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# Propagation of well-prepared states along Martinet singular geodesics

Yves Colin de Verdière\* and Cyril Letrouit<sup>†‡</sup>

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## Abstract

We prove that for the Martinet wave equation with “flat” metric, which a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along any singular geodesic. This is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacian.

## 1 Introduction

### 1.1 Propagation of singularities and singular curves

The celebrated propagation of singularities theorem describes the wave-front set  $WF(u)$  of a distributional solution  $u$  to a partial differential equation  $Pu = f$  in terms of the principal symbol  $p$  of  $P$ : it says that, if  $p$  is real, then  $WF(u) \setminus WF(f) \subset p^{-1}(0)$ , and that, if additionally the characteristics are simple ( $p = 0 \Rightarrow dp \neq 0$  outside the null section), then  $WF(u) \setminus WF(f)$  is invariant under the bicharacteristic flow induced by the Hamiltonian vector field of  $p$ .

This result was first proved in [DH72, Theorem 6.1.1] and [Hor71b, Proposition 3.5.1]. However, it leaves open the case where the characteristics of  $P$  are not simple. In a short and impressive paper [Mel86], Melrose sketched the proof of an analogous propagation of singularities result for the wave operator  $P = D_t^2 - A$  when  $A$  is a self-adjoint non-negative real second-order differential operator which is only subelliptic. Such operators  $P$  are typical examples for which there exist double characteristic points.

Restated in the language of sub-Riemannian geometry, Melrose’s result asserts that singularities of subelliptic wave equations propagate only along usual null-bicharacteristics (at speed 1) and along singular curves (see Definition 1.1). Along singular curves, Melrose writes in [Mel86] that the speed should be between 0 and 1, but nothing more. It is our purpose here to prove that for the Martinet wave

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equation, which is a subelliptic wave equation, singularities can propagate at any speed between 0 and 1 along the singular curves of the Martinet distribution. As explained in Remark 1.9, an analogous result also holds in the so-called quasi-contact case (the computations are easier in that case).

To state our main result, we consider the *Martinets sub-Laplacian*

$$\Delta = X_1^2 + X_2^2$$

on  $\mathbb{R}^3$ , where

$$X_1 = \partial_x, \quad X_2 = \partial_y + x^2 \partial_z.$$

Hörmander's theorem implies that  $\Delta$  is hypoelliptic since  $X_1, X_2$  and  $[X_1, [X_1, X_2]]$  span  $T\mathbb{R}^3$ . The Martinet half-wave equation is

$$i\partial_t u - \sqrt{-\Delta}u = 0 \tag{1}$$

on  $\mathbb{R}_t \times \mathbb{R}^3$ , with initial datum  $u(t = 0) = u_0$ . The vector fields  $X_1$  and  $X_2$  span the horizontal distribution

$$\mathcal{D} = \text{Span}(X_1, X_2) \subset T\mathbb{R}^3.$$

Let us recall the definition of singular curves. We use the notation  $\mathcal{D}^\perp$  for the annihilator of  $\mathcal{D}$  (thus a subcone of the cotangent bundle  $T^*\mathbb{R}^3$ ), and  $\bar{\omega}$  denotes the restriction to  $\mathcal{D}^\perp$  of the canonical symplectic form  $\omega$  on  $T^*\mathbb{R}^3$ .

**Definition 1.1** *A characteristic curve for  $\mathcal{D}$  is an absolutely continuous curve  $t \mapsto \lambda(t) \in \mathcal{D}^\perp$  that never intersects the zero section of  $\mathcal{D}^\perp$  and that satisfies*

$$\dot{\lambda}(t) \in \ker(\bar{\omega}(\lambda(t)))$$

*for almost every  $t$ . The projection of  $\lambda(t)$  onto  $\mathbb{R}^3$ , which is an horizontal curve<sup>1</sup> for  $\mathcal{D}$ , is called a singular curve, and the corresponding characteristic an abnormal extremal lift of that curve.*

We refer the reader to [Mon02] for more material related to sub-Riemannian geometry.

