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Generic numerical semigroups

Roland Bacher

May 8, 2021

Abstract

The use of compositions simplifies some aspects of the theory of numerical semigroups. We illustrate this by giving a new proof for the asymptotic number $C((1 + \sqrt{5})/2)^g$ of numerical semigroups of genus g and by describing the constant C explicitly¹.

1 Introduction

..., chacun appelant idées claires celles qui sont au même degré de confusion que les siennes propres.

Marcel Proust

A *numerical semigroup*² is a subgroup $S = S+S$ of the additive semigroup $\mathbb{N} = \mathbb{N} + \mathbb{N}$ such that the complementary set $\mathbb{N} \setminus S$ is finite. The *embedding dimension* $e = e(S)$ of S is the minimal cardinal of a generating set. The smallest non-zero element $m = m(S) = \min(S \setminus \{0\})$ of S is the *multiplicity* of S . The finite set $G = G(S) = \mathbb{N} \setminus S$ is the *set of gaps* and the number $g = g(S) = \#(G)$ of elements in G is the *genus* of S . The *Frobenius number* $f = f(S) = \max(G)$ is the maximal element of G .

Throughout the rest of the paper, the letter m will always denote the multiplicity $\min(S \setminus \{0\})$ of a numerical semigroup S .

¹Keywords: Numerical semigroup, composition, Fibonacci numbers, spin model, dihedral group. Math. class: 20M14, 05A16.

²Numerical semigroups are a hot topic during outbreaks of contagious diseases: You don't want to take change at grocery stores. (This paper was largely written during the Corona-virus lock-down.)

For $j = 0, \dots, m - 1$, we set

$$x_j = x_j(S) = \#\{G \cap j + m\mathbb{Z}\} . \quad (1)$$

We have $x_0 = 0$ and $x_1, \dots, x_{m-1} \geq 1$. The vector (x_1, \dots, x_{m-1}) is called the *Kunz coordinate vector* or simply the *Kunz vector* of S . It is also called the *Apéry tuple* of S , see e.g. [2]. The trivial identity

$$g = \sum_{j=1}^{m-1} x_j$$

shows that $x_1 + x_2 + \dots + x_{m-1}$ is a composition of g into $m - 1$ parts. The Frobenius number f of S is given by

$$\max_{j \in \{1, \dots, m-1\}} j + m(x_j - 1) .$$

The Frobenius number $f = l + m(x_l - 1)$ is equivalently defined in terms of the index and the value of the last maximal part x_l of the composition $x_1 + \dots + x_{m-2}$ with parts defined by (1).

We call the composition $x_1 + \dots + x_{m-1}$ the *composition of S* . It is closely related to the *Apéry set*

$$\text{Ap}(m, S) = \{0, x_1m + 1, x_2m + 2, \dots, x_{m-1}m + m - 1\}$$

consisting of minimal representatives in S for classes modulo the multiplicity $m = \min(S \setminus \{0\})$ of S .

Remark 1.1. *By a fortunate coincidence, the letter m (denoting the multiplicity of S) is also well-suited for denoting classes modulo $m\mathbb{Z}$ of \mathbb{Z} . The composition $x_1 + \dots + x_{m-1}$ can be considered as the image in the group algebra $\mathbb{Z}[\mathbb{Z}/m\mathbb{Z}]$ of the characteristic function $\sum_{\gamma \in \mathbb{N} \setminus S} [\gamma]$ for the gap-set in the group algebra $\mathbb{Z}[\mathbb{Z}]$ of \mathbb{Z} . The trivial character applied to the characteristic function of $\mathbb{N} \setminus S$ yields the genus of S .*

It is easy to show that a numerical semigroup is determined by its composition with parts defined by (1). We will recall a well-known characterisation of compositions associated to numerical semigroups. We can thus replace the tree-structure underlying numerical semigroups by the (essentially) linear structure of compositions when studying certain properties of numerical semigroups. This reduces the study of asymptotic growth (for numbers of numerical semigroups of given genus) to elementary considerations boiling often down to “generatingfunctionology”, after making use of simple geometric properties of (integers contained in) intervals.

We use this approach in a new proof for asymptotics conjectured by Maria Bras-Amorós in [1] and proven by Alex Zhai in [5] (a pleasant overview of this topic describing the gist of Zhai’s proof is given in [2]) for the number $n(g)$ of semigroups of genus g :

Theorem 1.2. *There exists a constant C such that*

$$\lim_{g \rightarrow \infty} \frac{n(g)}{\omega^g} = C \quad (2)$$

where $\omega = \frac{1+\sqrt{5}}{2} = 1.61803398874989\dots$ is the golden number.

The constant C is given by

$$C = \frac{5 + \sqrt{5}}{10} \left(1 + \tilde{C}(\omega^{-1})\right) \quad (3)$$

where $\tilde{C}(q)$ is the generating function

$$\tilde{C}(q) = \sum_{g=3}^{\infty} \tilde{c}_g q^g \quad (4)$$

enumerating the number \tilde{c}_g of numerical semigroups S of genus $g \geq 3$ satisfying the identity $f = 3m - 1$ linking their Frobenius number $f = \max(\mathbb{N} \setminus S)$ with their multiplicity $m = \min(S \setminus \{0\})$.

The series $\tilde{C}(q)$ defines a holomorphic function in an open disc of radius strictly larger than ω^{-1} .

Remark 1.3. *The convergency radius of \tilde{C} is at most equal to the positive real root $0.659982\dots$ of $1 - q^3 - 2q^4 - 2q^5 - q^6$, cf. Theorem 15.1. It is thus only slightly larger than $\omega^{-1} \sim 0.618$. Using this approach for determining a good numerical approximation of the constant C involved in (2) is thus quite tricky.*

Remark 1.4. *Formula (2) of Theorem 1.2 is sometimes given with ω^g replaced by the g -th Fibonacci number. This changes the value of the associated constant C' which depends also (up to a power of ω) on the convention for indices of Fibonacci numbers.*

Throughout the paper we use always ω for the golden number $\omega = \frac{1+\sqrt{5}}{2}$ with multiplicative inverse $\omega^{-1} = \frac{\sqrt{5}-1}{2}$.

The overall structure of our proof of Theorem 1.2 and of the proof in [5] are similar: Numerical semigroups whose Frobenius numbers are much larger than twice their multiplicity can be neglected when considering asymptotics.

Zhai proves this by analysing the tree-structure of numerical semigroups. We use instead compositions with parts defined by (1) which we call numerical semigroup compositions (called Kunz coordinate vectors or special cases of so-called Apéry sets by other authors) or NSG-compositions for short. The linear nature of compositions makes their study fairly elementary: It is essentially equivalent to the geometry of closed real intervals, endowed with partial actions by real reflections. Transfer matrix techniques and elementary properties of series expansions for holomorphic functions complete the proof.

Sections 2-14 are devoted to the proof of Theorem 1.2.

Section 2 introduces numerical semigroup compositions or NSG-compositions for short.

Section 3 illustrates the notion of NSG-compositions by describing all NSG-compositions of maximum at most 2, a result also contained in [5].

Sections 4 and 5 are digressions describing generalised compositions, algorithmic aspects and the tree-structure of NSG-compositions.

Section 6 contains a rough outline for the proof of Theorem 1.2.

Section 7 introduces pivot-factorisation, our main tool for obtaining enumerative results on NSG-compositions.

Section 8 recalls a few facts concerning generating series and growth-rates.

Section 9 defines weak admissibility for compositions. This is used in Section 10 for obtaining upper bounds on the growth-rate of NSG-compositions of maximum at least 6.

NSG-compositions of maximum 5 and 4 are treated in Sections 11 and 12.

Section 13 describes the Combinatorics of NSG-compositions of maximum 3.

Section 14 completes the proof of Theorem 1.2 by giving upper bounds on the growth-rate of NSG-compositions of maximum 3 ending with a maximal part enumerated by the series \tilde{C} occurring in Formula (4).

Sections 15-18 outline some (sometimes only conjectural) combinatorial or probabilistic aspects of numerical semigroups and certain types of compositions.

2 Numerical semigroup-compositions

We start this Section with a justification of our (non-standard) terminology. Numerical semigroup-compositions are equivalent to Kunz coordinate vectors, an unfortunate choice of terminology in our opinion: there is no underlying vector space and these “vectors” encode simply finite sequences of

strictly positive integers indexed by $1, 2, \dots, m-1$ (representing all non-zero classes of $\mathbb{Z}/m\mathbb{Z}$) summing up to the genus.

Our next result, which is folklore (see for example [2]), shows that numerical semigroups are encoded by their compositions:

Proposition 2.1. *A numerical semigroup S associated to a composition $x_1 + \dots + x_{m-1}$ with parts defined by formula (1) is uniquely determined by the formula*

$$S = \cup_{j=0}^{m-1} (j + mx_j + m\mathbb{N}) \quad (5)$$

using the convention $x_0 = 0$.

A numerical semigroup composition (or an NSG-composition, for short) is a composition $x_1 + \dots + x_{m-1}$ (with an omitted trivial part $x_0 = 0$) defining a numerical semigroup by (5). We identify henceforth numerical semigroups with their NSG-compositions.

Proof of Proposition 2.1. A composition $x_1 + \dots + x_{m-1}$ associated to a numerical semigroup S determines the multiplicity m of S (by adding 1 to the number $m-1$ of its summands). Every numerical semigroup associated to $x_1 + \dots + x_{m-1}$ has thus the same multiplicity m . Since m is an element of S , the intersection of S with an arithmetic progression $j + m\mathbb{N}$ (for j in $\{1, \dots, m-1\}$) lacks consecutive initial values of the arithmetic progression $j + m\mathbb{N}$. This shows that $S \cap (j + m\mathbb{Z})$ is given by $j + x_j m + m\mathbb{N}$ for every $j \in \{1, \dots, m-1\}$. Since $m \in S$ implies $S \cap m\mathbb{N} = m\mathbb{N}$, the composition $x_1 + \dots + x_{m-1}$ defines the intersection of S with $j + m\mathbb{Z}$ for all j . This determines S uniquely. \square

Corollary 2.2. *There are at most 2^{g-1} numerical semigroups of genus g .*

Proof. A strictly positive integer $g > 0$ has 2^{g-1} compositions³. \square

The next result is well-known (see for example Proposition 9 in [2]) and describes the set of all NSG-compositions:

³Write the word 1^g consisting of g identical letters 1. Starting with the second occurrence of 1, decide for each letter 1 if you add it to the precedent letter or if you do nothing. These 2^{g-1} possible choices produce all possible sequences of strictly positive integers adding up to g . Equivalently, write all 2^{g-1} words of $\{0, 1\}^{g-1}$ of length g , with letters in $\{0, 1\}$ and first letter 0: Consider lengths of factors for such a word after rewriting it in terms of 0, 01, 011, 0111, ...

For a proof with generating series observe that $(q/(1-q))^n$ counts the number of compositions with exactly n parts. We get the result by the identities $\sum_{n=1}^{\infty} (q/(1-q))^n = (q/(1-q)) \circ (q/(1-q)) = q/(1-2q)$ for the generating series enumerating all compositions of strictly positive integers.

Theorem 2.3. *A composition $x_1 + \dots + x_{m-1}$ is the composition of a numerical semigroup of multiplicity m if and only if we have the inequalities*

$$\begin{aligned} x_{s+t} &\leq x_s + x_t, \\ x_{m-s-t} &\leq x_{m-s} + x_{m-t} + 1 \end{aligned} \tag{6}$$

for all s, t in $\{1, \dots, m-2\}$ such that $s+t < m$.

The inequalities given by (6) are henceforth called *NSG-inequalities*.

Proof of Theorem 2.3. Let $x_1 + \dots + x_{m-1}$ be a composition to which we add a trivial part $x_0 = 0$. We define a subset S of \mathbb{N} by setting

$$S = \bigcup_{j=0}^{m-1} (j + x_j m + m\mathbb{N})$$

(cf. formula (5) of Proposition 2.1). We have to show that S is a numerical semigroup if and only if all NSG-inequalities (6) hold: Given two elements a and b of S we consider their representatives s, t modulo m in $\{0, \dots, m-1\}$. By construction of S , there exist two natural integers α and β such that $a = s + x_s m + \alpha m$ and $b = t + x_t m + \beta m$. We have thus $c = a + b = s + t + (x_s + x_t)m + (\alpha + \beta)m$.

If $s+t < m$ we have $u = s+t$ in $\{1, \dots, m-1\}$ representing $s+t$ modulo m . We get thus

$$\begin{aligned} c &= s + t + (x_s + x_t)m + (\alpha + \beta)m \\ &= u + x_u m + (\alpha + \beta + x_s + x_t - x_u)m \end{aligned}$$

and $\alpha + \beta + x_s + x_t - x_u$ is always a natural integer if and only if all inequalities of the first line in (6) hold.

If $s+t \geq m$, we get $u = s+t-m$ for u in $\{0, \dots, m-1\}$ and we have

$$\begin{aligned} c &= (s + t - m) + (x_s + x_t + 1)m + (\alpha + \beta)m \\ &= u + x_u m + (\alpha + \beta + x_s + x_t + 1 - x_u)m \end{aligned}$$

and $\alpha + \beta + x_s + x_t + 1 - x_u$ is always in \mathbb{N} if and only if we have the inequalities of the second line in (6). \square

Remark 2.4. *The NSG-inequalities (6) can be rewritten as*

$$x_i + x_j \geq x_{i+j \pmod{m}} + c(i, j)$$

where

$$c(i, j) = \begin{cases} 0 & \text{if } i + j < m, \\ 1 & \text{otherwise} \end{cases}$$

is the 2-cocycle of $H^2(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ corresponding to the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

defining \mathbb{Z} as a central extension of $\mathbb{Z}/m\mathbb{Z}$ by \mathbb{Z} . This observation can of course be explained by Remark 1.1.

2.1 Parameters

We describe without proofs how to recover basic parameters of a numerical semigroup $S = m\mathbb{N} \cup (\cup_{j=1}^{m-1} \{j + m(x_j + \mathbb{N})\})$ from its NSG-composition $x_1 + \dots + x_{m-1}$.

The multiplicity m of S is easily obtained from the number $m-1$ of parts of $x_1 + \dots + x_{m-1}$. Similarly, the genus is the sum $\sum_{i=1}^{m-1} x_i$ of all parts.

Minimal generators for S are given by the multiplicity m and by integers $j + x_j m$, $j \in \{1, \dots, m-1\}$ such that all NSG-inequalities (6) are strict for $s, t \in \{1, \dots, m-1\}$ with $s+t$ in $\{j, j+m\}$.

The Frobenius number $f = \max(\mathbb{N} \setminus S)$ of a numerical semigroup with composition $x_1 + \dots + x_{m-1}$ is given by

$$f = \max(x_1, \dots, x_{m-1})m - m + \max(\{j \mid x_j = \max(x_1, \dots, x_{m-1})\}) \quad (7)$$

or equivalently by $m(x_l - 1) + l$ where $x_1, \dots, x_{l-1} \leq x_l > x_{l+1}, \dots, x_{m-1}$ (i.e. x_l is the last summand of maximal value). We have the inequalities

$$m(\max(x_1, \dots, x_{m-1}) - 1) < f < m \max(x_1, \dots, x_{m-1})$$

and the equality $\max(x_1, \dots, x_{m-1}) = \lceil f/m \rceil$.

3 NSG-compositions with maximum 2

Results of this Section are well-known, see for example [5].

Proposition 3.1. *All compositions with parts in $\{1, 2\}$ are NSG-compositions.*

The generating function for the number of compositions $x_1 + \dots + x_{m-1}$ with genus g and all parts x_j in $\{1, 2\}$ is the generating function

$$\sum_{n=0}^{\infty} F_g q^g = \frac{1}{1 - q - q^2}$$

of Fibonacci numbers $F_g = \frac{5+\sqrt{5}}{10} \left(\frac{1+\sqrt{5}}{10}\right)^g + \frac{5-\sqrt{5}}{2} \left(\frac{1-\sqrt{5}}{2}\right)^g$ defined recursively by $F_0 = 1, F_1 = 1, F_2 = 2, \dots, F_n = F_{n-1} + F_{n-2}, \dots$

Proof. Compositions with all parts x_j in $\{1, 2\}$ satisfy obviously all NSG-inequalities (6) of Theorem 2.3 and are thus NSG-compositions.

The generating series for compositions with all parts in $\{1, 2\}$ is given by $\sum_{n=0}^{\infty} (q + q^2)^n = 1/(1 - (q + q^2))$. \square

Remark 3.2. *Two other proofs for the generating series enumerating compositions with all parts in $\{1, 2\}$ are as follows:*

There is a unique such composition of genus 0 or 1. Every such composition of genus ≥ 2 is obtained by adding a final part 1 to such a composition of genus $g - 1$ or by adding a final part 2 to such a composition of genus $g - 2$. Numbers of such compositions satisfy thus the initial conditions and recurrence relations of Fibonacci-numbers.

A composition of genus $g = \sum_{j=1}^{m-1} x_j$ with all parts in $\{1, 2\}$ and with multiplicity $m = g - k$ (for k in $\{0, \dots, \lfloor g/2 \rfloor\}$) has k parts equal to 2 and $g - 2k$ parts equal to 1. There are thus $\sum_{k \geq 0} \binom{g-k}{k}$ such compositions. The easy identity

$$\sum_{k=0}^{\lfloor g/2 \rfloor} \binom{g-k}{k} = F_g$$

for $F_0 = F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$ the sequence of Fibonacci numbers ends the proof.

