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QUADRATURE AND SYMMETRY ON THE CUBED SPHERE

JEAN-BAPTISTE BELLET[†], MATTHIEU BRACHET[‡], AND JEAN-PIERRE CROISILLE[†]

ABSTRACT. In the companion paper [6], a spherical harmonics subspace associated to the Cubed Sphere has been introduced. This subspace is further analyzed here. In particular, it permits to define a new Cubed Sphere based quadrature. This quadrature inherits the rotationally invariant properties of the spherical harmonics subspace. Contrary to Gauss quadratures, where the set of nodes and weights is solution of a nonlinear system, only the weights are unknown here. Despite this conceptual simplicity, the new quadrature displays an accuracy comparable to optimal quadratures, such as the Lebedev rules.

1. INTRODUCTION

We introduce a new quadrature rule defined on the equiangular Cubed Sphere grid.

Let $CS_N \subset \mathbb{S}^2$ denote the Cubed Sphere with parameter N . In [6], a finite dimensional subspace of spherical harmonics has been associated to the set of nodes CS_N . This subspace, called \mathcal{U}_N hereafter, provides existence and uniqueness to the Lagrange interpolation, with CS_N as set of nodes. Here our goal is to use the space \mathcal{U}_N to define a new quadrature rule on the sphere \mathbb{S}^2 , which uses CS_N as set of nodes.

Previous studies have considered the construction of Gauss quadratures on the sphere. In this case, the set of nodes and the associated weights are both unknown. Establishing the set of equations for the nodes and the weights is difficult. The modern analysis of this approach relies on the theory of t-designs, whose main purpose is to find an optimal distribution of nodes on the sphere. This theory has a mathematical and physical interest in itself. We refer to [9] for a review. The identification of a quadrature associated to a particular t-design is obtained as byproduct in a second time. Once the quadrature nodes are known, the weights are commonly identified by requiring exactness for a set of spherical harmonics. A general approach is to select this set as all spherical harmonics of degree smaller than a given value.

Here the Cubed Sphere nodes are selected from the beginning as “good” quadrature nodes, and therefore, only the weights have to be identified. In [15], two examples of weights have been suggested. The first one was based on some extended trapezoidal rule, attributing some area to each node. The second one was thought as a perturbation with a design based on some optimization principle. See also [7] for some other rules, including some Simpson formula. Here we come back to the general question of the “best choice” of weights associated to the Cubed Sphere nodes. As in the general approach, we require exactness of the quadrature for a particular set of spherical harmonics. Using the space \mathcal{U}_N in [6] immediately delivers a quadrature rule. This quadrature is different from the ones mentioned above. The space \mathcal{U}_N remarkably enjoys invariance under the action of the group of the cube. This is somehow expected, since the group of CS_N is in fact the group of the cube, (or of the octahedron) [5]. As will be shown below, the new quadrature rule inherits this invariance. This property is highly desirable. It is well known that group invariance is the backbone for the design of highly accurate spherical quadratures, [1, 13, 14, 17]. Refer for this to the review [12].

The paper is organized as follows. Section 2 gives the notation on the spherical harmonics, the Cubed Sphere and its group. Section 3 recalls how is defined the space \mathcal{U}_N and Section 4 establishes several of its invariance properties. In Section 5, the new quadrature is introduced. By construction, this quadrature is exact on the space \mathcal{U}_N . In addition, it is invariant under the octahedral group. This in particular implies that it is exact for a proportion of 15/16 of all (real) Legendre spherical harmonics. Finally in Section 6, we display numerical results for a large series of test cases. It is observed that the new rule is only slightly suboptimal, when compared to the optimal Lebedev rules. This somehow supports the main Ansatz of this study, namely that the Cubed Sphere nodes are good quadrature nodes on the sphere. Finally we report some comments on the accuracy of the new rule in Appendix A.

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2. BACKGROUND AND NOTATION

The real Legendre spherical harmonics of degree n are normalized as follows:

$$Y_n^m(x) = Y_n^m(\theta, \phi) = (-1)^{|m|} \sqrt{\frac{(n+1/2)(n-|m|)!}{\pi(n+|m|)!}} P_n^{|m|}(\sin \theta) \times \begin{cases} \sin |m| \phi, & m < 0, \\ \frac{1}{\sqrt{2}}, & m = 0, \\ \cos m \phi, & m > 0. \end{cases}$$

Here, $P_n^{|m|}(t) = (-1)^{|m|} (1-t^2)^{|m|/2} \frac{d^{|m|+n}}{dt^{|m|+n}} \frac{1}{2^n n!} (t^2-1)^n$ is the associated Legendre function, and we identify $x = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta)$ with (θ, ϕ) ; $\phi \in [-\pi, \pi]$ is the longitude, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the latitude. The family $(Y_n^m)_{-n \leq m \leq n}$ is an orthonormal basis of the subspace $\mathcal{Y}_n \subset L^2(\mathbb{S}^2)$, for the inner product

$$\langle u, v \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} u(x)v(x)d\sigma.$$

The infinite family $(Y_n^m)_{|m| \leq n, n \in \mathbb{N}}$ is a Hilbert basis of $L^2(\mathbb{S}^2)$.

The equiangular Cubed Sphere $\text{CS}_N \subset \mathbb{S}^2$, is defined by

$$\text{CS}_N := \left\{ \frac{1}{\sqrt{1+u^2+v^2}}(\pm 1, u, v), \frac{1}{\sqrt{1+u^2+v^2}}(u, \pm 1, v), \frac{1}{\sqrt{1+u^2+v^2}}(u, v, \pm 1); \right. \\ \left. u = \tan \frac{i\pi}{2N}, v = \tan \frac{j\pi}{2N}, -\frac{N}{2} \leq i, j \leq \frac{N}{2} \right\}.$$

For ease of notation, we denote the nodes of CS_N by x_i , $1 \leq i \leq \bar{N}$, where $\bar{N} = 6N^2 + 2$ is the cardinal number:

$$\text{CS}_N = \{x_i, 1 \leq i \leq \bar{N}\}.$$

The symmetry group of the Cubed Sphere is given in the following theorem [5].

Theorem 1 (Symmetry group of the Cubed Sphere). *An orthogonal matrix Q leaves the Cubed Sphere CS_N invariant if, and only if, it leaves the octahedron*

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$$

invariant. In other words, the symmetry group \mathcal{G} of CS_N is exactly the full octahedral group. Its matrix representation is given by

$$\mathcal{G} = \{[\epsilon_1 e_{\sigma_1} \quad \epsilon_2 e_{\sigma_2} \quad \epsilon_3 e_{\sigma_3}], \sigma \in \mathfrak{S}_3, \epsilon \in \{-1, 1\}^3\}, \quad \text{with } e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

The space of real functions defined on CS_N is denoted by

$$\mathcal{F}(\text{CS}_N) = \{f : \text{CS}_N \rightarrow \mathbb{R}\}.$$

The canonical basis $(\delta_{x_i})_{1 \leq i \leq \bar{N}}$ of $\mathcal{F}(\text{CS}_N)$ is defined by

$$\delta_{x_i}(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad 1 \leq i, j \leq \bar{N}.$$

In this basis, any $f \in \mathcal{F}(\text{CS}_N)$ is represented by the column vector $[f(x_i)]_{1 \leq i \leq \bar{N}} \in \mathbb{R}^{\bar{N}}$, due to the decomposition

$$f = \sum_{i=1}^{\bar{N}} f(x_i) \delta_{x_i}.$$

For any real function defined on the sphere, $u : x \in \mathbb{S}^2 \mapsto u(x) \in \mathbb{R}$, the restriction of u on CS_N is the function defined by

$$u|_{\text{CS}_N} := \sum_{i=1}^{\bar{N}} u(x_i) \delta_{x_i} \in \mathcal{F}(\text{CS}_N), \quad u|_{\text{CS}_N}(x_i) = u(x_i), \quad 1 \leq i \leq \bar{N}. \quad (1)$$

3. INTERPOLATION ON THE CUBED SPHERE WITH SPHERICAL HARMONICS

In [6], we have introduced a subspace \mathcal{U}_N of real spherical harmonics, dedicated to Lagrange interpolation on the Cubed Sphere CS_N . Our presentation here is deduced from intrinsic properties of \mathcal{U}_N proved in [6]¹.