The curve  $t \mapsto \gamma(t) = (0, t, 0) \in \mathbb{R}^3$  is a singular curve of the Martinet distribution  $\mathcal{D}$ . Denoting by  $(\xi, \eta, \zeta)$  the dual coordinates of  $(x, y, z)$ , this curve admits both an abnormal extremal lift, for which  $\xi(t) = \eta(t) = 0$ , and a normal extremal lift, for which  $\xi(t) = 0, \eta(t) = 1, \zeta(t) = 0$  (meaning that, if  $\tau = 1$  is the dual variable of  $t$ , this yields a null-bicharacteristic). Martinet-type distributions attracted a lot of attention since Montgomery showed in [Mon94] that they provide examples of singular curves which are geodesics of the associated sub-Riemannian structure, but which are not necessarily projections of bicharacteristics (in contrast with the Riemannian case, where all geodesics are obtained as projections of bicharacteristics).

In this note, all phenomena and computations are done (microlocally) near the abnormal extremal lift, and thus away (in the cotangent bundle  $T^*\mathbb{R}^3$ ) from the normal extremal lift, which plays no role.

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<sup>1</sup>i.e.,  $d\pi(\dot{\lambda}(t)) \in \mathcal{D}_{\lambda(t)}$  for almost every  $t$ , where  $\pi : T^*\mathbb{R}^3 \rightarrow \mathbb{R}^3$  denotes the canonical projection.

## 1.2 Main result

Let  $Y \in C^\infty(\mathbb{R}, \mathbb{R})$  be equal to 0 on  $(-\infty, 1)$  and equal to 1 on  $(2, \infty)$ . Take  $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R})$  with  $\phi \geq 0$  and  $\phi \not\equiv 0$ . Consider as Cauchy datum for the Martinet half-wave equation (1) the distribution  $u_0(x, y, z)$  whose Fourier transform<sup>2</sup> with respect to  $(y, z)$  is

$$\mathcal{F}_{y,z}u_0(x, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x). \quad (2)$$

Here,  $\psi_{\eta,\zeta}$  is the ground state of the  $x$ -operator

$$-d_x^2 + (\eta + x^2\zeta)^2$$

with  $\psi_{\eta,\zeta}(0) > 0$  and  $\|\psi_{\eta,\zeta}\|_{L^2} = 1$ , and  $\alpha_1$  is the associated eigenvalue. Thanks to the Fourier inversion formula applied to (2), we note that

$$\sqrt{-\Delta}u_0(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\sqrt{\alpha_1}(\eta, \zeta)\psi_{\eta,\zeta}(x)e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

We call  $u_0$  a *well-prepared Cauchy datum*. It yields a solution of (1), namely

$$(U(t)u_0)(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\phi(\eta/\zeta^{1/3})\psi_{\eta,\zeta}(x)e^{-it\sqrt{\alpha_1}(\eta,\zeta)}e^{i(y\eta+z\zeta)}d\eta d\zeta.$$

For  $\mu \in \mathbb{R}$ , we set  $H_\mu = -d_x^2 + (\mu + x^2)^2$  and we denote by  $\psi_\mu$  its normalized ground state

$$H_\mu\psi_\mu = \lambda_1(\mu)\psi_\mu,$$

whose properties are described at the beginning of Section 2. We also define

$$F(\mu) = \sqrt{\lambda_1(\mu)}.$$

We set  $\eta = \zeta^{1/3}\eta_1$  and we note that  $\psi_{\eta,\zeta}(x) = \zeta^{1/6}\psi_{\eta/\zeta^{1/3}}(\zeta^{1/3}x) = \zeta^{1/6}\psi_{\eta_1}(\zeta^{1/3}x)$  and  $\sqrt{\alpha_1} = \zeta^{1/3}F(\eta/\zeta^{1/3})$ . Hence,

$$(U(t)u_0)(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta)\zeta^{1/2}\phi(\eta_1)\psi_{\eta_1}(\zeta^{1/3}x)e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)}e^{iz\zeta}d\eta_1 d\zeta. \quad (3)$$

We denote by  $WF(f) \subset T^*\mathbb{R}^3 \setminus 0$  the wave-front set of  $f \in \mathcal{D}'(\mathbb{R}^3)$ , whose projection onto  $\mathbb{R}^3$  is the singular support  $\text{Sing Supp}(f)$  (see [Hor07, Definition 8.1.2]). Our main result states that the speed of propagation of the singularities of  $u_0$  is in some window determined by the support of  $\phi$ .