Remark 3.3. *A bijection between $\{0, 1, \dots, F_g - 2, F_g - 1\}$ and compositions of g with parts in $\{1, 2\}$ can be constructed as follows: The Zeckendorff expansion $\epsilon_g \epsilon_{g-1} \dots \epsilon_2 \epsilon_1$ of length g (defined by $n = \sum_{j=1}^g \epsilon_j F_j$ with $\epsilon_j \in \{0, 1\}$, $\epsilon_{j+1} \epsilon_j = 0$ for all j) of an integer $n < F_g$ starts with $\epsilon_g = 0$ and contains only isolated digits 1. Rewriting it in terms of 0 and 01 and considering lengths of factors yields the composition of g associated to n .*

We end this Section with a digression on the following well-known properties (not directly related to the topic of the paper) of compositions:

Remark 3.4. *Compositions of n having only parts ≤ 2 are equinumerous with compositions of all integers up to n having only parts ≥ 2 : Given a composition $x_1 + \dots + x_k$ of n with $x_j \in \{1, 2\}$, remove the final block (which is empty if $x_k = 2$) of consecutive parts 1, factor the resulting word $x_1 x_2 \dots x_l$ with factors in $\{1\}^* 2$ and replace $1^k 2$ by $k+2$. The resulting word corresponds to a composition of an integer $\leq n$ with summands ≥ 2 .*

Equivalently, compositions of n with all parts in $\{1, 2\}$ correspond to compositions of $n + 2$ with all parts ≥ 2 . (Add a final part 2 before applying the above algorithm.)

Combining the two previous bijections yields a bijection between all F_n compositions with parts ≥ 2 of integers up to n and all F_n compositions with parts ≥ 2 of the integer $n + 2$.

Subtracting 1 from the first part of partitions with all parts ≥ 2 shows finally that there are also F_n partitions of $n + 1$ with arbitrary first part and all other parts ≥ 2 . Such partitions encode Zeckendorff expansions of integers in $\{F_{n+1}, \dots, F_{n+2} - 1\}$ by associating to $x_1 + \dots + x_k$ the word $\epsilon_1 \epsilon_2 \dots \epsilon_{n+1}$ (ending with $\epsilon_{n+1} = 1$) obtained by replacing x_k by $0^{k-1}1$. The composition $1 + 3 + 2$ for example corresponds to $\epsilon_1 \dots \epsilon_6 = 1|001|01$ encoding the integer $\sum_{j=1}^6 \epsilon_j F_j = F_1 + F_4 + F_6 = 1 + 5 + 13 = 19$.

4 Generalised compositions

Let $\mathcal{N} \not\subset \{0\}$ be a non-trivial submonoid of the additive monoid $(\mathbb{N}, +)$. We associate to a strictly positive element M of \mathcal{N} the cardinals x_0, x_1, \dots, x_{M-1} in $\mathbb{N} \cup \{\infty\}$ counting the numbers

$$x_j = \#((j + M\mathbb{N}) \setminus \mathcal{N}) \in \mathbb{N} \cup \{\infty\}, \quad j = 0, \dots, M - 1$$

of elements in the complement (gap-set) $\mathbb{N} \setminus \mathcal{N}$ intersecting congruence classes modulo M .

This defines a generalised composition $x_0 + x_1 + \dots + x_{M-1}$ with parts in $\mathbb{N} \cup \{\infty\}$ and (perhaps infinite) sum $\sum_{j=0}^{M-1} x_j = \#\{\mathbb{N} \setminus \mathcal{N}\}$ counting the number of elements in the gap-set $\mathbb{N} \setminus \mathcal{N}$ of \mathcal{N} .

\mathcal{N} is a semigroup (i.e. \mathcal{N} contains 0) if and only if $x_0 = 0$. We work henceforth only with semigroups and omit the trivial summand $x_0 = 0$.

A semigroup \mathcal{N} is a numerical semigroup if and only if all parts x_1, \dots, x_{M-1} are natural integers.

The occurrence of infinite parts among x_1, \dots, x_{M-1} is equivalent to the existence of a divisor $d > 1$ of M such that $\mathcal{N} = d\mathcal{N}'$ with \mathcal{N}' a numerical semigroup. We have then $x_j < \infty$ if and only if d divides j .

All parts x_1, \dots, x_{M-1} are strictly positive (with ∞ being strictly positive by convention) if and only if M is the minimal non-zero element $\min(\mathcal{N} \setminus \{0\})$ of \mathcal{N} .

The generalised composition $x'_1 + \dots + x'_{M'-1}$ associated to another non-zero element M' of \mathcal{N} is defined by considering the smallest natural integer x'_j such that there exists $i \in \{0, \dots, M - 1\}$ with $x'_j M' + j = yM + i$ for $y \geq x_i$. We set $x'_j = \infty$ if \mathcal{N} contains no elements congruent to j modulo M' . (It is also possible to compute $x'_1, \dots, x'_{M'}$ using the algorithm sketched in Remark 4.1 below with generators $\{M, x_1 M + 1, \dots, x_{M-1} M + M - 1\} \cap \mathbb{N}$.)

Theorem 2.3 holds (except for the assertion concerning the multiplicity which is at most equal to M) for generalised compositions after extending the NSG-inequalities (6) to $\mathbb{N} \cup \{\infty\}$ by considering ∞ as a maximal element.

Remark 4.1. *Generalised compositions have interesting algorithmic aspects. They can easily be computed if $\mathcal{N} = \sum_{g \in \mathcal{G}} \mathbb{N}g$ (only finite sums are considered if \mathcal{G} is infinite) is defined in terms of a set $\mathcal{G} \subset \mathbb{N} \setminus \{0\}$ of non-zero generators:*

Choose an element M in \mathcal{G} (the choice $M = \min(\mathcal{G})$ is optimal).

For $j = 1, \dots, M - 1$, set $x_j = \infty$ if $\mathcal{G} \cap (j + M\mathbb{Z}) = \emptyset$ and

$$x_j = \min_{g \in \mathcal{G}, g \equiv j \pmod{M}} \frac{g - j}{M}$$

otherwise.

Iterate the following loop until stabilisation: For $j = 1, \dots, M - 1$ replace x_j by

$$\min \left(\{x_j\} \cup \left(\bigcup_{i=1}^{\lfloor j/2 \rfloor} \{x_i + x_{j-i}\} \right) \cup \left(\bigcup_{i=1}^{\lfloor (m-j)/2 \rfloor} \{1 + x_{j+i} + x_{m-i}\} \right) \right).$$

5 The tree of numerical semigroups

Numerical semigroups have a natural tree-structure: Adding the Frobenius number $f = \max(\mathbb{N} \setminus S)$ to a numerical semigroup S of strictly positive genus g yields a numerical semigroup $S \cup \{f\}$ of genus $g - 1$.

We discuss without proofs in this somewhat informal and colloquial Section how to recover the tree-structure from NSG-compositions. The content of this Section will not be explicitly used in the sequel. (The tree-structure is however implicitly used when discussing NSG-compositions with maximal parts of size 3, 4 or 5.)

The predecessor of a non-trivial NSG-composition $x_1 + \dots + x_{m-1}$ is given by $x_1 + \dots + x_{l-1} + (x_l - 1) + x_{l+1} + \dots + x_{m-1}$ if $x_l = \max(x_1, \dots, x_{m-1})$ is the last maximal part defining the Frobenius number $f = l + m(x_l - 1)$. A trailing part equal to zero is of course suppressed: The predecessor of the composition $g = 1 + 1 + \dots + 1 + 1$ with Frobenius number g is given by the composition $1 + 1 + \dots + 1$ of $g - 1$.

Children (immediate successors) of the NSG-composition $1 + 1 + \dots + 1$ (consisting of g parts $x_j = 1$) are given either by adding an additional part $x_{g+1} = 1$ or by replacing any part $x_j = 1$ by $x_j = 2$.

Children of a NSG-composition $x_1 + \dots + x_{m-1}$ with maximal parts of size $\max(x_1, \dots, x_{m-1}) \geq 2$ are given as follows: Let $f = l + m(x_l - 1)$ be the

Frobenius number associated to $x_1 + \dots + x_{m-1}$. Children of $x_1 + \dots + x_{m-1}$ are given by NSG-compositions

$$x_1 + \dots + x_{i-1} + (x_i + 1) + x_{i+1} + \dots + x_{m-1} \quad (8)$$

such that $x_i = x_l$ if $i \leq l$, respectively $x_i = x_l - 1$ if $i > l$. Observe however that compositions of the form (8) (with $x_i = x_l$ if $i \leq l$, respectively, $x_i = x_l - 1$ otherwise) do not necessarily satisfy NSG-inequalities (6) for indices s, t with $s + t \in \{i, i + m\}$.

The set of all children of a NSG-composition $x_1 + \dots + x_{m-1}$ with maximum at least 2 corresponds thus to a (perhaps empty) subset \mathcal{C} of $\{1, \dots, m-1\}$ with an element i of \mathcal{C} defining a child by formula (8).

Descendants of the NSG-composition $1 + 1 + \dots + 1$ with $m = g + 1$ are NSG-compositions of multiplicity at least m .

The set of all descendants of a NSG-composition $x_1 + \dots + x_{m-1}$ with $\max(x_1, \dots, x_{m-1}) > 1$ can be constructed as follows: Consider the generalised composition $\tilde{z}_1 + \dots + \tilde{z}_{m-1}$ defined by $\tilde{z}_i = \infty$ if i belongs to the set \mathcal{C} indexing children of $x_1 + \dots + x_{m-1}$ and $\tilde{z}_i = x_i$ if $i \notin \mathcal{C}$. Applying the NSG-algorithm of Remark 4.1 to $\tilde{z}_1 + \dots + \tilde{z}_{m-1}$ yields a generalised composition $z_1 + \dots + z_{m-1}$ encoding a smallest semigroup (missing perhaps infinitely many elements of \mathbb{N}) contained in all descendants of $x_1 + \dots + x_{m-1}$. Descendants are encoded by NSG-compositions $y_1 + \dots + y_{m-1}$ with $x_i \leq y_i \leq z_i$. (A composition $y_1 + \dots + y_{m-1}$ with $x_i \leq y_i \leq z_i$, $i = 1, \dots, m-1$ is however not necessarily a NSG-composition.) Observe that we have $z_i = x_i$ if i is not in \mathcal{C} . Observe also that the number of descendants is finite if and only if $\{1, \dots, m-1\} \setminus \mathcal{C}$ generates $\mathbb{Z}/m\mathbb{Z}$.

5.1 A combinatorial over-tree for successors

We have seen that descendants of a NSG-composition $x_1 + \dots + x_{m-1}$ with maximum $\max(x_1, \dots, x_{m-1})$ at least 2 correspond to all NSG-compositions $y_1 + \dots + y_{m-1}$ such that $x_i \leq y_i \leq z_i$ where $z_1 + \dots + z_{m-1}$ is a generalised NSG-composition with parts in $\{1, 2, \dots\} \cup \{\infty\}$ defined in terms of $x_1 + \dots + x_{m-1}$. We have $z_i = x_i$ except for i belonging to the set \mathcal{C} indexing all children of $x_1 + \dots + x_{m-1}$.

We define a sequence

$$\mathcal{D} = (z_{i_1} - x_{i_1}, z_{i_2} - x_{i_2}, \dots, z_{i_k} - x_{i_k}) \in (\{1, 2, \dots\} \cup \{\infty\})^{\mathcal{C}} \quad (9)$$

where the sequence of indices i_1, \dots, i_k corresponds to all elements of \mathcal{C} ordered by $i_a < i_b$ if either $x_{i_a} < x_{i_b}$ or if $x_{i_a} = x_{i_b}$ and $i_a < i_b$. (The

index i_k of the last element of \mathcal{D} corresponds thus to the Frobenius number $f = (x_{i_k} - 1)m + i_k$.

We associate to a sequence $(n_1, \dots, n_k) \in (\{1, 2, \dots\} \cup \{\infty\})^k$ recursively a decorated rooted plane tree $T(n_1, \dots, n_k)$ as follows: The root is decorated by the sequence (n_1, \dots, n_k) . It has k children defined as the roots of the trees given by $(n_{i+1}, n_{i+2}, \dots, n_k, n_i - 1)$ for $i = 1, \dots, k$ with the last coordinate $n_i - 1$ missing if $n_i = 1$. Leaves of T are associated to empty sequences.

A vertex labelled $(5, 1, 3, 1, 1, 3)$ for example has six children given by $(1, 3, 1, 1, 3, 4)$, $(3, 1, 1, 3)$, $(1, 1, 3, 2)$, $(1, 3)$, (3) and (2) .

Proposition 5.1. *The tree $T(n_1, \dots, n_k)$ has $\prod_{i=1}^k (n_i + 1)$ vertices.*

Proof. The formula holds obviously if $\sum_{i=1}^k n_i = \infty$. We can thus assume $n_1, \dots, n_k \in \{1, 2, \dots\}$.

The formula holds for the tree $T()$ reduced to its root.

The induction step reduces to the easy identity

$$\prod_{i=1}^k (n_i + 1) = 1 + \sum_{i=1}^k n_i \prod_{j=i+1}^k (n_j + 1)$$

with the right hand side obtained as a partial expansion of the product $(n_1 + 1)(n_2 + 1) \cdots (n_k + 1)$. \square

A rooted sub-tree T' of a rooted tree T is a sub-tree such that v in T' for a vertex v implies that the predecessor of v in T belongs also to T' .

Proposition 5.2. *The set of all successors of a NSG-composition $x_1 + \dots, x_{m-1}$ with maximum at least 2 is a rooted sub-tree of $T(\mathcal{D})$ with $\mathcal{D} = (z_{i_1} - x_{i_1}, z_{i_2} - x_{i_2}, \dots, z_{i_k} - x_{i_k})$ defined by (9).*

We leave the obvious proof to the reader.

Observe that the set of all successors of a NSG-composition defines in general a strict sub-tree of $T(\mathcal{D})$: The NSG-composition $3 + 1 + 2$ for example gives rise to $\mathcal{D} = (\infty, \infty)$. The corresponding combinatorial tree $T(\mathcal{D})$ can be embedded in \mathbb{R}^2 as follows: Vertices are all elements of \mathbb{N}^2 . Vertices in $\mathbb{N} \times \{0\}$ are labelled (∞, ∞) . All other vertices are labelled (∞) . A vertex $(x, 0)$ (labelled (∞, ∞)) has two successors $(x + 1, 0)$ (labelled (∞, ∞)) and $(x, 1)$ (labelled (∞)). A vertex (x, y) with $y > 0$ (labelled (∞)) has a unique successor $(x, y + 1)$ (labelled (∞)). Only vertices $(x, y) \in \mathbb{N}^2$ with $y \in \{0, 1\}$ correspond to NSG-compositions.

Remark 5.3. *It is tempting to use the combinatorial trees $T(\mathcal{D})$ for deriving bounds on NSG-compositions. This does not seem to pan out: It gives essentially only the trivial bounds obtained by considering NSG-compositions of genus g as a subset of all 2^{g-1} compositions of sum g .*

6 Road map for proving Theorem 1.2

We prove Theorem 1.2 by exploiting the linear structure of compositions. The proof has two essential ingredients:

First we show that NSG-compositions with maximal parts of size at least 4 have growth-rate strictly smaller than ω . Asymptotics are thus given by NSG-compositions with all parts in $\{1, 2, 3\}$.

In order to bound NSG-compositions with maximal parts larger than 3, we chose well-suited pivot-parts x_p of maximal value in such NSG-compositions. This cuts a composition $x_1 + \dots + x_{m-1}$ into a left composition $x_1 + \dots + x_{p-1}$, followed by the pivot-part x_p and a right composition $x_{p+1} + \dots + x_{m-1}$.

We construct then a useful upper bound on the number of possibilities for left compositions $x_1 + \dots + x_{p-1}$, retaining only suitably chosen NSG-inequalities given by the first line of (6) with indices s, t summing up at most to the pivot-index p .

Similarly, we construct an upper bound on the number of possibilities for right compositions $x_{p+1} + \dots + x_{m-1}$ such that NSG-inequalities of the second line of (6) hold for suitable choices of s, t such that $m - s, m - t > p$ and $m - s - t \geq p$. The additional summand $+1$ in the second line of (6) makes the study of right compositions a bit spicier.

The product of the two upper bounds for left, respectively right, compositions is now an upper bound for the number of all NSG-compositions. In terms of growth-rates, this translates into the fact that the maximum of the growth-rates for left, respectively right, compositions is at least equal to the growth-rate of all NSG-compositions. This inequality is sharp for NSG-compositions of maximum 3, see below.

by taking the maximum among the two upper bounds for left, respectively right, compositions.

This approach is easy for NSG-compositions with a maximal part of size at least 6: It is then enough to work with all inequalities (6) such that $s + t = l$, respectively $m - s - t = l$ where x_l is the last part of maximal size defining the Frobenius number $(x_l - 1)m + l$.

For maximum 5 and 4, things get more messy: we have to consider a few additional NSG-inequalities after a careful choice of pivot-parts. (We need also a technical condition on such NSG-compositions. NSG-compositions

not satisfying the condition are treated by the procrastinational technique of kicking the can down the road.)

Finally we have to study NSG-compositions with maximum at most 3. (The elementary case of maximum ≤ 2 has already been discussed in Section 3.) The crucial point for NSG-compositions of maximum 3 is the observation that pivoting with respect to the last maximal part works: all NSG-inequalities of the second line hold trivially for compositions with parts in $\{1, 2, 3\}$. We get thus a “factorisation”

$$(x_1 + \cdots + x_l) + (x_{l+1} + \cdots + x_{m-1})$$

where $x_l = 3$ is the last maximal part (defining the Frobenius number $2m+l$) of such a NSG-composition. This leads to equation (3) of Theorem 1.2.

Finally, we have to show that the generating series \tilde{C} enumerating NSG-compositions ending with a last maximal part $x_{m-1} = 3$ converges in an open disc of radius strictly larger than ω^{-1} . The proof is essentially analogous to the proof for an upper bound on the number of right compositions associated to NSG-compositions of maximum 4.

