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Propagation of well-prepared states along Martinet singular geodesics

Yves Colin de Verdière* and Cyril Letrouit^{†‡}

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Abstract

We prove that for the Martinet wave equation with “flat” metric, which is a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along any singular geodesic. This is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacian.

1 Introduction

1.1 Propagation of singularities and singular curves

The celebrated propagation of singularities theorem describes the wave-front set $WF(u)$ of a distributional solution u to a partial differential equation $Pu = f$ in terms of the principal symbol p of P : it says that, if p is real, then $WF(u) \setminus WF(f) \subset p^{-1}(0)$, and $WF(u) \setminus WF(f)$ is invariant under the bicharacteristic flow induced by the Hamiltonian vector field of p .

This result was first proved in [DH72, Theorem 6.1.1] and [Hor71b, Proposition 3.5.1]. However, it leaves open the case where the characteristics of P are not simple, i.e., when there are some points at which $p = dp = 0$. In a short and impressive paper [Mel86], Melrose sketched the proof of an analogous propagation of singularities result for the wave operator $P = D_t^2 - A$ when A is a self-adjoint non-negative real second-order differential operator which is only subelliptic. Such operators P are typical examples for which there exist double characteristic points.

Restated in the language of sub-Riemannian geometry (see [Let21]), Melrose’s result asserts that singularities of subelliptic wave equations propagate only along usual null-bicharacteristics (at speed 1) and along singular curves (see Definition 1.1). Along singular curves, Melrose writes in [Mel86] that the speed should be between 0 and 1, but nothing more. It is our purpose here to prove that for the

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Martinet wave equation, which is a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along the singular curves of the Martinet distribution. As explained in Point 5 of Section 2.1, an analogous result also holds in the so-called quasi-contact case (the computations are easier in that case).

To state our main result, we consider the *Martinet sub-Laplacian*

$$\Delta = X_1^2 + X_2^2$$

on \mathbb{R}^3 , where

$$X_1 = \partial_x, \quad X_2 = \partial_y + x^2 \partial_z.$$

Hörmander's theorem implies that Δ is hypoelliptic since X_1, X_2 and $[X_1, [X_1, X_2]]$ span $T\mathbb{R}^3$. The Martinet half-wave equation is

$$i\partial_t u - \sqrt{-\Delta}u = 0 \tag{1}$$

on $\mathbb{R}_t \times \mathbb{R}^3$, with initial datum $u(t = 0) = u_0$. The vector fields X_1 and X_2 span the horizontal distribution

$$\mathcal{D} = \text{Span}(X_1, X_2) \subset T\mathbb{R}^3.$$

Let us recall the definition of singular curves. We use the notation \mathcal{D}^\perp for the annihilator of \mathcal{D} (thus a subcone of the cotangent bundle $T^*\mathbb{R}^3$), and $\bar{\omega}$ denotes the restriction to \mathcal{D}^\perp of the canonical symplectic form ω on $T^*\mathbb{R}^3$.

Definition 1.1 *A characteristic curve for \mathcal{D} is an absolutely continuous curve $t \mapsto \lambda(t) \in \mathcal{D}^\perp$ that never intersects the zero section of \mathcal{D}^\perp and that satisfies*

$$\dot{\lambda}(t) \in \ker(\bar{\omega}(\lambda(t)))$$

for almost every t . The projection of $\lambda(t)$ onto \mathbb{R}^3 , which is an horizontal curve¹ for \mathcal{D} , is called a singular curve, and the corresponding characteristic an abnormal extremal lift of that curve.

We refer the reader to [Mon02] for more material related to sub-Riemannian geometry.

The curve $t \mapsto \gamma(t) = (0, t, 0) \in \mathbb{R}^3$ is a singular curve of the Martinet distribution \mathcal{D} . Denoting by (ξ, η, ζ) the dual coordinates of (x, y, z) , this curve admits both an abnormal extremal lift, for which $\xi(t) = \eta(t) = 0$, and a normal extremal lift, for which $\xi(t) = 0, \eta(t) = 1, \zeta(t) = 0$ (meaning that, if $\tau = 1$ is the dual variable of t , this yields a null-bicharacteristic). Martinet-type distributions attracted a lot of attention since Montgomery showed in [Mon94] that they provide examples of singular curves which are geodesics of the associated sub-Riemannian structure, but which are not necessarily projections of bicharacteristics (in contrast with the Riemannian case, where all geodesics are obtained as projections of bicharacteristics).

In this note, all phenomena and computations are done (microlocally) near the abnormal extremal lift, and thus away (in the cotangent bundle $T^*\mathbb{R}^3$) from the normal extremal lift, which plays no role.

¹i.e., $d\pi(\dot{\lambda}(t)) \in \mathcal{D}_{\lambda(t)}$ for almost every t , where $\pi : T^*\mathbb{R}^3 \rightarrow \mathbb{R}^3$ denotes the canonical projection.