**Theorem 1.2** *For any  $t \in \mathbb{R}$ , we have*

$$WF(U(t)u_0) = \{(0, y, 0; 0, 0, \lambda) \in T^*\mathbb{R}^3, \lambda > 0, y \in tF'(I)\}, \quad (4)$$

where  $I$  is the support of  $\phi$ . In particular,

$$\text{Sing Supp}(U(t)u_0) = \{(0, y, 0) \in \mathbb{R}^3, y \in tF'(I)\}. \quad (5)$$

---

<sup>2</sup>We take the convention  $\mathcal{F}f(p) = (2\pi)^{-d} \int_{\mathbb{R}^d} f(q)e^{-iqp}dq$  for the Fourier transform in  $\mathbb{R}^d$ .

Theorem 1.2 means that

$$\begin{aligned} & \text{singularities propagate along the singular curve } \gamma \\ & \text{at speeds given by } F'(I). \end{aligned} \tag{6}$$

Recall that in the Riemannian setting, when one says that singularities propagate at speed 1, this has to be understood with respect to the Riemannian metric. In the context of the Martinet distribution  $\mathcal{D}$ , there is also a metric, called sub-Riemannian metric, defined by

$$g_q(v) = \inf \{u_1^2 + u_2^2, \ v = u_1 X_1(q) + u_2 X_2(q)\}, \quad q \in \mathbb{R}^3, \ v \in T_q \mathbb{R}^3, \tag{7}$$

which is a Riemannian metric on  $\mathcal{D}$ . This metric  $g$  induces naturally a way to measure the speed of a point moving along an horizontal curve. Since  $g_q(\partial_y) = 1$  for any  $q$  of the form  $(0, y, 0)$ , the set of speeds  $F'(I)$  in (6) has to be understood as  $(g_q(F'(I)\partial_y))^{1/2}$ .

**Proposition 1.3** *There holds  $F'(\mathbb{R}) = [a, 1]$  for some  $-1 < a < 0$ .*

Together with (6), and choosing  $I$  adequately, this implies the following informal statement.

**“Corollary” 1.4** *Any value between 0 and 1 can be realized as a speed of propagation of singularities along the singular curve  $\gamma$ .*

The next remarks explain possible adaptations of the statement of Theorem 1.2.

**Remark 1.5** *Putting in the initial Fourier data (2) an additional phase  $e^{-iz_0\zeta}$  for some fixed  $z_0 \in \mathbb{R}$ , we obtain that the singularities of the corresponding solution propagate along the curve  $t \mapsto (0, t, z_0)$ , which is also a singular curve: for this new initial datum, we replace in (4) the 0 in the  $z$  coordinate by  $z_0$ .*

**Remark 1.6** *If we consider  $(u, D_t u)|_{t=0} = (u_0, 0)$  as initial data of the Martinet wave equation  $\partial_t^2 u - \Delta u = 0$ , the solution is given by*

$$u(t) = \frac{1}{2} (U(t)u_0 + U(-t)u_0).$$

*Hence, under the assumption that  $F'(I)$  and  $-F'(I)$  do not intersect, (4) must be replaced by*

$$WF(u(t)) = \{(0, y, 0; \ 0, 0, \lambda) \in T^*\mathbb{R}^3, \ \lambda > 0, \ y \in \pm tF'(I)\}.$$

