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# Rolling horizon optimal maintenance policy for a system subject to shocks and degradation under uncertain parameters

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**Abstract**

In this paper, we consider a single-unit system which is subject to shocks and deterioration over time. We assume that the shocks occur based on a counting process. Each incoming shock affects the system in two ways; the fatal shock causes the system failure whereas the effective shock weakens the system by increasing its age. The system degradation is modeled based on a general degradation path (GDP) model which depends on the impact of shocks received in the course of system operation and the system virtual age. Both soft and hard failures are considered and a new degradation-threshold-shock (DTS) model is proposed. The main focus of this paper is to provide a dynamic maintenance policy based on the current conditions of the system while both environmental and operational conditions of the system are uncertain. In this regard, a condition-based maintenance (CBM) policy with periodic inspection is developed. Both corrective and preventive replacements are taken into account and preventive replacement threshold is updated at each inspection time. The update is done in the light of new information on the number and magnitude of shocks using a rolling horizon approach within the Bayesian framework. A simulation study has been conducted to show the applicability and efficiency of the proposed method and the effect of prior information on the maintenance decisions has been studied.

*Keywords:* Rolling horizon approach, Parameter uncertainty, Optimal maintenance policy, Degradation-threshold-shock model

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## 1. Introduction

Maintenance is an important part of facilities management. The maintenance activities are related with inspection, service, repair and replacement of system in order to decrease system downtime, reduce the number of major repairs, improve system performance and safety and avoid poor quality products and customers dissatisfaction. Because maintenance actions are expensive, companies are looking to develop maintenance plans that ensure system performance at the lowest cost over time. Such maintenance plans are referred to as optimal maintenance policies. The search for optimal maintenance policies, resulting from the work of Barlow and Hunter [1], has been the subject of an enormous amount of research for at least five decades. A critical main step in determining a suitable maintenance plan is knowing why and how the system fails. In a dynamic environment, two major causes of system failure are deterioration and shocks. In such environments, to describe the failure behaviour of system a model is needed that jointly considers competing causes of degradation and shocks. Such a class of reliability models has been called degradation-threshold-shock (DTS) models by Lehmann [16]. In this paper, we first develop a DTS model in which

- System degradation is modeled based on the GDP model proposed by Lu and Meeker [19].
- Both fatal and effective shocks are taken into account.
- System degradation depends on the shocks, each incoming shock either leads to immediate system failure, or affects the system degradation by increasing the age of system according to the virtual age model proposed by Kijima [15], but the arrival of shock is not affected by the system degradation.

Today, thanks to development of sensor technology, it is often possible to collect information on the number and magnitude of shocks over time. For example, piezoelectric (PE) accelerometers with integral electronics, also called integrated

30 electronics piezoelectric (IEPE) accelerometers are vibration sensors designed for measurement of dynamic vibration signals at frequencies ranging from very low (near-dc) to 10 kHz, Levinzon [18]. The information collected by the sensors allows us to update prior information on the incoming shocks. Based on this new information, it is wise to reconsider the maintenance decisions which were  
35 previously made based on lesser information. According to Gertsbakh [10], the maintenance decisions made by taking to account new information that is provided during system operation are maintenance decisions with learning and the formal framework for maintenance with learning will be the so-called rolling horizon model. The rolling horizon approach has been used in many  
40 fields of science to tackle uncertainty in a dynamic stochastic environment. In this approach, the planning horizon is scheduled at periodic times and information/forecasts on uncertainty are updated at each discrete inspection/decision time. Updating is done based on a model that evolves over time. A decision is made for planning horizon based on the current information and is only im-  
45 plemented up to the next inspection/decision time. Updating is repeated at each inspection/decision time justifying the term of rolling horizon. The term of horizon refers to the number of periods in the future for which updating is done.

In this paper, we propose a dynamic maintenance policy for a single-unit  
50 system which is subject to shocks and deterioration. The maintenance decisions are made based on a new DTS model with uncertain parameters using the rolling horizon approach in a Bayesian framework. At any inspection time, the proposed method can choose the best maintenance action based on the current age and state of the system which minimizes the cost. To our knowledge, no  
55 work has been done in maintenance planning with DTS model where the model parameters are unknown. We believe that the assumptions made about failure and maintenance models in this paper make sense in practice, because in the real world most of systems operate under both deterioration and unexpected events while information about these two major causes of failure is often uncertain.  
60 Moreover, the proposed maintenance model provides an answer to the problem

of taking into account new information obtained during system operation in a condition-based-maintenance (CBM) with DTS model. It is worth noting that beside model-based maintenance policies, several maintenance policies have been proposed using model-free methods such as artificial intelligence or other  
65 machine learning algorithms, but very few of them consider the impact of new information on the decision-making procedure.

The remainder of this work is organized as follows. Section 2, introduces related previous research. In Section 3, model assumptions are introduced and a reliability model is derived. Section 4, proposes a CBM model with periodic  
70 inspection and a rolling horizon approach is adopted to tackle parameters uncertainty. Section 5, presents a numerical example illustrating the applicability of the proposed method. Section 6, presents the conclusion and future work.

## 2. Literature review

### 2.1. GDP model

75 The GDP model is a parametric model to describe degradation measurements. This model has been proposed to move reliability analysis from time-to-failure analysis to degradation measurements analysis. The GDP model has been considered by many researchers, e.g., Meeker and Escobar [21] for Alloy-A fatigue crack , Suzuki et al. [29] for a carbon film, and Tseng et al. [33]  
80 for fluorescent light bulbs. Further, extension of GDP methodology has been successfully applied in a variety of applications for prognostic and health management, Coble and Hines [5].

### 2.2. DTS model and maintenance

The class of DTS models have been examined by several authors, for example; Lemoine and Wenocur [17], Singpurwalla [28], Peng et al. [24], Wang et al. [35] and Fan et al. [6]. The problem of optimal maintenance policy under shock  
85 models was first studied by Taylor [31] and has become the subject of several subsequent generalization works, including CBM with DTS model. Existing

works on CBM with DTS model can be classified into two categories: first one,  
90 where system degradation is modeled by a stochastic process such as gamma  
process, Huynh et al. [11], Huynh et al. [12], Zhu et al. [44], Zhu et al. [45], Cas-  
tro et al. [2] and Yousefi et al. [42]; Wiener process, Yang et al. [38] and inverse  
Gaussian process, Wang et al. [34]. Second one, where system degradation is  
modeled by a GDP model; Peng et al. [24] and Yang et al. [39].  
95 In all the works mentioned above, the model parameters are assumed to be  
known.

### *2.3. Maintenance with parameter uncertainty*

The problem of parameter uncertainty in maintenance modeling has been  
considered by some authors. The common approach in such situations is to  
100 use the Bayesian theorem to update prior information about uncertain param-  
eters and to propose an adaptive maintenance policy. The existing works in  
this field could be classified into two parts based on the assumptions made  
on the system failure mechanism. The first one is those that consider a ran-  
dom distribution with uncertain parameters for system failure time as well as  
105 a maintenance model such as: age replacement model, Fox [9] and Sathe and  
Hancock [26]; age and block replacement models with minimal repair, Mazzuchi  
and Soyer [20]; age replacement model by taking into account the randomness  
of minimal repair cost, Sheu et al. [27]; age replacement model incorporating  
minimal repair, major repair, planned replacement, unplanned replacement and  
110 periodic scheduled maintenance, Juang and Anderson [13]. The second one is  
related to deteriorating systems in which a stochastic degradation process with  
unknown parameters is considered, failure mechanism is defined based on soft  
failure and a CBM model with periodic or aperiodic inspection times is consid-  
ered as a maintenance model, Kallen and van Noortwijk [14], Flage et al. [8]  
115 and Mosayebi et al. [22].

### *2.4. Dynamic maintenance model*

In the reliability literature, maintenance model with updating is called dy-  
namic maintenance model. Recently, Wu et al. [36] have proposed a dynamic

120 maintenance strategy with iteratively updated group information. They have  
formulated a new dynamic planning framework that captures economic depen-  
dence in both preventive and opportunistic replacement. The method proposed  
by them can be viewed as a dynamic version of the rolling horizon approach,  
which can be implemented iteratively until the end of the system service life  
without specifying and rolling the planning horizon. Yang et al. [40] have pro-  
125 posed an operations and maintenance (O&M) framework with fully considera-  
tion of both dynamic wind conditions and turbine operation conditions. In their  
work the impacts of dynamic wind conditions on both production and mainte-  
nance are integrated into decision-making process. A dynamic maintenance  
policy for multi-component systems with individually repairable components  
130 based on the reinforcement learning approach has been developed by Yousefi  
et al. [43], where each component experiences two competing failure processes  
of degradation and random shocks. Wu et al. [37] have proposed a dynamic  
CBM model based on an inverse Gaussian process with stochastic parameters  
where the distribution of drift parameters is updated over time and a dynamic  
135 maintenance threshold function (DMTF) has been proposed to replace the fixed  
maintenance threshold. Tang et al. [30] have formulated a dynamic maintenance  
policy in a semi-Markov decision process framework. In their work, a random-  
coefficient autoregressive model with time effect is developed to describe system  
degradation and a Bayesian approach for periodically updating the estimates of  
140 stochastic coefficients is proposed.