7 Pivot-factorisation

A part x_p of a composition $x_1 + \cdots + x_{m-1}$ determines a *pivot-factorisation* given by $x_1 + \cdots + x_{p-1}(+x_p)$ and $(x_p+) + x_{p-1} + \cdots + x_{m-1}$. We call p the *pivot-index* and x_p the *pivot-part*. We call $x_1 + \cdots (+x_p)$ the *left composition* and $(x_p+) \cdots + x_{m-1}$ the *right composition* (defined by the pivot-index p). The parentheses around x_p indicate that the inclusion (or exclusion) of x_p is a matter of convention. We omit generally pivot-parts in left or right compositions.

If $x_1 + \cdots + x_{m-1}$ is a NSG-composition, parts of the left composition $x_1 + \cdots (+x_p)$ (with respect to a pivot-part x_p) satisfy the NSG-inequalities

$$x_i + x_j \geq x_{i+j}$$

if $i + j \leq p$ and parts of the right composition satisfy

$$x_i + x_j + 1 \geq x_{i+j-m}$$

if $i + j \geq p + m$.

Pivot-factorisation with a pivot-part of maximal size are our main tool for getting useful upper bounds on NSG-compositions with genus g and maximum ≥ 4 . More precisely, canonical choices of maximal pivot-parts in compositions lead to factorisation LR (with L , respectively R , being generating

series accounting for left, respectively right, compositions) of power series giving useful upper bounds on NSG-compositions of certain types.

This idea works nicely for NSG-compositions of maximum $\mu \geq 6$ and needs a few technical refinements for $\mu = 4$ and 5 .

Pivot-factorisation with respect to a last maximal part with value 3 lead to Theorem 1.2: The left factor is given by $1+\tilde{C}$ (the summand 1 in $1+\tilde{C}(\omega^{-1})$ in (3) accounts for NSG-compositions of maximum at most 2), the right factor, given by the rational series $1/(1 - (q + q^2))$, induces the growth-rate ω .

8 Generating series, growth-rates

We define the (exponential) *growth-rate* of a sequence of strictly positive natural integers s_1, s_2, \dots by $\gamma = \limsup_{n \rightarrow \infty} \sqrt[n]{s_n}$. The sequence s_n has exponential growth, if $1 < \gamma < \infty$. We consider henceforth only sequences with exponential growth. Given $\epsilon > 0$, we have $s_n < (\gamma + \epsilon)^n$ for almost all integers n and $s_n > (\gamma - \epsilon)^n$ infinitely often. The inverse $\rho = 1/\gamma$ of the growth-rate γ for s_0, s_1, \dots is the radius of convergency for the power series $\sum_{n=0}^{\infty} s_n R^n$.

Remark 8.1. *Having exponential growth γ is slightly weaker than having an asymptotic growth of exponential rate γ (defined as $\gamma = \lim_{n \rightarrow \infty} \sqrt[n]{s_n}$).*

A non-constant power-series with real non-negative coefficients of growth-rate γ defines a holomorphic function in a neighbourhood of 0 which has always a smallest singularity at its convergency radius $\rho = 1/\gamma$. If such a series $\sum_{n=0}^{\infty} s_n P^n$ is rational, then its singularities are isolated and $\sum_{n=0}^{\infty} s_n q_0^n < \infty$ for some strictly positive q_0 implies $\gamma < 1/q_0$. Strict inequality does however generally not hold for series which are not rational: Coefficients of the series $\sum_{n=0}^{\infty} \lfloor \gamma^n / (1 + n^2) \rfloor q^n$ have growth-rate γ for $\gamma > 1$ and the series converges for q of absolute value $|q| = 1/\gamma$. The following result describes however a well-behaved class of generally irrational power series (with non-negative coefficients):

Lemma 8.2. *Let $S(q) = \sum_{n=n_0}^{\infty} A_n / (1 - B_n)$ be a power-series with coefficients in \mathbb{N} defined by sequences of polynomials $A_n, B_n \in \mathbb{N}[q]$ satisfying linear recursions with coefficients in $\mathbb{Q}[q]$.*

Suppose that there exists a strictly positive real number ρ_0 such that $S(q)$ converges for $q = \rho_0$ and such that the evaluations of A_n, B_n at $q = \rho_0$ decay exponentially fast. Then $S(q)$ has convergency radius strictly larger than ρ_0 .

Proof of Lemma 8.2. The hypotheses imply that evaluations at ρ_0 of the characteristic polynomials defining linear recursions for A_n and B_n have all roots in the open complex unit disc. This condition (which implies exponentially fast decay) holds by continuity for q close enough to ρ_0 .

A similar argument shows that all evaluations of B_n at q are strictly smaller than 1 for q close enough to ρ_0 .

Non-negativity of all involved coefficients shows now that the convergency radius of S is strictly larger than ρ_0 . \square

Remark 8.3. *Lemma 8.2 streamlines a few proofs. It can however easily be replaced by a few computations involving eigenvalues and eigenvectors of transfer matrices.*

Throughout the paper we will also use several times the trivial fact that the convergency radius of a finite product of power-series is at least equal to the minimal convergency radius among factors.

9 Weakly admissible compositions

A composition $x_1 + \dots + x_{m-1}$ with last maximal part $x_l = \max(x_1, \dots, x_l) > \max(x_{l+1}, \dots, x_{m-1})$ is *weakly admissible* if $x_l \leq \min(x_1 + x_{l-1}, x_2 + x_{l-2}, \dots, x_{l-1} + x_1)$ and $x_l \leq 1 + \min(x_{l+1} + x_{m-1}, x_{l+2} + x_{m-2}, \dots, x_{m-1} + x_{l+1})$. Otherwise stated, we require only NSG-inequalities (6) involving the last maximal part x_l of $x_1 + \dots + x_{m-1}$.

Since NSG-compositions are weakly admissible we get crude upper bounds for NSG-compositions by counting weakly admissible compositions.

Observe however that the composition $1 + 3 + 3$ for example is weakly admissible but is not a NSG-composition.

This section is devoted to generating series of weakly admissible compositions having maximal parts of given size μ .

Weakly admissible compositions of maximum at least 6 have growth-rate strictly smaller than ω and give thus useful upper bounds for NSG-compositions with maximal parts of size at least 6.

Refinements are needed for useful bounds on NSG-compositions with maximal parts of size 4 and 5. NSG-compositions with maximal parts of size at most 3 are asymptotically generic (their proportion among all NSG-compositions tends to 1 for $g \rightarrow \infty$) and induce the growth-rate for numerical semigroups.

For $n \geq 1$, we consider the generating polynomial

$$I_n = \sum_{1 \leq a, b \leq n \leq a+b} q^{a+b} \quad (10)$$

of all compositions $a + b$ with total sum at least n into two parts a, b not exceeding n .

The polynomial I_n can be computed by removing all contributions of degree strictly less than n (corresponding to compositions with two parts summing up to integers strictly smaller than n) from $(q + q^2 + \dots + q^n)^2$. This implies easily the closed formula

$$I_n = q^n \left(-2 + \sum_{i=0}^n (n+1-i)q^i \right). \quad (11)$$

The following table gives the first few polynomials I_n and their evaluations $I_n(\omega^{-1})$ in $\mathbb{Q}[\sqrt{5}]$, together with a decimal approximation, at the inverse ω^{-1} of the golden number $\omega = \frac{1+\sqrt{5}}{2}$:

k	I_k	$I_k(1/\omega)$	\sim
1	q^2	$(3 - \sqrt{5})/2$	0.3820
2	$q^2 + 2q^3 + q^4$	1	1
3	$2q^3 + 3q^4 + 2q^5 + q^6$	$(9 - 3\sqrt{5})/2$	1.1459
4	$3q^4 + 4q^5 + 3q^6 + 2q^7 + q^8$	$10 - 4\sqrt{5}$	1.0557
5	$4q^5 + 5q^6 + 4q^7 + 3q^8 + 2q^9 + q^{10}$	$21 - 9\sqrt{5}$	0.8754
6	$5q^6 + 6q^7 + 5q^8 + 4q^9 + 3q^{10} + 2q^{11} + q^{12}$	$70 - 31\sqrt{5}$	0.6819

Proposition 9.1. *The generating series $W_\mu(q)$ for weakly admissible compositions $x_1 + \dots + x_{m-1}$ of genus g with maximal parts $\max(x_1, \dots, x_{m-1}) = \mu \geq 2$ is given by*

$$W_\mu(q) = \frac{1 + \sum_{i=\lceil \mu/2 \rceil}^{\mu} q^i}{1 - I_\mu} q^\mu \frac{1 + \sum_{i=\lfloor \mu/2 \rfloor}^{\mu-1} q^i}{1 - I_{\mu-1}}. \quad (12)$$

Remark 9.2. *Since all compositions with parts in $\{1, 2\}$ are NSG-compositions, Propositions 3.1 and 9.1 imply the identity*

$$\frac{1}{1 - (q + q^2)} = \frac{1}{1 - q} + W_2(q).$$

Proof of Proposition 9.1. Let $x_1 + \dots + x_{l-1} + x_l + x_{l+1} + \dots + x_{m-1}$ be a weakly admissible composition with $x_1, \dots, x_{l-1} \leq x_l = \mu > x_{l+1}, \dots, x_{m-1}$. We have thus $1 \leq x_i, x_{l-i} \leq \mu \leq x_i + x_{l-i}$ for $i < l/2$. For odd l there are $I_\mu^{(l-1)/2}$ possibilities satisfying these inequalities. For l even, we have moreover to choose a coefficient $x_{l/2}$ in $\{\lceil \mu/2 \rceil, \dots, \mu\}$. Summing over l in $\mathbb{N} \setminus \{0\}$ we get the left factor $(1 + \sum_{i=\lceil \mu/2 \rceil}^{\mu} q^i)/(1 - I_\mu)$ of (12) enumerating

all possibilities for left compositions with respect to the pivot-part $x_l = \mu$ given by the last maximal part x_l .

The central factor q^μ accounts for the pivot-part $x_l = \mu$.

The final right factor corresponds to all possibilities involving the parts $x_{l+1}, \dots, x_{m-1} \in \{1, \dots, \mu-1\}$ of right compositions following the pivot-part $x_l = \mu$: We have $1 \leq x_{l+i}, x_{m-i} \leq \mu-1 \leq x_{l+i} + x_{m-i}$. Such pairs (x_{l+i}, x_{m-i}) are thus encoded by powers of $I_{\mu-1}$ and we have moreover a choice of $x_{(m+l)/2}$ in $\{\lfloor \mu/2 \rfloor, \dots, \mu-1\}$ if $m+l$ is even. \square

Remark 9.3. *Formula (12) gives crude upper bounds: It counts only compositions satisfying NSG-inequalities (6) involving the last maximal part. Only a small proportion of weakly admissible compositions with maximum strictly larger than 2 satisfy all NSG-inequalities.*

10 NSG-compositions with maximum ≥ 6

Proposition 9.1 gives useful upper bounds for the number of NSG-compositions with maximum at least 6, as suggested by the evaluations $I_n(\omega^{-1})$ of the first few polynomials I_n defined by (11):

Proposition 10.1. *Numbers of weakly admissible compositions with maximum at least 6 have growth-rate strictly smaller than ω .*

Proof. Proposition 9.1 shows that it is enough to prove that $\sum_{n=6}^{\infty} W_n(q)$ (for $W_n(q)$ defined by (12)) converges in an open disc of radius strictly larger than ω^{-1} .

This holds clearly for $W_6(q)$ which converges in the open disc of radius the strictly positive root $0.6318\dots > \omega^{-1}$ of $1 - I_5 = 1 - 4q^5 - 5q^6 - 4q^7 - 3q^8 - 2q^9 - q^{10}$.

Formula (11) shows that coefficients of the rational fraction

$$\mathbf{I}_6 = \sum_{j=6}^{\infty} (j-1)q^j = q^6 \frac{5-4q}{(1-q)^2}$$

yield upper bounds on the coefficients of I_n for $n \geq 6$.

The convergency radius of $\sum_{n=7}^{\infty} W_n(q)$ is thus at least as large as the convergency radius of the rational fraction

$$\sum_{n=7}^{\infty} \frac{\sum_{j=0}^{\infty} q^j}{1 - \mathbf{I}_6} q^n \frac{\sum_{j=0}^{\infty} q^j}{1 - \mathbf{I}_6} = \frac{q^7}{(1-q)^3(1 - \mathbf{I}_6)^2}$$

given by the positive root $0.6206\dots > \omega^{-1}$ of the polynomial $(1-q)^2(1 - \mathbf{I}_6) = 1 - 2q + q^2 - 5q^6 + 4q^7$. \square

11 NSG-compositions with maximum 5

The rational generating series $W_5(q)$, given by Proposition 9.1 and enumerating weakly admissible compositions of maximum 5, involves $1 - I_4(q)$ (accounting for right compositions) in its denominator. Since $1 - I_4$ has a root in $(0, \omega^{-1})$, the growth-rate of weakly admissible compositions of maximum 5 exceeds ω . Obtaining useful upper bounds for NSG-compositions of maximum 5 (trickier than obtaining the corresponding results for NSG-compositions of maximum at least 6, see Section 10) requires thus additional NSG-inequalities (6).

Proposition 11.1. *There exists a strictly positive constant $\kappa_5 < \omega$ such that NSG-compositions with maximum 5 have growth-rate at most equal to $\max(\gamma_4, \kappa_5)$ where γ_4 is the growth-rate of NSG-compositions with maximum 4.*

Proposition 11.1 follows trivially from the two following results:

Proposition 11.2. *The growth-rate of NSG-compositions with maximum 5 having a unique maximal part is at most equal to the growth-rate γ_4 of NSG-compositions with maximum 4.*

Proposition 11.3. *The growth rate of NSG-compositions having at least two maximal parts of size 5 is strictly smaller than ω .*

Remark 11.4. *We will show later that the constant γ_4 of Proposition 11.1 satisfies the inequality $\gamma_4 < \omega$ (see Proposition 12.1 and Theorem 13.3).*

11.1 Proof of Proposition 11.2

Proof of Proposition 11.2. A unique maximal part $x_l = 5$ of size 5 corresponds to the Frobenius number $f = 4m + l$ of such a NSG-composition $x_1 + \dots + x_{m-1}$. Adding the Frobenius element f to the associated numerical semigroup amounts to replacing $x_l = 5$ by $x_l = 4$ and results in a NSG-composition with maximum 4 (and genus decreased by 1). Such reductions yield any given NSG-composition of genus g and maximum 4 less than g times (since $m \leq g + 1$ with equality only for $1 + 1 + \dots + 1$). Numbers of NSG-compositions with a unique maximal part of size 5 are thus bounded by coefficients of $q^2 G'_4$ where G_4 is the generating series for all NSG-compositions with maximum 4. The result follows by observing that coefficients of G_4 and of its derivative G'_4 have identical growth-rates. \square

11.2 Proof of Proposition 11.3

Proposition 11.5. *NSG-compositions $x_1 + \dots + x_{m-1}$ with a last maximal part $x_l = 5$ such that $3l \geq m - 1$ have growth rate at most equal to $1/\rho < \omega$ where $\rho = 0.6189\dots > \omega^{-1}$ is the positive root of $1 - I_5 I_4^2$ for I_n given by (11).*

Proof of Proposition 11.5. Choosing the index l of the last maximal part $x_l = 5$ in such a NSG-composition as a pivot, the proof of Proposition 9.1 shows that the number of such NSG-compositions of genus g (with multiplicity m and last maximal part $x_l = 5$) is bounded by the coefficient of q^g in

$$(1 + q^3 + q^4 + q^5)(1 + q^2 + q^3 + q^4)I_5^{\lfloor (l-1)/2 \rfloor} I_4^{\lfloor (m-1-l)/2 \rfloor}$$

where

$$\begin{aligned} I_4 &= 3q^4 + 4q^5 + 3q^6 + 2q^7 + q^8, \\ I_5 &= 4q^5 + 5q^6 + 4q^7 + 3q^8 + 2q^9 + q^{10} \end{aligned}$$

(see Formula (11)).

The inequality $3l \geq m - 1$ implies that $(m - 1 - l)/2 - 1$ is at most twice as large as $(l - 1)/2$.

Neglecting polynomial factors, we have reduced the proof of Proposition 11.2 to the study of the convergency radius of

$$\sum_{a=0}^{\infty} \sum_{b=0}^{2a} I_5^a I_4^b. \quad (13)$$

Rewriting (13) by regrouping $I_5 I_4^2$ we can work with

$$\frac{1}{(1 - I_5 I_4^2)(1 - I_5)}$$

(up to neglecting a polynomial factor) which converges on the open disc of radius $\rho > \omega^{-1}$ the strictly positive root $0.6189\dots$ of $1 - I_5 I_4^2$. \square

Proof of Proposition 11.3. Given a NSG-composition $x_1 + \dots + x_{m-1}$ having at least two maximal parts of size 5 let k and l with $1 \leq k < l < m$ be the indices of the two last maximal parts $x_k = x_l = 5$.

The result holds by Proposition 11.5 for all NSG-compositions such that $3l \geq m - 1$.

We are now left with the case of NSG-compositions $x_1 + \dots + x_{m-1}$ with indices of the two last maximal parts satisfying $k < l < (m - 1)/3$.

We give again a factorised upper bound for all such NSG-compositions.

We use $\frac{1+q^3+q^4+q^5}{1-I_5}$ (see the proof of Proposition 9.1) for upper bounds on numbers of left compositions with respect to the pivot-part x_k given by the second-last maximum. (This works since coefficients of $1/(1 - I_5)$ have growth-rate strictly smaller than ω .)

