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## Efficient enumeration of rooted maps of a given orientable genus by number of faces and vertices

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#### Abstract

We simplify the recurrence satisfied by the polynomial part of the generating function that counts rooted maps of positive orientable genus g by number of vertices and faces. We have written an optimized program in C++ for computing this generating function and constructing tables of numbers of rooted maps, and we describe some of these optimizations here. Using this program we extended the enumeration of rooted maps of orientable genus g by number of vertices and faces to g = 4, 5 and 6 and by number of edges to g = 5and 6 and conjectured a further simplification of the generating function that counts rooted maps by number of edges. Our program is documented and available on request, allowing anyone with a sufficiently powerful computer to carry the calculations even further.

*Keywords: Efficient enumeration, rooted maps, orientable genus, generating functions Math. Subj. Class.: 05C30, 05A15* 

#### 1 Introduction: definitions and history

A *map* is defined topologically as a 2-cell imbedding of a connected graph, loops and multiple edges allowed, in a 2-dimensional surface. The *faces* of a map are the connected components of the complement of the graph in the surface. In this article the surface is assumed to be without boundary and orientable, with an orientation already attributed to it (clockwise, say), so that it is completely described by a non-negative integer g, its *genus*. For short, a map on a surface of genus g will be called a *genus-g map*. A *planar map* is a

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genus-0 map (a map on a sphere) and a *toroidal map* is a genus-1 map (a map on a torus or donut). If a map on a surface of genus g has v vertices, e edges and f faces, then by the Euler-Poincaré formula [7, chap. 9]

$$v - e + f = 2(1 - g).$$
 (1.1)

Two maps are equivalent if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges and faces of one map into the vertices, edges and faces of the other. A *dart* of a map or graph is a semi-edge. A loop is assumed to be incident twice with the same vertex, so that every edge, whether or not it is a loop, is incident to two darts. The *degree* of a vertex is the number of darts incident to it. The face incident to a dart d is the face incident to the edge containing d and on the left of an observer on d facing away from the vertex incident to d and the degree of a face is the number of darts incident to it. A rooted map is a map with a distinguished dart, its root. Two rooted maps are equivalent if there is an orientation-preserving homeomorphism between their imbedding surfaces that takes the vertices, edges, faces and the root of one map into the vertices, edges, faces and the root of the other. A *combinatorial map* is a connected graph with a cyclic order imposed on the darts incident to each vertex, representing the order in which the darts of a (topological) map are encountered during a rotation around the vertex according to the orientation of the imbedding surface. The darts incident to a face are encountered by successive application of the following pair of actions: go from the current dart to the dart on the other end of the same edge and then to the next dart incident to the same vertex according to the cyclic order. In this way the faces of a combinatorial map can be counted, so that its genus can be calculated from (1.1). Two combinatorial maps are equivalent if they are related by a map isomorphism - a graph isomorphism that preserves this cyclic order - with an analogous definition for the equivalence of two rooted combinatorial maps. By *enumerating* maps with a given set of properties, whether rooted or not, we mean counting the number of equivalence classes of maps with these properties. It was shown in [10] that each equivalence class of topological maps is uniquely defined by an equivalence class of combinatorial maps; so for the purposes of enumeration, the term "map" can be taken to mean "combinatorial map".

Let  $m_g(v, f)$  be the number of rooted genus-g maps with v vertices and f faces. By face-vertex duality, this number is equal to the number  $m_g(f, v)$  of rooted genus-g maps with f vertices and v faces. The generating function that counts rooted genus-g maps is the following formal power series in two variables u and w:

$$M_g(w, u) = \sum_{v, f \ge 1} m_g(v, f) w^v u^f.$$
 (1.2)

Rooted maps were introduced in [13] because they are easier to count than unrooted maps; this is because only the trivial map automorphism preserves the root [14], so that rooted maps can be counted without considering map automorphisms. In [13], W. T. Tutte found a closed-form formula for the number of rooted planar maps with n edges. In [14], he found a parametric system of equations defining  $M_0(w, u)$ . In [1] D. Arquès obtained the simpler expression

$$M_0(w,u) = pq(1 - 2p - 2q)$$
(1.3)

with the parameters p and q defined by

$$w = p(1 - p - 2q) \tag{1.4}$$

and

$$u = q(1 - 2p - q), \tag{1.5}$$

where p = q = 0 when w = u = 0. In [14], a recursive formula was found for the number of rooted planar maps given the number of vertices, the number of edges, and the degree of the face containing the root; these numbers of maps were then added over all possible degrees of this face and the result expressed in terms of generating functions. In [15], the first author generalized this method to obtain a recursive formula for the number of maps of genus g with a distinguished dart in each vertex given the number of vertices and the degree of each one; these numbers were then multiplied by the appropriate factor and added over all possible non-increasing sequences of vertex-degrees summing to 2e to obtain the number of rooted maps of genus g with e edges and v vertices. A table of these numbers of maps with up to 14 edges appears in [15] (see [17] for a published account of this work and a table of maps with up to 11 edges) but no attempt was made there to express this result in terms of generating functions. We note here that a similar generalization in which the degrees of all the faces are known but only some of them have a distinguished edge on their boundary, and these faces must be of degree at least 3, appears in [8], where it is attributed to Tutte under the name of Tutte's recursion equations.