First, we introduce a subspace \mathcal{W}_n of spherical harmonics of degree n , defined by

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_n := \{u \in \mathcal{Y}_n : \exists v \in \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_{n-1}, u|_{\text{CS}_N} = v|_{\text{CS}_N}\}, \quad n \geq 1. \quad (2)$$

¹The space \mathcal{U}_N was denoted by \mathcal{Y}'_N in [6].

Intuitively, \mathcal{W}_n contains spherical harmonics of degree n which are undersampled on CS_N . Second, we build an orthogonal decomposition based on this subspace,

$$\mathcal{Y}_n = \mathcal{V}_n \oplus \mathcal{W}_n, \quad \text{with } \mathcal{V}_n := (\mathcal{W}_n)^\perp = \{u \in \mathcal{Y}_n : \forall v \in \mathcal{W}_n, \langle u, v \rangle_{L^2(\mathbb{S}^2)} = 0\}. \quad (3)$$

In the orthogonal complement \mathcal{V}_n , any spherical harmonics can be reconstructed by interpolation on CS_N (unlike \mathcal{W}_n). This claim is part of the following theorem [6].

Theorem 2. *There exists an integer N' such that the Lagrange interpolation problem has always a unique solution in $\mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{N'}$, i.e.*

$$\forall f \in \mathcal{F}(\text{CS}_N), \quad \exists! u \in \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{N'}, \quad u|_{\text{CS}_N} = f, \quad (\text{i.e., } \forall 1 \leq j \leq \bar{N}, u(x_j) = f(x_j)). \quad (4)$$

This theorem justifies the following definition.

Definition 3 (Interpolation space). Let \mathcal{V}_n denote the orthogonal complement (3) of the *undersampled subspace* \mathcal{W}_n , defined by (2). The *interpolation space* on CS_N is defined as

$$\mathcal{U}_N := \mathcal{V}_0 \oplus \cdots \oplus \mathcal{V}_{N'},$$

where $N' \geq 0$ is the smallest integer such that (4) is realized.

We refer to [6] for a practical algorithm to compute a real orthonormal basis of \mathcal{U}_N , denoted here by $(u_j)_{1 \leq j \leq \bar{N}}$. Then,

$$u_j \in \mathcal{U}_N, \quad \langle u_i, u_j \rangle_{L^2(\mathbb{S}^2)} = \delta_{ij}, \quad 1 \leq i, j \leq \bar{N}.$$

Without loss of generality, the first basis function is $u_1(x) = \frac{1}{\sqrt{4\pi}} \in \mathcal{Y}_0$. By definition of \mathcal{U}_N , the following linear map is isomorphic:

$$\begin{aligned} T_N : \mathcal{U}_N &\longrightarrow \mathcal{F}(\text{CS}_N) \\ u &\longmapsto u|_{\text{CS}_N}. \end{aligned}$$

The matrix of T_N , in the canonical basis $(\delta_{x_i})_{1 \leq i \leq \bar{N}}$ of $\mathcal{F}(\text{CS}_N)$, and the basis $(u_j)_{1 \leq j \leq \bar{N}}$ of \mathcal{U}_N , is the *Vandermonde matrix* given by

$$A = [u_j(x_i)]_{1 \leq i, j \leq \bar{N}}, \quad (5)$$

where i is the row index, and j is the column index. In (4), the solution $u \in \mathcal{U}_N$ is given by $u = T_N^{-1} f \in \mathcal{U}_N$. This suggests to define an *interpolation operator*, by

$$\mathcal{I}_N := T_N^{-1}.$$

Then the unique element of \mathcal{U}_N interpolating $f \in \mathcal{F}(\text{CS}_N)$ is given by $\mathcal{I}_N f \in \mathcal{U}_N$, and is represented by the column vector $A^{-1}[f(x_i)] \in \mathbb{R}^{\bar{N}}$,

$$\mathcal{I}_N f(x) = [u_j(x)] A^{-1}[f(x_i)], \quad x \in \mathbb{S}^2.$$

4. ROTATIONAL INVARIANCE OF THE INTERPOLATION SPACE

In this section, we study the invariance of the interpolation space \mathcal{U}_N under the symmetry group \mathcal{G} of the Cubed Sphere. We call “rotated” a function defined as follows.

Definition 4 (“Rotated” function). Assume that $Q \in \mathcal{G}$ leaves a set E invariant, i.e. $Q^\top E = E$. Let $f : E \rightarrow \mathbb{R}$ be a function defined on E . The “rotated” function, denoted by $f(Q^\top \cdot)$, is the function

$$f(Q^\top \cdot) : x \in E \mapsto f(Q^\top x) \in \mathbb{R}.$$

Our main invariance result is the following theorem.

Theorem 5 (Invariance of the interpolation space). *Let $n \geq 0$.*

(i) *The undersampled subspace \mathcal{W}_n is invariant under \mathcal{G} , i.e.*

$$\forall Q \in \mathcal{G}, \forall u \in \mathcal{W}_n, u(Q^\top \cdot) \in \mathcal{W}_n.$$

(ii) *The subspace $\mathcal{V}_n = \mathcal{W}_n^\perp$ is invariant under \mathcal{G} , i.e.*

$$\forall Q \in \mathcal{G}, \forall u \in \mathcal{V}_n, u(Q^\top \cdot) \in \mathcal{V}_n.$$

(iii) *The interpolation space \mathcal{U}_N is invariant under \mathcal{G} , i.e.*

$$\forall Q \in \mathcal{G}, \forall u \in \mathcal{U}_N, u(Q^\top \cdot) \in \mathcal{U}_N.$$

Proof. Fix $Q \in \mathcal{G}$, i.e. $Q \in \mathbb{R}^{3 \times 3}$ is an orthogonal matrix such that $Q^\top \text{CS}_N = \text{CS}_N$.

(i) If $n = 0$, $\mathcal{W}_0 = \{0\}$ is invariant under \mathcal{G} . Fix now $u \in \mathcal{W}_n \subset \mathcal{Y}_n$ with $n \geq 1$. There exists $v \in \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_{n-1}$ such that $u|_{\text{CS}_N} = v|_{\text{CS}_N}$, or equivalently, $(u - v)|_{\text{CS}_N} = 0$. Firstly, $u(Q^\top \cdot) \in \mathcal{Y}_n$ and $v(Q^\top \cdot) \in \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_{n-1}$. Secondly, $(u(Q^\top \cdot) - v(Q^\top \cdot))|_{\text{CS}_N} = (u - v)|_{\text{CS}_N}(Q^\top \cdot) = 0$, and therefore $u(Q^\top \cdot)|_{\text{CS}_N} = v(Q^\top \cdot)|_{\text{CS}_N}$; here, the commutation between rotation and restriction is justified by the following lemma.

Lemma 6 (Rotation commutes with restriction). *For all $Q \in \mathcal{G}$, $n \geq 0$, and $u \in \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_n$,*

$$u(Q^\top \cdot)|_{\text{CS}_N} = u|_{\text{CS}_N}(Q^\top \cdot) \in \mathcal{F}(\text{CS}_N).$$

We postpone the proof of the lemma until the end of this section.

(ii) The result is a combination of (i) and $\mathcal{V}_n = \mathcal{W}_n^\perp$. Indeed, fix $u \in \mathcal{V}_n \subset \mathcal{Y}_n$ with $n \geq 0$. Then $u(Q^\top \cdot) \in \mathcal{Y}_n$. Furthermore, for every $v \in \mathcal{W}_n$,

$$\langle u(Q^\top \cdot), v \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} u(Q^\top x)v(x)d\sigma = \int_{\mathbb{S}^2} u(y)v(Qy)d\sigma = \langle u, v(Q \cdot) \rangle_{L^2(\mathbb{S}^2)}; \quad (y := Q^\top x).$$

\mathcal{W}_n is invariant under \mathcal{G} , so $v(Q \cdot) \in \mathcal{W}_n$. Then $v(Q \cdot)$ is orthogonal to u because $u \in \mathcal{W}_n^\perp$. We obtain $\langle u(Q^\top \cdot), v \rangle = \langle u, v(Q \cdot) \rangle = 0$, which proves $u(Q^\top \cdot) \in \mathcal{W}_n^\perp$.