1.2 Main result

Let $Y \in C^\infty(\mathbb{R}, \mathbb{R})$ be equal to 0 on $(-\infty, 1)$ and equal to 1 on $(2, \infty)$. Take $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with $\phi \geq 0$ and $\phi \not\equiv 0$. Consider as Cauchy datum for the Martinet half-wave equation (1) the distribution $u_0(x, y, z)$ whose Fourier transform² with respect to (y, z) is

$$\mathcal{F}_{y,z}u_0(x, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x). \quad (2)$$

Here, $\psi_{\eta,\zeta}$ is the ground state of the x -operator

$$-d_x^2 + (\eta + x^2\zeta)^2$$

with $\psi_{\eta,\zeta}(0) > 0$ and $\|\psi_{\eta,\zeta}\|_{L^2} = 1$, and α_1 is the associated eigenvalue. Thanks to the Fourier inversion formula applied to (2), we note that

$$\sqrt{-\Delta}u_0(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\sqrt{\alpha_1}(\eta, \zeta)\psi_{\eta,\zeta}(x)e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

We call u_0 a *well-prepared Cauchy datum*. It yields a solution of (1), namely

$$(U(t)u_0)(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x)e^{-it\sqrt{\alpha_1}(\eta,\zeta)}e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

For $\mu \in \mathbb{R}$, we set $H_\mu = -d_x^2 + (\mu + x^2)^2$ and we denote by ψ_μ its normalized ground state

$$H_\mu\psi_\mu = \lambda_1(\mu)\psi_\mu,$$

whose properties are described at the beginning of Section 3. We also define

$$F(\mu) = \sqrt{\lambda_1(\mu)}.$$

We assume that

$$F' \text{ is strictly monotonic on the support of } \phi, \quad (3)$$

which is no big restriction (choosing adequately the support of ϕ) since F is an analytic, non-affine, function³.

We set $\eta = \zeta^{1/3}\eta_1$ and we note that $\psi_{\eta,\zeta}(x) = \zeta^{1/6}\psi_{\eta/\zeta^{1/3}}(\zeta^{1/3}x) = \zeta^{1/6}\psi_{\eta_1}(\zeta^{1/3}x)$ and $\sqrt{\alpha_1} = \zeta^{1/3}F(\eta/\zeta^{1/3})$. Hence,

$$(U(t)u_0)(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}(\zeta^{1/3}x)e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)}e^{iz\zeta}d\eta_1 d\zeta. \quad (4)$$

We denote by $WF(f) \subset T^*\mathbb{R}^3 \setminus 0$ the wave-front set of $f \in \mathcal{D}'(\mathbb{R}^3)$, whose projection onto \mathbb{R}^3 is the singular support $\text{Sing Supp}(f)$ (see [Hor07, Definition 8.1.2]). Our main result states that the speed of propagation of the singularities of u_0 is in some window determined by the support of ϕ .

²We take the convention $\mathcal{F}f(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(q)e^{-iqp}dq$ for the Fourier transform in \mathbb{R}^d .

³See Point 1 of Lemma 3.1 and Proposition 6.1

Theorem 1.2 *For any $t \in \mathbb{R}$, we have*

$$WF(U(t)u_0) = \{(0, y, 0; 0, 0, \lambda) \in T^*\mathbb{R}^3, \lambda > 0, y \in tF'(I)\}, \quad (5)$$

where I is the support of ϕ . In particular,

$$\text{Sing Supp}(U(t)u_0) = \{(0, y, 0) \in \mathbb{R}^3, y \in tF'(I)\}. \quad (6)$$

Theorem 1.2 means that

$$\begin{aligned} & \text{singularities propagate along the singular curve } \gamma \\ & \text{at speeds given by } F'(I). \end{aligned} \quad (7)$$

Let us comment on the notion of “speed” used throughout this paper. In the Riemannian setting, when one says that singularities propagate at speed 1, this has to be understood with respect to the Riemannian metric. In the context of the Martinet distribution \mathcal{D} , there is also a metric, called sub-Riemannian metric, defined by

$$g_q(v) = \inf \{u_1^2 + u_2^2, v = u_1X_1(q) + u_2X_2(q)\}, \quad q \in \mathbb{R}^3, v \in T_q\mathbb{R}^3, \quad (8)$$

which is a Riemannian metric on \mathcal{D} . This metric g induces naturally a way to measure the speed of a point moving along an horizontal curve: if $\delta : J \rightarrow \mathbb{R}^3$ is an horizontal curve describing the time-evolution of a point, i.e., $\dot{\delta}(t) \in \mathcal{D}_{\delta(t)}$ for any $t \in J$, then the speed of the point is $(g_{\delta(t)}(\dot{\delta}(t)))^{1/2}$. In the case of the curve γ , since $g_q(\partial_y) = 1$ for any q of the form $(0, y, 0)$, we have $(g_q(F'(I)\partial_y))^{1/2} = F'(I)$. This is why the set $F'(I)$ is understood as a set of speeds in (7).

Proposition 1.3 *There holds $F'(\mathbb{R}) = [a, 1)$ for some $-1 < a < 0$.*

Together with (7), and choosing I adequately, this implies the following informal statement.

“Corollary” 1.4 *Any value between 0 and 1 can be realized as a speed of propagation of singularities along the singular curve γ .*

According to (5), the negative values in the range of F' yield singularities propagating backwards along the singular curve. This happens when $F'(I)$ contains negative values (see Proposition 1.3).

Organization of the paper. The paper is organized as follows. In Section 2, we explain in more details the geometrical meaning of the statement of Theorem 1.2, and we give several possible adaptations of this result. In Section 3, we prove some properties of the eigenfunctions ψ_μ which play a central role in the next sections. In Section 4, we compute the wave-front set of the Cauchy datum u_0 thanks to stationary phase arguments; this proves Theorem 1.2 at time $t = 0$. In Section 5, we complete the proof of Theorem 1.2 by extending the previous computation to any $t \in \mathbb{R}$. We could have directly done the proof for any $t \in \mathbb{R}$ (thus avoiding to distinguish the case $t = 0$), but we have chosen this presentation to improve readability. In Section 6, to illustrate Theorem 1.2, we prove Proposition 1.3, we provide plots of F and F' and compute their asymptotics.