### 3. Reliability model

For this paper, we use the following mathematical symbols and notations.



### 3.1. Notations

CBM	Condition-based maintenance
DTS	Degradation-threshold-shock
GDP	General degradation path
$\{N_i(t), t \geq 0\}, i = 1, 2$	Fatal and effective shock arrival processes
$T_v$	Virtual age
$W_i$	Random increment due to the $i$ th effective shock
$F$	Distribution function of $W$
$A$	Random vector related to system random properties
$g(t; A), t \geq 0$	General degradation path model
$T$	Lifetime of system
$z_0$	Degradation threshold
$h(z, A), z \geq 0$	Inverse function of $g$ with respect to $t$
$F^{k*}$	$k$ -fold convolution of $F$
$G(a)$	Distribution function of $A$
$M$	Preventive threshold
$k\tau, k = 0, 1, \dots$	Periodic inspection times
$P_p(k\tau), P_c(k\tau)$	Preventive and corrective replacement probabilities at $k\tau$
$c_p$	Cost of a preventive replacement
$c_c$	Cost of a corrective replacement
$c_0$	Cost of an inspection
$L(M)$	Average long-run maintenance cost per unit of time
$T_r$	A renewal cycle
$C_r$	Maintenance cost per a renewal cycle
$\Theta$	Model parameters vector
$\pi(\Theta)$	Prior distribution function of $\Theta$
$\Theta^{(1)}, \Theta^{(2)}$	Model parameters vectors related to external and internal sources
$\pi(\Theta^{(i)}), \pi_i(\Theta^{(i)} data)$	Prior and posterior distribution functions of $\Theta^{(i)}, i = 1, 2$
$HPP$	Homogeneous poisson process
$\zeta$	Arbitrary small positive number

Here, we present a reliability model for a single-unit system which is sub-  
 145 ject to random shocks and deterioration. The model is proposed based on the  
 following assumptions.

### 3.2. Shock assumptions

Each shock either causes immediate system failure or increases the age of  
 system. The first one is called a fatal shock and the second one is called an  
 150 effective shock. The shocks arrive according to the counting processes,  $N_1(t)$   
 and  $N_2(t)$ , where  $N_1(t)$  and  $N_2(t)$  count the number of fatal and effective shocks  
 that have occurred up to time  $t$ . The arrival of fatal and effective shocks are  
 independent.

### 3.3. Degradation assumptions

155 Suppose that the system starts operating at time  $t = 0$ . The time  $t$ , since  
 the system starts operating is called calendar age of the system. Because the  
 effective shocks affect the system by increasing its age, it is reasonable to con-  
 sider a real age that describes the system state at each time. This real age is  
 called virtual age in accordance with the established terminology on system age,  
 160 Finkelstein [7]. It is clear that virtual age should depend on calendar age and  
 both virtual and calendar ages coincide where there is no shock. Let  $T_v$  be the  
 system virtual age at time  $t$ . It is assumed that each effective shock increases  
 the system age by random increment  $W_i \geq 0$ , with distribution function  $F$ .

That is,

$$T_v = t + \sum_{i=0}^{N_2(t)} W_i. \quad (1)$$

165 where  $W_0 = 0$ . This assumption has been recently used by Qingan and Lirong  
 [25]. Moreover, it is assumed that the system deteriorates over time according  
 to a GDP model proposed by Meeker and Escobar [21] which takes to account  
 the system virtual age instead of the system calendar age. Accordingly, the  
 system degradation value at time  $t$ , is defined as follows:

$$g(t + \sum_{i=0}^{N_2(t)} W_i; A) = g(T_v; A), \quad (2)$$

170 where  $g$  is a non-decreasing invertible function with respect to  $t$  that in practice may be derived from physics-of-failure models or from degradation test data, Coble and Hines [5] and  $A$  is a random vector related to random properties of the system. According to Ye and Xie [41], other than the simplicity, there are advantages of using the GDP model: (a) it is very flexible in incorporating  
 175 random effects. In comparison, a stochastic process model has limited ways in incorporating covariates and random effects. (b) The GDP model is more robust than process-based models. Although, the GDP model is applicable only when the unexplained randomness due to environmental factors is sufficiently small, there are many degradation real cases with a very low volatility, for example  
 180 systems subject to friction or erosion, such as brake pads, see Figure (2) taken from Trilla et al. [32].

Note that given  $A$ , the system degradation is not deterministic; it is governed by an increasing piecewise-deterministic process that jumps at random arrival times of shocks. Figure (3) shows an example of degradation path under given  
 185 model.

### 3.4. Failure mechanism

A system failure occurs when 1) system degradation reaches a critical level  $z_0$  (threshold model) or 2) a fatal shock arrives (shock model). The failure due to the first cause is called soft failure and due to the second cause is called hard  
 190 failure.

### 3.5. Reliability modeling

Let  $T$  be the system lifetime and  $h$  denote the inverse function of  $g$  with respect to  $t$ . The conditional reliability function of system is given by

$$\begin{aligned}
P(T > t \mid N_1(t) = 0, N_2(t) = n, A = a) &= P(g(t + \sum_{i=0}^n W_i; a) \leq z_0) \\
&= F^{n*}(h(z_0; a) - t)
\end{aligned} \tag{3}$$

where  $F^{k*}$  is  $k$ -fold convolution of  $F$  with itself i.e,

$$F^{k*}(t) = \begin{cases} 1(t) & k = 0, \\ F(t) & k = 1, \\ \int_{-\infty}^{\infty} F^{(k-1)*}(t-x)f(x)dx & k \geq 2, \end{cases}$$

195  $f(\cdot) = F'(\cdot)$  and

$$1(t) = \begin{cases} 1 & t \geq 0, \\ 0 & t \leq 0. \end{cases}$$

From (3), it is concluded that

$$P(T > t \mid A = a) = P(N_1(t) = 0) \sum_{n=0}^{\infty} F^{n*}(h(z_0; a) - t)P(N_2(t) = n) \tag{4}$$

and then, the reliability function of  $T$  is given by

$$\begin{aligned}
\bar{F}_s(t) &= P(T > t) = P(N_1(t) = 0) \sum_{n=0}^{\infty} E_A\{F^{n*}(h(z_0; A) - t)\}P(N_2(t) = n) \\
&= P(N_1(t)=0)\{E_A\{F^{0*}(h(z_0;A)-t)\}P(N_2(t)=0)+\sum_{n=1}^{\infty} E_A\{F^{n*}(h(z_0;A)-t)\}P(N_2(t)=n)\} \\
&= P(N_1(t)=0)\{P(N_2(t)=0) \int_{h(z_0;a) \geq t} dG(a) + \sum_{n=1}^{\infty} (\int F^{n*}(h(z_0;a)-t)dG(a))P(N_2(t)=n)\}.
\end{aligned} \tag{5}$$

where  $E_A$  is expectation with respect to  $A$  and  $G(a)$  is the distribution function of  $A$ . The proposed model could be considered as a generalization of the models  
200 proposed by Cha and Finkelstein [3] and Nakagawa [23].

## 4. Maintenance modeling

### 4.1. Maintenance policy

Suppose that inspection is scheduled at periodic times  $k\tau, k = 1, 2, \dots$ . The time interval  $(k\tau - \tau, k\tau]$  will be called the  $k$ th period. We assume that (a) the

205 system failure is not self-announced, (b) inspection is performed at the end of each period and (c) the time needed for inspection and maintenance replacement is negligible. At the end of the  $k$ th period, a failure is announced if during this period a fatal shock arrives or the system degradation level attains a critical level  $z_0$ . The system is replaced by a new identical one if a failure is detected  
 210 or the system degradation level exceeds a preventive threshold  $M (M < z_0)$ . For the first case, the maintenance task is called corrective maintenance and for the second is called preventive maintenance. The time between two successive replacements is referred to as a (renewal) cycle. Let  $P_p(k\tau)$  and  $P_c(k\tau)$  denote preventive and corrective maintenance probabilities at the end of the  $k$ th period,  
 215 respectively.