In order to get upper bounds for right compositions, we introduce a graph Γ with vertices $k + 1, k + 2, \dots, m - 1$ and edges $\{i, j\}$ if $i + j - m \in \{k, l\}$. The graph Γ is a union of paths (trees with at most two leaves and interior vertices of degree 2). A connected component of Γ is *ordinary* if it contains no endpoint in $\{(k+m)/2, (l+m)/2\} \cap \mathbb{N}$. It is *exceptional* otherwise. Ordinary connected components of Γ have an even number of vertices and contain a central edge.

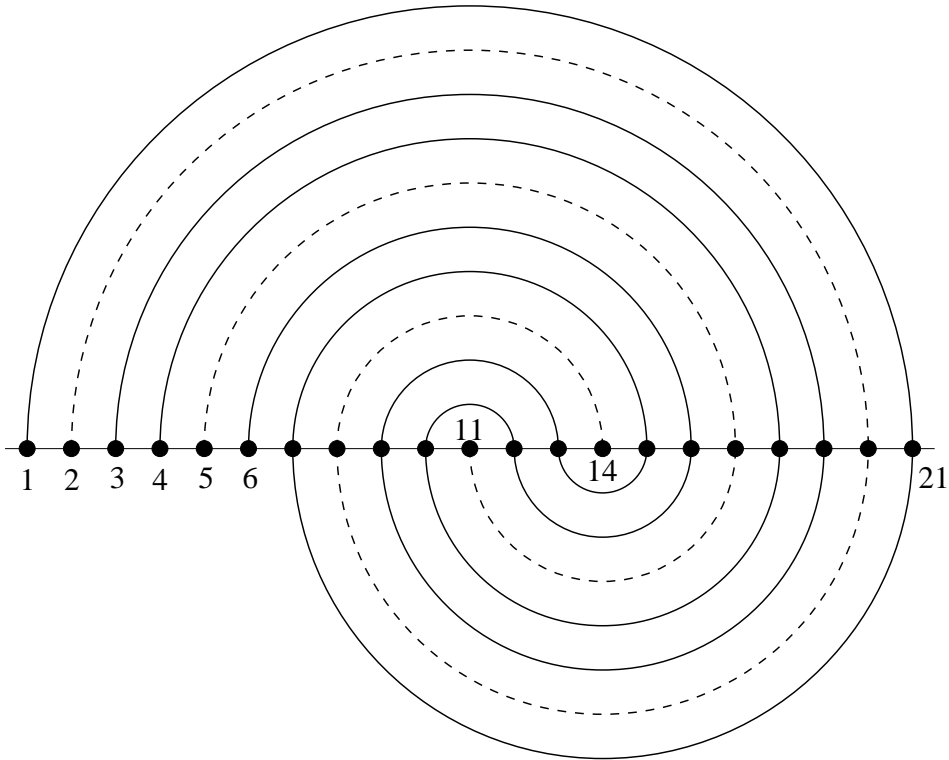


Figure 1: An example of a graph Γ .

Figure 1 shows an example with vertices $k + 1, k + 2, \dots$ simply numbered $1, 2$, where where $m = k + 22$ and $l = k + 6$. Edges $\{i, j\}$ with $i + j = k + m$, respectively $i + j = l + m$, are represented by upper, respectively lower, half-circles. Edges of exceptional components are dashed. (The horizontal coordinate-axis is not part of Γ .)

The first $l - k$ vertices $k + 1, k + 2, \dots, l$ of Γ are leaves of all connected components of Γ .

Ordinary connected components have length $2n \pm 1$ for some integer n (depending on k, l, m). Exceptional components have length n or $n - 1$.

The construction of Γ implies $x_i + x_j \geq 4$ if $\{i, j\}$ is an edge of Γ associated to a NSG-composition $x_1 + \dots + x_{m-1}$ as above. Generating functions of parts supported by connected components (maximal paths) of Γ can be considered as partition-functions of constrained spin models with spins $\{1, 2, 3, 4\}$ such that all pairs of adjacent spins have sum at least 4.

We can compute the generating series (partition-functions) of this model on paths by an easy application of the transfer-matrix method originating in statistical physics and widely used in enumerative combinatorics, see e.g. Chapter 4.7 of [4]. (Details are easy and can be understood without prior knowledge of spin models.)

We denote by A_n, B_n, C_n the generating series (with respect to the weight $q^{s_0 + s_1 + \dots + s_n}$) for sequences $(s_0, s_1, \dots, s_n) \in \{1, 2, 3, 4\}^{n+1}$ of length $n+1$ with coefficients in $\{1, 2, 3, 4\}$ such that $s_i + s_{i+1} \geq 4$ for $i = 0, \dots, n-1$ and ending with $s_n = 1$ for elements counted by A_n , $s_n = 2$ for B_n and $s_n \in \{3, 4\}$ for C_n . We have $A_0 = q, B_0 = q^2, C_0 = q^3 + q^4$ and

$$\begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & q \\ 0 & q^2 & q^2 \\ q^3 + q^4 & q^3 + q^4 & q^3 + q^4 \end{pmatrix}^n \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}, \quad (14)$$

by an easy induction. The transfer-matrix involved in (14) has eigenvalues $-0.3981, 0.2476, 0.9143$ at $q = \omega^{-1}$. This shows that the evaluations $P_n(\omega^{-1})$ of the polynomials $P_n = A_n + B_n + C_n$ decay exponentially fast to 0. The evaluations $P_n(\omega^{-1})$ are strictly smaller than 1 for $n \geq 3$.

Lemma 8.2 implies that the series

$$\left(1 + \sum_{n=0}^{\infty} P_n\right)^2 q \left(1 + \sum_{n=3}^{\infty} \frac{P_{2n-1} + P_{2n+1}}{(1 - P_{2n-1})(1 - P_{2n+1})}\right) \quad (15)$$

converges on an open disc of radius strictly larger than ω^{-1} .

Coefficients of (15) are upper bounds on the number of possibilities for right compositions of NSG-compositions with pivot-part the second last maximum $x_k = 5$: The squared first factor has convergency radius strictly larger than ω^{-1} by Lemma 8.2 and accounts for perhaps existing exceptional components, the isolated factor q corrects for the fact that $x_l = 5 \notin \{1, 2, 3, 4\}$. The final factor (which converges for q slightly larger than ω^{-1} , see Lemma 8.2) yields a coarse upper bound for contributions coming from ordinary

components of Γ . The summand 1 of the final factor is needed for NSG-compositions with graphs reduced to exceptional components. The sum of the final factor starts at $n = 3$ since the inequality $l < (m - 1)/3$ ensures length at least 5 for ordinary components of Γ . The numerator $P_{2n-1} + P_{2n+1}$ (necessary for convergency) is due to the assumed existence of at least one ordinary component of length $2n - 1$ or $2n + 1$. (NSG-compositions without ordinary components in the associated graph Γ are accounted for by the summand 1 of the last factor, see above). \square

12 NSG-compositions with maximum 4

This section contains a proof of the following result:

Proposition 12.1. *We assume the inequality $\tilde{\gamma} < \omega$ for the growth-rate of the generating series $\tilde{C}(q)$ enumerating NSG-compositions of maximum 3 ending with a maximal part.*

There exists then a positive constant κ_4 strictly smaller than ω such that NSG-compositions of maximum 4 have growth-rate at most equal to $\max(\kappa_4, 1/\rho, \sqrt[3]{\tilde{\gamma}\omega^2})$ where $\rho = 0.71667\dots > \omega^{-1}$ is the positive root of $1 - (2q^3 + q^4)$.

The inequality $\tilde{\gamma} < \omega$ will be proven in Theorem 13.3 whose proof uses parts (contained in Sections 12.4-12.6 and independent of the assumption $\tilde{\gamma} < \omega$) of the proof of Proposition 12.1.

We start by outlining the proof of Proposition 12.1, hopefully providing the reader with a “magic flute” guiding him through the foggier parts. The main difficulty is due to the fact that we have to work with NSG-inequalities involving three maximal parts when considering right compositions with respect to a suitable maximal pivot-part.

The proof remains however roughly similar to the proof of the corresponding result (given by Proposition 11.1) for NSG-compositions of maximum 5.

We start by studying the growth-rate of NSG-compositions with maximum 4 having a bounded number of maximal parts. This is done procrastinationally in Section 12.1 whose main result, Proposition 12.2 is the analogue of Proposition 11.2.

Section 12.2 is devoted to left compositions with respect to an arbitrary maximal pivot-part.

Section 12.3 is the analogue of Proposition 11.5: It deals with NSG-compositions whose Frobenius numbers are close to $4m$.

Section 12.4 discusses Cayley and Schreier graphs of (some) groups.

Section 12.5 discusses spin models of lanes, used in the proof of Section 12.6

Section 12.6 contains the core of the proof: It introduces the ϵ -condition and bounds the growth rate of right compositions satisfying the ϵ -condition.

The content of Sections 12.3–12.6 corresponds more or less to the statement and proof of Proposition 11.3 in the case of NSG-compositions of maximum 5.

Section 12.7 ties up loose ends.

12.1 NSG-compositions with a bounded number of maximal parts of size 4

Proposition 12.2. *Given a natural integer A , the growth-rate of NSG-compositions with maximum 4 having at most A maximal parts is at most equal to $\max(1/\rho, \sqrt[3]{\tilde{\gamma}\omega^2})$ where $\rho = 0.71667\dots > \omega^{-1}$ is the positive root of $1 - (2q^3 + q^4)$ and where $\tilde{\gamma} < \omega$ is the growth-rate of the generating series $\tilde{C}(q)$ enumerating NSG-compositions with maximum 3 ending with a part of maximal size.*

A straightforward modification of the proof of Proposition 11.2 does unfortunately not work: NSG-compositions with a maximal part of size 3 are generic and have growth-rate ω due to the possibility of arbitrary long 'tails' involving only summands 1 and 2. In order to circumvent this difficulty, we discuss first NSG-compositions with 'short tails'.

A NSG-composition $x_1 + \dots + x_{m-1}$ is *short-tailed* if it involves a summand $x_l \geq 3$ with index $l \geq m/2$.

Lemma 12.3. *The generating series S for short-tailed NSG-compositions with maximum 3 has growth rate at most $\sqrt[3]{\tilde{\gamma}\omega^2}$ where $\tilde{\gamma} < \omega$ is the growth-rate of the generating series $\tilde{C}(q)$ enumerating NSG-compositions with maximum 3 ending with a part of maximal size.*

Proof of Lemma 12.3. Let $\mathbf{x} = x_1 + \dots + x_l + \dots + x_{m-1}$ be a short-tailed NSG-composition with last maximum $x_l = 3$ for $2l \geq m$. Since $x_1, \dots, x_l \geq 1$ and $x_{l+1}, \dots, x_{m-1} \leq 2$ with $l > m - 1 - l$ we have $2(x_1 + \dots + x_l) = 2l > 2(m - 1 - l) \geq x_{l+1} + \dots + x_{m-1}$. In particular, the NSG-composition $x_1 + \dots + x_l$ contributes at least $g/3$ to the genus $g = x_1 + \dots + x_{m-1}$ of \mathbf{x} .

Denoting by \tilde{c}_g the number of NSG-compositions of genus g ending with a final maximal part of size 3 (with growth-rate $\tilde{\gamma}$, involved in the generating

series $\tilde{C} = \sum_{g=3}^{\infty} \tilde{c}_g q^g$, see (4)), the series

$$U = \sum_{g=3}^{\infty} \left(\sum_{k=\lceil g/3 \rceil}^g \tilde{c}_k F_{g-k} \right) q^g \quad (16)$$

(where the Fibonacci numbers F_{g-k} count possibilities for the final NSG-composition $x_{l+1} + \dots + x_{m-1}$ with all parts in $\{1, 2\}$, see Proposition 3.1) with growth rate $\sqrt[3]{\tilde{\gamma}\omega^2}$ gives upper bounds on the coefficients of S counting short-tailed NSG-compositions with maximum 3. \square

Proof of Proposition 12.2. Replacing all parts of size 4 in such a NSG-composition $\mathbf{x} = x_1 + \dots + x_{m-1}$ with parts of size 3, we get a NSG-composition $\underline{\mathbf{x}} = \underline{x}_1 + \dots + \underline{x}_{m-1}$ with parts $\underline{x}_i = \min(x_i, 3)$ of maximal size 3.

We count first all such NSG-compositions \mathbf{x} (having at most A maximal parts of size 4) such that $\underline{\mathbf{x}}$ is short-tailed. Since such a short-tailed NSG-composition of genus g can be lifted into at most $\binom{g}{a}$ NSG-compositions \mathbf{x} having a maximal parts of size 4, we can bound the number of such NSG-compositions \mathbf{x} by applying the same trick used with $a = 1$ in the proof of Proposition 11.2. More precisely, numbers of such NSG-compositions \mathbf{x} (with maximum 4 arising at most A times, associated to a short-tailed NSG-composition $\underline{\mathbf{x}}$ as above) are bounded above by coefficients of

$$U_A = \sum_{a=1}^A q^{2a} \frac{d^a}{dq^a} U = \sum_{a=1}^A q^{2a} \frac{d^a}{dq^a} \left(\sum_{g=3}^{\infty} \left(\sum_{k=\lceil g/3 \rceil}^g \tilde{c}_k F_{g-k} \right) q^g \right) \quad (17)$$

with U the series of upper bounds for short-tailed NSG-compositions of maximum 3 given by (16). Since finite sums of derivatives do not increase growth-rates, the growth rate of the series U_A is still given by $\sqrt[3]{\tilde{\gamma}\omega^2}$.

We consider now NSG-compositions \mathbf{x} having at most A maximal parts of size 4 giving rise to NSG-compositions $\underline{\mathbf{x}}$ which are not short-tailed. Such a NSG-composition $\mathbf{x} = x_1 + \dots + x_l + x_{l+1} + \dots + x_{m-1}$ has thus a last part x_l of size 3 or 4 indexed by $l < m/2$. Since it has maximum 4, it contains a part x_k (not necessarily distinct from x_l) of size 4 with $k \leq l$. Since $k \in \{1, \dots, l\}$ with $l < m/2$, the two (not necessarily distinct) indices $l+1$ and $m+k-l-1$ belong to $\{l+1, \dots, m-1\}$ and the NSG-inequalities (6) involving $x_k = 4$ yield $x_{l+i} + x_{m+k-l-i} \geq 3$ with $x_{l+i}, x_{m+k-l-i}$ in $\{1, 2\}$ for $i \geq 1$ such that $l+i \leq m+k-l-i$. The contribution of all such (not necessarily distinct pairs) to the generating series involved in Proposition 12.2 can be bounded above by $\frac{1+q^2}{1-(2q^3+q^4)}$. Removing all summands $x_{l+1}, x_{l+2}, \dots, x_{m+k-l-1}$ involved in

such pairs from \mathbf{x} and replacing maximal parts of size 4 by parts of size 3, we get a NSG-composition

$$\underline{\tilde{\mathbf{x}}} = \min(x_1, 3) + \cdots + \min(x_l, 3) + x_{m+k-l} + x_{m+k-l+1} + \cdots + x_{m-1}$$

with tail-length (given by the number of final parts in $\{1, 2\}$ following $\underline{x}_l = 3$)

$$m - 1 - (m + k - l - 1) = l - k < l .$$

This implies that $\underline{\tilde{\mathbf{x}}}$ is short-tailed.

The generating series of NSG-compositions having at most A parts of maximal size 4 can thus be bounded above by the generating series

$$U_A \frac{1 + q^2}{1 - (2q^3 + q^4)}$$

(the constant coefficient 1 of $\frac{1+q^2}{1-(2q^3+q^4)}$ accounts for NSG-compositions discussed at the start of the proof) with growth-rate given by Proposition 12.2. \square

12.2 Left compositions

We set

$$\tilde{P}_n = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & q \\ 0 & q^2 & q^2 \\ q^3 + q^4 & q^3 + q^4 & q^3 + q^4 \end{pmatrix}^n \begin{pmatrix} q \\ q^2 \\ q^3 \end{pmatrix} \quad (18)$$

for $n \geq 1$.

Proposition 12.4. *The coefficients of the generating series for distinct left compositions with respect to a maximal pivot-part 4 are bounded above by coefficients of*

$$(1 + q) \left(1 + q^2 + q^3 + q^4 + \sum_{n=1}^{\infty} \tilde{P}_n \right)^2 \left(1 + \sum_{n=1}^{\infty} \frac{\tilde{P}_{2n-1} + \tilde{P}_{2n+1}}{(1 - \tilde{P}_{2n-1})(1 - \tilde{P}_{2n+1})} \right) \quad (19)$$

which converges on an open disc of radius strictly larger than ω^{-1} .

Proof. In order to study the generating series associated to left compositions $x_1 + \cdots + x_{l-1} (+x_l)$ with respect to a maximal pivot-part $x_l = 4$, we consider the graph Γ with vertices $1, \dots, l - 1$ and edges $\{i, j\}$ if $i + j \in \{k, l\}$ where

$k < l \leq 2k$ is maximal such that $x_k = 4$ if such an index $k \geq l/2$ exists. The graph Γ is a union of disjoint edges $\{i, l - i\}$ (and of the isolated vertex $l/2$ if l is even) if the index k does not exist. Connected components of Γ are paths having at least one end-point in $k, k + 1, \dots, l - 1$ (respectively $\lceil l/2 \rceil, \dots, l - 1$ if k does not exist). (More precisely, they all start and end at points in $k, \dots, l - 1$, except for at most two paths starting at elements of $k, \dots, l - 1$ and ending in $\{k/2, l/2\} \cap \mathbb{N}$.)