**Remark 1.7** *If we take  $\zeta < 0$  instead of  $\zeta > 0$  in the (Fourier) initial data*

$$Y(|\zeta|)\phi(\eta/|\zeta|^{1/3})\psi_{\eta,\zeta}(x),$$

*then we must replace  $F'(I)$  by  $-F'(-I)$  in the Theorem 1.2. The same if we replace  $X_2$  by  $\partial_y - x^2\partial_z$  and keep  $\zeta > 0$  in the Fourier initial data. This is due to the “orientation” of the singular curve  $\gamma$ : for Theorem 1.2 to hold without any change, we have to take  $(0, 0, \zeta)(X_2) > 0$ .*

**Remark 1.8** *Instead of  $\psi_{\eta,\zeta}$ , we can use in the Fourier initial datum (2) the  $k$ -th eigenfunction of  $-d_x^2 + (\eta + x^2\zeta)^2$ . This yields a function  $F_k$  and the associated velocity  $F'_k$ , instead of  $F$  and  $F'$ . Theorem 1.2 also holds for this initial datum with the same proof, just replacing  $F'$  by  $F'_k$  in the statement.*

**Remark 1.9** *It is possible to establish an analogue of Theorem 1.2 for the half-wave equation associated to the quasi-contact sub-Laplacian*

$$\Delta = \partial_x^2 + \partial_y^2 + (\partial_z - x\partial_s)^2$$

on  $\mathbb{R}^4$ . For that, we take Fourier initial data of the form

$$\mathcal{F}_{y,z,s}u_0(x, \eta, \zeta, \sigma) = \phi(\eta/\sigma^{1/2}, \zeta/\sigma^{1/2})\psi_{\eta,\zeta,\sigma}(x)$$

where  $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ ,  $\eta, \zeta, \sigma$  denote the dual variables of  $y, z, s$ , and  $\psi_{\eta,\zeta,\sigma}$  is the normalized ground state of the  $x$ -operator  $-d_x^2 + \eta^2 + (\zeta - x\sigma)^2$ . The computations are simpler than in the Martinet case since, instead of quartic oscillators, they involve usual harmonic oscillators. Note that the (non-flat) quasi-contact case has also been investigated in [Sav19], with other methods.

### 1.3 Comments and organization of the paper

The singular curve  $t \mapsto \gamma(t) = (0, t, 0) \in \mathbb{R}^3$  of the Martinet distribution  $\mathcal{D}$  has played an important role in the last decades in the development of sub-Riemannian geometry. This role is due to the fact that  $\gamma$  is a minimizing geodesic for the sub-Riemannian distance induced by the metric  $g$  defined in (7). However, we insist on the fact that in the present work,

*the minimizing character of the singular curve  $\gamma$  plays no role.*

For example, as explained in Remark 1.9, our computations can be adapted to the quasi-contact case, where singular curves are not minimizing.

It follows from Definition 1.1 that the existence of singular curves is a property of the distribution  $\mathcal{D}$ , and does not depend on the metric  $g$  on  $\mathcal{D}$  (or on the vector fields  $X_1, X_2$  which span  $\mathcal{D}$ ). Besides, it was proved in [Mar70, Section II.6] that generically, a rank 2 distribution  $\mathcal{D}_0$  in a 3D manifold  $M_0$  is of contact type outside a surface  $\mathcal{S}$ , called the Martinet surface, and near any point of  $\mathcal{S}$  except a finite number of them, the distribution is isomorphic to  $\mathcal{D} = \ker(dz - x^2dy)$ , which is exactly the distribution under study in the present work. Therefore, we expect to be able to generalize Theorem 1.2 to more generic situations.

To explain further the importance of singular curves, let us provide more context about sub-Riemannian geometry. A sub-Riemannian manifold is a triple  $(M, \mathcal{D}, g)$  where  $M$  is a smooth manifold,  $\mathcal{D}$  is a smooth sub-bundle of  $TM$  which is assumed to satisfy the Hörmander condition  $\text{Lie}(\mathcal{D}) = TM$ , and  $g$  is a Riemannian metric on  $\mathcal{D}$  (which naturally induces a distance  $d$  on  $M$ ). Sub-Riemannian manifolds are thus a generalization of Riemannian manifolds (for which  $\mathcal{D} = TM$ ), and they have been studied in depth since the years 1980, see [Mon02] and [ABB19] for surveys.