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2 Comments on the main result

2.1 Possible adaptations of Theorem 1.2

We describe several possible adaptations of the statement of Theorem 1.2:

1. Putting in the initial Fourier data (2) an additional phase $e^{-iz_0\zeta}$ for some fixed $z_0 \in \mathbb{R}$, we obtain that the singularities of the corresponding solution propagate along the curve $t \mapsto (0, t, z_0)$, which is also a singular curve: for this new initial datum, we replace in (5) the 0 in the z coordinate by z_0 .
2. If we consider $(u, D_t u)|_{t=0} = (u_0, 0)$ as initial data of the Martinet wave equation $\partial_t^2 u - \Delta u = 0$, the solution is given by

$$u(t) = \frac{1}{2} (U(t)u_0 + U(-t)u_0).$$

Hence, under the assumption that $F'(I)$ and $-F'(I)$ do not intersect, (5) must be replaced by

$$WF(u(t)) = \{(0, y, 0; 0, 0, \lambda) \in T^*\mathbb{R}^3, \lambda > 0, y \in \pm tF'(I)\}.$$

3. If we take $\zeta < 0$ instead of $\zeta > 0$ in the (Fourier) initial data

$$Y(|\zeta|)\phi(\eta/|\zeta|^{1/3})\psi_{\eta,\zeta}(x),$$

then we must replace $F'(I)$ by $-F'(-I)$ in the Theorem 1.2. The same if we replace X_2 by $\partial_y - x^2\partial_z$ and keep $\zeta > 0$ in the Fourier initial data. This is due to the ‘‘orientation’’ of the singular curve γ : for Theorem 1.2 to hold without any change, we have to take $(0, 0, \zeta)(X_2) > 0$.

4. Instead of $\psi_{\eta,\zeta}$, we can use in the Fourier initial datum (2) the k -th eigenfunction of $-d_x^2 + (\eta + x^2\zeta)^2$. This yields a function F_k and the associated velocity F'_k , instead of F and F' . Theorem 1.2 also holds for this initial datum with the same proof, just replacing F' by F'_k in the statement.
5. It is possible to establish an analogue of Theorem 1.2 for the half-wave equation associated to the quasi-contact sub-Laplacian

$$\Delta = \partial_x^2 + \partial_y^2 + (\partial_z - x\partial_s)^2$$

on \mathbb{R}^4 . For that, we take Fourier initial data of the form

$$\mathcal{F}_{y,z,s}u_0(x, \eta, \zeta, \sigma) = \phi(\eta/\sigma^{1/2}, \zeta/\sigma^{1/2})\psi_{\eta,\zeta,\sigma}(x)$$

where $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$, η, ζ, σ denote the dual variables of y, z, s , and $\psi_{\eta,\zeta,\sigma}$ is the normalized ground state of the x -operator $-d_x^2 + \eta^2 + (\zeta - x\sigma)^2$. Then, the singularities propagate along the curve $t \mapsto (0, t, 0, 0)$ which is a singular curve of the quasi-contact distribution $\text{Span}(\partial_x, \partial_y, \partial_z - x\partial_s)$. The proof of this fact requires simpler computations than in the Martinet case since, instead of *quartic* oscillators, they involve usual *harmonic* oscillators. Note that the (non-flat) quasi-contact case has also been investigated in [Sav19], with other methods.

2.2 Geometric comments

Motivations. The result [Mel86, Theorem 1.8] already mentioned in the introduction (and revisited in [Let21]) implies that in the absence of singular curves, singularities of solutions of the wave equation only propagate along null-bicharacteristics. It is in particular the case when the sub-Laplacian has an associated distribution of contact type, since the orthogonal of the distribution is in this case symplectic (see [Mon02, Section 5.2.1]). Another reference for the contact case is [Mel84]. Our paper arose from the following questions: in the presence of singular curves, can singularities effectively propagate along these singular curves? If yes, at which speed(s)?

Together with the quasi-contact distribution mentioned in Point 5 of Section 2.1, the Martinet distribution is one of the simplest distributions to exhibit singular curves, and this is why we did our computations in this setting.

We now explain that the presence of singular curves for rank 2 distributions in 3D manifolds is generic. First, it follows from Definition 1.1 that the existence of singular curves is a property of the distribution \mathcal{D} , and does not depend on the metric g on \mathcal{D} (or on the vector fields X_1, X_2 which span \mathcal{D}). Besides, it was proved in [Mar70, Section II.6] that generically, a rank 2 distribution \mathcal{D}_0 in a 3D manifold M_0 is of contact type outside a surface \mathcal{S} , called the Martinet surface, and near any point of \mathcal{S} except a finite number of them, the distribution is isomorphic to $\mathcal{D} = \ker(dz - x^2 dy)$, which is exactly the distribution under study in the present work. Therefore, we expect to be able to generalize Theorem 1.2 to more generic situations of rank 2 distributions in 3D manifolds.

Singular curves as geodesics. To explain further the importance of singular curves, let us provide more context about sub-Riemannian geometry. A sub-Riemannian manifold is a triple (M, \mathcal{D}, g) where M is a smooth manifold, \mathcal{D} is a smooth sub-bundle of TM which is assumed to satisfy the Hörmander condition $\text{Lie}(\mathcal{D}) = TM$, and g is a Riemannian metric on \mathcal{D} (which naturally induces a distance d on M). Sub-Riemannian manifolds are thus a generalization of Riemannian manifolds (for which $\mathcal{D} = TM$), and they have been studied in depth since the years 1980, see [Mon02] and [ABB19] for surveys.

Singular curves arise as possible geodesics for the sub-Riemannian distance, i.e. absolutely continuous horizontal paths for which every sufficiently short subarc realizes the sub-Riemannian distance between its endpoints. Indeed, it follows from Pontryagin's maximum principle (see also [Mon02, Section 5.3.3]) that any sub-Riemannian geodesic is

- either *normal*, meaning that it is the projection of an integral curve of the normal Hamiltonian vector field ⁴;
- or *singular*, meaning that it is the projection of a characteristic curve (see Definition 1.1).