Parts (spins) associated to initial vertices in $\{k, k + 1, \dots, l - 1\}$ are elements of $\{1, 2, 3\}$, except for the path starting at $x_k = 4$ if k exists. All other spins are in $\{1, 2, 3, 4\}$ and we have $x_i + x_j \geq 4$ for edges $\{i, j\}$ of Γ .

The generating series of a path of length $n \geq 1$ not starting at x_k is thus given by the polynomial \tilde{P}_n (obtained by the transfer matrix method, compare with formula 14) defined by formula (18).

Formula (19) is analogous to Formula (15): The factor $(1 + q)$ of formula (19) accounts for the correction $x_k = 4 \notin \{1, 2, 3\}$ of the (not necessarily existing) path starting at k with initial spin $x_k = 4$. The squared factor takes into account the possible existence of exceptional paths ending at $\{l/2, k/2\} \cap \mathbb{N}$. The final factor deals with all ordinary paths (not containing a vertex of $\{l/2, k/2\} \cap \mathbb{N}$) of Γ .

Convergency of the series (19) for q slightly larger than ω^{-1} follows from Lemma 8.2: Elementary (and somewhat lengthy) computations involving the transfer matrix of (18) show that evaluations at $q = \omega^{-1}$ of \tilde{P}_n decay exponentially fast and are strictly smaller than 1 for $n \geq 1$. \square

12.3 Frobenius numbers close to $4m$

Proposition 12.5. *There exists $\delta > 0$ such that NSG-compositions $x_1 + \dots + x_l + \dots + x_{m-1}$ satisfying the inequality $(m - 1 - l) \leq \delta(m - 1)$ for the index l of their last maximal part $x_l = 4$ have growth-rate strictly smaller than ω .*

Remark 12.6. *The Frobenius number $3m + l$ of NSG-compositions described by Proposition 12.5 is close to $4m$ if δ is small.*

Proof of Proposition 12.5. We consider the pivot-factorisation of such a NSG-composition $x_1 + \dots + x_{m-1}$ with pivot-part the last maximal part x_l . Using the NSG-inequalities $x_l = 4 \leq 1 + x_{l+i} + x_{m-i}$ we see that the right composition $x_{l+1} + \dots + x_{m-1}$ (with $m - 1 - l$ parts in $\{1, 2, 3\}$) contributes at most a factor of $\sum_{n=0}^{3(m-1-l)} r_n q^n$ to the generating series where $\sum_{n=1}^{\infty} r_n q^n = (1 + q^2 + q^3)/(1 - I_3)$ (for $I_3 = 2q^3 + 3q^4 + 2q^5 + q^6$ given by Formula (11)). Since $(m - 1 - l) \leq \delta(m - 1) < \delta g$ (with $g = x_1 + \dots + x_{m-1}$ denoting the

genus) we get the upper bound

$$\sum_{n=0}^{\lceil 3\delta g \rceil} r_n q^n \quad (20)$$

for contributions coming from right-compositions. Coefficients of the rational series $\sum_{n=0}^{\infty} r_n q^n$ have however growth-rate $1/\rho > \omega$ for $\rho = 0.596\dots$ the real positive root of $1 - I_3$.

Contributions coming from left compositions are bounded by the series $A = \sum_{n=0}^{\infty} a_n q^n$ defined by formula (19) (which yields upper bounds on left compositions with respect to a maximal pivot-part of size 4). Combining this with the contribution (20) we get the upper bound

$$\sum_{g=4}^{\infty} q^g \sum_{i=\lceil g(1-3\delta) \rceil}^g a_i r_{g-i}. \quad (21)$$

on the generating series for NSG-compositions described by Proposition 12.5. Denoting by α the growth-rate of the series A we get the upper bound $\alpha^{1-3\delta}(1/\rho)^{3\delta}$ on the growth-rate of (21). Proposition 12.4 shows the inequality $\alpha < \omega$ which ends the proof since $\lim_{\delta \rightarrow 0} \alpha^{1-3\delta}(1/\rho)^{3\delta} = \alpha$. \square

12.4 Groups generated by reflections of \mathbb{R}

A group Γ acting properly by affine isometries on the d -dimensional Euclidean space \mathbb{E}^d is *crystallographic* if it has a bounded fundamental domain. Crystallographic groups are considered up to equivalence under conjugation by affine bijections. A crystallographic group is a *Bravais group* if it is the full group of all affine isometries of an Euclidean lattice.

The simplest crystallographic group is \mathbb{Z}^d (acting by translations). The group \mathbb{Z}^d is not a Bravais group.

The simplest non-commutative example is the simplest Bravais group $\{\pm 1\} \ltimes \mathbb{Z}^d$ with ± 1 acting by $x \mapsto \pm x$ on the Euclidean space (and with -1 conjugating a translation to its inverse). It is the full group of affine isometries of a generic d -dimensional Euclidean lattice having a trivial automorphism group reduced to ± 1 . It consists of all translations by elements of \mathbb{Z}^d and of all involutions $x \mapsto z - x$, $z \in \mathbb{Z}^d$ with fix-points in the super-lattice $\frac{1}{2}\mathbb{Z}^d$ containing \mathbb{Z}^d with index 2^d .

Proposition 12.7. *The group $\langle \mathcal{R} \rangle$ generated by a finite set \mathcal{R} of real reflections of \mathbb{R} is isomorphic to the simplest Bravais group $\{\pm 1\} \ltimes \mathbb{Z}^d$ where d is the dimension of the \mathbb{Q} -vector-space generated by all translations (given by $\sigma \circ \rho$ for $\sigma, \rho \in \mathcal{R}$) of $\langle \mathcal{R} \rangle$.*

Proof. Products of an even number of generators in \mathcal{R} generate the normal commutative subgroup Γ^e (of index 2) consisting of all translations in $\Gamma = \langle \mathcal{R} \rangle$. The torsion-free subgroup Γ^e is thus isomorphic to \mathbb{Z}^d where d is the dimension of the \mathbb{Q} -vector space spanned by all translations defined by elements in Γ^e . (The group Γ acts of course by affine bijections on the underlying \mathbb{Q} -affine space.)

Up to conjugation of Γ by the translation of \mathbb{R} which sends the fix-point of ρ_0 to the origin, we can assume that the first element ρ_0 of \mathcal{R} is given by $x \mapsto -x$. This element acts on Γ^e by conjugating each element of Γ^e to its inverse.

Finally, the reflection $x \mapsto t - x$ with fix-point $t/2$ for $t \in \mathbb{Z}^d$ is the composition of the reflection $x \mapsto -x$ followed by the translation $x \mapsto x + t$. \square

Given a group $\Gamma = \langle \mathcal{G} \rangle$ generated by a finite symmetric set $\mathcal{G} = \mathcal{G}^{-1}$ containing all inverses of its elements, the group Γ indexes the set of vertices of its *Cayley graph* (with respect to the generating set \mathcal{G}) with edges given by $\{r, gr\}$ for $(g, r) \in \mathcal{G} \times \Gamma$. Observe that edges of Cayley graphs are coloured by $g^{\pm 1}$ for g in \mathcal{G} .

The structure of Cayley graphs is compatible with left-cosets giving rise to Schreier graphs: The *Schreier graph* with respect to a subgroup H of Γ has vertices given by left cosets rH for r in Γ and edges given by $\{rH, grH\}$ for g in \mathcal{G} . This turns sets with actions of Γ into graphs: connected components are orbits and correspond to Schreier graphs with respect to stabilisers of base-points chosen in each orbit.

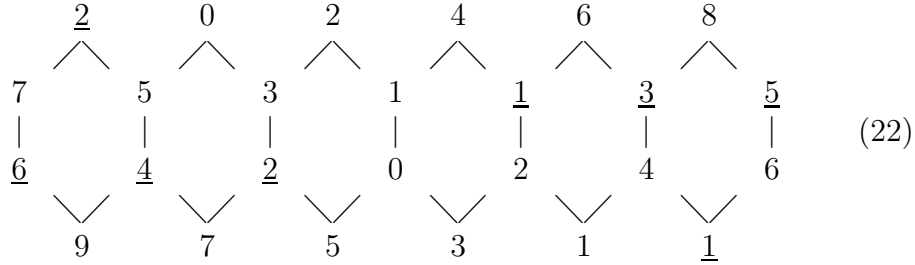
We apply this to groups $\Gamma = \langle \alpha, \beta, \gamma \rangle$ generated by three reflections $\alpha(x) = a - x$, $\beta(x) = b - x$, $\gamma(x) = c - x$ where $a, b, c \in \mathbb{Z}$ are three distinct integers. Γ is always the infinite dihedral group with translation-subgroup $\gcd(b - a, c - b)\mathbb{Z}$. Orbits of its obvious action (by affine isometries) on \mathbb{Z} are either isomorphic to its Cayley graph or are isomorphic to Schreier graphs defined by two-element subgroups fixing some integer⁴.

More precisely, since compositions of all three generators define reflections, the Cayley graph (which is a bipartite 3-regular graph with edges of three ‘colours’ corresponding to the sum in $\{a, b, c\}$ of their two endpoints) is a hexagonal tiling of a cylinder $\mathbb{S}^1 \times \mathbb{R}$. Orbits of its obvious action on \mathbb{Z} (with vertices \mathbb{Z} and edges $\{x, y\}$ for $x + y \in \{a, b, c\}$) are either hexagonal tilings of cylinders (for orbits with free action) or of half-cylinders (for orbits with stabilisers), up to neglecting a finite set of points near fix-points of generators forming a somewhat messy ‘cap’. The exact nature of the tiling

⁴For graphical representations of such orbits the reader can look at pictures of single-walled carbon nanotubes.

(and of the extremal cap in the half-cylinder case) depends on arithmetic properties of $b - a, c - b$ and the orbit.

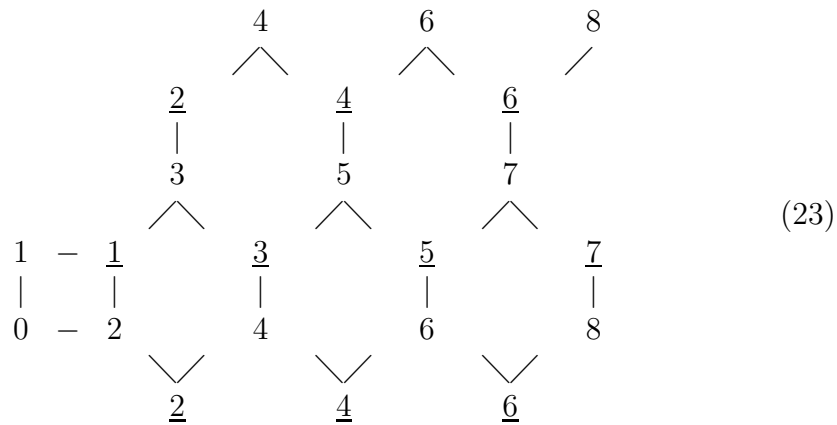
The example $\{a, b, c\} = \{1, 3, 5\}$ leads to a free transitive action on \mathbb{Z} (which can hence be identified with the Cayley graph of Γ). The tiled cylinder is given by



with \underline{x} representing $-x$. Endpoints of vertical edges sum up to 1 (corresponding to the generator $x \mapsto 1 - x$). The corresponding sum for edges \searrow , respectively \swarrow , is 3, respectively 5. Edges with identical endpoints are of course identified in the representation given above.

Remark 12.8. *Identifying even integers with even translations, the group-law on Γ is given by $x \cdot y = x + (-1)^x y$ for $x, y \in \mathbb{Z}$ (this works if and only if a, b, c are all odd and $\gcd(b - a, c - b) = 2$, or equivalently, if the action of Γ on \mathbb{Z} is simply transitive). Observe that this group-law is compatible with classes modulo $2N$ (leading to finite dihedral groups).*

An example of an orbit defining a Schreier graph is given by $\{a, b, c\} = \{0, 1, 2\}$:



(with conventions as above).

Each generator g in $\{\alpha, \beta, \gamma\}$ of such a group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ defines parallel lanes given by bi-infinite sequences $\dots, H_{-1}, H_0, H_1, \dots$ of hexagons in the

Cayley graph of Γ with H_i, H_{i+1} sharing a common edge coloured by the generator g (i.e. of the form $\{r, gr\}$). The three generators of Γ correspond thus to three directions (or types) or parallel lanes. A lane is of colour g if its interior edges (separating adjacent hexagons) are coloured by the generator g in $\{\alpha, \beta, \gamma\}$. A lane is *embedded* if there are no identifications along its boundaries, i.e. if it injects into the Cayley graph. A lane is embedded if and only if it misses some edges (of its proper colour) in the full Cayley graph of Γ . Embedded lanes have parallel lanes of the same direction. Non-embedded lanes of a given colour (direction) are unique if they exist.

All lanes are infinite (otherwise we get a non-trivial translation of finite order defined by the composition of the two reflections associated to its boundary-edges). This implies that there is always a colour corresponding to embedded lanes. Indeed, all three directions of lanes go off to infinity and lanes with fastest escape-rates are embedded since they cover a strictly smaller proportion of the total area than lanes of other colours (which wind more around the tiled cylinder).

In example (22), the two parallel lanes defined by $x \mapsto 3 - x$ (corresponding to edges \setminus) are embedded. Both remaining generators define a unique lane which is not embedded.

The notion of a lane makes sense for Schreier graphs defining hexagonal tilings of half-cylinders by considering infinite sequences H_0, H_1, H_2, \dots of distinct hexagons moving only in one direction. Embeddedness is defined by neglecting the messy end capping off the tiled half-cylinder. Example (23) gives rise to two embedded parallel lanes associated to the generator $x \mapsto 1 - x$ (corresponding to vertical edges) and to (unique) non-embedded lanes associated to each of the remaining generators.

12.5 Spin models on lanes

Recall that a lane is isomorphic to an infinite sequence of consecutively adjacent hexagons with centres on a straight line. The following picture shows the i -th hexagon H_i of an upgoing vertical lane.

$$\begin{array}{ccccc}
 & & l_{i+1} & - & r_{i+1} \\
 & / & & & \backslash \\
 l'_i & & & H_i & & r'_i \\
 & \backslash & & & / \\
 & & l_i & - & r_i
 \end{array} \tag{24}$$

A *finite (embedded) lane* is a connected finite graph defined by a finite number of consecutively adjacent hexagons in a (embedded) lane. Finite lanes of

vertical direction are obtained by vertically stacking (24) in the obvious way. We use the notations of (24): Left and right boundaries form paths with vertices $\dots, l_i, l'_i, l_{i+1}, l'_{i+1}, \dots$, respectively $\dots, r_i, r'_i, r_{i+1}, r'_{i+1}, \dots$.

We consider the constrained spin model with spins in $\{1, 2, 3\}$ on vertices of finite embedded lanes such that $x_i + x_j \geq 3$ for spins x_i, x_j on adjacent vertices. We have the following result:

Proposition 12.9. *The partition function of a finite embedded lane containing n hexagons is given by the polynomial*

$$L_n = (q^2 + q^3)^{2n} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} T^n \begin{pmatrix} q(q^2 + q^3) \\ (q^2 + q^3)q \\ (q^2 + q^3)(q^2 + q^3) \end{pmatrix} \quad (25)$$

where

$$T = \begin{pmatrix} q(q + q^2 + q^3) & q(q^2 + q^3) & q(q + q^2 + q^3) \\ q(q^2 + q^3) & q(q + q^2 + q^3) & q(q + q^2 + q^3) \\ (q^2 + q^3)(q + q^2 + q^3) & (q^2 + q^3)(q + q^2 + q^3) & (q + q^2 + q^3)^2 \end{pmatrix} \quad (26)$$

Evaluations at $q = \omega^{-1}$ of the polynomials L_n decay exponentially fast to 0.

Proof. We use transfer-matrices with respect to bases given by given by

$$\begin{array}{c|c|c} & l_i & r_i \\ \hline b_1 & q & q^2 + q^3 \\ b_2 & q^2 + q^3 & q \\ b_3 & q^2 + q^3 & q^2 + q^3 \end{array}$$

encoding all possibilities for spins at two horizontally adjacent vertices l_i, r_i belonging to the left and right boundary path of a lane, see (24).

The reader should convince himself that $(q^2 + q^3)^2 T$ is the transfer matrix with respect to the basis b_1, b_2, b_3 . Denoting by $1, \geq 2$, respectively $\geq 2, 1$ and $\geq 2, \geq 2$ possibilities of spins at l_i, r_i for b_1, b_2, b_3 , the transfer matrix can be recovered from the following table with columns indexed by spins at l_i, r_i , rows indexed by spins at l_{i+1}, r_{i+1} and entries giving by all possibilities (using the notation $*$ for arbitrary spins in $\{1, 2, 3\}$ and ≥ 2 for spins in $\{2, 3\}$) for spins at the intermediary vertices l'_i, r'_i :

$$\begin{array}{c|ccc} & 1, \geq 2 & \geq 2, 1 & \geq 2, \geq 2 \\ \hline 1, \geq 2 & \geq 2, * & \geq 2, \geq 2 & \geq 2, * \\ \geq 2, 1 & \geq 2, \geq 2 & *, \geq 2 & *, \geq 2 \\ \geq 2, \geq 2 & \geq 2, * & *, \geq 2 & *, * \end{array} \quad .$$

Initial conditions are encoded by the final column-vector of (25). The initial row vector sums over all possible states for the two final vertices.