(iii) The space $\mathcal{U}_N = \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_N$ is a sum of invariant subspaces due to (ii). \square

Corollary 7 (Interpolation and symmetry). *(i) The interpolation operator commutes with any symmetry of the group \mathcal{G} :*

$$\forall f \in \mathcal{F}(\text{CS}_N), \forall Q \in \mathcal{G}, [\mathcal{I}_N f](Q^\top \cdot) = \mathcal{I}_N[f(Q^\top \cdot)].$$

(ii) The interpolation operator preserves the invariance property; in other words, if $f \in \mathcal{F}(\text{CS}_N)$ is invariant under \mathcal{G} , i.e. $\forall Q \in \mathcal{G}, f(Q^\top \cdot) = f$, then $\mathcal{I}_N f$ is invariant under \mathcal{G} , i.e. $\forall Q \in \mathcal{G}, [\mathcal{I}_N f](Q^\top \cdot) = \mathcal{I}_N f$.

Proof. (i) Firstly, $f(Q^\top \cdot) \in \mathcal{F}(\text{CS}_N)$ and $u = \mathcal{I}_N[f(Q^\top \cdot)] \in \mathcal{U}_N$ is the unique element of \mathcal{U}_N such that $u|_{\text{CS}_N} = f(Q^\top \cdot)$. Secondly, $v = \mathcal{I}_N f \in \mathcal{U}_N$ is the unique element of \mathcal{U}_N such that $v|_{\text{CS}_N} = f$. Due to Theorem 5.(iii), $v(Q^\top \cdot) \in \mathcal{U}_N$. By Lemma 6, $v(Q^\top \cdot)|_{\text{CS}_N} = v|_{\text{CS}_N}(Q^\top \cdot) = f(Q^\top \cdot)$, which proves $u = v(Q^\top \cdot)$. (ii) is an immediate consequence of (i). \square

Proof of Lemma 6. Firstly, $\mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_n$ is left invariant by Q , so $[u(Q^\top \cdot) : x \in \mathbb{S}^2 \mapsto u(Q^\top x)] \in \mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_n$, and $u(Q^\top \cdot)|_{\text{CS}_N}$ is defined by

$$u(Q^\top \cdot)|_{\text{CS}_N} = \sum_{i=1}^{\bar{N}} u(Q^\top x_i) \delta_{x_i} \in \mathcal{F}(\text{CS}_N).$$

On the other hand, $[u|_{\text{CS}_N} : x \in \text{CS}_N \mapsto u(x)] \in \mathcal{F}(\text{CS}_N)$, with CS_N left invariant by Q . Then the function $u|_{\text{CS}_N}(Q^\top \cdot)$ is well-defined and is given by

$$u|_{\text{CS}_N}(Q^\top \cdot) : x \in \text{CS}_N \mapsto u(Q^\top x) \in \mathcal{F}(\text{CS}_N).$$

At every $x = x_i \in \text{CS}_N$, the two functions have the same value, $u(Q^\top x_i)$. \square

5. A NEW QUADRATURE ON THE CUBED SPHERE

In this section, we study a new quadrature rule on CS_N ; it is defined by interpolation as follows.

Theorem 8 (Quadrature rule). *Let $u : \mathbb{S}^2 \rightarrow \mathbb{R}$ be a given function. The quadrature rule \mathcal{Q}_N is defined by*

$$\mathcal{Q}_N u := \int_{\mathbb{S}^2} \mathcal{I}_N[u|_{\text{CS}_N}](x) d\sigma.$$

(i) The formula \mathcal{Q}_N can be written as follows:

$$\mathcal{Q}_N u = \sum_{j=1}^{\bar{N}} \omega_N(x_j) u(x_j), \quad \text{with } \omega_N \in \mathcal{F}(\text{CS}_N) \text{ such that } [\omega_N(x_j)] = (A^\top)^{-1}[\sqrt{4\pi} \ 0 \ \cdots \ 0]^\top; \quad (6)$$

here, A is the Vandermonde matrix (5).

(ii) The formula \mathcal{Q}_N is exact on \mathcal{U}_N , i.e.

$$\forall u \in \mathcal{U}_N, \quad \mathcal{Q}_N u = \int_{\mathbb{S}^2} u(x) d\sigma.$$

(iii) The rule \mathcal{Q}_N and the weight ω_N are invariant under \mathcal{G} , i.e.

$$\forall Q \in \mathcal{G}, \quad \forall u \in \mathcal{U}_N, \quad \mathcal{Q}_N(u(Q^\top \cdot)) = \mathcal{Q}_N(u), \quad \text{and} \quad \omega_N(Q^\top \cdot) = \omega_N.$$

Proof. (i-ii) Firstly, if $u \in \mathcal{U}_N$, \mathcal{Q}_N exactly integrates u , since u coincides with $\mathcal{I}_N u$. In particular, for each basis function $u_j \in \mathcal{U}_N$, $\mathcal{Q}_N u_j = \int_{\mathbb{S}^2} u_j(x) d\sigma$. For $u_1 = \frac{1}{\sqrt{4\pi}}$, $\int_{\mathbb{S}^2} u_1(x) d\sigma = \sqrt{4\pi}$. For every $1 \leq j \leq \bar{N}$, $u_j \perp u_1$, which means $\int_{\mathbb{S}^2} u_j(x) d\sigma = 0$. Then, $[\mathcal{Q}_N u_j]_{1 \leq j \leq \bar{N}} = [\sqrt{4\pi} \ 0 \ \cdots \ 0]^\top$. Secondly, fix $\omega_N \in \mathcal{F}(\text{CS}_N)$ such that

$$\omega_N(x_i) = \int_{\mathbb{S}^2} \mathcal{I}_N[\delta_{x_i}] d\sigma, \quad 1 \leq i \leq \bar{N}. \quad (7)$$

By linearity, we deduce from (1) that

$$\mathcal{Q}_N u = \sum_{i=1}^{\bar{N}} \omega_N(x_i) u(x_i) = [u(x_i)]^\top [\omega_N(x_i)].$$

Using the basis functions, we obtain

$$A^\top [\omega_N(x_i)]_{1 \leq i \leq \bar{N}} = [\mathcal{Q}_N u_j]_{1 \leq j \leq \bar{N}} = [\sqrt{4\pi} \ 0 \ \cdots \ 0]^\top,$$

where the matrix $A^\top \in \mathbb{R}^{\bar{N} \times \bar{N}}$ is non singular.

Remark 9. In [6], a LQ-factorization of A^\top is computed, $A^\top = LQ$ where L is lower triangular, whereas Q is orthogonal. So, ω_N can be deduced from a linear system whose matrix is L : $[\omega_N(x_j)] = Q^\top L^{-1} [\sqrt{4\pi} \ 0 \ \cdots \ 0]^\top$.

(iii) Fix $Q \in \mathcal{G}$ and $u \in \mathcal{U}_N$. By Theorem 5, $u(Q^\top \cdot) \in \mathcal{U}_N$. Thus, by (ii) and change of variable,

$$\mathcal{Q}_N(u(Q^\top \cdot)) = \int_{\mathbb{S}^2} u(Q^\top x) d\sigma = \int_{\mathbb{S}^2} u(x) d\sigma = \mathcal{Q}_N(u) \quad (x := Q^\top x).$$

Fix now $1 \leq i \leq \bar{N}$ and $u = \mathcal{I}_N \delta_{x_i} \in \mathcal{U}_N$. By Corollary 7, $u(Q^\top \cdot) = \mathcal{I}_N [\delta_{x_i}(Q^\top \cdot)] \in \mathcal{U}_N$, with $\delta_{x_i}(Q^\top \cdot) = \delta_{Qx_i}$. Therefore by (7), $\mathcal{Q}_N(u) = \omega_N(x_i)$ and $\mathcal{Q}_N(u(Q^\top \cdot)) = \omega_N(Qx_i)$. Then, by invariance of \mathcal{Q}_N ,

$$\omega_N(x_i) = \mathcal{Q}_N(u) = \mathcal{Q}_N(u(Q^\top \cdot)) = \omega_N(Qx_i).$$

□

The quadrature rule exactly integrates the \bar{N} spherical harmonics of \mathcal{U}_N . Taking benefit from the rotational invariance, we prove furthermore that it exactly integrates an infinite number of spherical harmonics.