In [5] an improvement on the method of [15] was introduced: to count rooted genusg maps it is sufficient to know the degree of the first g + 1 vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered. Using doubly-rooted maps, D. Arquès [2] obtained the analogue of (1.3) for toroidal maps:

$$M_1(w,u) = \frac{pq (1-p-q)}{\left[(1-2p-2q)^2 - 4pq\right]^2}.$$
(1.6)

From this result, he obtained a closed-form formula for the number of rooted toroidal maps with e edges and another one for the number of rooted toroidal maps with v vertices and f faces. In [6] a generating function was obtained for the number of rooted maps of genus 2 and 3 with e edges.

In [9] the second author generalized (1.6) and obtained a general form for the generating function  $M_g(w, u)$  counting rooted maps of any genus g > 0:

$$M_g(w,u) = \frac{pq\left(1-p-q\right)P_g(p,q)}{\left[\left(1-2p-2q\right)^2 - 4pq\right]^{5g-3}},$$
(1.7)

where  $P_g(p,q)$  is a symmetric polynomial in p and q of total degree bounded by 6g - 6 with integral coefficients (in what follows, unless otherwise specified, all the polynomials defined here are polynomials in p and q). The polynomial  $P_g$  is defined in terms of another polynomial  $T_q$  of degree bounded by 10g - 8 by

$$P_g = \frac{T_g}{\left(1 - p\right)^{4g - 2}},\tag{1.8}$$

and that polynomial, in turn, is defined in terms of a family of polynomials  $R_g(n_1, \ldots, n_r)$ in p and q by

$$T_g = R_{g-1}(0,0) + \sum_{j=1}^{g-1} q(1-p-q)R_j(0)R_{g-j}(0).$$
(1.9)

The degree of the polynomial  $R_q(n_1, \ldots, n_r)$  is defined by the equation

deg 
$$R_g(n_1, \dots, n_r) = 2(n_1 + \dots + n_r) + 7r + 10g - 12.$$
 (1.10)

The polynomials  $R_g(n_1, \ldots, n_r)$  are defined recursively in terms of several other families of polynomials and a recursively-defined family of rational functions of p and q. We have two finite families of polynomials in p alone defined by the following two sets of equations.

$$K_0(p) = -p; \quad K_1(p) = -1 - p; \quad K_2(p) = -1; \\ K_m(p) = 0 \text{ for all } m \ge 3.$$
(1.11)

$$L_0(p) = -p; \qquad L_1(p) = -1 - 2p; L_2(p) = -2 - p; \qquad L_3 = -1; L_k(p) = 0 \text{ for all } k \ge 4.$$
(1.12)

In what follows, the parameter p will be omitted, so that these polynomials will be referred to as  $K_m$  and  $L_k$ . We then have two polynomials H and J (in p and q) defined by

$$J = q(1 - p - q) \tag{1.13}$$

and

$$H = (1 - 2p - 2q)^2 - 4pq.$$
(1.14)

Finally we have an infinite family  $(E_k)_{k\geq 1}$  of rational functions of p and q, all but the first two of which are polynomials, defined recursively by

$$E_{1} = \frac{1}{2J(1-p)^{2}};$$

$$E_{2} = \frac{-p - 4q + 2p^{2} + 4q^{2} + 4pq}{2J(1-p)^{2}};$$

$$E_{3} = -1;$$

$$E_{k} = -J(1-p)^{2} \sum_{i=2}^{i=k-1} E_{i}E_{k+1-i} \text{ for all } k \ge 4.$$
(1.15)

To make the recursive definition of the polynomials  $R_g(n_1, \ldots, n_r)$  comprehensible, we first explain the abbreviations we use. For any positive integer r, [r] denotes the sequence  $(2, \ldots, r)$  if  $r \ge 2$  and the empty sequence if r = 1. For any subsequence X of [r], [r] - X denotes the subsequence of the elements of [r] that are not in X. For any sequence  $(n_2, \ldots, n_r)$  of integers,  $N_X$  denotes the sequence of those  $n_i$  such that i is in X and  $N_j$  denotes the sequence  $(n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_r)$ .