Exponentially fast decay to 0 of L_n at $q = \omega^{-1}$ follows from the three eigenvalues (given approximately by 0.0916, 0.1459 and 0.9297) smaller than 1 of the evaluation at $q = \omega^{-1}$ of $(q^2 + q^3)^2 T$. \square

We set

$$M_n = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & q \\ q^2 + q^3 & q^2 + q^3 \end{pmatrix}^{2n} \begin{pmatrix} q \\ q^2 + q^3 \end{pmatrix} \quad (27)$$

and

$$S_n = \frac{L_n(1 + M_n)}{1 - L_n}. \quad (28)$$

Proposition 12.10. *Up to contributions of boundary vertices, the series S_n defined by formula (28) gives upper bounds for coefficients of the partition function (of the constrained spin model with spins in $\{1, 2, 3\}$) for finite hexagonal tilings of bounded cylinders tiled by a finite number of parallel embedded lanes consisting of n hexagons.*

Evaluations at $q = \omega^{-1}$ of S_n decay exponentially fast to 0.

Proof. We consider a finite graph Γ on a hexagonally tiled cylinder which consists of k embedded parallel finite lanes all consisting of n hexagons. Since the family of k lanes consists of embedded lanes we have $k \geq 2$. Up to a few boundary vertices, vertices of Γ are contained in the union of $k/2$ embedded disjoint lanes if k is even and they are contained in the union of $(k-1)/2 \geq 1$ disjoint embedded lanes and of a simple path (parallel to the boundaries of the previous lanes) of length $2n$ if k is odd. The fraction $\frac{L_n}{1-L_n}$ gives upper bounds for the contribution coming from the $\lfloor k/2 \rfloor$ disjoint embedded lanes (consisting of n hexagons). The factor $(1 + M_n)$ accounts for the possible presence (for k odd) of a simple path of length $2n$.

The evaluation at $q = \omega^{-1}$ of the transfer matrix involved in the definition of M_n given by formula (27) has eigenvalues 1 and $-\omega^{-2}$. Evaluations of M_n at $q = \omega^{-1}$ define thus a bounded sequence.

Since L_n decays exponentially fast to 0, the final part of Proposition 12.10 holds. \square

Remark 12.11. *The transfer-matrix $\begin{pmatrix} 0 & q \\ q^2 + q^3 & q^2 + q^3 \end{pmatrix}$ involved in the definition of M_n has eigenvalue 1 at $q = \omega^{-1}$. This is of course the reason for working with more complicated graphs (given by hexagonal tilings and associated to three maximal parts) when dealing with NSG-compositions of maximum 4.*

Remark 12.12. *The spin models considered in this section can be considered as a variation of the hard-square model (with spins in $\{0, 1\}$ on vertices of the square lattice such that vertices with maximal spin 1 are isolated). We work essentially with spins in $\{1, 2, 3\}$ on vertices of quotients of the 3-regular graph defined by a hexagonal tiling such that vertices with minimal spin 1 are isolated.*

12.6 The ϵ -condition

Given $\epsilon > 0$, a NSG-composition $x_1 + \dots + x_{m-1}$ of maximum 4 satisfies the ϵ -condition if it has at least three maximal parts and if we have $l_3 - l_1 \leq \epsilon(m - 1 - l_1)$ for the indices $l_1 < l_2 < l_3$ of the last three maximal parts.

Proposition 12.13. *There exists $\epsilon > 0$ such that NSG-compositions of maximum 4 satisfying the ϵ -condition have growth-rate strictly smaller than ω .*

Proof. We consider a NSG-composition $x_1 + \dots + x_{l_1} + \dots + x_{m-1}$ satisfying the ϵ -condition (for some small enough positive ϵ) with $x_{l_1} = x_{l_2} = x_{l_3} = 4$ for $l_1 < l_2 < l_3$ the last three maximal parts, as in the definition of the ϵ -condition. We use pivot-factorisation with respect to the third-last maximal part x_{l_1} .

Proposition 12.4 shows that associated left compositions have growth-rate strictly smaller than ω .

In order to study right compositions, we consider the graph Γ with $m - 1 - l_1$ vertices labelled $l_1 + 1, \dots, m - 1$ and edges $\{i, j\}$ if $i + j - m \in \{l_1, l_2, l_3\}$. More precisely, the graph Γ is the restriction to $\{l_1 + 1, \dots, m - 1\}$ of the orbit-graph underlying the group generated by the three real reflections $x \mapsto m + l_i - x$ with half-integral fix-points $(m + l_i)/2$ in $\frac{1}{2}\mathbb{N}$. Removing from Γ the central set $\{[(l_1 + m)/2], \dots, [(l_3 + m)/2]\}$ (called the set of *central vertices*) containing all fix-points, the graph Γ is a finite union of hexagonal tilings on cylinders (up to neglecting a few incomplete hexagons at boundaries), see Section 12.4. These hexagonal tilings contain parallel lanes of proper type consisting of a roughly equal number of at least $(m - l_1)/(2(l_3 - l_1)) - c$ hexagons for some small constant c (choosing $c = 10$ certainly works). For ϵ small enough, we have thus a lower bound of $\frac{1}{4\epsilon}$ for the number of hexagons of all lanes.

Proposition 12.10 shows thus that the contribution of most vertices of Γ (except central vertices and a few vertices near the other boundaries of these finite cylinder-tilings) can be bounded by $\sum_{n=\lfloor 1/(4\epsilon) \rfloor}^{\infty} \frac{S_n}{1 - S_n}$ (for S_n given by (28)) which has growth-rate strictly smaller than ω by Proposition 12.10 and Lemma 8.2.

The contribution of the remaining $a = O(l_3 - l_1)$ vertices can be bounded trivially by $(q + q^2 + q^3)^a q^2$ (with the factor q^2 accounting for the values $x_{l_2} = x_{l_3} = 4 \notin \{1, 2, 3\}$). Such vertices contribute thus at most a proportion of ϵ to the (degree of) the total contribution of such a NSG-composition. We can thus apply the arguments used for the series given by (21) during the proof of Proposition 12.5. \square

12.7 Proof of Proposition 12.1

The main ingredient for proving Proposition 12.1 is the following result:

Proposition 12.14. *There exists a natural integer A such that NSG-compositions with at least A parts of maximal size 4 have growth-rate strictly smaller than ω .*

Proof. We δ such that Proposition 12.5 holds. We chose ϵ such that Proposition 12.13 holds for $\epsilon' = \frac{\epsilon}{1-\epsilon}$. We choose now a natural integer N such that $(1 - \epsilon)^N < \delta$. We are going to prove that $A = 2N + 1$ works.

Let $\mathbf{x} = x_1 + \dots + x_{m-1}$ be a NSG-composition having at least $A = 2N + 1$ maximal parts of size 4. For $i = 0, \dots, N - 1$ we denote by I_i the set of all integers of the real interval

$$[[m - 1 - (1 - \epsilon)^i(m - 1)], \lfloor m - 1 - (1 - \epsilon)^{i+1}(m - 1) \rfloor] .$$

We set $I_N = [[m - 1 - (1 - \epsilon)^N(m - 1)], m - 1] \cap \mathbb{N}$. We have of course $\cup_{i=0}^N I_i = \{1, \dots, m - 1\}$. We denote by \mathcal{L} the set of the $A = 2N + 1$ largest elements in the set $\{i \mid 1 \leq i \leq m - 1, x_i = 4\}$ indexing maximal parts of \mathbf{x} . Proposition 12.5 and the definition of I_N show that the set of such NSG-compositions with \mathcal{L} intersecting I_N has growth-rate smaller than ω .

We can thus assume that \mathcal{L} does not intersect I_N . The pigeon-hole principle implies thus that there exists a set I_j among the N sets I_0, \dots, I_{N-1} containing at least three elements of \mathcal{L} . Let $l_1 < l_2 < l_3$ be the three largest elements of $\mathcal{L} \cap I_j$. We denote by $a = \#\{i \in \mathcal{L} \mid i > l_3\} \leq A - 3$ the number of elements of \mathcal{L} which are larger than l_3 . Adjoining a times the Frobenius element to (the numerical semigroup associated to) \mathbf{x} , we get a NSG-composition $\underline{\mathbf{x}}$ of genus $g - a$ with parts $\underline{x}_i = x_i$ if $i \leq l_3$ and $\underline{x}_i = \min(3, x_i)$ if $i > l_3$. Since three last maximal parts $\underline{x}_{l_1} = \underline{x}_{l_2} = \underline{x}_{l_3} = 4$ of $\underline{\mathbf{x}}$ have indices l_1, l_2, l_3 in I_j we get

$$l_3 - l_1 \leq \frac{(1 - (1 - \epsilon))}{1 - \epsilon} (1 - \epsilon)^{i+1} (m - 1) \leq \frac{\epsilon}{1 - \epsilon} (m - 1 - l_1)$$

which implies that \underline{x} satisfies the ϵ' -condition for $\epsilon' = \frac{\epsilon}{1-\epsilon}$. The generating series of such NSG-compositions is bounded by the series

$$\sum_{k=0}^{A-3} q^{2k} \frac{d^k}{dq^k} G_{\epsilon'}(q)$$

with $G_{\epsilon'}$ denoting the generating series for all NSG-compositions of maximum 4 satisfying the ϵ' -condition. Since $G_{\epsilon'}$ has growth-rate strictly smaller than ω for ϵ' small enough (see Proposition 12.13) and since algebraic differential operators do not increase the growth-rate we get the result. \square

Proof of Proposition 12.1. Combine Propositions 12.14 and 12.2. \square

13 NSG-compositions with maximum 3

Proposition 13.1. *The generating series for the number of all NSG-compositions with maximum 3 is given by*

$$\frac{\sum_{j=3}^{\infty} \tilde{c}_j q^j}{1 - q - q^2}$$

where $\tilde{C} = \sum_{g=3}^{\infty} \tilde{c}_g q^g$ is the generating series of all NSG-compositions with maximum 3 ending with a maximal part.

Remark 13.2. *The number \tilde{c}_g is the number of numerical semigroups S of genus $g = \sharp(\mathbb{N} \setminus S)$ with Frobenius number $f = 3m - 1$.*

Proof of Proposition 13.1. All NSG-inequalities $x_{m-s-t} \leq x_{m-s} + x_{m-t} + 1$ (with $1 \leq s, t \leq s + t < m$) of the second line in (6) are satisfied for compositions with maximum at most 3.

Any NSG-composition $x_1 + \dots + x_{m-1}$ of maximum 3 has thus a unique pivot-factorisation

$$(x_1 + \dots + x_{l-1} + 3) + (x_{l+1} + \dots + x_{m-1})$$

with pivot-part the last occurrence $x_l = 3$ of a maximal part into a left composition defining a NSG-composition with parts in $\{1, 2, 3\}$ ending with the pivot part $x_l = 3$ (defined as the last maximal part) and a right composition defining a NSG-composition with all parts in $\{1, 2\}$. This decomposition is bijective: Concatenating a NSG-composition ending with a maximal part 3 with a composition having all parts in $\{1, 2\}$ yields an NSG-composition of maximum 3.

The result follows since the numerator accounts for the initial NSG-composition ending with 3. Possibilities for the final composition with all parts in $\{1, 2\}$ are enumerated by the denominator, see Proposition 3.1. \square

13.1 Convergencency of \tilde{C} at ω^{-1}

Theorem 13.3. *The generating series $\tilde{C} = \sum_{j=3}^{\infty} \tilde{c}_j q^j$ for NSG-compositions ending with a maximal part of size 3 has growth-rate strictly smaller than ω .*

Proof. Given $\epsilon > 0$, a NSG-composition $x_1 + \dots + x_{m-1}$ ending with a maximal part $x_{m-1} = 3$ satisfies the ϵ -condition if it has at least three maximal parts and if we have $(m - k) < \epsilon m$ for the third last maximal part $x_k = 3$.

A slight modification of the proof of Proposition 12.13 shows that NSG-compositions satisfying the ϵ -condition have growth-rate smaller than ω for ϵ small enough. We fix ϵ to a strictly positive value which is small enough.

We consider now NSG-compositions ending with a maximal part $x_{m-1} = 3$ which do not satisfy the ϵ -condition (for our fixed value of ϵ). Adding to (the numerical semigroup of) such a composition the second largest gap if it corresponds to an index l at least equal to $(1 - \epsilon)m$, we are reduced to consider NSG-compositions $x_1 + \dots + x_{m-1}$ of maximum 3 ending with a maximal part such that we have $k < (1 - \epsilon)m$ for the (perhaps non-existing) index k of the second last maximal part.

We consider the graph Γ with vertices $1, \dots, m - 2$ and edges $\{i, j\}$ if the sum $i + j$ belongs to $\{k, m - 1\}$ (with k missing if x_{m-1} is the unique maximal part of $x_1 + \dots + x_{m-1}$). Connected components are paths of lengths at most equal to $2/\epsilon + 1$ with endpoints in $k, k + 1, \dots, m - 2$, except for (perhaps existing) exceptional components ending at an element of $\{k/2, (m - 1)/2\} \cap \mathbb{N}$. The associated constrained Ising model has spins in $\{1, 2, 3\}$ except for endpoints $k + 1, \dots, m - 2$ corresponding to spins in $\{1, 2\}$. (If such a NSG-composition has a unique part of maximal size 3, these paths are simply of length 1, given by edges $\{m - i, i\}$, together with the (not necessarily existing) isolated vertex $(m - 1)/2$.) The generating function for such paths of length n is given by

$$\tilde{P}_n = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & q \\ q^2 + q^3 & q^2 + q^3 \end{pmatrix}^n \begin{pmatrix} q \\ q^2 \end{pmatrix}$$

and we have $\tilde{P}_n(\omega^{-1}) \leq \tilde{P}_2(\omega^{-1}) = \frac{13-5\sqrt{5}}{2} < 1$ for all $n \geq 1$. Observe now that the number of NSG-compositions under consideration is bounded by the coefficients of the rational function

$$\left(1 + q^2 \frac{d}{dq}\right) \left((1 + q^2 + q^3)(1 + q) \left(\sum_{j=1}^{\lceil 2/\epsilon + 1 \rceil} \frac{1}{1 - \tilde{P}_j} \right)^4 q^3 \right)$$

with convergency radius strictly larger than ω^{-1} . (The differential operator accounts for a perhaps suppressed second-last maximal part, the factor $(1 +$

$q^2 + q^3$) takes into account a perhaps existing isolated vertex $(m - 1)/2$ of Γ , the factor $(1 + q)$ corrects for a (perhaps non-existing) path with initial vertex starting at the second-last maximal part with index at least $(1 - \epsilon)m$, the fourth power accounts for all possible paths of length at least 1 (there are at most four different possible lengths, all bounded by $2/\epsilon + 1$), the final factor q^3 corresponds of course to the part $x_{m-1} = 3$. \square

Remark 13.4. *The upper bound $2/\epsilon + 1$ on lengths of paths in Γ is crucial in the previous proof: The evaluations $\tilde{P}_n(\omega^{-1})$ tend to a strictly positive limit for $n \rightarrow \infty$ (the corresponding transfer-matrix has an eigenvalue 1 at $q = \omega^{-1}$).*

14 Proof of Theorem 1.2

Proposition 14.1. *The generating series for all NSG-compositions with maximum at most 3 is given by*

$$\frac{1 + \tilde{C}}{1 - q - q^2}$$

where \tilde{C} is the generating series for all NSG-compositions ending with a maximal part of size 3.

Proof. Follows from Proposition 3.1 and Proposition 13.1. \square

Corollary 14.2. *We have*

$$\lim_{g \rightarrow \infty} \frac{n_{\leq 3}(g)}{\left(\frac{1+\sqrt{5}}{2}\right)^g} = \frac{5 + \sqrt{5}}{2} \left(1 + \tilde{C}((\sqrt{5} - 1)/2)\right)$$

for the number $n_{\leq 3}(g)$ of NSG-compositions with genus g and maximum at most 3.

Proof of Corollary 14.2. The algebraic identity

$$\frac{1}{1 - q - q^2} = \frac{5 + \sqrt{5}}{10} \frac{1}{1 - \frac{1+\sqrt{5}}{2}q} + \frac{5 - \sqrt{5}}{10} \frac{1}{1 - \frac{1-\sqrt{5}}{2}q}$$

and Theorem 13.3 show that

$$\frac{1 + \tilde{C}(q)}{1 - q - q^2} = \frac{5 + \sqrt{5}}{10} \frac{1 + \tilde{C}(\omega^{-1})}{1 - \omega q}$$

is holomorphic in a open disc of radius strictly larger than ω^{-1} . We have thus

$$n_{\leq 3}(g) = \frac{5 + \sqrt{5}}{10} \left(1 + \tilde{C}(\omega^{-1})\right) \omega^g (1 + o(1/g))$$

for coefficients $n_{\geq 3}(g)$ of $(1 + \tilde{C}(q))/(1 - q - q^2)$ enumerating NSG-partitions of genus g with parts in $\{1, 2, 3\}$. \square

Proof of Theorem 1.2. Corollary 14.2 implies that it is enough to show that NSG-compositions of maximum ≥ 4 have growth-rate strictly smaller than ω . Theorem 13.3 and Proposition 12.1 ensure a growth-rate strictly smaller than ω for NSG-compositions with maximum 4. By Proposition 11.1 we get then the result for NSG-compositions with maximum 5 and Proposition 10.1 completes the proof. \square

15 NSG-compositions ending with a maximal part of size 3

A good understanding of NSG-compositions ending with a maximal part of size 3 associated to the generating series \tilde{C} is desirable in view of Theorem 1.2 and Proposition 13.1. This Section has four distinct parts:

In a first part we prove the following result:

Theorem 15.1. *The coefficient \tilde{c}_g of the generating series \tilde{C} is at most equal to the coefficient α_g of the rational series $A = \sum_{g=3}^{\infty} \alpha_g q^g$ defined by*

$$A = \frac{1 + q^2 + q^3}{1 - (q^2 + q^3)(q + q^2 + q^3)} q^3. \quad (29)$$

In particular, the convergency radius of \tilde{C} is at most equal to the convergency radius $\rho_A = 0.659982\dots$ of A given by the positive real root ρ of $1 - (q^2 + q^3)(q + q^2 + q^3)$.