Corollary 10. *The quadrature rule \mathcal{Q}_N exactly integrates $\frac{15}{16}$ of all real Legendre spherical harmonics. More precisely, for all $|m| \leq n$,*

$$\mathcal{Q}_N(Y_n^m) = \int_{\mathbb{S}^2} Y_n^m(x) d\sigma, \quad \text{if} \quad \begin{cases} n \equiv 1 \ (2), \\ m < 0, \\ m \geq 0 \text{ and } m \equiv 1, 2, 3 \ (4); \end{cases}$$

equivalently, $\mathcal{Q}_N(Y_n^m) \neq \int_{\mathbb{S}^2} Y_n^m(x) d\sigma \Rightarrow n \equiv 0 \ (2), m \geq 0$ and $m \equiv 0 \ (4)$.

Proof. Fix $n \geq 1$ and $|m| \leq n$. Then $\int_{\mathbb{S}^2} Y_n^m(x) d\sigma = 0$. For well chosen n, m , we build a symmetry $Q \in \mathcal{G}$ such that $Y_n^m(Q^\top \cdot) = -Y_n^m$. In such cases, we obtain $\mathcal{Q}_N(Y_n^m) = \mathcal{Q}_N(Y_n^m(Q^\top \cdot)) = -\mathcal{Q}_N(Y_n^m)$, which proves $\mathcal{Q}_N(Y_n^m) = 0 = \int_{\mathbb{S}^2} Y_n^m(x) d\sigma$. Recall that $x(\theta, \phi) = (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta)$, $\phi \in [-\pi, \pi]$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $Y_n^m(x(\theta, \phi)) := Y_n^m(\theta, \phi)$.

Case 1: $n \equiv 1 \ (2)$ and $m \equiv 0 \ (4)$. Then $\theta \mapsto P_n^{(|m|)}(\sin \theta)$ is odd, so is $\theta \mapsto Y_n^m(x(\theta, \phi))$; hence,

$$Y_n^m(Q^\top x(\theta, \phi)) = Y_n^m(x(-\theta, \phi)) = -Y_n^m(x(\theta, \phi)), \quad \text{for} \quad Q := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Case 2: $m < 0$. Then $\phi \mapsto Y_n^m(x(\theta, \phi))$ is odd, so,

$$Y_n^m(Q^\top x(\theta, \phi)) = Y_n^m(x(\theta, -\phi)) = -Y_n^m(x(\theta, \phi)), \quad \text{for} \quad Q := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Case 3: $m \equiv 1, 3 \ (4)$. Then $m(\phi + \pi) \equiv m\phi + \pi \ (2\pi)$, and

$$Y_n^m(Q^\top x(\theta, \phi)) = Y_n^m(x(\theta, \phi + \pi)) = -Y_n^m(x(\theta, \phi)), \quad \text{for} \quad Q := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Case 4: $m \equiv 2 \ (4)$. Then $m(\phi + \frac{\pi}{2}) \equiv m\phi + \pi \ (2\pi)$, and

$$Y_n^m(Q^\top x(\theta, \phi)) = Y_n^m(x(\theta, \phi + \frac{\pi}{2})) = -Y_n^m(x(\theta, \phi)), \quad \text{for} \quad Q := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

Remark 11. The ratio 15/16 of the real Legendre basis is asymptotic. In [7, 15], a similar approach based on invariance properties provided an asymptotic ratio of 7/8 of the complex Legendre basis. We observe here that using the real Legendre basis highlights further exact results.

6. NUMERICAL RESULTS

6.1. Symmetry invariance assessment. We begin by two numerical assessments related to interpolation in \mathcal{U}_N .

First, we illustrate that the interpolation operator \mathcal{I}_N preserves the invariance property, as stated in Corollary 7.(ii). Fix $N = 6$ and consider the series of symmetric functions $g_i \in \mathcal{F}(\text{CS}_N)$, described in Table 1. By construction, each function g_i , $1 \leq i \leq 6$, is constant along any orbit, *i.e.* $\forall Q \in \mathcal{G}$, $g_i(Q^\top \cdot) = g_i$, and is supported by a set of symmetric nodes. For $i \leq 5$, g_i takes the value 1 along the orbit of $a_i \in \text{CS}_N$, and the value 0 otherwise. The orbit of a_1 contains the vertices of an octahedron. The orbit of a_2 contains the vertices of a cube. The orbit of a_3 contains the vertices of a cuboctahedron. The orbit of a_4 is included in the edges of an octahedron. The orbit of a_5 is “generic”, with cardinal number 48. In Figure 1, we visualize how the symmetry is reflected in the interpolating functions $\mathcal{I}_N g_i \in \mathcal{U}_N$, $1 \leq i \leq 6$. The octahedral symmetry predicted by Corollary 7.(ii) can be observed; the functions $\mathcal{I}_N g_i$ are constant along any orbit.

i	g_i	a_i	$ \text{supp } g_i $
1	$\frac{1}{8} \sum_{Q \in \mathcal{G}} \delta_{Qa_1}$	$[1\ 0\ 0]^\top$	6
2	$\frac{1}{6} \sum_{Q \in \mathcal{G}} \delta_{Qa_2}$	$\frac{1}{\sqrt{3}} [1\ 1\ 1]^\top$	8
3	$\frac{1}{4} \sum_{Q \in \mathcal{G}} \delta_{Qa_3}$	$\frac{1}{\sqrt{2}} [1\ 0\ 1]^\top$	12
4	$\frac{1}{2} \sum_{Q \in \mathcal{G}} \delta_{Qa_4}$	$(1 + \tan^2 \frac{\pi}{6})^{-1/2} [1\ 0\ \tan \frac{\pi}{6}]^\top$	24
5	$\sum_{Q \in \mathcal{G}} \delta_{Qa_5}$	$(1 + \tan^2 \frac{\pi}{12} + \tan^2 \frac{\pi}{6})^{-1/2} [1\ \tan \frac{\pi}{12}\ \tan \frac{\pi}{6}]^\top$	48
6	$g_1 + g_2 + g_3 + g_4 + g_5$		98

TABLE 1. Grid functions with octahedral symmetry. The function g_i , $1 \leq i \leq 6$, takes the value 1 on its support and is invariant under \mathcal{G} .

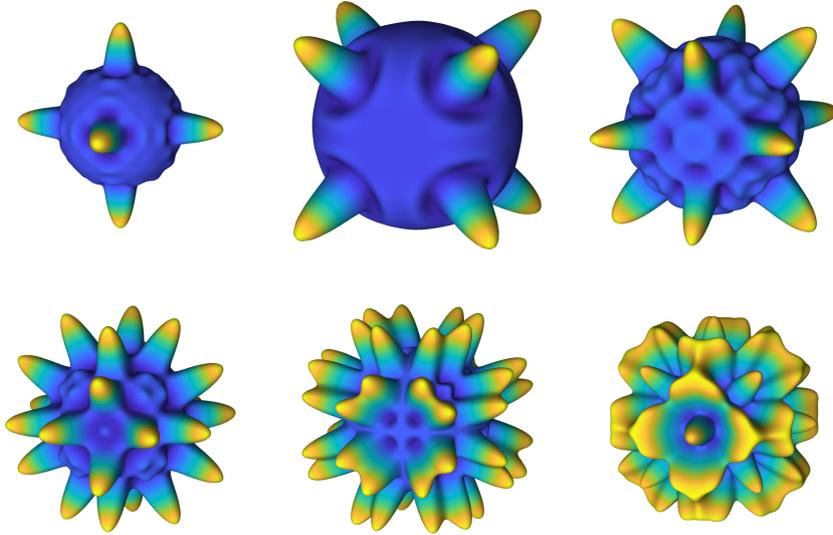


FIGURE 1. Interpolation with octahedral symmetry. For every $1 \leq i \leq 6$, the symmetric function $\mathcal{I}_N g_i$ is represented by the surface $(1.5 + \mathcal{I}_N g_i(x))x$, $x \in \mathbb{S}^2$.