The polynomials  $R_0(n_1)$  are not defined. The anchor of this recursive definition is

$$R_0(0,0) = (1-p)^2.$$
(1.16)

If g = 0 and r = 2 but  $(n_1, n_2) \neq (0, 0)$ , then we have

$$R_{0}(n_{1}, n_{2}) = (1-p)^{2} (-n_{2}HE_{n_{1}+n_{2}+2} - (n_{2}+1)E_{n_{1}+n_{2}+3}) +2J(1-p)^{2} \sum_{\substack{i+j+k=n_{1}+1\\i>0,\,k< n_{1}}} (-1)^{j+1} H^{j}E_{i}R_{0}(k, n_{2}).$$
(1.17)

We note that (1.16) is a special case of (1.17) where  $n_1 = n_2 = 0$ .

If  $(g, r) \neq (0, 2)$ , then

$$R_g(n_1,\ldots,n_r) = \operatorname{term}_1 + \operatorname{term}_2 + \operatorname{term}_3 + \operatorname{term}_4, \tag{1.18}$$

where

$$\operatorname{term}_{1} = 2J(1-p)^{2} \sum_{\substack{i+j+k=n_{1}+1\\i>0,\,k< n_{1}}} (-1)^{j+1} H^{j} E_{i} R_{g}(k, n_{2}, \dots, n_{r})$$
(1.19)

(we note that the second line of (1.17) is a special case of (1.19) where g = 0 and r = 2),

$$\operatorname{term}_{2} = J \sum_{\substack{k+l+m=n_{1}+1\\0 \le j \le g\\X \subseteq [r]\\(j,X) \ne (0,\emptyset)\\(j,X) \ne (g,[r])}} K_{m} H^{m} R_{j}(k, N_{X}) R_{g-j}(l, N_{[r]-X}),$$
(1.20)

$$\operatorname{term}_{3} = \sum_{i+j+m=n_{1}+1} K_{m} H^{m} R_{g-1}(i, j, N_{[r]})$$
(1.21)

and

$$\operatorname{term}_{4} = \sum_{j=2}^{r} \left( \begin{array}{c} n_{j} \sum_{k+l=n_{1}+n_{j}+2} L_{k} H^{k+1} R_{g}(l, N_{j}) \\ + (n_{j}+1) \sum_{k+l=n_{1}+n_{j}+3} L_{k} H^{k} R_{g}(l, N_{j}) \end{array} \right).$$
(1.22)

It was shown in [9] that each polynomial  $R_g(n_1, \ldots, n_r)$  is symmetric in all its variables. This was made possible by distinguishing a dart incident to each of the vertices whose degree is considered, which increases the size of the coefficients but does not increase the number of polynomials that have to be calculated.

We note here that in the account of these results published in [3] formula (1.17) and the sum in (1.15) are missing; the formulas are presented correctly in [9]. At that time the second author, programming in Maple, calculated the polynomial  $P_g$  and the generating function  $M_g(w, u)$  for g = 2 and g = 3 (these results are published in [3]) and also computed the generating function that counts rooted maps of genus 4 by number of edges. This result was recently included in [11], where it was used to count both rooted and unrooted maps of genus 4 by number of edges.

Recently, the second author extended his enumeration results to genus 5. The first author, programming mainly in C, optimized the calculation of the polynomials  $R_g(n_1, \ldots, n_r)$  and thus extended the enumeration by number of vertices and faces, as well as by number of edges, to genus 6. Although each author used a different algorithm and a different programming language, we both obtained the same answers, and the numbers of rooted maps we calculated agree with the tables in [15], providing evidence of the correctness of our results. An account of these extensions is given in Sections 2 and 3 and the polynomials  $P_g(p,q)$  appear in Appendix A. A discussion of the enumeration of rooted genus-g maps by number of edges appears in Section 4 and the polynomial part of each of the corresponding generating functions appears in Appendix B. Finally, a discussion of some open problems appears in Section 5.

#### 2 Results from the Maple program

A first version of the Maple code written in 1998 implemented recurrence relations between the rational functions introduced in [4] for the computation of the generating functions  $M_g$ . It was not designed for efficiency but for validating formulas from [4]. That code has also been used for validating the formulas from (1.2) through (1.22) for the first values of g, rand  $n_1, \ldots, n_r$  (these formulas were first obtained from a long computation that was done by hand and is thus error-prone). When executed in 1998 with Maple V for computing  $M_4(w, u)$  that code ran into a fundamental limitation (wired into the Maple kernel) of a maximal number of 65,535 terms in any polynomial.

That old code has been recently replaced by a simpler code implementing directly the recursion between polynomials described by the formulas from (1.2) through (1.22). The code is short (less than 400 lines) and resembles the mathematical formulas as much as possible in order to detect errors. All the results obtained by this new code match known results in rooted map enumeration. For all these reasons, it can be considered as a reference for the debugging of optimized implementations.