A preventive replacement is carried out at the end of the  $k$ th period when

$$\bullet g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau-\tau)} W_i; A) < M \ \& \ N_1(k\tau) = 0 \ \& \ M < g(k\tau + \sum_{i=0}^{N_2(k\tau)} W_i; A) < z_0.$$

Based on the failure mechanism, corrective replacement is performed at the end  
 220 of the  $k$ th period if

$$\bullet g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau-\tau)} W_i; A) < M \ \& \ N_1(k\tau) = 0 \ \& \ z_0 < g(k\tau + \sum_{i=0}^{N_2(k\tau)} W_i; A)$$

or

$$\bullet g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau-\tau)} W_i; A) < M \ \& \ N_1(k\tau - \tau) = 0 \ \& \ N_1(k\tau) = 1$$

225 The following costs are considered. A cost  $c_c$  occurs when a failed system is replaced (corrective maintenance cost). A cost  $c_p$  is incurred when a non-failed system is exchanged before failure (preventive maintenance cost) and  $c_0$  is the cost of inspection. Let  $E(C_r)$  and  $E(T_r)$  denote the expected maintenance cost per cycle and expected cycle length, respectively. According to the classical  
 230 renewal theorem, the average long-run maintenance cost per unit of time,  $L(M)$ , equals to the expected maintenance cost per cycle divided by the expected cycle

length, that is

$$L(M) = \frac{E(C_r)}{E(T_r)} \quad (6)$$

where

$$E(C_r) = \sum_{k=1}^{\infty} ((kc_0 + c_c)P_c(k\tau) + (kc_0 + c_p)P_p(k\tau))$$

and

$$E(T_r) = \sum_{k=1}^{\infty} k\tau(P_c(k\tau) + P_p(k\tau)).$$

Our objective is to search  $M$ , such that minimizes the average long-run main-  
 235 tenance cost per unit of time.

Let  $\Theta = (\theta_1, \dots, \theta_m)$  denote the vector of model parameters. We assume  
 that there is an uncertainty about  $\Theta$ . In other words, there exists a range of  
 potential values of  $\theta_i$  over which  $\theta_i$  varies unpredictably and thus randomly.  
 Assume that the uncertainty about the actual value of  $\Theta$  can be represented by  
 240 a prior distribution function  $\pi(\Theta)$ . In this case, the values of  $P_p(k\tau)$  and  $P_c(k\tau)$   
 are obtained as follows:

$$P_p(k\tau) = E_{\Theta}(P_p(k\tau|\Theta)) = \int_{\Theta} P_p(k\tau|\Theta)d\pi(\Theta) \quad (7)$$

and

$$P_c(k\tau) = E_{\Theta}(P_c(k\tau|\Theta)) = \int_{\Theta} P_c(k\tau|\Theta)d\pi(\Theta) \quad (8)$$

where  $P_p(k\tau|\Theta)$  and  $P_c(k\tau|\Theta)$  are preventive and corrective maintenance prob-  
 abilities given  $\Theta$  and are derived as follows:

$$P_p(k\tau|\Theta) = P_{\Theta_2}(N_1(k\tau) = 0) \sum_{x=0}^{\infty} \sum_{y \leq x} \left[ P_{\Theta_3}(N_2(k\tau) = x, N_2(k\tau - \tau) = y) \times \right. \\ \left. \left( \int_0^{\infty} \int_0^{h(M,a)-k\tau+\tau} (F_{\Theta_1}^{(x-y)*}(h(z_0, a) - k\tau - u) - F_{\Theta_1}^{(x-y)*}(h(M, a) - k\tau - u)) dF_{\Theta_1}^{y*}(u) dG_{\Theta_4}(a) \right) \right], \quad (9)$$

where  $\Theta_1, \Theta_2, \Theta_3$  and  $\Theta_4$  are the model parameters related to distribution functions of  $W, N_1(t), N_2(t)$  and  $A$ , respectively. Further,

$$P_c(k\tau|\Theta) = P_{c1}(k\tau|\Theta) + P_{c2}(k\tau|\Theta),$$

245

$$P_{c1}(k\tau|\Theta) = P_{\Theta_2}(N_1(k\tau) = 0) \sum_{x=0}^{\infty} \sum_{y \leq x} \left[ P_{\Theta_3}(N_2(k\tau) = x, N_2(k\tau - \tau) = y) \times \left( \int_0^{\infty} \int_0^{h(M,a) - k\tau + \tau} \left( 1 - F_{\Theta_1}^{(x-y)*}(h(z_0, a) - k\tau - u) \right) dF_{\Theta_1}^{y*}(u) dG_{\Theta_4}(a) \right) \right], \quad (10)$$

and

$$P_{c2}(k\tau|\Theta) = P_{\Theta_2}(N_1(k\tau - \tau) = 0, N_1(k\tau) = 1) \times \sum_{y=0}^{\infty} \left[ \left( \int_0^{\infty} F_{\Theta_1}^{y*}(h(M, a) - k\tau + \tau) dG_{\Theta_4}(a) \right) P_{\Theta_3}(N_2(k\tau - \tau) = y) \right]. \quad (11)$$

For more details, see appendices A1 and A2.

#### 4.2. Rolling horizon and information updating

The preventive and corrective maintenance probabilities given in (7-8) are  
 250 obtained by integrating the product of two parts: a permanent part,  $\{P_p(k\tau|\Theta), P_c(k\tau|\Theta)\}$ , and a variable part,  $\pi(\Theta)$ . Using the rolling horizon approach, the variable part is changed and updated based on the information received during the system operation. Set  $\Theta = (\Theta^{(1)}, \Theta^{(2)})$ , where  $\Theta^{(1)} = (\theta_1, \dots, \theta_j)$  is the model parameters related to external (environmental) source, i.e. the shocks, and  
 255  $\Theta^{(2)} = (\theta_{j+1}, \dots, \theta_m)$  is the model parameters related to internal source, i.e. the system degradation.

Suppose that  $\Theta^{(1)}$  and  $\Theta^{(2)}$  are independent and  $\pi_1(\Theta^{(1)})$  and  $\pi_2(\Theta^{(2)})$  represent prior distribution functions of  $\Theta^{(1)}$  and  $\Theta^{(2)}$ , respectively. In order to simplify and make tractable the formal part of our investigation, some assumptions are made as follows:  
 260

- Before the system starts operating, some initial information about the variable part is available and could be summarized in the form of prior distribution functions denoted by  $Data(0) = \{\pi_1(\Theta^{(1)}), \pi_2(\Theta^{(2)})\}$ .
- $W = \kappa W^*$ , where  $W^*$  is the effective shock magnitude and  $\kappa$  is known. This assumption means that the effective shock impact on the system age is proportional to its magnitude, Fan et al. [6].
- Number of shocks and their magnitude could be observed at the end of each inspection time.
- Information received in the course of system operation does not depend on the system properties and comes from external sources.

Based on  $Data(0)$  and equations (7) and (8), an optimal value of  $M$ , which minimizes (6) is obtained. This value is the optimal preventive threshold without updating the information. If the system does not fail during  $(0, \tau]$ , new information about the number of effective shocks,  $n_2(\tau)$ , and their magnitude  $\{w_1, \dots, w_{n_2(\tau)}\}$  is received at the end of the first period. Set  $data_1 = \{n_2(\tau), w_1, \dots, w_{n_2(\tau)}\}$ . This new information,  $data_1$ , is applied to recompute the prior distributions into the posterior distributions according to the Bayesian theorem. Therefore, at the end of the first period, the variable part is partially updated and denoted by

$$Data(1) = \{\pi_1(\Theta^{(1)}|data_1), \pi_2(\Theta^{(2)})\}$$

where  $\pi_1(\Theta^{(1)}|data_1)$  is the posterior distribution function of  $\Theta^{(1)}$ . Since  $\Theta^{(1)}$  is the model parameters related to shocks, its prior distribution function could be updated based on the information received in the course of system operation while this is not the case for  $\Theta^{(2)}$ , consequently  $\pi_2(\Theta^{(2)})$  is not changed. Now, we look for an optimal value of  $M$  given  $Data(1)$ . First, the probabilities of preventive and corrective maintenance are updated as follows:

$$P_p(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_p(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_1) d\pi_2(\Theta^{(2)}) \quad (12)$$



and

$$P_c(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_c(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_1) d\pi_2(\Theta^{(2)}) \quad (13)$$