A sub-Riemannian geodesic can be normal and singular at the same time, and it is indeed the case of the singular curve $t \mapsto (x, y, z) = (0, t, 0)$ in the Martinet

⁴By this, we mean the Hamiltonian vector field of g^* , the semipositive quadratic form on T_q^*M defined by $g^*(q, p) = \|p\|_{\mathcal{D}_q}^2$, where the norm $\|\cdot\|_q$ is the norm on \mathcal{D}_q^* dual of the norm g_q .

distribution described above. But it was proved in [Mon94] that there also exist sub-Riemannian manifolds which exhibit geodesics which are singular, but not normal (they are called strictly singular).

We insist on the fact that in the present work,

the minimizing character of the singular curve γ plays no role,

since our computations can be adapted to the quasi-contact case (see Point 5 of Section 2.1), where singular curves are not minimizing.

Spectral effects of singular curves. The study of the spectral consequences of the presence of singular minimizers was initiated in [Mon95], where it was proved that in the situation where strictly singular minimizers show up as zero loci of two-dimensional magnetic fields, the ground state of a quantum particle concentrates on this curve as e/h tends to infinity, where e is the charge and h is the Planck constant. In [CHT-21?], it is proved that, for 3D compact sub-Riemannian manifolds with Martinet singularities, the support of the Weyl measure is the 2D Martinet manifold: most eigenfunctions concentrate on it.

The present work gives a new illustration of the intuition that singular curves play a role “at the quantum level”, this time at the level of propagation for a wave equation. However, the fact that the propagation speed is not 1, but can take any value between 0 and 1 was unexpected, since it is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacians.

3 Some properties of the eigenfunctions ψ_μ

Let us recall that H_μ is the essentially self-adjoint operator⁵ $H_\mu = -d_x^2 + (\mu + x^2)^2$ on $L^2(\mathbb{R}, dx)$ and ψ_μ is the ground state eigenfunction with $\int_{\mathbb{R}} \psi_\mu(x)^2 dx = 1$ and $\psi_\mu(0) > 0$. We denote by $\lambda_1(\mu)$ the associated eigenvalue, $\lambda_1(\mu) = F(\mu)^2$.

Lemma 3.1 *The domain of the essentially self-adjoint operator H_μ is independent of μ . It is denoted by $D(H_0)$. Moreover, the following assertions hold:*

1. *The map $\mu \mapsto \lambda_1(\mu)$ is analytic on \mathbb{R} , and the map $\mu \mapsto \psi_\mu$ is analytic from \mathbb{R} to $D(H_0)$;*
2. *The function ψ_μ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of \mathbb{R} ⁶;*
3. *Any derivative in $D(H_0)$ of the map $\mu \mapsto \psi_\mu$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of \mathbb{R} .*

Proof. The domain of H_μ is given by

$$\begin{aligned} D(H_\mu) &= \{\psi \in L^2(\mathbb{R}), -\psi'' + (\mu + x^2)^2\psi \in L^2(\mathbb{R})\} \\ &= \{\psi \in L^2(\mathbb{R}), -\psi'' + x^4\psi \in L^2(\mathbb{R}), x^2\psi \in L^2(\mathbb{R})\} \\ &= \{\psi \in L^2(\mathbb{R}), -\psi'' \in L^2(\mathbb{R}), x^4\psi \in L^2(\mathbb{R}), x^2\psi \in L^2(\mathbb{R})\} \end{aligned}$$

⁵The operator H_μ has already been studied for example in [Mon95] and [HP10].

⁶This means that for any compact $K \subset \mathbb{R}$, in the definition of $\mathcal{S}(\mathbb{R})$, the constants in the semi-norms can be taken independent of $\mu \in K$.

(see [Sim70] and [EG74]). We have hence $D(H_\mu) = D(H_0)$. The map $\mu \mapsto H_\mu$ is analytic from \mathbb{R} into $\mathcal{L}(D(H_0), L^2(\mathbb{R}))$. Moreover, by [BS12, Theorem 3.1], the eigenvalues of H_μ are non-degenerate (simple). This implies (see [Kat13, Chapter VII.2] or [CR19, Proposition 5.25]) that the eigenvalues $\lambda_1(\mu)$ and eigenfunctions ψ_μ are analytic functions of μ , respectively with values in \mathbb{R} and in $D(H_0)$. This proves Point 1.

Point 2 follows from Agmon estimates (precisely, [Hel88, Proposition 3.3.4] with $h = h_0 = 1$), which are uniform with respect to μ on any compact subset of \mathbb{R} .

This allows to start to prove Point 3 by induction. Assume that Point 3 is true for the derivatives of order $0, \dots, k-1$. Then, taking the derivatives with values in the domain $D(H_0)$ with respect to μ in the equation $(H_\mu - \lambda_1(\mu))\psi_\mu = 0$, we get

$$(H_\mu - \lambda_1(\mu)) \frac{d^k}{d\mu^k} \psi_\mu = v_{k,\mu} \quad (9)$$

and we know, by the induction hypothesis, that $v_{k,\mu} \in \mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of \mathbb{R} . We now use the results of [Shu87, Section 25] (see also [Shu87, Section 23] for the notations, and [HR82] for similar results). We check that $\xi^2 + x^4$ is a symbol in the sense of Definition 25.1 of [Shu87], with $m = 4$, $m_0 = 2$ and $\rho = 1/2$. Its standard quantization (i.e., $\tau = 0$ in Equation (23.31) of [Shu87]) is H_μ . By [Shu87, Theorem 25.1], $H_\mu - \lambda_1(\mu)$ admits a parametrix B_μ ; in particular, $B_\mu(H_\mu - \lambda_1(\mu)) = \text{Id} + R_\mu$ where R_μ is smoothing. Hence, composing on the left by B_μ in (9), and noting that $B_\mu v_{k,\mu} \in \mathcal{S}(\mathbb{R})$, we obtain that $\frac{d^k}{d\mu^k} \psi_\mu \in \mathcal{S}(\mathbb{R})$ uniformly with respect to μ on any compact subset of \mathbb{R} , which concludes the induction and the proof of Point 3.