With a personal computer running under Windows XP with an Intel Core 2 Duo CPU at 2.19 GHz and 3.5 Gb of memory, and a Maple 14 release supporting larger objects, the next two generating functions  $M_4(w, u)$  and  $M_5(w, u)$  were successfully computed in 4 minutes and 5 hours, respectively. It was, however, not sensible to continue using this inefficient prototype for computing the next generating functions. A better idea was to write an independent implementation optimizing memory space and execution time.

#### **3** Optimizations and the C program

Aside from the advantage in execution speed that C has over Maple, the first author optimized the calculation of the polynomials  $R_g(n_1, \ldots, n_r)$ . One of these optimizations was made possible by the following observation.

**Proposition 1.** For any  $(g,r) \neq (0,1)$  and any sequence  $n_1, \ldots, n_r$ , the polynomial  $R_g(n_1, \ldots, n_r)$  is divisible by  $(1-p)^2$ .

*Proof.* (by generalized induction on the degree of a polynomial of the form  $R_g(n_1, \ldots, n_r)$ , which we call an *R*-polynomial).

**Basic step (degree 2)** The only *R*-polynomial of degree 2 is  $R_0(0,0) = (1 - p)^2$ : see (1.16).

**Induction step** Suppose that the degree d of a given R-polynomial  $R_g(n_1, \ldots, n_r)$ , as defined by (1.10), is greater than 2 and that every R-polynomial of degree < d is divisible by  $(1 - p)^2$ . We show that  $R_g(n_1, \ldots, n_r)$  is also divisible by  $(1 - p)^2$ . Since every R-polynomial on the right side of equations (1.17), (1.19), (1.20), (1.21) and (1.22) is of degree < d, it follows from the induction hypothesis that each such polynomial is divisible by  $(1 - p)^2$ . We examine each of these equations in turn. Equation (1.17). The first line contains a factor  $(1-p)^2$ . The term  $E_{n_1+n_2+3}$  is a polynomial for any non-negative  $n_1$  and  $n_2$ . The term  $E_{n_1+n_2+2}$  is a polynomial unless  $n_1 = n_2 = 0$ , but in this case  $E_{n_1+n_2+2}$  is multiplied by  $n_2 = 0$ ; so the first line of (1.17) is divisible by  $(1 - p)^2$ . In the second line, each term of the sum contains a polynomial  $R_0(k, n_2)$ , which, by the induction hypothesis,

is divisible by  $(1-p)^2$ . This factor of  $(1-p)^2$  could be cancelled by  $E_1$  or  $E_2$ , but the sum is nevertheless a polynomial, and the factor  $(1-p)^2$  by which the sum is multiplied ensures that the second line of (1.17) too is divisible by  $(1-p)^2$ ; so the right side of (1.17) is divisible by  $(1-p)^2$ . Equation (1.19). By an argument similar to the one used for the second line of (1.17), the right side of (1.19) is divisible by  $(1-p)^2$ .

Equations (1.20)-(1.22). Each term in the sum contains at least one *R*-polynomial that is divisible by  $(1-p)^2$ ; so the right side of each of these equations is divisible by  $(1-p)^2$ . It follows from (1.18) that  $R_g(n_1, \ldots, n_r)$  is divisible by  $(1-p)^2$ , which completes the proof.

We now modify equations (1.8)-(1.10), and (1.16)-(1.22) in the light of Proposition 1. We introduce a new family of polynomials (which we call *S*-polynomials) defined by

$$S_g(n_1, \dots, n_r) = R_g(n_1, \dots, n_r)/(1-p)^2$$
 (3.1)

and we also let

$$U_g = T_g / (1 - p)^2. ag{3.2}$$

Then  $U_g$  is a polynomial of degree 10(g - 1) and (1.8)-(1.10) become (3.3)-(3.5), respectively.

$$P_g = \frac{U_g}{\left(1 - p\right)^{4g - 4}},\tag{3.3}$$

$$U_g = S_{g-1}(0,0) + q(1-p-q)(1-p)^2 \sum_{j=1}^{g-1} S_j(0) S_{g-j}(0).$$
(3.4)

deg 
$$S_g(n_1, \dots, n_r) = 2(n_1 + \dots + n_r) + 7(r-2) + 10g.$$
 (3.5)

Also, (1.16)-(1.22) become (3.6)-(3.12), respectively.

$$S_0(0,0) = 1, (3.6)$$

$$S_{0}(n_{1}, n_{2}) = (-n_{2}HE_{n_{1}+n_{2}+2} - (n_{2}+1)E_{n_{1}+n_{2}+3}) + 2J(1-p)^{2} \sum_{\substack{i+j+k=n_{1}+1\\i>0, \ k< n_{1}}} (-1)^{j+1} H^{j}E_{i}S_{0}(k, n_{2}).$$
(3.7)

