strong order 1/2 method for multidimensional SDEs with discontinuous drift*. Annals of Applied Probability, Vol.27(4), pp.2382–2418, 2017.
- [36] G. Leobacher and M. Szölgényi. *Convergence of the Euler-Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient*. Numerische Mathematik, Vol.138(1), pp.219–239, 2017.
- [37] A. Makhlof. *Representation and Gaussian bounds for the density of Brownian motion with random drift*. Commun. Stoch. Anal., 10, n°2, Article 2, 151–162, 2016.
- [38] S. Menozzi, A. Pesce and X. Zhang. *Density and gradient estimates for non degenerate Brownian SDEs with unbounded measurable drift*. J. Differential Equations, 272:330–369, 2021.
- [39] S. Menozzi and X. Zhang. *Heat kernel of supercritical nonlocal operators with unbounded drifts*. J. Éc. Polytech. Math. 9:537–579, 2022.
- [40] R. Mikulevicius and E. Platen. *Rate of convergence of the Euler approximation for diffusion processes*. Mathematische Nachrichten, Vol.151, pp.233–239, 1991.
- [41] T. Müller-Gronbach and L. Yaroslavl'tseva. *On the performance of the Euler-Maruyama scheme for SDEs with discontinuous drift coefficient*. Annales de l'Institut Henri Poincaré - Probabilités et Statistiques, Vol.56(2), pp.1162–1178, 2020.
- [42] T. Müller-Gronbach and L. Yaroslavl'tseva. *A strong order 3/4 method for SDEs with discontinuous drift coefficient*. IMA J. Num. Anal. 42(1):229–259, 2022.
- [43] T. Müller-Gronbach and L. Yaroslavl'tseva. *Sharp lower error bounds for strong approximation of SDEs with discontinuous drift coefficient by coupling of noise*. The Annals of Applied Probability 33(2):1102–1135, 2023.
- [44] A. Neuenkirch, M. Szölgényi and L. Szpruch. *An Adaptive Euler-Maruyama Scheme for Stochastic Differential Equations with Discontinuous Drift and its Convergence Analysis*. SIAM Journal on Numerical Analysis, Vol.57(1), pp.378–403, 2019.
- [45] A. Neuenkirch and M. Szölgényi. *The Euler-Maruyama Scheme for SDEs with Irregular Drift: Convergence Rates via Reduction to a Quadrature Problem*. IMA J. Num. Anal., 41(2):1164–1196, 2021.
- [46] H. Ngo and D. Taguchi. *Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients*. Mathematics of Computation, Vol.85, pp.1793–1819, 2016.
- [47] H. Ngo and D. Taguchi. *On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients*. IMA Journal of Numerical Analysis, Vol.37(4), pp.1864–1883, 2017.
- [48] G. Pagès. *Numerical Probability*, Springer, 2018.
- [49] N. Perkowski and W. van Zuijlen. *Quantitative heat kernel estimates for diffusions with distributional drift*. ArXiv preprint arXiv:2009.10786, 2020.
- [50] N.I. Portenko. *Generalized diffusion processes*. Vol. 83, American Math. Soc, 1990.
- [51] M. Röckner and G. Zhao. *SDEs with critical time dependent drifts: weak solutions*. ArXiv preprint arXiv:2012.04161, 2020.
- [52] M. Röckner and G. Zhao. *SDEs with critical time dependent drifts: strong solutions*. ArXiv preprint arXiv:2103.05803, 2021.
- [53] S. Sabanis. *A note on tamed Euler approximations*. Electron. Commun. Probab. 18(47), 2013.
- [54] Y. Suo, C. Yuan and S. Zhang. *Weak convergence of Euler scheme for SDEs with low regular drift*. Numer. Algorithms 90(2):731–747, 2022.
- [55] D. Talay and L. Tubaro. *Expansion of the global error for numerical schemes solving stochastic differential equations*. Stoch. Anal. and App., 8-4:94–120, 1990.
- [56] X. Zhang. *Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients*. Electron. J. Probab., Vol. 16, No. 38, 1096–1116, 2011.
- [57] X. Zhang. *Stochastic differential equations with Sobolev drifts and driven by  $\alpha$ -stable processes*. Ann. Inst. Henri Poincaré, B. Vol. 49, No. 4, 1057–1079, 2013.

- [58] X. Zhang. *Stochastic differential equations with Sobolev diffusion and singular drift and applications*. Ann. Appl. Probab. 26(5):2697–2732, 2016.
- [59] X. Zhang. *A discretized version of Krylov’s estimate and its applications*. Elect. Journal of Probability, 24-131, p. 1-17, 2019.
- [60] X. Zhang and G. Zhao. *Heat kernel and ergodicity of SDEs with distributional drifts*. ArXiv preprint arXiv:1710.10537, 2017.
- [61] X. Zhang and G. Zhao. *Stochastic Lagrangian Path for Leray’s Solutions of 3D Navier-Stokes Equations*. Communications in Mathematical Physics, 381(2):1-35.
- [62] L. Yan, *The Euler scheme with irregular coefficients*. The Annals of Probability, Vol.30(3), pp.1172–1194, 2002.