Then, based on (12) and (13), an optimal value of  $M$ , which minimizes (6), is obtained. This optimal value is denoted by  $M_1$ , and used to make decision about the system preventive replacement at inspection time  $\tau$ . If the system does not fail during  $(\tau, 2\tau]$ , the second inspection takes place at time  $2\tau$ . Now, we reconsider all information including  $Data(1)$  and the new information received during the second period. This new information is denoted by  $data_2 = \{n_2(2\tau), w_1, \dots, w_{n_2(2\tau)}\}$  and applied to construct  $Data(2)$  as follows:

$$Data(2) = \{\pi_1(\Theta^{(1)}|data_2), \pi_2(\Theta^{(2)})\}$$

where  $\pi_1(\Theta^{(1)}|data_2)$  is the posterior distribution function of  $\Theta^{(1)}$  at the second stage of rolling horizon approach. Given  $Data(2)$ , the probabilities of preventive and corrective maintenance are updated once again at the second inspection time as follows:

$$P_p(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_p(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_2) d\pi_2(\Theta^{(2)}) \quad (14)$$

and

$$P_c(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_c(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_2) d\pi_2(\Theta^{(2)}) \quad (15)$$

Based on (14) and (15), an optimal value of  $M$  which minimizes (6) is obtained. This optimal value is denoted by  $M_2$ , and used to make decision about the system preventive replacement at inspection time  $2\tau$ . For next inspections, the information is updated again and the algorithm proceeds in a similar way resulting in  $M_3, M_4, \dots$

As we will see later, the values of  $M_1, M_2, M_3, \dots$  tend to a constant value which is the optimal preventive threshold where the model parameters are known. Therefore, the rolling horizon approach stops when a convergence in the successive obtained optimal values of  $M$  is observed.

## 5. Numerical example

The following numerical example shows how the rolling horizon approach is  
 310 used to obtain an optimal preventive threshold. In this example, a single-unit  
 system is considered and the following assumptions have been made:

1. The system degradation follows a linear degradation path model  $g(t, A) = \frac{T}{A}$ , where  $T$  is the system virtual age at time  $t$ ,  $h(z, A) = Az$ ,  $A \in \{7.25, 9.25\}$ ,  $P(A = 7.25) = \theta$  and  $P(A = 9.25) = 1 - \theta$ .
- 315 2. The system is subject to shocks and the shocks arrive according to homogeneous poisson process (HPP) with intensity  $\lambda$ . Each shock is fatal with probability  $p$  and is effective with complementary probability  $q = 1 - p$ . Thus, we have a thinning of the process that results in two HPP processes such that the process of effective shock is HPP with rate  $(1 - p)\lambda$  and the  
 320 process of fatal shock is HPP with rate  $p\lambda$ , [4].
3. The magnitude of effective shock,  $W^*$ , follows a gamma distribution with shape parameter  $\alpha$  and scale parameter  $\eta$ , i.e,  $f(w) = \frac{w^{\alpha-1}}{\eta^\alpha \Gamma(\alpha)} e^{-\frac{w}{\eta}}$ . The quantities  $\alpha$  and  $\eta$  are independent.
4.  $\kappa = 1$ , that is  $W = W^*$ .
- 325 5. The number and magnitude of shocks are observable.

Without loss of generality, we suppose that inspections are scheduled at  
 periodic times  $k, k = 1, 2, \dots$ . At each inspection, a preventive replacement  
 is carried out if the system degradation level reaches a specified value and a  
 corrective replacement is performed if a hard or soft failure occurs. We set  
 330  $z_0 = 10, c_c = 1000, c_p = 500$  and  $c_0 = 50$ .

### 5.1. Optimal value of $M$ when the model parameters are known

We suppose that the model parameters  $\Theta = (p, \lambda, \alpha, \eta, \theta)$  are known and we have  $(p, \lambda, \alpha, \eta, \theta) = (0.01, 4, 2, 1, 0.3)$ . Let  $M^*$  denote the optimal value of  $M$ . Based on the given parameters and using numerical methods, the following values are obtained for  $M^*$  and  $L(M^*)$  :

$$M^* = 7.8, \quad \text{and} \quad L(M^*) = 146.86.$$

These values are used to compare the accuracy of the following methods.

5.2. *Optimal value of  $M$  when the model parameters are unknown and there is no learning*

335 In this method, the probabilities of corrective and preventive maintenance are obtained from (7) and (8) without updating the prior distributions. Then, the optimal value of  $M$  is obtained by minimizing (6) with respect to  $M$ . A critical main step in this method is to specify the prior distributions in order to quantify the expert belief about the model parameters. In this regard, physical  
 340 meaning of the parameters could be sometimes helpful. We consider an informative prior for  $\lambda$  in terms of its physical meaning as follows. It is known that for a HPP with intensity  $\lambda$ , the mean number of shocks in any interval of length one is  $\lambda$ . This physical meaning of  $\lambda$  might be usefully exploited to formulate the prior knowledge about  $\lambda$  regarding to a mean value  $\mu$  and a standard deviation value  $\sigma$ . That is, if a gamma distribution is chosen as prior distribution  
 345 for  $\lambda$ ,  $\pi_\lambda(\lambda) = \frac{\lambda^{a-1}}{b^a \Gamma(a)} e^{-\frac{\lambda}{b}}$ , the hyperparameters  $a, b$  can be related to the prior knowledge by  $a = \frac{\mu^2}{\sigma^2}$  and  $b = \frac{\sigma^2}{\mu}$ . Now, suppose that based on the engineer's experience, we know that  $\mu = 4$  and  $\sigma^2 = 3.2$ . As a result, the hyperparameters are obtained as  $a = 5$  and  $b = 0.8$ .

350 Further, suppose that according to the existing information, the fatal shock rate is 0.01. We choose a beta distribution  $\pi_p(p) = \frac{p^{e-1}(1-p)^{f-1}}{B(e,f)}$  with  $e = 1$  and  $f = 99$  as prior distribution for  $p$  that supports a strong belief of the rarity of fatal shocks. For the shape parameter  $\alpha$  of density function  $f(w)$ , a uniform distribution on the interval (1.5, 2.5) is used as prior distribution. Note that  
 355 the uniform distribution is a special case of discrete beta distribution which is commonly used as prior distribution for the shape parameter.

As prior distribution for  $\eta$ , a gamma distribution  $\pi_\eta(\eta) = \frac{\eta^{c-1}}{d^c \Gamma(c)} e^{-\frac{\eta}{d}}$  is chosen. We set the hyperparameters as  $c = 4$  and  $d = 0.75$ , resulting in  $E(\eta) = 3$  and  $Var(\eta) = 2.25$ . Assume that there is no information about  $\theta$ . In such a case, we believe that it is reasonable to present prior knowledge by a flat prior, i.e., uniform distribution on the interval (0, 1). We refer to the prior distribu-

tions given in this section as Prior I. Let  $M_0^I$  denote the optimal value of  $M$  under Prior I. Using numerical methods, the following values are obtained for  $M_0^I$  and  $L(M_0^I)$  :

$$M_0^I = 7.1, \quad \text{and} \quad L(M_0^I) = 234.69.$$

It is observed that the results obtained in this section are significantly different from the similar results in section 5.1. Therefore, using the Bayesian method without learning leads to a significant bias in results.

360 *5.3. Dynamic optimal value of  $M$  when the model parameters are unknown through the rolling horizon approach*

In this method, the prior distribution of  $\lambda$  and the joint prior distribution of  $(\alpha, \eta)$  are updated at the end of the  $k$ th period. Updating is done based on the information received on the number and magnitude of effective shocks up to the  
 365  $k$ th inspection through the rolling horizon approach. After updating the prior distributions, both probabilities of corrective and preventive maintenance are subsequently updated and applied in (6). At each inspection time, an optimal value of  $M$  is obtained by minimizing (6) with respect to  $M$ , resulting in a dynamic maintenance policy throughout the monitoring procedure. The detailed  
 370 steps of this method are given in appendix A3 and Figure (1).

The information on the number and magnitude of shocks and the observed degradation values were generated as follows. The number of incoming shocks in each period with length one was generated from a Poisson distribution with parameter  $\lambda = 4$ , and the magnitude of effective shocks was generated from a  
 375 gamma distribution with shape parameter  $\alpha = 2$ , and scale parameter  $\eta = 1$ . According to the GDP model, observed degradation value is equal to real degradation value plus measurement error, i.e.,  $g_{obs}(t, a) = g(t, a) + \xi$  where  $g_{obs}(t, a)$  is the observed degradation value at time  $t$ . Moreover, based on the simulation assumptions, the real degradation value at time  $t$  is  $g(T, a) = \frac{T}{a} = \frac{t + \sum_{i=0}^{N_2(t)} W_i}{a}$ .  
 380 We set  $a = 9.25$  as real value of  $a$ , and we suppose that the measurement error  $\xi$  follows a normal distribution with mean  $\mu_\xi = 0$  and variance  $\sigma_\xi^2 = 0.3$ . The generated data have been reported in Table (1).