If  $(g, r) \neq (0, 2)$ , then

$$S_g(n_1,\ldots,n_r) = \operatorname{term}_5 + \operatorname{term}_6 + \operatorname{term}_7 + \operatorname{term}_8, \tag{3.8}$$

where

$$\operatorname{term}_{5} = 2J(1-p)^{2} \sum_{\substack{i+j+k=n_{1}+1\\i>0,\,k< n_{1}}} (-1)^{j+1} H^{j} E_{i} S_{g}(k, n_{2}, \dots, n_{r}),$$
(3.9)

$$\operatorname{term}_{6} = J(1-p)^{2} \sum_{\substack{k+l+m=n_{1}+1\\0\leq j\leq g\\X\subseteq [r]\\(j,X)\neq (0,\emptyset)\\(j,X)\neq (g,[r])}} K_{m}H^{m}S_{j}(k,N_{X})S_{g-j}(l,N_{[r]-X}),$$
(3.10)

$$\operatorname{term}_{7} = \sum_{i+j+m=n_{1}+1} K_{m} H^{m} S_{g-1}(i, j, N_{[r]}), \qquad (3.11)$$

and

$$\operatorname{term}_{8} = \sum_{j=2}^{r} \left( \begin{array}{c} n_{j} \sum_{k+l=n_{1}+n_{j}+2} L_{k} H^{k+1} S_{g}(l, N_{j}) \\ +(n_{j}+1) \sum_{k+l=n_{1}+n_{j}+3} L_{k} H^{k} S_{g}(l, N_{j}) \end{array} \right).$$
(3.12)

Since  $R_g(n_1, \ldots, n_r)$  is symmetric in all its variables, so is  $S_g(n_1, \ldots, n_r)$ ; so only those polynomials  $S_g(n_1, \ldots, n_r)$  with  $n_1 \leq \ldots \leq n_r$  are treated. In all the S-polynomials on the right side of each of the equations (3.9)-(3.12), only the first two variables can violate these inequalities; so they are inserted into their proper slots among the remaining variables to preserve the inequalities. Also, equation (3.4) is symmetric in j and g - j, equation (3.10) is symmetric in k and l and equation (3.11) is symmetric in i and j; so the calculations there can be cut almost in half. In equation (3.12), each polynomial  $S_g(l, N_j)$ is calculated only once and then used twice. The following easily proved observations can be used to avoid calculating a polynomial that is identically 0: term<sub>5</sub> = 0 if  $n_1 = 0$ , term<sub>6</sub> = 0 if  $g + r \leq 2$ , term<sub>7</sub> = 0 if g = 0, term<sub>8</sub> = 0 if r = 1 or (g, r) = (0, 2). From these observations, it follows that the only term that could possibly contribute to  $S_0(n_1)$ is term<sub>5</sub>. From (3.9) it follows by generalized induction on  $n_1$  that  $S_0(n_1) = 0$  for all  $n_1 \geq 0$ ; so these polynomials do not have to be defined.

All the S-polynomials are stored in a single one-dimensional array s. A preliminary recursion does not calculate any of these polynomials. Instead, it calculates all the quadruples (d, q, r, c) of parameters of the S-polynomials that will later be calculated, where  $d = \deg S_q(n_1, \ldots, n_r)$  and c is an integer coding the sequence  $(n_1, \ldots, n_r)$ , and stores the list of quadruples in four parallel arrays, one array for each of the four parameters d, g, r, c and one element of all four arrays for each quadruple (d, g, r, c). The program then sorts the four parallel arrays by degree d using bucket sort, computes the number of Spolynomials that have to be calculated and the total number of terms in these polynomials and stores in two arrays the index in s and the one in the four parallel arrays of the first term for each degree d. Then the S-polynomials are calculated in increasing order of their degree and stored in s. This can be done non-recursively because all the S-polynomials that need to be used will have already been stored and need only be found by searching the four parallel arrays, starting with the first index for the appropriate degree d, for the appropriate parameters, and adding (d+1)(d+2)/2 to the index in s each time the index in the four parallel arrays is increased by 1. Once the last polynomial  $S_{q-1}(0,0)$  has been calculated, first (3.4) is used to calculate  $U_g$  and then (3.3) is used to calculate  $P_g$  and its coefficients are stored in a text file, which is available from the first author on request. The polynomials  $P_2$  and  $P_3$  appear in [3]. The polynomials  $P_4$ ,  $P_5$  and  $P_6$  appear in Appendix A. Because these polynomials are symmetric in p and q, to save space we include only those terms in which the exponent of p is at least as great as the exponent of q.