4 Wave-front of the Cauchy datum

The goal of this section is to compute the wave-front set of u_0 . In other words, we prove Theorem 1.2 for $t = 0$. Recall that (see (4))

$$u_0(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta) \zeta^{1/2} \phi(\eta_1) \psi_{\eta_1}(\zeta^{1/3} x) e^{i(y\zeta^{1/3}\eta_1 + z\zeta)} d\eta_1 d\zeta. \quad (10)$$

Lemma 4.1 *The function u_0 is smooth on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$.*

Proof. We prove successively that u_0 is smooth outside $x = 0$, $y = 0$ and $z = 0$. Any derivative of (10) in x, y, z is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta) \zeta^\alpha \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x) \phi(\eta_1) \eta_1^\beta e^{i(y\zeta^{1/3}\eta_1 + z\zeta)} d\eta_1 d\zeta \quad (11)$$

for some $\alpha, \beta, \gamma \geq 0$. By the dominated convergence theorem, locally uniform (in x, y, z) convergence of these integrals implies smoothness. Recalling that ϕ has compact support, we see that the main difficulty for proving smoothness comes from the integration in ζ in (11).

For $x \neq 0$ it follows from Lemma 3.1 (Point 2) that the integrand in (11) has a fast decay in ζ . This proves that u_0 is smooth outside $x = 0$.

If $y \neq 0$, we use the fact that the phase $y\zeta^{1/3}\eta_1 + z\zeta$ is non critical with respect to η_1 to get the decay in ζ . More precisely, (11) is equal to

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha(y\zeta^{1/3})^{-N}D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta)e^{i(y\zeta^{1/3}\eta_1+z\zeta)}d\eta_1d\zeta$$

after integration by parts in η_1 (where $D_{\eta_1} = i^{-1}\partial_{\eta_1}$). Taking N sufficiently large and using that $D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta)$ is bounded thanks to Lemma 3.1 (Point 3), we obtain that this integral converges when $y \neq 0$, and that this convergence is locally uniform with respect to x, y, z . This proves that u_0 is smooth outside $y = 0$.

Finally, let us study the case $z \neq 0$. We can also assume that $y \leq 1$ due to the previous point.

Claim. The function

$$\zeta \mapsto Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)e^{iy\zeta^{1/3}\eta_1} \quad (12)$$

is a symbol (see Definition A.1) uniformly on every compact in (y, η_1) .

Proof. The functions $\zeta \mapsto \zeta^{1/2}\phi(\eta_1)$ and $\zeta \mapsto Y(\zeta)e^{iy\zeta^{1/3}\eta_1}$ are symbols (with $\rho = 1$ and $\rho = 2/3$ respectively, see Definition A.1) uniformly on every compact in (y, η_1) . Besides, $\zeta \mapsto \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)$ is also a symbol (of degree 0 with $\rho = 1$): we notice for example that the first derivative with respect to ζ writes $(1/3)\zeta^{-1}(\zeta^{1/3}x)\psi_{\eta_1}^{(\gamma+1)}(\zeta^{1/3}x)$ which is uniformly $O(1/\zeta)$ thanks to Lemma 3.1 (Point 2). Finally, since the space of symbols is an algebra for the pointwise product, we get the claim.

Integrating (12) in $\eta_1 \in \mathbb{R}$ and using Lemma A.2 (in the variable ζ), we obtain that (10) is smooth outside $z = 0$, which concludes the proof of Lemma 4.1.

The following lemma proves Theorem 1.2 at time $t = 0$.

Lemma 4.2 *There holds $WF(u_0) = \{(0, 0, 0; 0, 0, \lambda) \in T^*\mathbb{R}^3, \lambda > 0\}$.*

Proof. The Fourier transform of u_0 is

$$U_0(\xi, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3}) \quad (13)$$

where Ψ_μ is the Fourier transform of the eigenfunction ψ_μ . By Lemma 3.1 (Point 2), for any $N \in \mathbb{N}$ we get

$$|U_0(\xi, \eta, \zeta)| \leq C_N|\phi(\eta/\zeta^{1/3})|(1 + |\xi/\zeta^{1/3}|)^{-N}. \quad (14)$$

We show that U_0 is fastly decaying in any cone $C := \{|\xi| + |\eta| \geq c|\zeta|\}$ for c small. We split the cone into $C = C_1 \cup C_2$ with $C_1 = C \cap \{|\xi| \leq |\eta|\}$ and $C_2 = C \cap \{|\eta| \leq |\xi|\}$. In C_1 , we have $|\eta/\zeta^{1/3}| \geq c_1|\eta^{2/3}|$. This implies that $\phi(\eta/\zeta^{1/3})$ vanishes for η large enough. Hence, U_0 has fast decay in C_1 .

In C_2 , we have $|\xi/\zeta^{1/3}| \geq c_2|\xi|^{2/3} \geq c_3(1 + \xi^2 + \eta^2 + \zeta^2)^{1/3}$, hence, plugging into (14), we get that U_0 has fast decay in C_2 .

This proves that no point of the form $(x, y, z; \xi, \eta, \zeta) \in T^*\mathbb{R}^3$ with $(\xi, \eta) \neq (0, 0)$ can belong to $WF(u_0)$. Moreover, due to the factor $Y(\zeta)$, necessarily $WF(u_0) \subset \{\zeta > 0\}$. Combining with Lemma 4.1, we get the inclusion \subset in Lemma 4.2.