As already mentioned, a particular interest has been devoted to the understanding of geodesics, i.e. absolutely continuous horizontal paths for which every sufficiently short subarc realizes the sub-Riemannian distance between its endpoints. It follows from Pontryagin's maximum principle (see also [Mon02, Section 5.3.3]) that any sub-Riemannian geodesic is

- either *normal*, meaning that it is the projection of an integral curve of the normal Hamiltonian vector field <sup>3</sup>;
- or *singular*, meaning that it is the projection of a characteristic curve (see Definition 1.1).

A sub-Riemannian geodesic can be normal and singular at the same time, and it is indeed the case of the singular curve  $t \mapsto (x, y, z) = (0, t, 0)$  in the Martinet distribution described above. But it was proved in [Mon94] that there also exist sub-Riemannian manifolds which exhibit geodesics which are singular, but not normal (they are called strictly singular).

The study of the spectral consequences of the presence of singular minimizers was initiated in [Mon95], where it was proved that in the situation where strictly singular minimizers show up as zero loci of two-dimensional magnetic fields, the ground state of a quantum particle concentrates on this curve as  $e/h$  tends to infinity, where  $e$  is the charge and  $h$  is the Planck constant. In [CHT-21?], it is proved that, for 3D compact sub-Riemannian manifolds with Martinet singularities, the support of the Weyl measure is the 2D Martinet manifold: most eigenfunctions concentrate on it.

The present work gives a new illustration of the intuition that singular curves play a role “at the quantum level”, this time at the level of propagation for a wave equation. However, the fact that the propagation speed is not 1, but can take any value between 0 and 1 was unexpected, since it is in strong contrast with the usual propagation of singularities at speed 1 for wave equations with elliptic Laplacians.

The paper is organized as follows. In Section 2, we prove some properties of the eigenfunctions  $\psi_\mu$  which play a central role in the next sections. In Section 3, we compute the wave-front set of the Cauchy datum  $u_0$  thanks to stationary phase arguments; this proves Theorem 1.2 at time  $t = 0$ . In Section 4, we complete the proof of Theorem 1.2 by extending the previous computation to any  $t \in \mathbb{R}$ . We could have directly done the proof for any  $t \in \mathbb{R}$  (thus avoiding to distinguish the case  $t = 0$ ), but we have chosen this presentation to improve readability. In Section 5, to illustrate Theorem 1.2, we prove Proposition 1.3, we provide plots of  $F$  and  $F'$  and compute their asymptotics.

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## 2 Some properties of the eigenfunctions $\psi_\mu$

Let us recall that  $H_\mu$  is the essentially self-adjoint operator  $H_\mu = -d_x^2 + (\mu + x^2)^2$  on  $L^2(\mathbb{R}, dx)$  and  $\psi_\mu$  is the ground state eigenfunction with  $\int_{\mathbb{R}} \psi_\mu(x)^2 dx = 1$  and  $\psi_\mu(0) > 0$ . We denote by  $\lambda_1(\mu)$  the associated eigenvalue,  $\lambda_1(\mu) = F(\mu)^2$ .

**Lemma 2.1** *The domain of the essentially self-adjoint operator  $H_\mu$  is independent of  $\mu$ . It is denoted by  $D(H_0)$ . Moreover, the following assertions hold:*

---

<sup>3</sup>By this, we mean the Hamiltonian vector field of  $g^*$ , the semipositive quadratic form on  $T_q^*M$  defined by  $g^*(q, p) = \|p\|_{\mathcal{D}_q}^2$ , where the norm  $\|\cdot\|_q$  is the norm on  $\mathcal{D}_q^*$  dual of the norm  $g_q$ .