Based on the given prior distributions in previous section and new information received on the number of shocks, a gamma distribution with shape parameter  $n + a$  and scale parameter  $\frac{b}{kb+1}$  is concluded as updated prior distribution of  $\lambda$  given data (posterior distribution),  $\pi_\lambda(\lambda|data)$ , when  $n$  is the number of effective shocks up to the  $k$ th period. The updated joint prior distribution of  $(\alpha, \eta)$  given data (joint posterior distribution) is obtained as follows:

$$\pi_{(\alpha, \eta)}(\alpha, \eta|data) \propto L(data|\alpha, \eta)\pi_\eta(\eta)\pi_\alpha(\alpha) = \frac{w_p^{\alpha-1}e^{-\left(\frac{w_s}{\eta} + \frac{\eta}{d}\right)}}{\eta^{n\alpha-c+1}\Gamma^n(\alpha)},$$

$$1.5 < \alpha < 2.5, \quad 0 < \eta < \infty. \quad (16)$$

where  $\pi_\alpha(\alpha)$  is prior distribution of  $\alpha$ ,  $w_p = \prod_{i=1}^n w_i$  and  $w_s = \sum_{i=1}^n w_i$ .

Let  $M_k^I, k = 1, 2, \dots$ , denote the optimal value of  $M$  at the  $k$ th inspection time obtained by the rolling horizon approach under Prior I. The prior distributions, and then  $P_p(k\tau)$  and  $P_p(k\tau)$ , are updated based on the information received at the  $k$ th inspection time via the rolling horizon approach and applied in (6). Then,  $M_k^I$  is obtained by minimizing (6) with respect to  $M$ . Table (3, first row) gives  $M_k^I$  and  $L(M_k^I)$  for  $k = 0, \dots, 8$  and Figure (8) describes the evolution of  $M_k^I$  over successive inspection times. From the results, it is observed that at the first inspection time with  $n = 3$ , the value of  $M_1^I$  is the same as the value of  $M_0^I$ . This means that the number of observations,  $n = 3$ , is not enough to affect  $M_1^I$ . At the second inspection time with  $n = 8$ , the value of  $M_2^I$  begins to increase and from the 3th inspection time to the 8th inspection time, the value of  $M_k^I$  moves between two values: 7.6 and 7.7. This means that  $M_k^I$  has a stationary behaviour after the 3th inspection time. That is, the value of  $M_k^I$  moves toward the desired value of  $M^*$  over inspection times and eventually its mean value becomes constant and its variance decreases. Moreover, the value of  $L(M_k^I)$  has a decreasing rate from 169.21 at the first inspection time to 148.01 at the 8th inspection time that the latter is close to  $L(M^*) = 146.86$ . From the numerical results obtained here, it is concluded that the rolling horizon approach is efficient as it is accurate enough for providing optimal value of  $M$  over time while working under uncertainty.

410 The number of inspections considered here for applying the rolling horizon approach has been limited to 8. The reason is that based on the simulated degradation values, Table (1) and the obtained optimal values of  $M$ , Table(3, first row), the system should be replaced at the 8th inspection time as a preventive act. Therefore, it is not available after the 8th inspection time .

415 *5.4. The effect of prior information*

In order to study the effect of prior information on the obtained results, besides Prior I, three other priors denoted by Prior II, Prior III and Prior IV have been considered. Prior III is less informative than Prior II and Prior IV is less informative than Prior I. Table (2) provides details about the priors. The evolution of updated prior distributions given data (posterior distributions) for  $\lambda$  and  $(\alpha, \eta)$  has been shown in Figures (4-7). The Figures reveal that the posterior distributions are highly influenced by the prior distributions at the first stage of the rolling horizon approach. This happens because posterior distribution is affected by likelihood and prior distribution (Posterior distribution  $\propto$  Likelihood  $\times$  Prior distribution). At the first stage of the rolling horizon approach, the number of observations is small ( $n = 3$ ). Therefore, the influence of prior distribution on the posterior distribution is greater than the likelihood function. But, as the number of observations increases, the posterior distribution is more affected by the likelihood function. Moreover, by increasing the number of observations, the variations decrease and then the tail of univariate distributions of  $\alpha$  and  $\eta$  becomes shorter resulting in a narrow and compact posterior contour plot.

425 Let  $M_k^m, k = 0, 1, \dots, 8, m = I, II, III, IV$  denote the optimal value of  $M$  under Prior  $m$  at the  $k$ th inspection time. Table (3) and Figures (8-9) reveal some facts concerning evolution of  $M_k^m$ , as follows:

- The evolution behaviour of optimal value of  $M$  is similar for all priors. That is, for each  $m = I, II, III, IV$ , there is no significant difference between the values of  $M_0^m$  and  $M_1^m$ , the value of  $M_k^m$  increases from  $k = 1$  up to  $k = 4$  and then moves around a constant value. This means that

440 the evolution behavior of optimal value of  $M$  depends on the information received in the course of system operation, i.e., the number and magnitude of shocks.

- All values of  $M_k^m, k = 0, 1, \dots, 8, m = I, II, III, IV$  are less than  $M^*$ , and are influenced by the prior information, particularly at the first stage of the rolling horizon. At the first stage of the rolling horizon, the evolution paths related to the less informative priors have a downward shift compared to the informative priors. This is because at the initial inspection times due to lack of information, the prior distribution has more impact on the posterior distribution than the likelihood function while the opposite is true for the next inspection times with enough information.
- Under the less informative priors, the values of  $M_k^m, k = 0, 1, \dots$ , are more spread out.

From a statistical point of view, since the degradation parameters are not updated over time, even for  $k \rightarrow \infty$ , the value of  $M_k^m$  is biased. Its bias depends on the existing information about  $\Theta^{(2)}$ . Therefore, a particular attention should be paid to collect information about  $\Theta^{(2)}$ , as far as possible.

From an economic point of view, the results show that the preventive replacement decision made at the initial inspection times is more costly. This is due to the conservatism of the preventive maintenance decision to take into account the uncertainty about the model parameters. Therefore, it is recommended to continue the rolling horizon till the obtained preventive threshold reaches to a constant value.

## 6. Conclusion

In this work, we first developed a DTS model for a single-unit system where  
465 i) both fatal and effective shocks are considered, ii) the system degradation  
follows a GDP model and depends on the effective shocks, iii) the impact of  
effective shocks on the system degradation is expressed through the virtual age  
concept, iv) the system fails due to soft or hard failure. Then, we investigated an  
optimal maintenance strategy for the system based on the proposed DTS model  
470 where there is an uncertainty about the model parameters. A rolling horizon  
approach in the Bayesian framework has been adopted for this purpose. Accord-  
ing to this method the system learns about environmental conditions through  
the information received at each inspection time and in the light of this knowl-  
edge the maintenance decisions are improved. The results obtained from the  
475 numerical simulation studies reveal the applicability and efficiency of the pro-  
posed method. The obtained preventive threshold by this method is improved  
over time and tends to a constant value which is close to the optimal value  
of preventive threshold where the model parameters are known. Based on the  
sensitivity analysis, it is observed that prior knowledge highly influences on the  
480 early stage of rolling horizon approach and the evolution behaviour of optimal  
preventive threshold depends on the information received at each inspection.

This work could be developed at least in three following aspects as future  
work. First, in the maintenance model represented in this work the inspection  
times are scheduled at periodic times while it is more convenient to consider ape-  
485 riodic inspection times and update both the next inspection time and preventive  
degradation level by the rolling horizon approach. It is more complicated but it  
is less costly plan. Second, preventive act here is the system replacement while  
there are another types of preventive maintenance which are less costly such  
as imperfect or minimal repair. Although, this work considers the uncertainty  
490 about the degradation model parameters, proposing a maintenance model with  
possibility of updating degradation model parameters as well as shock model  
parameters could be third aspect of developing current work.



## Appendix

### A1. Preventive replacement probability

495 Suppose that  $\Theta = (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$  is the vector of model parameters where  $\Theta_1, \Theta_2, \Theta_3$  and  $\Theta_4$  are the model parameters related to distribution functions of  $W, N_1(t), N_2(t)$  and  $A$ , respectively.