Second, we assess the invariance of the interpolation space, stated in Theorem 5, and the commutation between interpolation and rotation, stated in Corollary 7.(i). For that purpose, for each basis function $u_j \in \mathcal{U}_N$, we compare $u_j(Q^\top \cdot) = [\mathcal{I}_N u_j](Q^\top \cdot)$ with $\mathcal{I}_N[u_j(Q^\top \cdot)]$. Indeed, by linearity, Corollary 7.(i) is equivalent to:

$$\forall 1 \leq j \leq \bar{N}, \forall Q \in \mathcal{G}, [\mathcal{I}_N u_j](Q^\top \cdot) = \mathcal{I}_N[u_j(Q^\top \cdot)].$$

If this condition is achieved, then for each basis function u_j , $u_j(Q^\top \cdot) = [\mathcal{I}_N u_j](Q^\top \cdot) \in \text{Ran } \mathcal{I}_N = \mathcal{U}_N$, which implies Theorem 5.(iii) by linearity. This also implies Theorem 5.(ii), due to $u_j(Q^\top \cdot) \in \mathcal{V}_n \cap \mathcal{U}_N = \mathcal{V}_n$. We compare the functions on a fine grid CS_M ($M = 33$), by computing the relative error

$$\epsilon_{N,j}(Q) := \frac{\max_{x \in \text{CS}_M} |u_j(Q^\top x) - \mathcal{I}_N[u_j(Q^\top \cdot)](x)|}{\max_{x \in \text{CS}_M} |u_j(x)|}.$$

Then we compute the maximal error ϵ_N , and we repeat the procedure for several values of N :

$$\epsilon_N := \max\{\epsilon_{N,j}(Q), Q \in \mathcal{G}, 1 \leq j \leq \bar{N}\}, \quad 1 \leq N \leq 16. \quad (8)$$

The results reported in Table 2 are in agreement with the invariance stated in Theorem 5 and Corollary 7.(i).

N	1	2	3	4	5	6	7	8
ϵ_N	2.5e-15	3.4e-15	7.8e-15	1.4e-14	9.7e-15	9.3e-15	1.3e-14	1.1e-14
N	9	10	11	12	13	14	15	16
ϵ_N	1.4e-14	1.5e-14	1.9e-14	1.7e-14	3.4e-14	2.2e-14	1.8e-14	2.9e-14

TABLE 2. Numerical invariance: $u_j(Q^\top) = \mathcal{I}_N[u_j(Q^\top)]$, $Q \in \mathcal{G}$, up to relative error ϵ_N (8).

6.2. Quadrature weight. We have computed the quadrature weight $\omega_N \in \mathcal{F}(\text{CS}_N)$ for $1 \leq N \leq 32$, and $N = 64$. Some of them are displayed in Figure 2. As can be observed, the weight is positive, $\omega_N > 0$, and the maximum value is reached at the center of a panel. Moreover, some statistics are displayed in Figure 3. They reveal that the weight ω_N is quasi-uniform. In particular,

$$\frac{\max \omega_N}{\min \omega_N} \approx \sqrt{2}.$$

We recognize the ratio between the surface element at the center of a panel of CS_N , and the smallest surface element of a panel, as it is derived in [16, Eq. (20)].

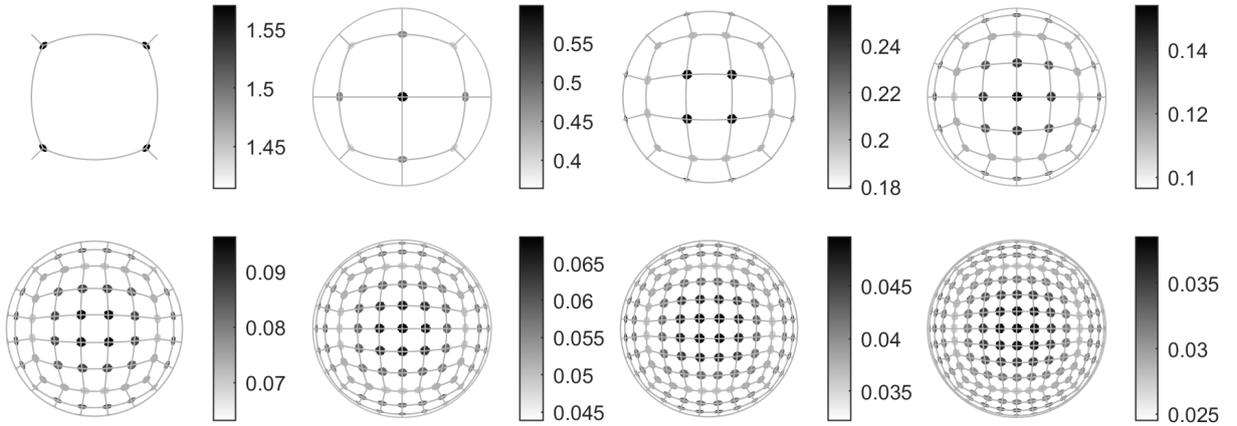


FIGURE 2. Quadrature weight ω_N , for $1 \leq N \leq 8$.

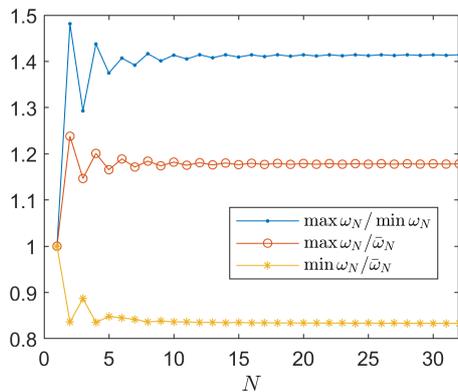


FIGURE 3. Statistical distribution of the quadrature weight ω_N in (6), $1 \leq N \leq 32$. The maximum, minimum, and mean values satisfy $\max \omega_N \approx 1.41 \min \omega_N$, $\max \omega_N \approx 1.18 \bar{\omega}_N$, and $\min \omega_N \approx 0.83 \bar{\omega}_N$.

i	$f_i(x, y, z)$	$\int_{\mathbb{S}^2} f_i(x, y, z) d\sigma$	Ref.
1	$\exp(x)$	14.7680137457653...	[3, 10]
2	$\frac{3}{4} \exp\left[-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} - \frac{(9z-2)^2}{4}\right]$ $+ \frac{3}{4} \exp\left[-\frac{(9x+1)^2}{49} - \frac{9y+1}{10} - \frac{9z+1}{10}\right]$ $+ \frac{1}{2} \exp\left[-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} - \frac{(9z-5)^2}{4}\right]$ $- \frac{1}{5} \exp\left[-(9x-4)^2 - (9y-7)^2 - (9z-5)^2\right]$	6.6961822200736179523...	[2, 4, 7, 11, 15]
3	$\frac{1}{10} \frac{\exp(x+2y+3z)}{(x^2+y^2+(z+1)^2)^{1/2}} \mathbf{1}(z > -1)$	4.090220018862976...	[2]
4	$\cos(3 \arccos z) \mathbf{1}(3 \arccos z \leq \frac{\pi}{2})$	$\frac{\pi}{8}$	inspired from [2]
5	$\mathbf{1}(z \geq \frac{1}{2})$	$\frac{\pi}{9}$	
6	$\frac{1}{9} [1 + \text{sign}(-9x - 9y + 9z)]$	$\frac{4\pi}{9}$	[4, 7, 11, 15]

TABLE 3. Test functions and exact integration values.