In a second part we list a few initial coefficients of \tilde{C} .

In a third part we give generating sequences for all contributions to \tilde{C} with only one or two maximal parts (of size 3).

The fourth part is speculative and non-rigorous: We believe that the upper bound ρ_A given by Theorem 15.1 on the convergency radius on \tilde{C} is sharp and we describe (in a non-rigorous way) a family of NSG-contributions ending with a maximal part 3 which should give the bulk of contributions to \tilde{c}_g for $g \rightarrow \infty$.

15.1 Proof of Theorem 15.1

Proof. We consider the set of all compositions $x_1 + \dots + x_{m-1}$ ending with a maximal part $x_{m-1} = 3$, with $x_1, x_2, \dots, x_{\lfloor (m-1)/2 \rfloor}$ in $\{2, 3\}$ and with $x_{\lfloor (m-1)/2 \rfloor + 1}, \dots, x_{m-2}$ in $\{1, 2, 3\}$. If i and j are two strictly positive integers such that $i + j \leq m - 1$, then $\min(i, j) \leq \lfloor (m - 1)/2 \rfloor$ and we have thus $x_i + x_j \geq 2 + 1 = 3$ showing that all such compositions are NSG-compositions ending with a maximal part 3. Regrouping parts x_i, x_{m-1-i} and taking into account the central part $x_{(m-1)/2}$ (which exists only for odd m) we get the generating series

$$A = \frac{1 + (q^2 + q^3)}{1 - (q^2 + q^3)(q + q^2 + q^3)} q^3$$

for the set of all such NSG-compositions. This proves the inequality $\alpha_g \leq \tilde{c}_g$.

The convergency radius of A is given by the smallest pole at the positive real root ρ_A of the denominator of the rational series A . Since all real positive coefficients of \tilde{C} are bounded below by the corresponding coefficients of A , the convergency radius of \tilde{C} is at most equal to the convergency radius ρ_A of A . \square

15.2 A few initial coefficients for \tilde{C}

The following tables list a few numbers \tilde{c}_g counting NSG-compositions of genus g ending with a maximal part of size 3. The first table contains also the corresponding compositions, encoded as words in $\{1, 2, 3\}^*$ (with omitted addition-signs):

g	\tilde{c}_g	compositions
3	1	3
4	0	
5	1	23
6	3	33, 123, 213
7	2	223, 313
8	4	233, 323, 1223, 2213
9	9	333, 1233, 2223, 2313, 3213, 11223, 12123, 21213, 22113
10	12	2233, 2323, 3223, 3313, 12223, 21223, 21313, 22123, 22213, 23113, 31213, 32113

Coefficients $\tilde{c}_1, \dots, \tilde{c}_{50}$ of \tilde{C} are given by

1 – 10	0	0	1	0	1	3	2	4	9	12
11 – 20	20	32	50	84	132	208	331	526	841	1333

21 – 25	2145	3401	5314	8396	13279
26 – 30	20952	33029	51927	81527	128102
31 – 35	201700	317461	498911	782868	1226255
36 – 40	1919070	3000905	4687213	7315975	11419861
41 – 45	17833383	27857264	43511423	67908811	105857661
46 – 50	164837336	256493732	398937594	620308837	964299016

A sequence starting with initial coefficients of \tilde{C} does presently not appear in [3].

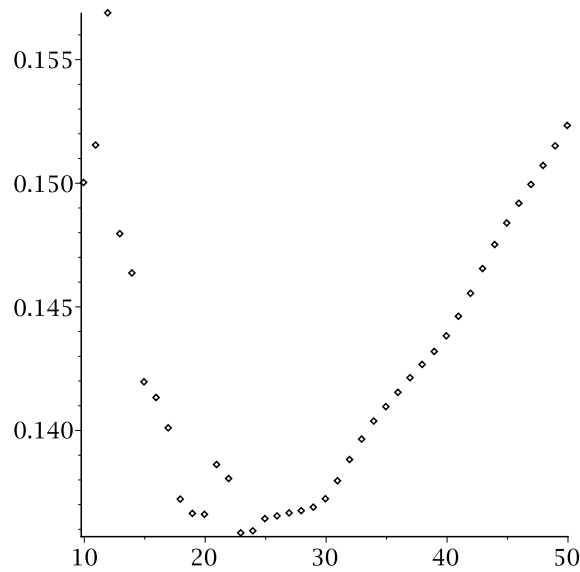


Figure 2: Quotients $\tilde{c}_i/(ia_i)$ for $i = 10, \dots, 50$.

These coefficients were computed using the algorithm outlined in Section 16.2. Figure 2 displays some quotients $\tilde{c}_i/(ia_i)$ for $\sum_{n=3}^{\infty} a_n q^n$ the rational series defined by Formula (29) of Theorem 15.1.

Initial coefficients for \tilde{C} suggest the inequalities

$$3.57 < C < 3.93$$

for the constant C occurring in Formula (2) of Theorem 1.2. The left inequality is proven since it was obtained by considering a large subset of NSG-compositions contributing to \tilde{C} . The right side is somewhat conjectural: It was obtained by assuming $\tilde{c}_i/\tilde{c}_{i-1} < \tilde{c}_{50}/\tilde{c}_{49}$ and by replacing the missing coefficients \tilde{c}_i for $i > 50$ with $\tilde{c}_{50} \left(\frac{\tilde{c}_{50}}{\tilde{c}_{49}}\right)^{i-50}$.

15.3 Contributions to \tilde{C} with k maximal parts

We denote by $\tilde{c}_{g,k}$ the number of NSG-compositions of genus ending with a maximal part of size 3 and having a total number of k maximal parts.

Setting

$$\tau(k) = \sum_g \tilde{c}_{g,k} \omega^{-g} \quad (30)$$

we have $\tilde{C}(\omega^{-1}) = \sum_{k=1}^{\infty} \tau(k)$ and Theorem 1.2 implies immediately the following result:

Proposition 15.2. *The proportion of NSG-compositions with k maximal parts of size 3 is asymptotically given by*

$$\frac{\tau(k)}{1 + \sum_{k=1}^{\infty} \tau(k)} .$$

We describe in this section the easy value of $\tau(1)$ and we give a useful formula for the already complicated case of $\tau(2)$.

Proposition 15.3. *We have*

$$\sum_{g=3}^{\infty} \tilde{c}_{g,1} q^g = \frac{1 + q^2}{1 - (2q^3 + q^4)} q^3 . \quad (31)$$

We get in particular the evaluation

$$\tau(1) = \frac{1}{\omega} + \frac{1}{\omega^3} = 0.85410196624968454461376 \dots .$$

Proof. We consider graphs with vertices $1, \dots, m-2$ and edges $i, m-1-i$. Connected components are isolated edges and perhaps an isolated vertex $(m-1)/2$. We consider the spin model with spins 1 or 2 summing up at least to 3 along edges. The isolated vertex $(m-1)/2$ (which exists only for odd m) is required to have spin 2. The corresponding partition function, corrected by the final factor q^3 in order to account for the last maximal part $x_{m-1} = 3$ is easily seen to be given by (31). \square

Proposition 15.4. *The generating series $\sum_{g=6}^{\infty} \tilde{c}_{g,2} q^g$ counting all NSG-compositions ending with a last maximal part of size 3 and having a unique*

additional maximal part is given by

$$\begin{aligned}
& \frac{1 + q^3 + q^4}{1 - (3q^6 + 4q^7 + q^8)} q^3 \left(1 + (q + q^2) \frac{1 + q^2}{1 - (2q^3 + q^4)} \right) q^3 \\
& + q^4 \sum_{n=1}^{\infty} P_{n,0,0} \\
& + q^4 \sum_{n=2}^{\infty} \frac{P_{2n-1,0,1}(1 + P_{n-1,1,0})}{1 - P_{2n-1,1,1}} \\
& + q^4 \sum_{n=3}^{\infty} \left(\frac{P_{2n-1,1,1} + P_{n-1,1,0}}{1 - P_{2n-1,1,1}} \right) P_{n-2,0,0} \\
& + q^4 \sum_{n=3}^{\infty} \left(\frac{P_{2n-1,1,1} + P_{n-1,1,0}}{1 - P_{2n-1,1,1}} \right) \frac{P_{2n-3,0,1}(1 + P_{n-2,1,0})}{1 - P_{2n-3,1,1}}
\end{aligned}$$

where

$$P_{n,\epsilon_\alpha,\epsilon_\omega} = \begin{pmatrix} \epsilon_\omega & 1 \end{pmatrix} \begin{pmatrix} 0 & q \\ q^2 & q^2 \end{pmatrix}^n \begin{pmatrix} \epsilon_\alpha q \\ q^2 \end{pmatrix}.$$

We have in particular

$$\tau(2) = .7628736853796206184361443135239953344793590 \dots$$

Coefficients $\tilde{c}_{g,2}$ for $g = 1, \dots, 20$ are given by

$$0, 0, 0, 0, 0, 1, 1, 2, 3, 7, 10, 11, 25, 38, 43, 75, 123, 153, 233, 383.$$

Sketch of proof for Proposition 15.4. Let $x_1 + \dots + x_k + \dots + x_{m-1}$ be a NSG-composition with $x_k = x_{m-1} = 3$ and all remaining parts in $\{1, 2\}$. We assume first that $2k \leq m - 1$. Removing all 'central' parts $x_{k+1}, \dots, x_{m-2-k}$ following x_k reduces such a NSG-composition to a NSG-composition with $m - 1 = 2k$. The associated graph with vertices $1, \dots, m - 2$ and edges $\{i, j\}$ such that $i + j \in \{k, 2k\}$ gives rise to a spin-model with partition function $(1 + q^3 + q^4)q^6 / (1 - 3q^6 + 4q^7 + q^8)$. Possibly removed central parts account for a factor of $(1 + (q + q^2)(1 + q^2) / (1 - (2q^3 + q^4)))$.

We can now suppose $2k > m - 1$. We consider the associated graph Γ with vertices $1, \dots, m - 2$ and edges $\{i, j\}$ with $i + j \in \{k, m - 1\}$. Connected components of this graph are line-segments of length $2n - 1, 2n - 3$ and perhaps an exceptional line-segment of length $n - 1$ and an exceptional line-segment of length $n - 2$. More precisely, connected components of length $2n - 1$ start at the last vertices of Γ and are 'wrapped around' a (not necessarily existing) exceptional component of length $n - 1$. Components of length

$2n - 3$ wrapped around the (not necessarily existing) exceptional component of length $n - 2$ have the same structure.

Exceptional line-segments end at a vertex of spin 2. The connected component with endpoint k corresponds also to a spin 2 point after diminishing $x_k = 3$ by 1. Endpoints with spins restricted to 2 are accounted for by setting one or both of the parameters $\epsilon_\alpha, \epsilon_\beta$ in the polynomials $P_{n, \epsilon_\alpha, \epsilon_\omega}$ (counting possibilities for spins in $\{1, 2\}$ summing up at least to 3 along edges in a line-graph of length n) to 0. The different combinatorial possibilities correspond to the next four summands. The somewhat lengthy but straightforward details are left to the reader. \square

15.4 Speculations on typical asymptotic contributions to \tilde{C}

We say that a class of NSG-compositions ending with a last maximal part of size 3 is a *asymptotically \tilde{C} -typical* if the proportion of NSG-compositions of the class contributing to \tilde{c}_g tends to 1 for $g \rightarrow \infty$.

The aim of this Section is to describe a class which should be asymptotically \tilde{C} -typical. The class is given by the set of all NSG-compositions close to all NSG-compositions occurring in the proof of Theorem 15.1. Somewhat informally, an asymptotically \tilde{C} -typical NSG-composition $x_1 + \dots + x_{m-1}$ has two large regular chunks: The first chunk starts with x_1 and ends somewhere slightly before $x_{\lfloor (m-1)/2 \rfloor}$. Its parts are independent random variables, equal to 2 with probability $1/(1 + \rho_A)$ (for ρ_A as in Theorem 15.1) and equal to 3 with probability $\rho_A/(1 + \rho_A)$. It contains no parts of size 1. This part is followed by a small transitional region centered around $(m-1)/2$ (containing parts 1 with gradually increasing density) ending at the beginning of the second large chunk with parts given by independent random variables equal to $\alpha \in \{1, 2, 3\}$ with probabilities $\rho_A^{\alpha-1}/(1 + \rho_A + \rho_A^2)$. The very last parts of an asymptotically \tilde{C} -typical NSG-composition form again a transitional region containing parts of size 3 with gradually decreasing density. (The presence of pairs $x_i = x_j = 1$ of parts 1 with indices $i + j < m - 1$ in the central transitional region forces $x_{i+j} \leq 2$.)

An asymptotically \tilde{C} -typical NSG-composition $x_1 + \dots + x_{m-1}$ of large multiplicity m has genus $\frac{1}{2} \left(\frac{2+3\rho_A}{1+\rho_A} + \frac{1+2\rho_A+3\rho_A^2}{1+\rho_A+\rho_A^2} \right) m + O(\sqrt{m})$.

We ignore the typical size of the central and final transitional parts.

16 Generic numerical semigroups

We call NSG-compositions with maximum at most 3 *asymptotically generic* or simply *generic* since they prevail proportionally in large genus.

16.1 A combinatorial model

Generic NSG-compositions (NSG-compositions of maximum at most 3) can be extended by adding a last part 1 or 2. Depending on the configuration of parts equal to 1 an extension by a last part of size 3 is sometimes possible. We encode this by a rooted binary tree with left descendants corresponding to parts 1, right descendants corresponding to parts 2 or (sometimes) 3. We represent this by drawing thin edges for right descendants corresponding only to extensions by a part of size 2 and by drawing fat edges for right descendants corresponding to extensions by an additional last part of size 2 or 3. A finite downward path starting at the root represents 2^f generic NSG-compositions if it contains f fat edges. All contributions to \tilde{C} can be computed as follows. A given fat edge (representing a final part of size 3) joined by f (different) fat edges, l left edges and s slim right edges to the root corresponds to 2^f generic NSG-compositions ending with a last part 3 yielding a total contribution of $q^{3+l+2s+2f}(1+q)^f$ to \tilde{C} .

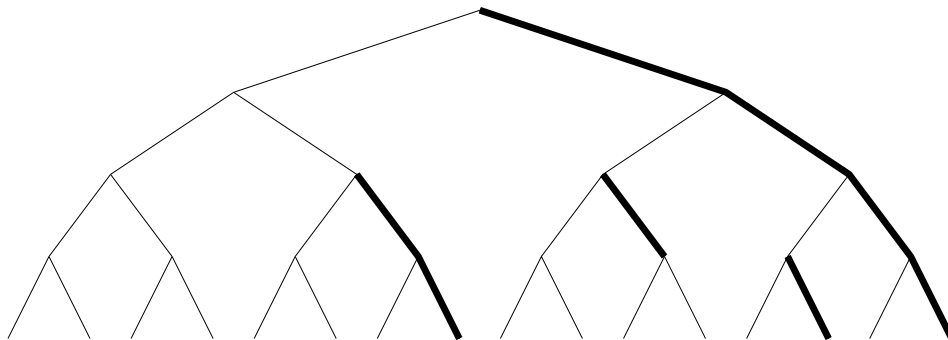


Figure 3: The tree of generic NSG-compositions.

16.2 An algorithm of complexity $3^{2g/3}$ for \tilde{c}_g

The tree of generic NSG-compositions suggests the following elementary algorithm of complexity $3^{2g/3}$ for computing \tilde{c}_g .

Let $x_1 + \dots + x_{m-1}$ be an NSG-composition ending with a maximal part $x_{m-1} = 3$. Since $x_i + x_{m-1-i} \geq 3$, we have

$$g = x_1 + \dots + x_{m-1} \geq 3 + 3(m-2)/2$$

equivalent to the inequality $m \leq 2g/3$.

Computing all initial coefficients of \tilde{C} up to \tilde{c}_g can thus be achieved by considering all NSG-compositions of multiplicity $m \leq 2g/3$ which end with a maximal part of size 3.

This can be achieved as follows: We fix $m \leq 2g/3$. We restrict first x_1, \dots, x_{m-2} to values in $\{1, 2\}$. Since $x_i + x_{m-1-i} \geq 3$, pairs (x_i, x_{m-1-i}) of distinct parts belong to the set $\{(1, 2), (2, 1), (2, 2)\}$. If $m \geq 3$ is odd, we set $x_{(m-1)/2} = 2$. Given such a fixed choice of x_1, \dots, x_{m-2} , we compute the number

$$f = \{k \leq m - 1 \mid x_k = 2 < x_i + x_{k-i}, i = 1, \dots, \lfloor k/2 \rfloor\}$$

of corresponding fat edges (representing parts of size 2 which can be replaced by parts of size 3) in the tree of generic NSG-compositions. Such a choice yields a total contribution of

$$q^{\sum_{i=1}^{m-1} x_i} (1+q)^f$$

to \tilde{C} .

The computation of f is quadratic in g and does thus not increase the exponential complexity which comes from the roughly $3^{g/3}$ possible choices for the pairs (x_i, x_{m-1-i}) .

17 Asymptotic properties for special compositions

The k -th part x_k of a composition $x_1 + \dots$ with very large sum $g = \sum_j x_j$ with respect to the index k , chosen uniformly among all compositions of g has obviously an asymptotic limit-distribution⁵: It is equal to n with asymptotic probability 2^{-n} for $g \rightarrow \infty$.

Such asymptotic limit-distributions for parts exist more generally for partitions with parts satisfying suitable restrictions (satisfying some mild hypotheses) depending at most on the index of parts (the resulting limit-distributions depend then also on the index of the part under consideration). An example is given by compositions $x_1 + \dots$ with a non-zero part x_k in $\{1, \dots, k\}$. Such compositions are enumerated by the sequence A8930 of [3].