Given values of  $\Theta, A, N_1(k\tau), N_2(k\tau)$ , and  $N_2(k\tau - \tau)$ , the conditional preventive replacement probability at the  $k$ th period  $P_p(k\tau|\Theta, A, N_1(k\tau), N_2(k\tau), N_2(k\tau - \tau))$  is given by

$$\begin{aligned}
 & P_p(k\tau|\Theta, A, N_1(k\tau) = 0, N_2(k\tau), N_2(k\tau - \tau)) = \\
 & P \left( M \leq g(k\tau + \sum_{i=0}^{N_2(k\tau)} W_i; A) < z_0, g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau - \tau)} W_i; A) < M \right) = \\
 & P \left( h(M, A) - k\tau \leq \sum_{i=0}^{N_2(k\tau)} W_i < h(z_0, A) - k\tau, \sum_{i=0}^{N_2(k\tau - \tau)} W_i < h(M, A) - k\tau + \tau \right) = \\
 & \int_0^{h(M, A) - k\tau + \tau} P \left( h(M, A) - k\tau \leq \sum_{i=0}^{N_2(k\tau)} W_i < h(z_0, A) - k\tau, \sum_{i=0}^{N_2(k\tau - \tau)} W_i = u \right) du = \\
 & \int_0^{h(M, A) - k\tau + \tau} \left[ P \left( h(M, A) - k\tau - u \leq \sum_{i=N_2(k\tau - \tau) + 1}^{N_2(k\tau)} W_i < h(z_0, A) - k\tau - u \right) \times \right. \\
 & \left. f_{\Theta_1}^{N_2(k\tau - \tau)*}(u) \right] du = \\
 & \int_0^{h(M, A) - k\tau + \tau} \left( F_{\Theta_1}^{\Delta N(k\tau)*}(h(z_0, A) - k\tau - u) - F_{\Theta_1}^{\Delta N(k\tau)*}(h(M, A) - k\tau - u) \right) dF_{\Theta_1}^{N_2(k\tau - \tau)*}(u).
 \end{aligned} \tag{17}$$

where  $f_{\Theta_1}^{n*}(u)$  is the probability density function of  $\sum_{i=0}^n W_i$  and  $\Delta N(k\tau) = N_2(k\tau) - N_2(k\tau - \tau)$ .

By taking expectation with respect to  $A, N_1(k\tau), N_2(k\tau)$ , and  $N_2(k\tau - \tau)$ , the preventive replacement probability at the  $k$ th period given  $\Theta$ ,  $P_p(k\tau|\Theta)$  is

505 obtained as follows:

$$\begin{aligned}
P_p(k\tau|\Theta) = & \\
& \sum_{x=0}^{\infty} \sum_{y \leq x} \left[ P_{\Theta_2}(N_1(k\tau) = 0) P_{\Theta_3}(N_2(k\tau) = x, N_2(k\tau - \tau) = y) \times \right. \\
& \left. \left( \int_0^{\infty} \int_0^{h(M,a)-k\tau+\tau} (F_{\Theta_1}^{(x-y)*}(h(z_0, a) - k\tau - u) - F_{\Theta_1}^{(x-y)*}(h(M, a) - k\tau - u)) dF_{\Theta_1}^{y*}(u) dG_{\Theta_4}(a) \right) \right].
\end{aligned}$$

### A2. Corrective replacement probability

Given values of  $\Theta, A, N_1(k\tau), N_2(k\tau)$  and  $N_2(k\tau - \tau)$ , the conditional corrective replacement probability at the  $k$ th period  $P_c(k\tau|\Theta, A, N_1(k\tau) = 0, N_2(k\tau), N_2(k\tau - \tau))$  is obtained as follows:

$$\begin{aligned}
P_c(k\tau|\Theta, A, N_1(k\tau) = 0, N_2(k\tau), N_2(k\tau - \tau)) = & \\
P_{c1}(k\tau|\Theta, A, N_1(k\tau) = 0, N_2(k\tau), N_2(k\tau - \tau)) + P_{c2}(k\tau|\Theta, A, N_1(k\tau) = 1, N_2(k\tau), N_2(k\tau - \tau)) & \\
(18) &
\end{aligned}$$

510 where

$$\begin{aligned}
P_{c1}(k\tau|\Theta, A, N_1(k\tau) = 0, N_2(k\tau), N_2(k\tau - \tau)) = & \\
P \left( z_0 \leq g(k\tau + \sum_{i=0}^{N_2(k\tau)} W_i; A), g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau - \tau)} W_i; A) < M \right) = & \\
P \left( h(z_0, A) - k\tau \leq \sum_{i=0}^{N_2(k\tau)} W_i, \sum_{i=0}^{N_2(k\tau - \tau)} W_i < h(M, A) - k\tau + \tau \right) = & \\
\int_0^{h(M,A)-k\tau+\tau} P \left( h(z_0, A) - k\tau \leq \sum_{i=0}^{N_2(k\tau)} W_i, \sum_{i=0}^{N_2(k\tau - \tau)} W_i = u \right) du = & \\
\int_0^{h(M,A)-k\tau+\tau} \left[ P \left( h(z_0, A) - k\tau - u \leq \sum_{i=N_2(k\tau - \tau) + 1}^{N_2(k\tau)} W_i \right) \times \right. & \\
\left. f_{\Theta_1}^{N_2(k\tau - \tau)*}(u) \right] du = & \\
\int_0^{h(M,A)-k\tau+\tau} \left( 1 - F_{\Theta_1}^{\Delta N(k)*}(h(z_0, A) - k\tau - u) \right) dF_{\Theta_1}^{N_2(k\tau - \tau)*}(u). & \\
(19) &
\end{aligned}$$

and

$$\begin{aligned}
& P_{c2}(k\tau|\Theta, A, N_1(k\tau - \tau) = 0, N_1(k\tau) = 1, N_2(k\tau - \tau)) = \\
& P\left(g(k\tau - \tau + \sum_{i=0}^{N_2(k\tau - \tau)} W_i; A) < M\right) = P\left(\sum_{i=0}^{N_2(k\tau - \tau)} W_i < h(M, A) - k\tau + \tau\right) = \\
& = F_{\Theta_1}^{(N_2(k\tau - \tau))^*}(h(M, A) - k\tau + \tau).
\end{aligned} \tag{20}$$

By taking expectation with respect to  $A, N_1(k\tau), N_2(k\tau)$ , and  $N_2(k\tau - \tau)$ , it follows that

$$\begin{aligned}
P_{c1}(k\tau|\Theta) &= \sum_{x=0}^{\infty} \sum_{y \leq x} \left[ P_{\Theta_2}(N_1(k\tau) = 0) P_{\Theta_3}(N_2(k\tau) = x, N_2(k\tau - \tau) = y) \times \right. \\
&\quad \left. \left( \int_0^{\infty} \int_0^{h(M, a) - k\tau + \tau} \left(1 - F_{\Theta_1}^{(x-y)^*}(h(z_0, a) - k\tau - u)\right) dF_{\Theta_1}^{y*}(u) dG_{\Theta_4}(a) \right) \right]
\end{aligned}$$

and

$$\begin{aligned}
P_{c2}(k\tau|\Theta) &= P_{\Theta_2}(N_1(k\tau - \tau) = 0, N_1(k\tau) = 1) \times \\
&\quad \left( \sum_{y=0}^{\infty} \left( \int_0^{\infty} F_{\Theta_1}^{y*}(h(M, a) - k\tau + \tau) dG_{\Theta_4}(a) \right) P_{\Theta_3}(N_2(k\tau - \tau) = y) \right).
\end{aligned}$$

Finally, we have

$$P_c(k\tau|\Theta) = P_{c1}(k\tau|\Theta) + P_{c2}(k\tau|\Theta).$$

515 *A3. Procedure of obtaining the optimal value of  $M$  via the rolling horizon approach*

1. Set initial values  $\tau, c_0, c_c, c_p, z_0, \zeta$  and initial prior distributions  $Data(k) = \{\pi_1(\Theta^{(1)}|data_k), \pi_2(\Theta^{(2)})\}, k = 0$ .
  2. Update the prior distributions and then the probabilities of preventive and
- 520 corrective replacement based on the  $Data(k)$  according to:

$$P_p(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_p(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_k) d\pi_2(\Theta^{(2)}). \tag{21}$$

and

$$P_c(k\tau) = \int_{\Theta^{(2)}} \int_{\Theta^{(1)}} P_c(k\tau|\Theta^{(1)}, \Theta^{(2)}) d\pi_1(\Theta^{(1)}|data_k) d\pi_2(\Theta^{(2)}). \quad (22)$$

3. Obtain the optimal preventive degradation level  $M_k^*$ , such that

$$M_k^* = \operatorname{argmin}_M L(M).$$

4. If  $|M_k^* - M_{k-1}^*| < \zeta$ , go to the next step. Otherwise, set  $k = k + 1$  and go back to step 2.
5. Output the optimal preventive degradation level  $M_k^*$  and  $L(M_k^*)$ .