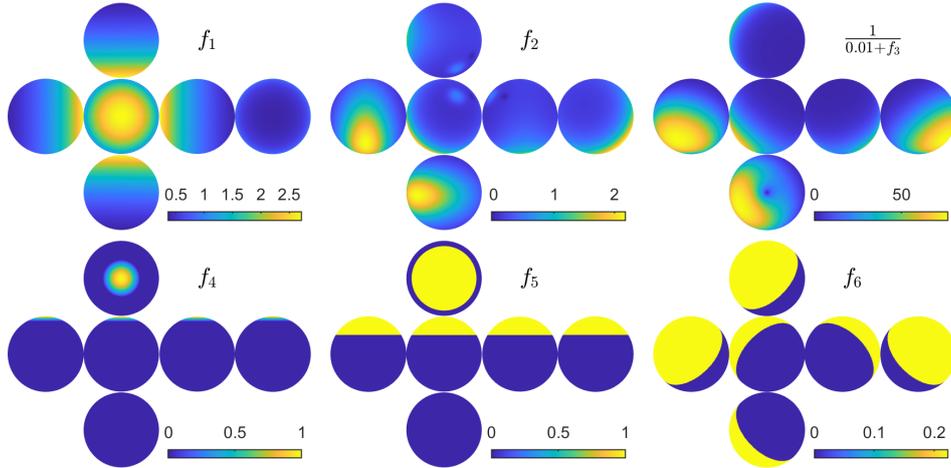


FIGURE 4. Test functions of Table 3.

6.3. Quadrature of test functions. We test the accuracy of the quadrature formula \mathcal{Q}_N on the series of functions reported in Table 3. They are displayed in Figure 4. Most of these functions serve as testing functions for quadrature assessment. References are indicated in Table 3. The exponential function f_1 is a smooth, non trivial function. The Franke function f_2 is a standard test case. The function f_3 is smooth, except near the South pole, where it has an infinite spike. The cosine cap function f_4 is continuous but is not differentiable on the circle $z = \frac{\sqrt{3}}{2}$. The function f_5 is the characteristic function of a spherical cap; it is not continuous. Similarly, the discontinuous function f_6 represents an hemisphere; it is a standard test function.

We report in Table 4 the quadrature error

$$\eta_N(f_i) = \left| \int_{\mathbb{S}^2} f_i d\sigma - \mathcal{Q}_N f_i \right|, \quad N = 1, 2, 4, 8, 16, 32, 64, \quad 1 \leq i \leq 6.$$

Moreover, Table 5 reports a rate of convergence $r_N(f_i)$, defined by the equation

$$\eta_{2N}(f_i) = \frac{\eta_N(f_i)}{2^{r_N(f_i)}}.$$

Note that the computations have been performed with Matlab, in double precision. In particular the machine epsilon is approximately 2.2×10^{-16} ; we do not compute the rate when the relative error is close to this value. For the smooth function f_1 , the error rapidly reaches a value which is about 10^{-14} . For the Franke function f_2 , a thinner grid is required to reach such values, but a very fast convergence is still observed. For the spike function f_3 , the convergence rate is $r_N(f_3) \approx 1$. For the continuous cap function f_4 , and the discontinuous one f_5 , the error slowly decrease, at a convergence rate which depends on the grid size. For the cap function f_6 , which is discontinuous and “symmetric” (supported by a hemisphere), the error is close to the machine epsilon, independently of the grid size.

N	\bar{N}	$\eta_N(f_1)$	$\eta_N(f_2)$	$\eta_N(f_3)$	$\eta_N(f_4)$	$\eta_N(f_5)$	$\eta_N(f_6)$
1	8	4.8e-02	8.2e-01	2.4e-01	3.9e-01	3.1e+00	6.7e-16
2	26	2.0e-06	1.5e-02	1.7e-02	2.1e-01	9.9e-01	0.0e+00
4	98	1.2e-14	2.2e-03	7.8e-03	2.0e-02	6.7e-02	2.2e-16
8	386	1.8e-15	9.0e-06	3.8e-03	4.8e-03	6.4e-02	6.7e-16
16	1538	7.1e-15	5.5e-09	1.9e-03	3.0e-04	1.5e-02	6.7e-16
32	6146	5.3e-15	1.8e-15	9.5e-04	3.1e-04	7.9e-03	6.7e-16
64	24578	3.6e-15	1.8e-15	4.8e-04	1.9e-07	2.6e-03	4.4e-16

 TABLE 4. Quadrature error $\eta_N(f_i) = \left| \int_{\mathbb{S}^2} f_i d\sigma - \mathcal{Q}_N f_i \right|$.

N	$r_N(f_1)$	$r_N(f_2)$	$r_N(f_3)$	$r_N(f_4)$	$r_N(f_5)$	$\bar{r}_N(f_1)$	$\bar{r}_N(f_2)$	$\bar{r}_N(f_3)$	$\bar{r}_N(f_4)$	$\bar{r}_N(f_5)$
1	15	5.8	3.9	0.93	1.7	15	2.8	5.4	2.5	-0.0033
2	27	2.7	1.1	3.4	3.9	26	3.3	1.1	3.3	1.8
4	2.8	7.9	1	2.1	0.077	2.1	4.1	1	1.9	1.3
8		11	1	4	2.1		13	1.2	2.8	1.7
16		22	1	-0.047	0.92		24	0.92	2.2	1.4
32			1	11	1.6			0.93	2.7	1.6

 TABLE 5. Convergence rate $r_N(f_i)$ of the error $\eta_N(f_i)$, and convergence rate $\bar{r}_N(f_i)$ of the average error $\bar{\epsilon}_N(f_i)$, over 1000 random orthogonal transformations of the grid.

6.4. Sensitivity to the grid orientation.

Here we consider more closely the accuracy of the rule \mathcal{Q}_N : we modify randomly the orientation of the grid, [11, 15]. We compute

$$\epsilon_N(f_i, Q) = \left| \int_{\mathbb{S}^2} f_i d\sigma - \mathcal{Q}_N f_i(Q^\top \cdot) \right|,$$

where Q browses a set of 1000 randomly selected orthogonal matrices (uniform law in $[0, 2\pi]$ for the Euler angles, and uniform law in $\{-1, 1\}$ for the orientation). The worst error, the average error and their ratio are defined by

$$\epsilon_N(f_i) = \max_Q \epsilon_N(f_i, Q), \quad \bar{\epsilon}_N(f_i) = \frac{1}{1000} \sum_Q \epsilon_N(f_i, Q), \quad \rho_N(f_i) = \frac{\bar{\epsilon}_N(f_i)}{\epsilon_N(f_i)}.$$

The worst error $\epsilon_N(f_i)$ and the ratio $\rho_N(f_i)$ are displayed² in Figure 5. We report in Table 5 a convergence rate $\bar{r}_N(f_i)$ of the average error $\bar{\epsilon}_N(f_i)$, defined by

$$\bar{e}_{2N}(f_i) = \frac{\bar{\epsilon}_N(f_i)}{2\bar{r}_N(f_i)}.$$

The worst errors $\epsilon_N(f_1)$ and $\epsilon_N(f_2)$ fastly decrease, and $\epsilon_N(f_6)$ is zero, up to rounding errors. This indicates that the quadrature rule \mathcal{Q}_N efficiently integrates the smooth functions f_1 , f_2 , and the symmetric cap function f_6 , independently of the grid orientation. For the function f_4 , which is continuous and non differentiable, the worst error $\epsilon_N(f_4)$ decreases at constant rate. The decrease of the worst error $\epsilon_N(f_5)$ of the “generic” cap function f_5 , which is discontinuous, is slower. And for the spike function f_3 , the worst error $\epsilon_N(f_3)$ slowly decreases, with oscillations.

Roughly speaking, Figure 5 indicates that

$$\bar{\epsilon}_N(f_i) \approx 0.25\epsilon_N(f_i), \quad i \neq 3, \quad \bar{\epsilon}_N(f_3) \approx 0.025\epsilon_N(f_3).$$

Except for f_3 , the worst error is not very large in comparison with the average error (factor 4). This indicates that the result is almost insensitive to the grid orientation. For the function f_3 with a spike, the situation is different (factor 40); the error is sensitive to the grid orientation. Concerning the speed of convergence, we note $\bar{r}_N(f_3) \approx 1$ for the spike function, $\bar{r}_N(f_5) \approx 1.4$ for the discontinuous cap function, $\bar{r}_N(f_4) \approx 2.6$ for the continuous one. The average errors for f_1 , f_2 and f_6 converge fastly, since it was already the case for the worst errors.