The number of S-polynomials that have to be calculated is roughly the total number of partitions of all the positive integers up to 10(g - 1). For each of these polynomials, the most expensive calculation is term<sub>6</sub>, because the sum there runs over all the partitions of the sequence  $[r] = (2, \ldots, r)$ , where r can be as great as g + 1, and involves multiplying two S-polynomials. The time-complexity of calculating  $P_g$  is therefore exponential in g, but the optimizations made here nevertheless made it possible to calculate  $P_g$  for a greater value of g than was possible previously. Another program computes a table of numbers of rooted genus-g maps counted by number of vertices and faces by reading this file and

using (1.2) if  $g \ge 1$  or (1.3) if g = 0. Tables of numbers of rooted genus-g maps for any  $g \le 6$  and with up to any reasonable number of edges are available from the first author on request.

The programs were written mainly in C. The one that computes the polynomials is about 2000 lines long and the one that computes the tables is about 300 lines long. They both use the C++ library CLN to do arithmetic on big integers because CLN reads arithmetic expressions in C that use only addition, multiplication and subtraction; only statements involving quotients, remainders, input/output of big integers and file management had to be modified. Since CLN requires a GNU compiler, XCODE was downloaded and installed by Jerome Tremblay, a computer technician at UQAM, who also downloaded and installed CLN and wrote sample C++ statements for input/output of big numbers and file management.

The programs were executed on a 2004 Macintosh GR4 computer. The time taken to compute the polynomial  $P_g$  varied from run to run. In Table 1 we show, for each g from 1 to 6, the number of S-polynomials that were calculated, the total number of terms in all these S-polynomials, and a typical execution time. Once the S-polynomials had been computed and stored, it took the computer only 48 seconds to make a table of numbers of genus-6 maps with up to 42 edges counted by number of vertices and faces. Source codes of both programs are available from the first author on request.

g	number of $S$ -polynomials	total number of terms	execution time
1	1	1	instantaneous
2	16	507	1 second
3	67	7407	10 seconds
4	205	49796	2 minutes
5	543	235410	20 minutes
6	1314	900114	3,5 hours

Table 1: Evaluation of the computation cost

#### 4 Counting by number of edges

To compute the generating function  $M_g(z) = z^{2g-2}M_g(z, z)$  that counts rooted genus-g maps by number of edges alone, we use the substitution obtained in [11], which is a more compact form of the one obtained in [9] and published in [3]. Let

$$p = q = m, \tag{4.1}$$

where

$$z = m(1 - 3m)$$
 and  $m = 0$  when  $z = 0$ . (4.2)

By substituting from (4.1) into (1.4) and (1.5) to express w and u in terms of m and then substituting into (1.7), we obtain the following equation for  $g \ge 1$ :

$$M_g(z) = \frac{m^{2g}(1-3m)^{2g-2}P_g(m,m)}{(1-6m)^{5g-3}(1-2m)^{5g-4}}.$$
(4.3)

For g = 0, we substitute into (1.3) instead of (1.7) and obtain

$$M_0(z) = (1 - 3m)^{-2}(1 - 4m).$$
(4.4)

The first author computed  $M_g(z)$  from the computed values of  $M_g(w, u)$  for  $g \leq 6$ . The program divides the polynomial  $P_g(m, m)$  by 1 - 2m as often as possible. The program then divides the resulting polynomial by 2 and by 3 as often as possible, extracts the appropriate constant factor and then stores the resulting generating function in another text file, also available from the first author. The second author computed  $M_g(z)$  directly for  $g \leq 6$ . We then compared our formulas and verified that they agree. The formulas for  $P_g(m,m)$  for  $1 \leq g \leq 6$  appear in Appendix B. Now  $P_g(m,m)$  is of degree 6g - 6, but we found experimentally that for  $1 \leq g \leq 6$ ,  $P_g(m,m)$  is divisible by  $(1-2m)^{2g-2}$ , so that the quotient is only of degree 4g - 4, and we conjecture that this will be the case for any positive integer g.

#### 5 Some interesting open problems

The recurrences satisfied by the R- and S-polynomials both result from proofs by induction. After the right conjecture has been guessed by observing the first computed terms, these proofs are not difficult to find, but they are tedious and error-prone due to the length of the expressions involved. Thus they are good candidates for automation. We plan to develop a suitable formal framework for assisting this kind of proofs with a computer algebra system. The challenge is to shorten the chain of conjectures and proofs about the general pattern of generating functions for counting rooted maps. This chain has been initiated in [6] and continues in the present work with a new conjecture that the polynomial  $P_g(m,m)$  is divisible by  $(1-2m)^{2g-2}$  for all  $g \ge 1$ . Once the numbers of rooted maps of genus up to g are known, the number of unrooted maps up to genus g can be calculated using the methods presented in [12]. As was mentioned above, the second author collaborated with A. Mednykh to count rooted and unrooted maps of genus 4 by number of edges [11]. It would be interesting to count unrooted genus-g maps by number of vertices and faces for as many values of g as possible (see [16] for an account of the progress made on this problem).