1. The map  $\mu \mapsto \lambda_1(\mu)$  is analytic on  $\mathbb{R}$ , and the map  $\mu \mapsto \psi_\mu$  is analytic from  $\mathbb{R}$  to  $D(H_0)$ ;
2. The function  $\psi_\mu$  is in the Schwartz space  $\mathcal{S}(\mathbb{R})$  uniformly with respect to  $\mu$  on any compact subset of  $\mathbb{R}^4$ ;
3. Any derivative in  $D(H_0)$  of the map  $\mu \mapsto \psi_\mu$  is in the Schwartz space  $\mathcal{S}(\mathbb{R})$  uniformly with respect to  $\mu$  on any compact subset of  $\mathbb{R}$ .

*Proof.* The domain of  $H_\mu$  is given by

$$D(H_\mu) = \{\psi \in L^2(\mathbb{R}), -\psi'' + x^4\psi \in L^2(\mathbb{R}), x^2\psi \in L^2(\mathbb{R})\},$$

the last property coming from the finiteness of the associated quadratic form  $Q(\psi) = \int_{\mathbb{R}} ((\psi')^2 + (\mu + x^2)^2\psi^2)dx$ . We have hence  $D(H_\mu) = D(H_0)$ . The map  $\mu \mapsto H_\mu$  is analytic from  $\mathbb{R}$  into  $\mathcal{L}(D(H_0), L^2(\mathbb{R}))$ . Moreover, by [BS12, Theorem 3.1], the eigenvalues of  $H_\mu$  are non-degenerate (simple). This implies (see [Kat13, Chapter VII.2] or [CR19, Proposition 5.25]) that the eigenvalues  $\lambda_1(\mu)$  and eigenfunctions  $\psi_\mu$  are analytic functions of  $\mu$ , respectively with values in  $\mathbb{R}$  and in  $D(H_0)$ . This proves Point 1.

Point 2 follows from Agmon estimates (precisely, [Hel88, Proposition 3.3.4] with  $h = h_0 = 1$ ), which are uniform with respect to  $\mu$  on any compact subset of  $\mathbb{R}$ .

This allows to start to prove Point 3 by induction. Assume that Point 3 is true for the derivatives of order  $0, \dots, k-1$ . Then, taking the derivatives with values in the domain  $D(H_0)$  with respect to  $\mu$  in the equation  $(H_\mu - \lambda_1(\mu))\psi_\mu = 0$ , we get

$$(H_\mu - \lambda_1(\mu)) \frac{d^k}{d\mu^k} \psi_\mu = v_{k,\mu} \tag{8}$$

and we know, by the induction hypothesis, that  $v_{k,\mu} \in \mathcal{S}(\mathbb{R})$  uniformly with respect to  $\mu$  on any compact subset of  $\mathbb{R}$ . We now use the results of [Shu87, Section 25] (see also [Shu87, Section 23] for the notations, and [HR82] for similar results). We check that  $\xi^2 + x^4$  is a symbol in the sense of Definition 25.1 of [Shu87], with  $m = 4$ ,  $m_0 = 2$  and  $\rho = 1/2$ . Its standard quantization (i.e.,  $\tau = 0$  in Equation (23.31) of [Shu87]) is  $H_\mu$ . By [Shu87, Theorem 25.1],  $H_\mu - \lambda_1(\mu)$  admits a parametrix  $B_\mu$ ; in particular,  $B_\mu(H_\mu - \lambda_1(\mu)) = \text{Id} + R_\mu$  where  $R_\mu$  is smoothing. Hence, composing on the left by  $B_\mu$  in (8), and noting that  $B_\mu v_{k,\mu} \in \mathcal{S}(\mathbb{R})$ , we obtain that  $\frac{d^k}{d\mu^k} \psi_\mu \in \mathcal{S}(\mathbb{R})$  uniformly with respect to  $\mu$  on any compact subset of  $\mathbb{R}$ , which concludes the induction and the proof of Point 3.