From an enumerative point of view, one can ask for generating series of compositions satisfying some restrictions. The answer, given by $1/(1 -$

⁵Disclaimer: Probability theory is a branch of mathematics where I tend to be absolutely sure and completely wrong.

$\sum_{a \in \mathcal{A}} q^a$ is easy for compositions with all parts x_i in a common subset \mathcal{A} of non-zero elements in \mathbb{N} . More generally, for x_i restricted to non-empty subsets \mathcal{A}_i of $\mathbb{N} \setminus \{0\}$, we get

$$\sum_{n=0}^{\infty} \prod_{k=1}^n \left(\sum_{a \in \mathcal{A}_k} q^a \right).$$

Remark 17.1. *It is possible to consider more generalised compositions where parts of equal size can have a finite number of different 'colours' (depending perhaps on the index of the part). We leave the details to the reader.*

Putting restrictions on parts depending not only on the index of parts but also of all previous parts is more challenging: Let us consider compositions $x_1 + \dots$ whose k -th non-zero part belongs to some non-empty subset $\mathcal{A}(x_1, \dots, x_{k-1})$ of strictly positive integers. The existence of limit-distributions for parts x_1, x_2, \dots is probably no longer easy to decide: $\mathcal{A}(x_1, \dots, x_{k-1}) = \{1, 2, \dots, x_{k-1}\}$ (and no restriction for the first part) leads for example to partitions (compositions with decreasing parts) which have no asymptotic limit-distribution. Ordering parts of partitions in increasing order (by considering compositions with finitely many parts such that $x_1 \leq x_2 \leq x_3 \dots$) yields to a trivial asymptotic distribution: $x_k = 1$ asymptotically for almost all such compositions of sufficiently large integers.

A pseudo-example is given by $x_1 = 1$ (if it exists) and $x_k \in \{1, x_{k-1} + 1\}$. Regrouping suitable terms of such compositions yields a bijection with compositions having all parts x_i in the set $\{1, 1 + 2, 1 + 2 + 3, \dots, \binom{k}{2}, \dots\}$ of triangular numbers.

Since the set $\mathcal{A}(x_1, \dots, x_{k-1})$ of possible values for x_k depends on x_1, \dots, x_{k-1} , perhaps existing limit-distributions for different parts x_i, x_j (in compositions with x_k in $\mathcal{A}(x_1, \dots, x_{k-1})$) are no longer independent and we can also consider asymptotic probabilities that a given random compositions starts with $x_1 + \dots + x_k$. The corresponding asymptotic probabilities (and related quantities) will be discussed in the next Section for generic NSG-compositions.

An example where an asymptotic limit-distribution exists is given by generic NSG-compositions: $\mathcal{A}(x_1, \dots, x_{k-1}) = \{1, 2\}$ if $x_i = x_{k-i} = 1$ for some $i < k$ and $\mathcal{A}(x_1, \dots, x_{k-1}) = \{1, 2, 3\}$ otherwise. We will give a few more details below.

An asymptotic limit distribution should also exist for

$$\mathcal{A}(x_1, \dots, x_{k-1}) = \{1, 2, \dots, \min_{i, 1 \leq i < k} x_i + x_{i-k}\}, \quad (32)$$

i.e. for composition satisfying only the NSG-inequalities given by the first line of (6). The methods of this paper give however no rigorous proof for the exis-

tence of limit-distributions in this case since our proofs for compositions with parts of size larger than 3 involve also right factors in pivot-factorisations.

In particular, the numbers ν_g (defined as the number of compositions $x_1 + \dots + x_m$ with $\sum_i x_i = g$ and $x_i + x_j \geq x_{i+j}$ whenever $i + j \leq m$) of such compositions of g have probably nice asymptotics. The first 25 values ν_1, \dots, ν_{25} are

1, 2, 4, 7, 13, 25, 43, 79, 142, 254, 449, 800, 1407, 2475, 4339, 7590, 13222,
23009, 39898, 69068, 119353, 205842, 354267, 608805, 1044528.

A fairly easy example with respect to enumeration is given by $\mathcal{A}_1 = \{1, 2, 3, \dots\}$ and $\mathcal{A}_k(x_1, \dots, x_{k-1}) = \{1, \dots, x_{k-1} + 1\}$. Denoting by G_k the generating series of all such compositions ending with a last part of size k we have

$$G_k = x^k \left(1 + \sum_{i=\max(k-1,1)}^{\infty} G_i \right).$$

This allows to compute finite series-expansions of G_1, G_2, \dots by 'bootstrapping'. The generating series for all such compositions (with $x_i \leq x_{i-1} + 1$) is of course defined by $1 + \sum_{i=1}^{\infty} G_i$. Its coefficients define the series A3116 (by definition) of [3] and seem to have asymptotics of the form $\gamma \cdot \lambda^n$ with $\gamma = 0.52893714\dots$ and $\lambda = 1.7356628\dots$

I ignore if parts of such compositions have asymptotic limit-distributions.

A last example with probably rather small exponential growth is given by compositions with arbitrary x_1 and with $x_k \in \{1, 1 + x_{k-1}, 1 + x_{k-1} + x_{k-2}, \dots, 1 + \sum_{j=i}^{k-1} x_j, \dots, 1 + \sum_{j=1}^{k-1} x_j\}$ (which forbids two consecutive parts of identical size larger than 1).

18 Probabilities related to generic NSG-compositions

Given a composition $\mathbf{x} = x_1 + x_2 + \dots + x_k$, we denote by $P_g(\mathbf{x})$ the proportion of NSG-compositions of genus $g \geq \sum_{i=1}^k x_i$ starting with \mathbf{x} (among all NSG-compositions of genus g). This proportion tends to a limit $P(\mathbf{x}) = \lim_{g \rightarrow \infty} P_g(\mathbf{x})$ defining a natural probability law on generic compositions with k parts in $\{1, 2, 3\}$. The limit-probability satisfies

$$P(\mathbf{x}) = P(\mathbf{x} + 1) + P(\mathbf{x} + 2) + P(\mathbf{x} + 3)$$

and is non-zero on a composition $\mathbf{x} = x_1 + \dots + x_n$ if and only if all parts x_1, \dots, x_n are elements of $\{1, 2, 3\}$ and $x_{i+j} \leq 2$ whenever $x_i = x_j = 1$. Equivalently, $P(\mathbf{x})$ is non-zero if and only if \mathbf{x} encodes a path starting at

the root in the combinatorial model of Section 16.1 where parts of size 1 correspond to left edges, parts of size 2 to (slim or fat) right edges and parts of size 3 to fat right edges. For simplicity, we call compositions \mathbf{x} such that $P(\mathbf{x}) > 0$ henceforth *generic NSG-compositions*.

The limit-probability P encodes some aspects of the generic behaviour of (uniformly distributed) NSG-compositions of large genus.

An interesting feature of these probabilities P is the following result:

Proposition 18.1. *Given an arbitrary generic NSG-composition \mathbf{x} , we have*

$$\frac{P(\mathbf{x} + 3)}{P(\mathbf{x} + 2)} \in \{0, \omega^{-1}\} .$$

Proof. This ratio is obviously 0 if no additional part of size 3 can be appended to \mathbf{x} . Otherwise we get a bijection between NSG-compositions of genus g ending with a last part 2 (by appending an additional part 2) and some NSG-composition of genus $g + 1$ ending with a last part 3 (by appending an additional part 3).

Theorem 1.2 ends the proof. \square

Proposition 18.1 shows that

$$\rho(\mathbf{x}) = \frac{P(\mathbf{x} + 1)}{P(\mathbf{x} + 1) + P(\mathbf{x} + 2) + P(\mathbf{x} + 3)} \quad (33)$$

are essentially the only interesting values: The combinatorics of \mathbf{x} determines if $P(\mathbf{x} + 3) = 0$. Proposition 18.1 determines then $P(\mathbf{x} + 1)$, $P(\mathbf{x} + 2)$ and $P(\mathbf{x} + 3)$ uniquely in terms of $\rho(\mathbf{x})$ and $P(\mathbf{x}) = P(\mathbf{x} + 1) + P(\mathbf{x} + 2) + P(\mathbf{x} + 3)$.

We have obviously $\rho(\mathbf{x}) \rightarrow \omega^{-1}$ for most generic NSG-compositions \mathbf{x} with multiplicity (or genus) tending to ∞ .

Using the tree model of Section 16.1 and identifying infinite geodesics starting at the root of the the binary tree with binary expansions of elements in $[0, 1]$, the probability laws P correspond to a continuous distribution function on $[0, 1]$.

A random-variable related to these probabilities is the asymptotic number of maximal parts equal to 3 in generic NSG-compositions: Let $A_g(n)$ be the proportion of generic NSG-compositions of genus g having exactly n parts of size 3. We get asymptotic limit-probabilities

$$A(n) = \lim_{g \rightarrow \infty} A_g(n) = \frac{\tau(n)}{1 + \tilde{C}(\omega^{-1})}$$

for $\tau(0) = 1$ and $\tau(k)$ defined by (30) for $k \geq 1$. The number $A(n)$ is the asymptotic proportion of NSG-composition with n parts of size 3 among all

NSG-compositions of very large genus. (It is not necessarily to require that the n parts of size 3 are of maximal size: NSG-compositions with larger parts can be neglected when considering asymptotics.) Particularly interesting is the value of $A(0)$ (i.e. the proportion of NSG-compositions having all parts in $\{1, 2\}$) since we have obviously the identity

$$1 + \tilde{C}(\omega^{-1}) = \frac{1}{A(0)}$$

linking the probability $A(0)$ to the value of the constant $C = \frac{5+\sqrt{5}}{10A(0)}$, cf. Formula (3) in Theorem 1.2.

Similarly, we have

$$\frac{A(1)}{A(0)} = \frac{1 + \omega^{-2}}{1 - (2\omega^{-3} + \omega^{-4})} \omega^{-3} = \frac{1}{\omega} + \frac{1}{\omega^3}$$

since $\frac{1+q^2}{1-(2q^3+q^4)}q^3$ is the generating series for all NSG-compositions $x_1 + \dots + x_{m-1}$ with $x_{m-1} = 3$ and x_1, \dots, x_{m-2} in $\{1, 2\}$ (satisfying $x_i + x_{m-1-i} \geq 3$), see Proposition 15.3.

Remark 18.2. *Typical NSG-compositions of high genus have only very few parts of size 3. They behave thus very differently from \tilde{C} -typical contributions to \tilde{C} which should have many maximal parts.*

A similar random-variable (on $\mathbb{Z} \setminus \{0\}$) defined by generic NSG-compositions is given by $f - 2m$ (for f the Frobenius number and m the multiplicity).

Last parts of generic NSG-compositions have also an asymptotic limit distribution, simply given by independent Bernoulli distributions yielding final parts of size 1 with asymptotic probability ω^{-1} and final parts of size 2 with asymptotic probability ω^{-2} . In particular, we have most of the time

$$\lim_{|\mathbf{x}|, P(\mathbf{x}) > 0} \rho(\mathbf{x}) = \omega^{-1} \quad (34)$$

with $|\mathbf{x}|$ denoting the length (number of summands) of \mathbf{x} . (Exceptions can occur if the repartition of parts of size 1 in the first half of \mathbf{x} is atypical.)

More precisely, a generic NSG-composition of large genus g has typically multiplicity $\frac{5+\sqrt{5}}{10}g + O(\sqrt{g})$. It consists of $g/\sqrt{5} + O(\sqrt{g})$ parts of size 1, of $\frac{5-\sqrt{5}}{10}g + O(\sqrt{g})$ parts of size 2 and of a small number (given by the random variable $A(n)$ considered above) of parts 3 among its initial parts.

Remark 18.3. *One can also consider probability laws corresponding to largest gaps in generic semigroups. The probability that an element $f - a$ at distance a of the Frobenius element (largest gap) $f = \max(\mathbb{N} \setminus S)$ of a generic numerical semigroup S does not belong to S tends to ω^{-2} for $a \rightarrow \infty$.*

18.1 A toy generator for NSG-compositions using un-fair coin tosses

A naive way to generate NSG-composition with given multiplicity m is to choose generators in $\{m+1, m+2, \dots\}$ independently with probability λ in $(0, 1)$. This results in NSG-compositions with maximum 2 or 3 having Frobenius numbers close to $2m$ and genus $(2-\lambda)m + O(\sqrt{m})$.

For $\lambda = \frac{\sqrt{5}-1}{2} = \omega^{-1}$ this should lead to more or less uniform random NSG-compositions for large m , as suggested by (33) and (34).

The corresponding probabilities $\tilde{P}_\lambda(\mathbf{x})$ are easy to compute: Given $\mathbf{x} = x_1 + \dots + x_k$ with x_1, \dots, x_k in $\{1, 2, 3\}$, the asymptotic probability (for $m \rightarrow \infty$) to generate a NSG-composition starting with \mathbf{x} can be computed as follows: $\tilde{P}_\lambda(\mathbf{x}) = 0$ if and only if there exists i, j (not necessarily distinct) with $i+j \leq k$ such that $x_i = x_j = 1$ and $x_{i+j} = 3$ (i.e. if \mathbf{x} is not a generic NSG-composition). Otherwise, the probability $\tilde{P}_\lambda(\mathbf{x})$ is a product of k factors in $\{\lambda, (1-\lambda), \lambda(1-\lambda), (1-\lambda)^2\}$ defined as follows:

- Every summand $x_i = 1$ contributes a factor λ .
- A summand $x_i = 2$ contributes a factor $(1-\lambda)$ if there exists $j < i$ such that $x_j = x_{i-j} = 1$. It contributes a factor $\lambda(1-\lambda)$ otherwise.
- A summand $x_i = 3$ contributes a factor $(1-\lambda)^2$.

It is easy to check that the probabilities $\tilde{P}_{\omega^{-1}}$ defined in this way satisfy Proposition 18.1. We have moreover

$$\frac{\tilde{P}_{\omega^{-1}}(\mathbf{x}+1)}{\tilde{P}_{\omega^{-1}}(\mathbf{x}+1) + \tilde{P}_{\omega^{-1}}(\mathbf{x}+2) + \tilde{P}_{\omega^{-1}}(\mathbf{x}+3)} = \omega^{-1}, \quad (35)$$

cf. (33) and (34).

NSG-compositions sampled in this way (for a given fixed λ in $(0, 1)$) have typically only a small number of summands 3. Setting

$$\mu_h(x_1 + \dots + x_k) = \sum_{i, x_i=3} i^h,$$

the limit-expectancy of μ_h (with respect to $\tilde{P}_\lambda(\mathbf{x})$) is easy to compute and is given by

$$\mu_h = (1-\lambda)^2 \sum_{n=0}^{\infty} (1-\lambda^2)^n ((2n+1)^h + (1-\lambda)(2n+2)^h)$$

(which is a rational function and can be rewritten in terms of dilogarithms). Indeed, a part x_{2n+1} , respectively x_{2n+2} , can be equal to 3 (with probability $(1 - \lambda)^2$) if and only if $\{x_i, x_{2n+1-i}\} \neq \{1\}$ (which happens with probability $1 - \lambda^2$), respectively $\{x_i, x_{2n+2-i}\} \neq \{1\}$ which happens with probability $1 - \lambda$ in the case $i = n + 1$. In both cases, there are n such distinct pairs $\{i, 2n+1-i\}$, respectively $\{i, 2n+2-i\}$ containing two indices. Independency of choices among parts of size 2, 3 whenever possible leads to the formula.

In particular, the expected asymptotic number of parts of size 3 in large random NSG-compositions sampled accordingly to $\tilde{P}_\lambda(\mathbf{x})$ is equal to

$$\mu_0 = \frac{(2 - \lambda)(1 - \lambda)^2}{\lambda^2} \quad (36)$$

which evaluates to $5 - 2\sqrt{5} = 0.52786\dots$ at $\lambda = \omega^{-1}$.

The following variation generates NSG-compositions of genus g accordingly to the law \tilde{P}_λ : We do not fix m but consider it as an unknown, to be fixed later. We add generators $m+i$ with independent uniform probability λ and generators $2m+j$ again with uniform independent probability λ (most of them will be of the form $(m+i_1)+(m+i_2)$ for generators $m+i_1, m+i_2$ already chosen). This defines the beginning of an NSG-composition $x_1 + x_2 + \dots$. Stop if $x_1 + \dots + x_k \geq g$. Set $m = k + 1$ and accept $\mathbf{x} = x_1 + \dots + x_k$ if $x_1 + \dots + x_k = g$. Reject it and restart if $x_1 + \dots + x_k > g$ (which happens asymptotically with probability $(1 - \lambda)/2$). The expected multiplicity m is asymptotically given by

$$\frac{g}{2 - \lambda} - \mu_0$$

with μ_0 given by (36) denoting the asymptotic expectation for the number of parts of size 3.

I ignore how to compute the asymptotic probability $\tilde{A}_\lambda(n) = \lim_{g \rightarrow \infty} \tilde{A}_{\lambda,g}(n)$ with $A_{\lambda,g}(n)$ denoting the proportion of NSG-compositions of genus g sampled as above which have n parts of size 3.

NSG-compositions generated by this algorithm with $\lambda = \omega^{-1}$ are not (asymptotically) uniformly sampled. Indeed, the expected number of parts of size 3 in uniformly sampled NSG-compositions is asymptotically equal to

$$\frac{\sum_{n=1}^{\infty} n\tau(n)}{1 + \sum_{n=1}^{\infty} \tau(n)} \geq \frac{\tau(1) + 2\tau(2)}{1 + \tau(1) + \tau(2)} = 0.909389\dots$$

which is larger than the corresponding expectation μ_0 for the toy generator with $\lambda = \omega^{-1}$.

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