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#### Appendix A

Coefficients of the polynomial  $P_g(p,q)$  in the generating function  $M_g(w,u)$ . The exponent of p is i and the exponent of q is j.

g = 6	g = 5	g = 4	j	i
24325703325	59520825	225225	0	0
-467684337375	-1052857260	-3447873	0	1
331572725235	2996675136	13149279	0	2
15552480037725	29236662630	75740652	1	1
78629111673210	84903326508	99376849	0	3
-164732858445258	-278088909474	-601290577	1	2
-1115786805002100	-1167635102634	-1364702625	0	4
-197193320467497	314258237049	1323954013	1	3
4114016654215968	5139190834104	7798852738	2	2
7679132084813280	7383190634820	7278085815	0	5
22225108689398171	16626928198725	11669094136	1	4
-38011804449745265	-38825216035101	-44518090957	2	3
-25194145114920600	-25517944452000	-21915530637	0	6
-258660509951405369	-176205495337005	-111691406491	1	5
46817554992783206	71414891373744	87176823147	2	4
698821084235510238	521245727579217	394951357187	3	3
-34634849450319300	27486911684232	34368350016	0	7
1599994018015673343	944779067359587	467365399116	1	6
2325052326558898989	1065131939557443	369316330580	2	5
-5305278050309690436	-3109774180465479	-1653054602996	3	4
875431778419760250	199121845688766	5493029256	0	8
-5427079882725409196	-2978944697593944	-1150226915904	1	7
-26281328772579796556	-10502687980853538	-3099366738272	2	6
11612050266758295299	6485662227087078	2799989555248	3	5
67234802590102445596	28864236434784750	9770469656312	4	4
-5250824393143736550	-1290553718021568	-168971030800	0	9
2631424256603826266	4319747300224332	1603000133328	1	8
151752654232148634876	49447100947421712	10375535302688	2	7
134785360586359808415	33443054101099032	4675439367040	3	6
-403081200640447102025	-131655002707242864	-29062591554512	4	5
19107855057709768350	4175411121161280	446623903440	0	10
79781498509512839622	7557377263434480	-531396192624	1	9
-514882253213415808290	-140196978107516880	-20288443296752	2	8
-1507879430637581935244	-322195198691448864	-37157221449760	3	7
891983836146643438886 3712198575809542074306	248581069034660208 864416573906739888	38459634403216 113455708544944	4 5	6 5
-44994838722276643200	-8761889037807360	-669316766976	0	11
-502584153034345687008	-62049749603864880	-2677064058240	1	10
728298352379130773279	217239133515491184	23203491496832	2	9
7881173581586749085040	1301723014819664880	95704575490304	3	8
4757786800382576737632	573069029641873824	25495111214592	4	7
-17695909946707419577712	-2999701307315295744	-232473613207168	5	6
53771085509744694000	11812486910550144	636872524288	0	12
1776396207720770666496	186781204204153728	6233151448832	1	11
2455921681097348996048	18744511957965024	-10474170075648	2	10
-24856837687770483118400	-3164855575142421984	-139300678503936	3	9
-51467124780997381162872	-5372851934394547296	-194528089368832	4	8
35482577674056890378656	4735380833365872192	221773985029376	5	7
123284812684751523348352	14241231802471091712	587209040349952	6	6
			0	13
63590534180019957600 -4153379048094569728032	-7139224937714688 -352660752052764480	-363019168512 -7182531345920	0	12