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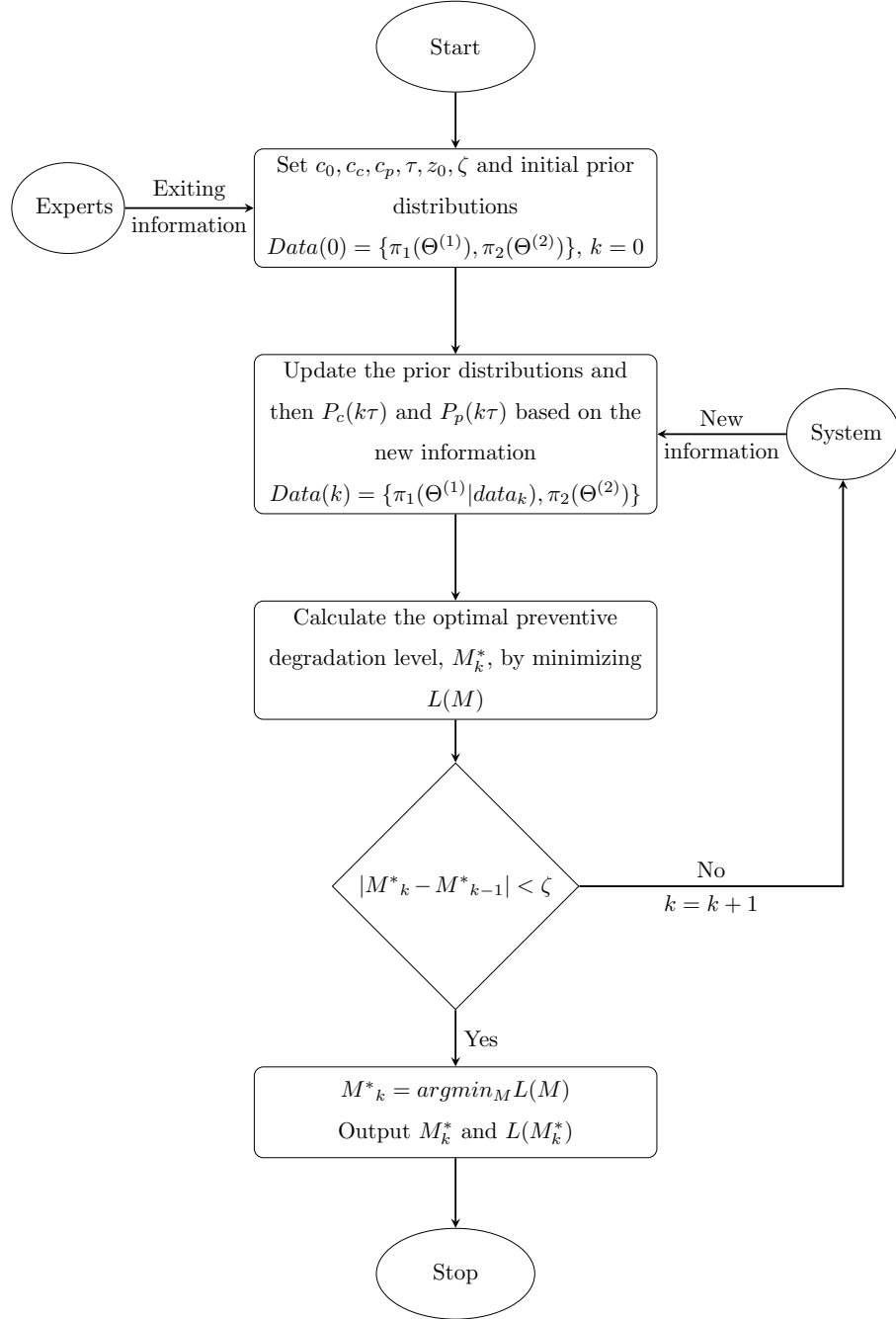


Figure 1: Flowchart of the optimization procedure under the proposed maintenance strategy

Table 1: Generated data including number ( $n$ ) and magnitude of effective shocks ( $W$ ) as well as observed degradation at each inspection time

$k$	$n_k$	$W_i$ $i=1,\dots,n_k$	Observed degradation at the $k$ th inspection time
1	3	1.530, 2.986, 1.165	0.3287
2	5	3.594, 1.084, 0.912, 1.043, 2.416	1.708
3	4	2.918, 1.753, 1.949, 2.900	2.723
4	5	0.743, 5.217, 0.865, 1.185, 3.768	4.274
5	2	1.993, 0.702	4.678
6	4	2.091, 0.411, 2.080, 2.072	4.827
7	3	6.387, 3.079, 1.080	6.452
8	4	0.625, 0.756, 0.652, 1.617	7.611
9	0	—	7.848
10	2	3.044, 1.374	7.943

Table 2: Different configurations for priors

Prior	a	b	c	d	$E(\lambda)$	$Var(\lambda)$	$E(\eta)$	$Var(\eta)$
I	5	0.8	4	0.75	4	3.2	3	2.25
II	5	0.6	2	1	3	1.8	2	2
III	5	0.6	0.04	50	3	1.8	2	100
IV	2.7	1.5	4	0.75	4.05	6.07	3	2.25

Table 3: Optimal values of preventive threshold (M) and corresponding minimum costs over time under different priors

	Inspection time	0	1	2	3	4	5	6	7	8
Prior I	Optimal value of M	7.1	7.1	7.4	7.6	7.7	7.6	7.6	7.7	7.6
	Cost	234.69	169.21	158.7	157.06	158.99	151.72	149.88	150.70	148.01
Prior II	Optimal value of M	6.4	6.5	7.1	7.4	7.6	7.5	7.4	7.6	7.5
	Cost	181.90	154.31	150.9	150.46	152.46	147.78	146.81	147.43	145.72
Prior III	Optimal value of M	6.0	6.0	6.8	7.3	7.6	7.4	7.4	7.6	7.4
	Cost	271.07	153.66	150.21	149.71	151.63	147.28	146.45	146.99	145.42
Prior IV	Optimal value of M	6.9	7.0	7.4	7.5	7.7	7.6	7.6	7.6	7.6
	Cost	230.26	168.02	159.41	157.52	159.65	151.85	150.02	150.65	148.05

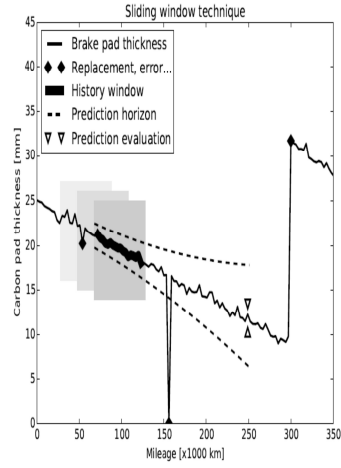


Figure 2: Evaluation of brake pad prognostic with the sliding window prediction technique  
(Figure taken from reference [32])

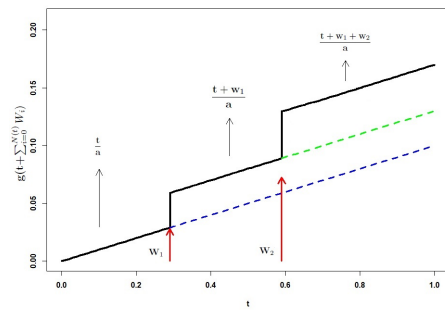


Figure 3: Degradation path under two effective shocks with increments  $w_1$  and  $w_2$  where  $g$  is a linear function and  $A = a$