Let us finally prove that $(0, 0, 0; 0, 0, \lambda) \in WF(u_0)$ for $\lambda > 0$. We pick $a, b \in \mathbb{R}$ such that $\phi(a) \neq 0$ and $\Psi_a(b) \neq 0$. Then, we note that $U_0(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$ is independent of ζ and $\neq 0$, thus it is not fastly decaying as $\zeta \rightarrow +\infty$. Since $(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$ converges to the direction $(0, 0, +\infty)$ as $\zeta \rightarrow +\infty$, we get that there exists at least one point of the form $(x, y, z; 0, 0, \lambda) \in T^*\mathbb{R}^3$ which belongs to $WF(u_0)$. By Lemma 4.1, we necessarily have $x = y = z = 0$, which concludes the proof.

5 Wave front of the propagated solution

In this Section, we complete the proof of Theorem 1.2. We set

$$\mathcal{G}_t = \{(0, y, 0; 0, 0, \lambda), \lambda > 0, y \in tF'(\text{Support}(\phi))\}.$$

In Section 5.1, we prove the inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$, and then in Section 5.2 the converse inclusion $\mathcal{G}_t \subset WF(U(t)u_0)$. This completes the proof of Theorem 1.2.

5.1 The inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$

For this inclusion, we follow the same arguments as in Section 4: we adapt Lemma 4.1 to find out the singular support of $U(t)u_0$, and then we adapt Lemma 4.2 to determine the full wave-front set.

Lemma 5.1 *For any $t \in \mathbb{R}$, $U(t)u_0$ is smooth outside $\{(0, y, 0) \in \mathbb{R}^3, y \in tF'(I)\}$.*

Proof. As in Lemma 4.1, we prove successively that $U(t)u_0$ is smooth outside $x = 0$, $y \notin tF'(I)$ and $z = 0$. Any derivative of $U(t)u_0$ is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta) \zeta^\alpha \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x) \phi(\eta_1) \eta_1^\beta e^{-i\zeta^{1/3}(tF(\eta_1) - y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta \quad (15)$$

for some $\alpha, \beta, \gamma \geq 0$.

For $x \neq 0$, it follows from Lemma 3.1 (Point 2) that the integrand in (15) has a fast decay in ζ (locally uniformly in x, y, z). This proves that $U(t)u_0$ is smooth outside $x = 0$.

If $y \notin tF'(I)$, we use the fact that the phase $\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta$ is non critical with respect to η_1 to get decay in ζ . We set $R_{\eta_1}H = D_{\eta_1}(Q^{-1}H)$ where $Q = D_{\eta_1}(-i(\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta)) = -\zeta^{1/3}(tF'(\eta_1) - y)$. Note that $Q \neq 0$ since $y \notin tF'(I)$. Doing N integration by parts, the above expression becomes

$$\iint_{\mathbb{R}^2} Y(\zeta) \zeta^\alpha R_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x) \phi(\eta_1) \eta_1^\beta) e^{-i\zeta^{1/3}(tF(\eta_1) - y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta. \quad (16)$$

We set $H(x, \eta_1, \zeta) = \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x) \phi(\eta_1) \eta_1^\beta$.

Claim. For any N , there exists C_N such that $|R_{\eta_1}^N H(x, \eta_1, \zeta)| \leq C_N |\zeta|^{-N/3}$ for any $\zeta \in \mathbb{R}$, any $\eta_1 \in I = \text{Support}(\phi)$ and any $x \in \mathbb{R}$.

Taking N sufficiently large, the claim implies that (16), and thus (15), converge (locally uniformly), which proves the smoothness when $y \notin tF'(I)$ thanks to the dominated convergence theorem.

Proof of the claim. We prove it first for $N = 1$. We have

$$R_{\eta_1}H = \frac{D_{\eta_1}H}{Q} - H \frac{D_{\eta_1}Q}{Q^2}. \quad (17)$$

Since H is bounded (thanks to Point 2 of Lemma 3.1) and $|Q| \geq c|\zeta|^{1/3}$ with $c > 0$ and $|D_{\eta_1}Q| \leq C|\zeta|^{1/3}$ on the support of ϕ , we have $|H \frac{D_{\eta_1}Q}{Q^2}| \leq c|\zeta|^{-1/3}$. For the first term in the right-hand side of (17), we only need to prove that $D_{\eta_1}H$ is bounded. When D_{η_1} falls on $\phi(\eta_1)$ or η_1^β , it is immediate. When D_{η_1} falls on $\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)$, we use Lemma 3.1 (Point 3) and also get the result. This ends the proof of the

case $N = 1$. Now, we notice that our argument works not only for H , but for any function of the form $\psi_{\eta_1}^{(\gamma')}(\zeta^{1/3}x)\phi^{(\delta)}(\eta_1)\eta_1^{\beta'}$ where $\phi^{(\delta)}$ is any derivative of ϕ and $\beta', \gamma' \geq 0$. Hence, applying the previous argument recursively, we obtain the claim for any N .

Finally, the case $z \neq 0$ is checked in the same way as in the case $t = 0$, just shifting the phase by $it\zeta^{1/3}F(\eta_1)$ in (12).

Let us finish the proof of the inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$.

The Fourier transform of $U(t)u_0$ is

$$\mathcal{F}(U(t)u_0)(\xi, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3})e^{-it\sqrt{\alpha_1}(\eta, \zeta)}. \quad (18)$$

The change of phase with respect (13) has no influence on the properties of decay at infinity. Hence, the proof of Lemma 4.2 allows to conclude that $WF(U(t)u_0) \subset \mathcal{G}_t$ for any $t \in \mathbb{R}$.