1         2         -1071330625238         -106253493246902304         -1918736418875001945312           10         3         1177704783344         45169002399191232         413410118470017461125           9         4         33800427309950         179944883026107552         22607648384255468601760           7         6         -786362512663296         -37365766731378686976         -470331494499995012997600           13         1         4062558501792         -470331494499995012997600         -8879362292376576         -50326427081623151700702           12         2         22091079382272         2948070190097081856         65776115947137517559992         -318307238061159471375206           10         4         -35144511752398         -3662786418291175296         -60051570417721787575259922           10         4         -35144511752398         -3582909907760         12853994412052951424         22205366072827414572256           15         0         45405634560         30437536586096640         1405897564579656192000         -313898934629182720         -30206856213467445336           12         1         -198713050006         -313898934629182720         -30206856213465714416489           13         170305130241         -313898934629182720         -3020685621444646400          13         <	i	j	g = 4	g = 5	g = 6	
10         3         117776047833344         451689062899195132         4134410118476021791296           8         5         55421194708224         5366219028182108736         1027805280519486509760           7         6         -786362512663296         -37365766731378686976         -470331494469995012997600           13         1         4962585601792         439159836169197312         600511720227491109702           12         2200709788277         53585909682771061760         -60615714073706256305248           10         4-35144511752396         -16039150741375025503248           10         4-35144511752396         -1603915074137062556305248           8         6         491093270238720         -1730736887168         +3820363589503225892801440           12         1977155699797760         12863594421962951424         252265360728373415372256           15         0         45405634560         30437353684096404         14058975645790651044           13         2         -1730738687168         -4811256134778833008         -145683381592057141541640           14         19034542081520         -13138984515917472         -2202685562130647246336           13         2         -1730738687168         -48123604381037802           14         190345420815120						
9         4         358004273099520         17994448820561075320         22607748584525548609152           8         5         55542104708224         536621920812108736         -47033149449999501997600           14         0         78087609600         -8879362292376576         -5032642708162315760           13         14965558601792         2439159836169197312         600511720274911079072           12         2         22001079382272         2944070190097081856         6577611594717357559022           13         -4506543874432         18007782379165888         2380742660187432918496           9         5         -40139307038720         4719424142517734752         8206614960187432918496           14         1         -198710500906         -31389834629182720         -20206856619742030           13         -198730800404900         -313889834629182720         -202068566192000           14         1         -198710500906         -313889834629182720         -202068566192000           14         1         -198710500906         -31389834629182720         -202068566192000           151         30383454208120         -70202683764185991820         -14588378418592087145164380           16         0         -4461717883008         -147778434791320         -31313726484479						
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10         4         -351445117232968         -3558909962271001760         -6601571407437025305248           9         5         -40198904040960         -4862986044983717529         82063658963522580144           7         7         1207155699797760         128635994421962951424         2522653660728373415372256           15         0         45405634560         3043735585609660         14058975645795564579055192000           14         1         -198713050096         -313889834629182720         -2020685626130647246336           12         3         -1095800010432         -79902071415917281741516480         -4461115419177472           10         5         4540551362304         4317529814874091320         1006653307052589581622228           10         5         45510351362204         255735090088         134852658810680591082008           8         7         -948072226163712         -2568226190899984192         -796266335481053280103280           15         1         33538363328         -5940163278028880         -152745019190573180596           14         2         7138023596448         2121507330103302656         12205533769009020555264           14         3         1951924685184         20212507330103302656         12205533769009020555264           15						
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			1	1

#### **Appendix B**

Polynomial  $P_a(m,m)/(1-2m)^{(2g-2)}$  in the generating function  $M_a(z)$ .  $P_g(m,m)/(1-2m)^{(2g-2)}$ g 1  $\mathbf{2}$  $3(7 - 70m + 295m^2 - 636m^3 + 588m^4)$  $165 - 2596m + 19835m^2 - 102138m^3 + 397742m^4$  $3^2$ 3  $-1162744m^5 + 2360496m^6 - 2918016m^7 + 1642656m^8$  $25025 - 465894m + 4245462m^2 - 28633200m^3$  $+178608786m^4 - 1025233956m^5 + 4855070265m^6$  $3^2$ 4  $-17709582732m^7 + 48202134300m^8 - 95026128096m^9$  $+128766120048m^{10} - 107657028288m^{11} + 41956066368m^{12} \\ 6613425 - 128153480m + 1123286598m^2 - 7641539820m^3$  $+68489369190m^4 - 681945904584m^5 + 5453799804351m^6$  $-33175983024306m^7 + 157025924018370m^8$  $3^2$  $-590662433458296m^9 + 1778501684246544m^{10}$ 5 $-4258112783048352m^{11} + 7946769062433024m^{12}$  $-11156448512891520m^{13} + 11087677481748480m^{14}$  $-6955529138076672m^{15} + 2071316467035648m^{16}$  $900951975 - 16624244750m + 105922471285m^2$  $-402327939748m^3 + 9014122899102m^4$  $-183050473605084m^5 + 2152106046117936m^6$  $-17716916701552824m^7 + 113738504396139378m^8$  $-602051461456822740m^9 + 2694620167659984726m^{10}$  $3^3$ 6  $-10264333975933057272m^{11} + 33144207748349404248m^{12}$  $-89851078246171110912m^{13} + 201700042332545251008m^{14}$  $-368052722019205320960m^{15} + 531966143515513800960m^{16}$ 

 $-586003188281237388288m^{17} + 462270648384927677952m^{18}$  $232608604432295245824m^{19} + 56102738197832792064m^{20}$