### 3 Wave-front of the Cauchy datum

The goal of this section is to compute the wave-front set of  $u_0$ . In other words, we prove Theorem 1.2 for  $t = 0$ . Recall that (see (3))

$$u_0(x, y, z) = \iint_{\mathbb{R}^2} Y(\zeta) \zeta^{1/2} \phi(\eta_1) \psi_{\eta_1}(\zeta^{1/3}x) e^{i(y\zeta^{1/3}\eta_1 + z\zeta)} d\eta_1 d\zeta. \tag{9}$$

**Lemma 3.1** *The function  $u_0$  is smooth on  $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ .*

---

<sup>4</sup>This means that for any compact  $K \subset \mathbb{R}$ , in the definition of  $\mathcal{S}(\mathbb{R})$ , the constants in the semi-norms can be taken independent of  $\mu \in K$ .



*Proof.* We prove successively that  $u_0$  is smooth outside  $x = 0$ ,  $y = 0$  and  $z = 0$ . Any derivative of (9) in  $x, y, z$  is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta) \zeta^\alpha \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x) \phi(\eta_1) \eta_1^\beta e^{i(y\zeta^{1/3}\eta_1 + z\zeta)} d\eta_1 d\zeta \quad (10)$$

for some  $\alpha, \beta, \gamma \geq 0$ . By the dominated convergence theorem, convergence of these integrals implies smoothness. Recalling that  $\phi$  has compact support, we see that the main difficulty for proving smoothness comes from the integration in  $\zeta$  in (10).

For  $x \neq 0$  it follows from Lemma 2.1 (Point 2) that the integrand in (10) has a fast decay in  $\zeta$ . This proves that  $u_0$  is smooth outside  $x = 0$ .

If  $y \neq 0$ , we use the fact that the phase  $y\zeta^{1/3}\eta_1 + z\zeta$  is non critical with respect to  $\eta_1$  to get the decay in  $\zeta$ . More precisely, (10) is equal to

$$\iint_{\mathbb{R}^2} Y(\zeta) \zeta^\alpha (y\zeta^{1/3})^{-N} D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x) \phi(\eta_1) \eta_1^\beta) e^{i(y\zeta^{1/3}\eta_1 + z\zeta)} d\eta_1 d\zeta$$

after integration by parts in  $\eta_1$  (where  $D_{\eta_1} = i^{-1}\partial_{\eta_1}$ ). Taking  $N$  sufficiently large and using that  $D_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x) \phi(\eta_1) \eta_1^\beta)$  is bounded thanks to Lemma 2.1 (Point 3), we obtain that this integral converges when  $y \neq 0$ . This proves that  $u_0$  is smooth outside  $y = 0$ .

Finally, let us study the case  $z \neq 0$ . We can also assume that  $y \leq 1$  due to the previous point.

*Claim.* The function

$$\zeta \mapsto Y(\zeta) \zeta^{1/2} \phi(\eta_1) \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x) e^{iy\zeta^{1/3}\eta_1} \quad (11)$$

is a symbol (see Definition A.1) uniformly on every compact in  $(y, \eta_1)$ .

*Proof.* The functions  $\zeta \mapsto \zeta^{1/2} \phi(\eta_1)$  and  $\zeta \mapsto Y(\zeta) e^{iy\zeta^{1/3}\eta_1}$  are symbols. Besides,  $\zeta \mapsto \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3} x)$  is also a symbol (of degree 0 with  $\rho = 1$ ): we notice for example that the first derivative with respect to  $\zeta$  writes  $(1/3)\zeta^{-1}(\zeta^{1/3} x) \psi_{\eta_1}^{(\gamma+1)}(\zeta^{1/3} x)$  which is uniformly  $O(1/\zeta)$  thanks to Lemma 2.1 (Point 2). Finally, since the space of symbols is an algebra for the pointwise product, we get the claim.

Integrating (11) in  $\eta_1 \in \mathbb{R}$  and using Lemma A.2 (in the variable  $\zeta$ ), we obtain that (9) is smooth outside  $z = 0$ , which concludes the proof of Lemma 3.1.