²In order to clarify the figure, we have eliminated the following ratios: $\rho_N(f_1)$, $N > 3$, and $\rho_N(f_6)$. Indeed, these ratios are “large”, because the associated errors are almost zero.

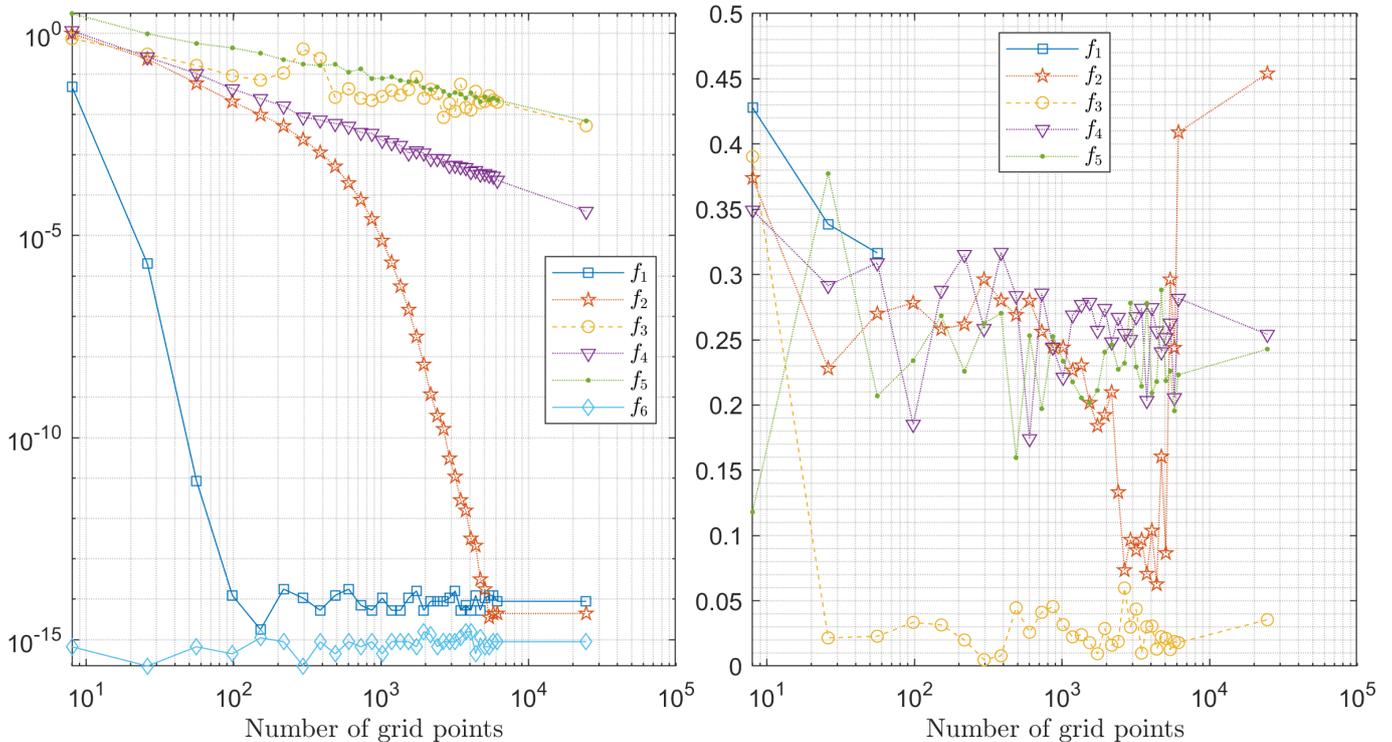


FIGURE 5. Statistics of the quadrature error $\varepsilon_N(f_i, Q) = |\int_{\mathbb{S}^2} f_i d\sigma - \mathcal{Q}_N f_i(Q^\top \cdot)|$, where Q scans a set of 1000 random orthogonal matrices. Left: worst error $\varepsilon_N(f_i)$. Right: ratio $\rho_N(f_i) = \bar{\varepsilon}_N(f_i)/\varepsilon_N(f_i)$ of the average error divided by the worst one.

6.5. Comparison with other quadrature rules. We compare our quadrature rule \mathcal{Q}_N with some spherical quadrature rules of the literature, summarized in Table 6.

Abbr.	Description	Ref.
CS-BBC21	Interpolation on the Cubed Sphere by spherical harmonics	This article (\mathcal{Q}_N)
CS-CP18	Octahedral quadrature on the Cubed Sphere, by least-square	[15]
CS-BC18	Corrected bivariate trapezoidal on the Cubed Sphere	[7]
Lebedev	Gauss quadrature, invariant under the octahedral group	[13, 14]
t-design	Spherical t -design	[18, 19]

TABLE 6. Quadrature of the literature used for comparison.

- (1) We use two other rules on the Cubed Sphere; CS-BC18 is a correction of some bivariate trapezoidal rule, CS-CP18 is an octahedral rule which minimizes some least-square error concerning the integration of Legendre spherical harmonics.
- (2) We also use “optimal” quadrature rules, whose distribution of nodes is “optimized”. Firstly, our rule is invariant under the octahedral group \mathcal{G} , so we compare with the Lebedev rule, which is an optimal octahedral rule. Indeed, the optimal grids/weights of Lebedev maximize the degree of precision, under the constraint of invariance under \mathcal{G} . Secondly, our weights are quasi-uniform, so we compare with spherical t -designs. This rules have equal weights and degree of precision t ; the associated spherical grids have $\sim \frac{t^2}{2}$ nodes, which is the optimal order.

Remark 12. For the Lebedev rules, we have used the Matlab function `getLebedevSphere` (by R.M. Parrish). The code is available on

<https://fr.mathworks.com/matlabcentral/fileexchange/27097-getlebedevsphere>

The t -designs have been found on R.S. Womersley webpage

<https://web.maths.unsw.edu.au/~rsw/Sphere/EffSphDes/sf.html>

Results are given on Figure 6. The worst error after 1000 random orthogonal matrices is plotted related to the number of grid points using different quadrature rules.