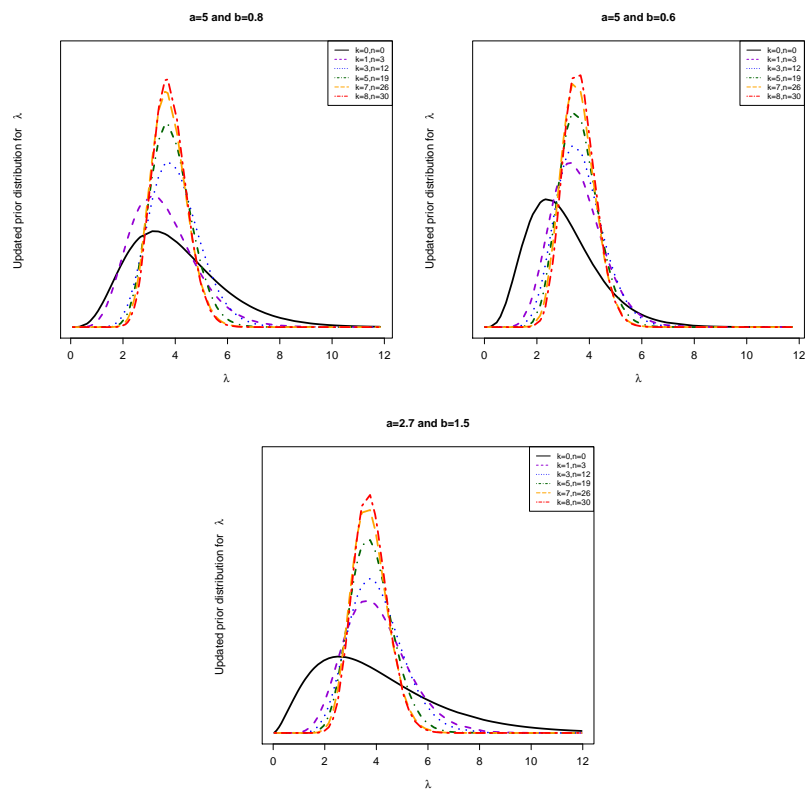


Figure 4: Prior and posterior distribution functions for  $\lambda$  with different hyperparameters

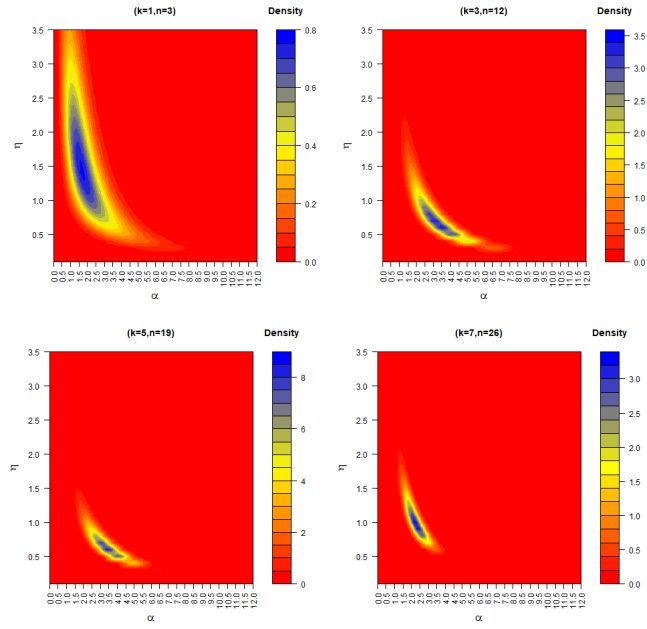


Figure 5: Evolution of posterior distribution function for  $(\alpha, \eta)$  where  $c = 4$  and  $d = 0.75$

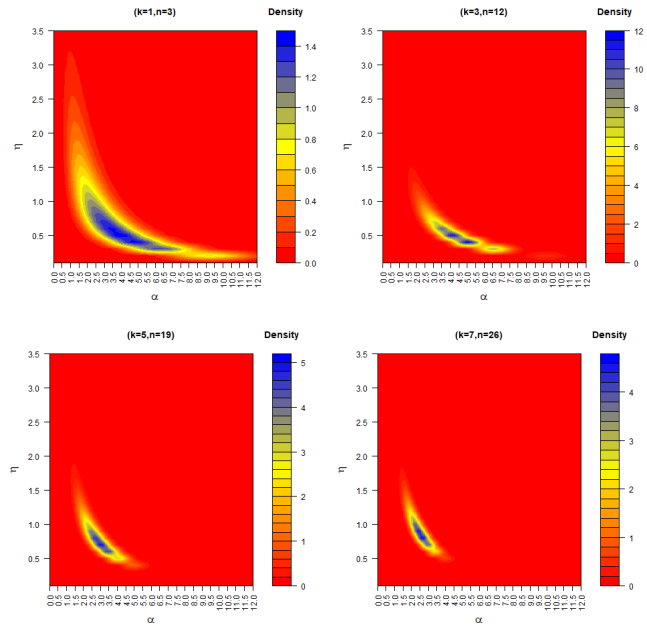


Figure 6: Evolution of posterior distribution function for  $(\alpha, \eta)$  where  $c = 2$  and  $d = 1$

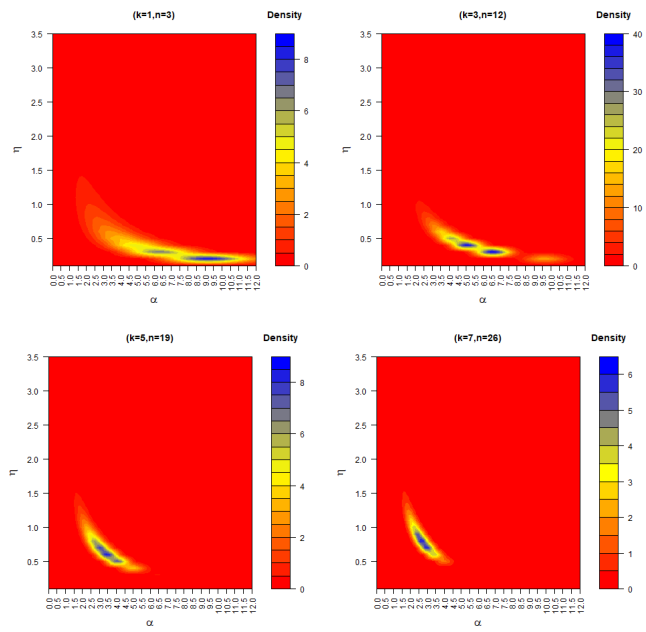


Figure 7: Evolution of posterior distribution function for  $(\alpha, \eta)$  where  $c = 0.04$  and  $d = 50$



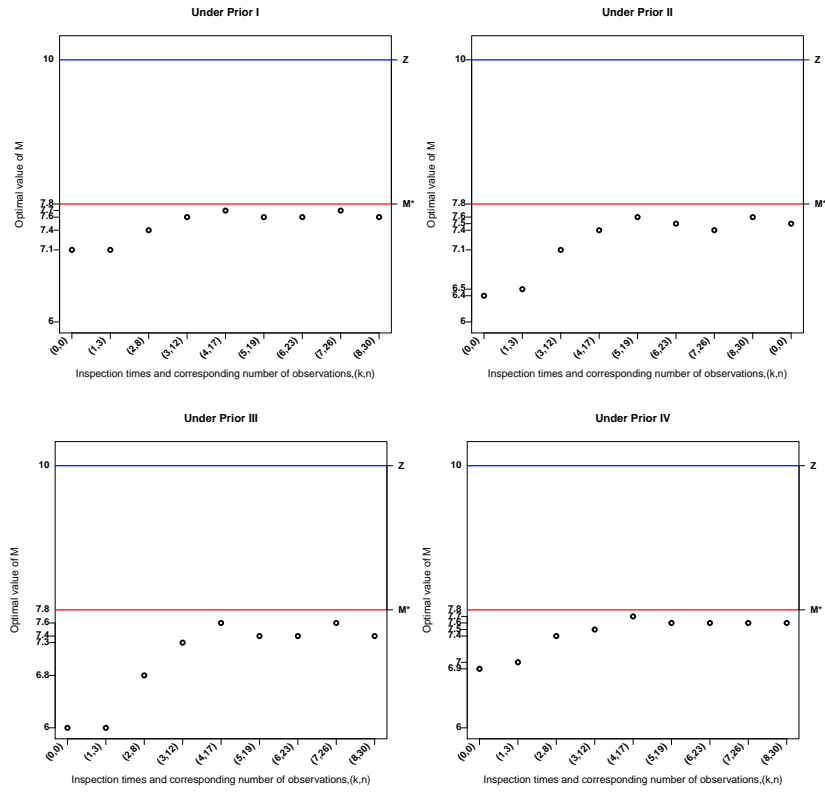


Figure 8: Evolution of optimal preventive threshold over time under different priors

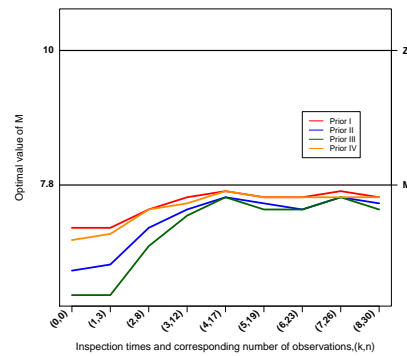


Figure 9: Comparison of optimal preventive threshold evolution paths under different priors