5.2 The inclusion $\mathcal{G}_t \subset WF(U(t)u_0)$

We fix $t \in \mathbb{R}$ and we prove the non smoothness at $(0, tF'(c), 0)$ for any $c \in I$. We can assume that c is in the interior of I and that $\phi(c) \neq 0$. This implies thanks to (3) that $F''(c) \neq 0$. We want to show non-smoothness with respect to z at $x = 0$, $y = tF'(c)$ and $z = 0$. We set $v(z) := (U(t)u_0)(0, tF'(c), z)$. We will show that the Fourier transform of v is not fastly decaying.

Starting from (4), we get the explicit formula for the Fourier transform of v ,

$$\mathcal{F}v(\zeta) = Y(\zeta)\zeta^{1/2}K(\zeta)$$

where

$$K(\zeta) = \int_{\mathbb{R}} \phi(\eta_1)\psi_{\eta_1}(0)e^{-i\zeta^{1/3}t(F(\eta_1) - F'(c)\eta_1)}d\eta_1.$$

The only critical point of the phase $\eta_1 \mapsto -i\zeta^{1/3}t(F(\eta_1) - F'(c)\eta_1)$ located in I is c thanks to (3). Applying the stationary phase theorem with respect to η_1 , we obtain

$$K(\zeta) = e^{-i\zeta^{1/3}t(F(c) - F'(c)c)} \sum_{j \geq 1} a_j(\zeta^{1/3}|t|)^{-j/2}$$

where

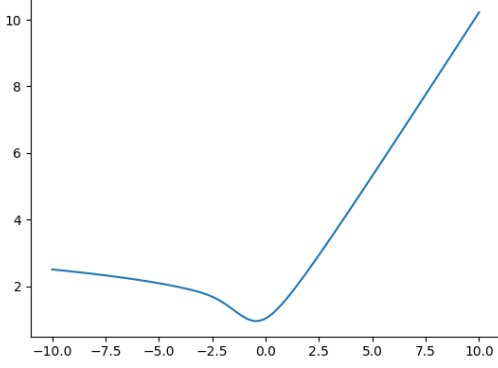
$$a_1 = \phi(c)\psi_c(0) \left(\frac{2\pi}{|F''(c)|} \right)^{1/2} \exp(-i\frac{\pi}{4}\text{sgn}(F''(c))) \neq 0.$$

Since $\phi(c) > 0$ and $\psi_c(0) > 0$, we have $K(\zeta) \sim c_0(\zeta^{1/3}|t|)^{-1/2}$ where $c_0 \neq 0$, and $\mathcal{F}v(\zeta)$ is not fastly decaying as $\zeta \rightarrow +\infty$. Applying Lemma A.2 to $a = \mathcal{F}v$ which is a symbol in ζ , this implies that v is not smooth at $z = 0$, thus $U(t)u_0$ is not smooth at $(0, tF'(c), 0)$.

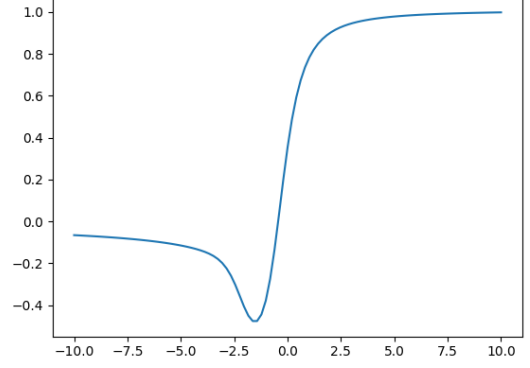
6 The function $F_k(\mu) = \sqrt{\lambda_k(\mu)}$

In this Section, we illustrate Theorem 1.2 with some plots and asymptotics of the functions F_k defined by $\mu \rightarrow \sqrt{\lambda_k(\mu)}$. As shown by Theorem 1.2 (and Point 4 in Section 2.1), the speeds of the propagation of singularities along the singular

curve are determined by the derivative $F'_k(\mu)$. Below, we plot $F = F_1$ and F' for $\mu \in (-10, 10)^7$.



(a) Plot of $F(\mu)$ for $\mu \in (-10, 10)$



(b) Plot of $F'(\mu)$ for $\mu \in (-10, 10)$

Recall that the F_k 's are analytic (see Point 1 of Lemma 3.1). We state a more precise version of Proposition 1.3:

Proposition 6.1 *For any $k \in \mathbb{N} \setminus \{0\}$, there holds $F'_k(\mu) \rightarrow 1^-$ as $\mu \rightarrow +\infty$, $F'_k(\mu) \rightarrow 0^-$ as $\mu \rightarrow -\infty$, and F'_k is minimal for some value $\mu_k^* < 0$. There exists $a_k \in (-1, 0)$ such that the range of F'_k is $[a_k, 1)$.*

Proposition 6.1 will be a consequence of the following result:

Proposition 6.2 *Denote by $\lambda_k(\mu)$ the k -th eigenvalue of $H_\mu = -d_x^2 + (\mu + x^2)^2$. Then, for $k \in \mathbb{N} \setminus \{0\}$, as $\mu \rightarrow +\infty$,*

$$\lambda_k(\mu) = \mu^2 + \sqrt{2}(2k-1)\sqrt{\mu} + \sum_{\ell=2}^{\infty} b_{\ell,k} \mu^{2-3\ell/2} \quad (19)$$

and

$$\frac{d}{d\mu} \sqrt{\lambda_k(\mu)} = 1 - \frac{2k-1}{2\sqrt{2}} \mu^{-3/2} + O(\mu^{-3}) \quad (20)$$

These derivatives are > 0 and converge to 1.