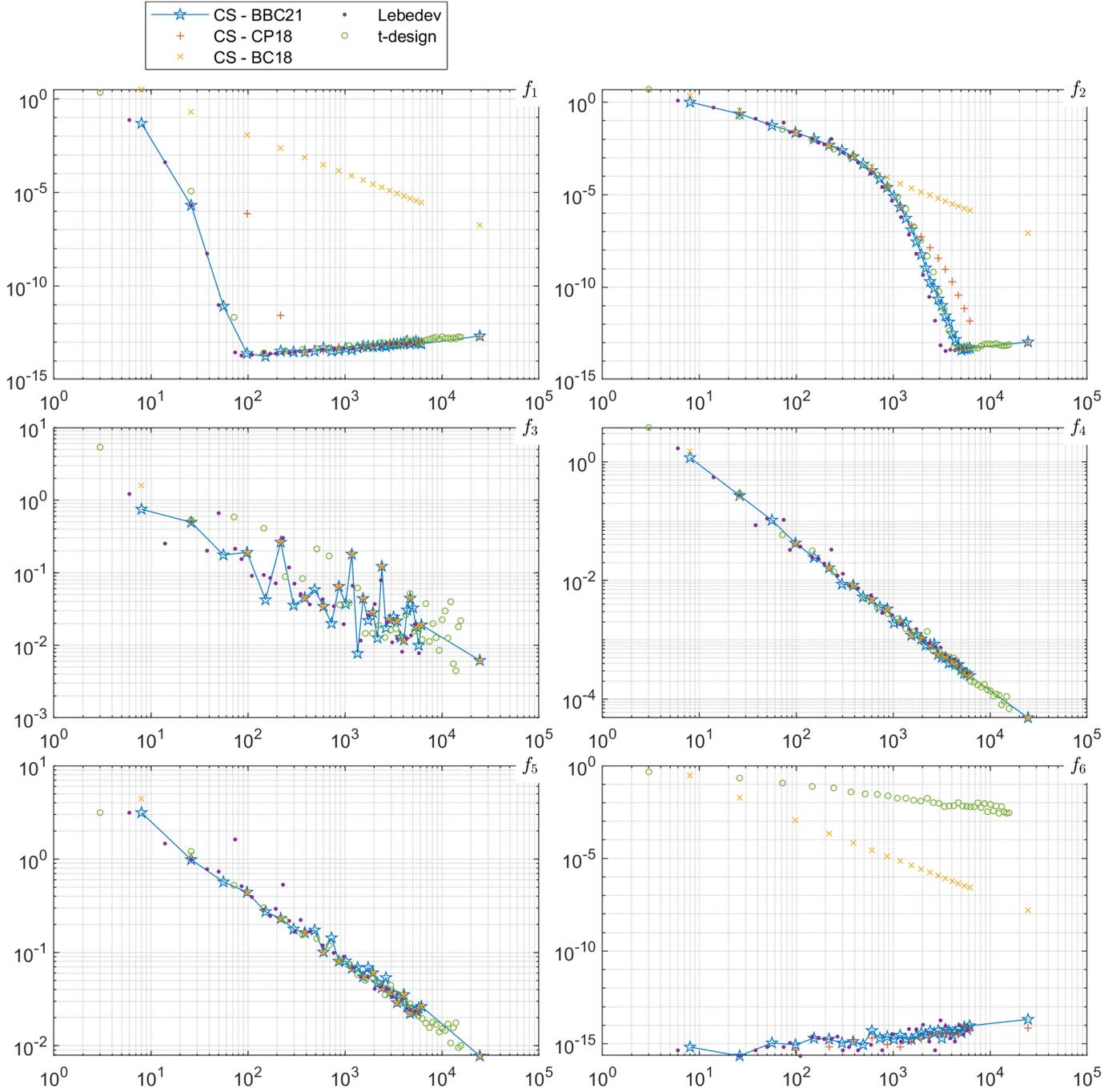


FIGURE 6. Worst quadrature error (for 1000 random orthogonal transformations of the grid), versus the number of grid points.

- (1) Among the quadrature rules on CS_N , the new rule \mathcal{Q}_N outperforms CS-BC18 for the smooth functions f_1 and f_2 , and for the symmetric cap function f_6 . The rules \mathcal{Q}_N and CS-CP18 give similar accuracy for most of the cases, with the following exceptions. The rule \mathcal{Q}_N integrates f_1 more accurately than CS-CP18 before convergence, and \mathcal{Q}_N converges slightly faster than CS-CP18 for f_2 .
- (2) For the smooth functions f_1 and f_2 , the rules \mathcal{Q}_N and t-design have similar accuracy, whereas the Lebedev rule converges slightly faster. For the function f_3 with a spike, the worst errors are almost similar; they decay slowly with oscillations. For the cap functions f_4 and f_5 , the methods converge slowly with similar accuracy. For the “symmetric” cap function f_6 , \mathcal{Q}_N and the Lebedev rule are exact (up to rounding errors) and give better accuracy than the t-design rule.

Overall, the rule \mathcal{Q}_N on a fixed grid CS_N displays remarkable accuracy, compared to “optimal” quadrature methods, which require “optimal” grids (Lebedev and t-design rules).

7. CONCLUSION

Quadrature on the sphere is important in many respects. It arises in a variety of contexts in physics and chemistry: quantum physics, crystallography, gravimetry, astrophysics, kinetic theory (Boltzmann collision kernel approximations, neutron transport), to quote some of them. Our interest in this question stemmed from numerical experiments with a Cubed Sphere finite difference (FD) solver for the spherical shallow water equations, [8]. Important mean quantities must be accurately preserved with time, one of them being the mean potential vorticity. Since a FD solver is not *ab initio* conservative, a conservativity analysis must be performed a posteriori, and therefore an accurate quadrature is needed.

The new rule \mathcal{Q}_N has the property to be quasi-uniform with positive weights. The numerical results can be compared in accuracy with optimal rules such as t-designs and Lebedev rules. This supports the claim of the “approximation power” of the Cubed Sphere. Among the questions open, a better convergence analysis must be performed. Proving the positivity of the weights is also an important goal. This is deferred to a future study. Overall, the symmetry properties of the Cubed Sphere as a support for quadrature seems a promising topic.

APPENDIX A. ACCURACY OF THE QUADRATURE RULE

The quadrature rule \mathcal{Q}_N is expected to exactly integrate any spherical harmonics of \mathcal{U}_N , and 15/16 of the Legendre spherical harmonics. In this section, we go further by measuring the accuracy for various tolerances $\epsilon > 0$. More precisely, we compute $d_N(\epsilon)$, defined as the largest integer such that:

$$\forall |m| \leq n \leq d_N(\epsilon), \left| \int_{S^2} Y_n^m d\sigma - \mathcal{Q}_N Y_n^m \right| \leq \epsilon.$$

The results are reported in Table 7. Except for $N = 3, 4, 64$, the degree $d_N(10^{-14})$ is $2N + 1$ if N is odd, and $2N + 3$ if N is even. For the particular cases, $d_N(10^{-14}) = 4N - 1$ for $N = 3, 4$, and $d_{64}(10^{-14}) = 2 \cdot 64 + 11$.

The table contains implicit information about the default of precision of the first Legendre spherical harmonics that is not well integrated at a fixed accuracy. For instance with $N = 8$, the first error which is above 10^{-14} is between 10^{-4} and 10^{-6} , and it is reached for the degree 20 (since the method is exact for the odd degree). Such defaults of precision are strong for small values of N , but decrease as N grows. For the extreme case $N = 1$, there exists a spherical harmonics of degree 4 which is not well integrated, even the tolerance is set to 10^{-4} . For the extreme case $N = 64$, each Legendre function of degree between 139 and 156 is well integrated if the tolerance is 10^{-8} .

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N	N	$d_N(10^{-14})$	$d_N(10^{-12})$	$d_N(10^{-10})$	$d_N(10^{-8})$	$d_N(10^{-6})$	$d_N(10^{-4})$
1	8	3	3	3	3	3	3
2	26	7	7	7	7	7	7
3	56	11	11	11	11	11	11
4	98	15	15	15	15	15	15
5	152	11	11	11	11	11	11
6	218	15	15	15	15	15	17
7	296	15	15	15	15	15	17
8	386	19	19	19	19	19	21
9	488	19	19	19	19	19	23
10	602	23	23	23	23	23	27
11	728	23	23	23	23	23	27
12	866	27	27	27	27	29	33
13	1016	27	27	27	27	29	33
14	1178	31	31	31	31	33	39
15	1352	31	31	31	31	33	41
16	1538	35	35	35	35	39	45
17	1736	35	35	35	35	39	47
18	1946	39	39	39	41	43	51
19	2168	39	39	39	41	45	55
20	2402	43	43	43	45	49	59
21	2648	43	43	43	45	49	65
22	2906	47	47	47	49	53	65
23	3176	47	47	47	51	55	71
24	3458	51	51	51	55	59	75
25	3752	51	51	51	55	61	79
26	4058	55	55	57	59	65	85
27	4376	55	55	57	59	65	89
28	4706	59	59	61	65	69	95
29	5048	59	59	61	65	71	97
30	5402	63	63	65	69	75	101
31	5768	63	63	65	69	77	105
32	6146	67	67	71	73	81	111
64	24578	139	143	147	155	183	255

TABLE 7. Quadrature rule \mathcal{Q}_N : observed degree of precision $d_N(\epsilon)$ for various tolerances ϵ .

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[†] UNIVERSITÉ DE LORRAINE, CNRS, IECL, F-57000 METZ, FRANCE
Email address: jean-baptiste.bellet@univ-lorraine.fr, jean-pierre.croisille@univ-lorraine.fr

[‡] UNIVERSITÉ DE POITIERS, CNRS, LMA, F-86000 POITIERS, FRANCE
Email address: matthieu.brachet@math.univ-poitiers.fr