As $\mu \rightarrow -\infty$, for $k \in \mathbb{N} \setminus \{0\}$,

$$\lambda_{2k-1}(\mu) = 2(2k-1)\sqrt{-\mu} + \sum_{\ell=2}^{\infty} c_{\ell,k} (-\mu)^{2-3\ell/2} \quad (21)$$

$$\lambda_{2k}(\mu) = \lambda_{2k-1} + o(\mu^{-\infty}) \quad (22)$$

and

$$\frac{d}{d\mu} \sqrt{\lambda_{2k-1}(\mu)} = -\frac{\sqrt{2(2k-1)}}{4} (-\mu)^{-3/4} + O(\mu^{-3/2}) \quad (23)$$

and the same for $\frac{d}{d\mu} \sqrt{\lambda_{2k}(\mu)}$. These derivative are < 0 and converge to 0.

⁷We thank Julien Guillod for his help in making the first numerical experiments.

Proof of Proposition 6.2. For $\mu > 0$, we consider the operator $T_\mu : \psi \mapsto \psi(\cdot/\mu^{1/4})$. Then $H_\mu = T_\mu^{-1}G_\mu T_\mu$ where $G_\mu = \mu^2 + \mu^{1/2}(-d_x^2 + 2x^2 + x^4/\mu^{3/2})$. The eigenvalues of $-d_x^2 + 2x^2 + hx^4$ for $h \rightarrow 0$ can be computed with the usual perturbation theory (see [RS78, Chapter XII.3]), and this yields (19) with $h = \mu^{-3/2}$. Moreover the formal expansion can be differentiated with respect to μ , hence we get (20).

For $\mu = -\mu_0 < 0$, we see that the transformation $x \mapsto \mu_0^{1/4}(x \mp \mu_0^{1/2})$ conjugates H_μ to the operator $\mu_0^{1/2}(-d_x^2 + 4x^2 \pm 4\mu_0^{-3/4}x^3 + \mu_0^{-3/2}x^4)$. Using again perturbation theory and the separation into pairs of eigenvalues in double wells (see [HS84]), we get (21) and (22), and (23) follows.

Proof of Proposition 6.1. The convergences at $\pm\infty$ are proved by Proposition 6.2. This behaviour at $\pm\infty$ implies the existence of μ_k^* such that $F'_k(\mu_k^*) = a_k$ is minimal. We denote by ψ_μ^k the normalized eigenfunction corresponding to $\lambda_k(\mu)$. Taking the first derivative (with value in the domain $D(H_0)$) with respect to μ of the eigenfunction equation $(H_\mu - \lambda_k(\mu))\psi_\mu^k = 0$, and then integrating against ψ_μ^k , we obtain $\lambda'_k(\mu) = \int_{\mathbb{R}}(\mu + x^2)\psi_\mu^k(x)^2 dx$. Thus,

$$F'_k(\mu) = \frac{1}{\sqrt{\lambda_k(\mu)}} \int_{\mathbb{R}}(\mu + x^2)\psi_\mu^k(x)^2 dx$$

which is positive for $\mu \geq 0$, hence $\mu_k^* < 0$.

It remains to show that $|F'_k(\mu)| < 1$ for every μ : by the Cauchy-Schwarz inequality, we get

$$F'_k(\mu)^2 \leq \frac{1}{\lambda_k(\mu)} \int_{\mathbb{R}}(\mu + x^2)^2\psi_\mu^k(x)^2 dx \int_{\mathbb{R}}\psi_\mu^k(x)^2 dx$$

and, from the quadratic form associated to H_μ ,

$$\int_{\mathbb{R}}(\mu + x^2)^2\psi_\mu^k(x)^2 dx < \lambda_k(\mu),$$

which concludes the proof.

Appendix

A Fourier transform of symbols

Definition A.1 A smooth function $a : \mathbb{R}^d \rightarrow \mathbb{C}$ is called a symbol of degree $\leq m$ if there exists $0 < \rho \leq 1$ so that the partial derivatives of a satisfy

$$\forall \alpha \in \mathbb{N}^d, \quad |D^\alpha a(\xi)| \leq C_\alpha(1 + |\xi|)^{m-\rho|\alpha|}.$$

The space of symbols is an algebra for the pointwise product. If a is a real valued symbol of degree $m < 1$ and $\rho > m$, e^{ia} is a symbol of degree 0 (with a different ρ).

We will need the

Lemma A.2 If a is a symbol, the Fourier transform $\mathcal{F}a$ of a is smooth outside $x = 0$ and all derivatives of $\mathcal{F}a$ decay fastly at infinity. If moreover a does not belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then $\mathcal{F}a$ is non smooth at $x = 0$.

Proof. For $x \neq 0$ and for any $\alpha, \beta \in \mathbb{N}^d$, we have

$$(\mathcal{F}a)^{(\beta)}(x) = C_\beta \int_{\mathbb{R}^d} \xi^\beta a(\xi) e^{-ix\xi} d\xi = \frac{c_\beta^\alpha}{x^\alpha} \int_{\mathbb{R}^d} D_\xi^\alpha (\xi^\beta a(\xi)) e^{-ix\xi} d\xi. \quad (24)$$

The multi-index $\beta \in \mathbb{N}^d$ being fixed, this last integral converges for $|\alpha|$ sufficiently large since a is a symbol. By the dominated convergence theorem, this implies that $\mathcal{F}a$ is smooth outside $x = 0$. Moreover, (24) also implies that all derivatives of $\mathcal{F}a$ decay fastly at infinity.

Finally, if $\mathcal{F}a$ were smooth at 0, then $\mathcal{F}a$ would be in the Schwartz space as well as a .

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