The following lemma proves Theorem 1.2 at time  $t = 0$ .

**Lemma 3.2** *There holds  $WF(u_0) = \{(0, 0, 0; 0, 0, \lambda) \in T^*\mathbb{R}^3, \lambda > 0\}$ .*

*Proof.* The Fourier transform of  $u_0$  is

$$U_0(\xi, \eta, \zeta) = Y(\zeta) \phi(\eta/\zeta^{1/3}) \Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3}) \quad (12)$$

where  $\Psi_\mu$  is the Fourier transform of the eigenfunction  $\psi_\mu$ . By Lemma 2.1 (Point 2), for any  $N \in \mathbb{N}$  we get

$$|U_0(\xi, \eta, \zeta)| \leq C_N |\phi(\eta/\zeta^{1/3})| (1 + |\xi/\zeta^{1/3}|)^{-N}. \quad (13)$$

We show that  $U_0$  is fastly decaying in any cone  $C := \{|\xi| + |\eta| \geq c|\zeta|\}$  for  $c$  small. We split the cone into  $C = C_1 \cup C_2$  with  $C_1 = C \cap \{|\xi| \leq |\eta|\}$  and  $C_2 = C \cap \{|\eta| \leq |\xi|\}$ .

In  $C_1$ , we have  $|\eta/\zeta^{1/3}| \geq c_1|\eta^{2/3}|$ . This implies that  $\phi(\eta/\zeta^{1/3})$  vanishes for  $\eta$  large enough. Hence,  $U_0$  has fast decay in  $C_1$ .

In  $C_2$ , we have  $|\xi/\zeta^{1/3}| \geq c_2|\xi|^{2/3} \geq c_3(1 + \xi^2 + \eta^2 + \zeta^2)^{1/3}$ , hence, plugging into (13), we get that  $U_0$  has fast decay in  $C_2$ .

This proves that no point of the form  $(x, y, z; \xi, \eta, \zeta) \in T^*\mathbb{R}^3$  with  $(\xi, \eta) \neq (0, 0)$  can belong to  $WF(u_0)$ . Moreover, due to the factor  $Y(\zeta)$ , necessarily  $WF(u_0) \subset \{\zeta > 0\}$ . Combining with Lemma 3.1, we get the inclusion  $\subset$  in Lemma 3.2.

Let us finally prove that  $(0, 0, 0; 0, 0, \lambda) \in WF(u_0)$  for  $\lambda > 0$ . We pick  $a, b \in \mathbb{R}$  such that  $\phi(a) \neq 0$  and  $\Psi_a(b) \neq 0$ . Then, we note that  $U_0(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$  is independent of  $\zeta$  and  $\neq 0$ , thus it is not fastly decaying as  $\zeta \rightarrow +\infty$ . Since  $(\zeta^{1/3}a, \zeta^{1/3}b, \zeta)$  converges to the direction  $(0, 0, +\infty)$  as  $\zeta \rightarrow +\infty$ , we get that there exists at least one point of the form  $(x, y, z; 0, 0, \lambda) \in T^*\mathbb{R}^3$  which belongs to  $WF(u_0)$ . By Lemma 3.1, we necessarily have  $x = y = z = 0$ , which concludes the proof.

## 4 Wave front of the propagated solution

In this Section, we complete the proof of Theorem 1.2. We set

$$\mathcal{G}_t = \{(0, y, 0; 0, 0, \lambda), \lambda > 0, y \in tF'(\text{Support}(\phi))\}.$$

In Section 4.1, we prove the inclusion  $WF(U(t)u_0) \subset \mathcal{G}_t$ , and then in Section 4.2 the converse inclusion  $\mathcal{G}_t \subset WF(U(t)u_0)$ . This completes the proof of Theorem 1.2.

### 4.1 The inclusion $WF(U(t)u_0) \subset \mathcal{G}_t$

For this inclusion, we follow the same arguments as in Section 3: we adapt Lemma 3.1 to find out the singular support of  $U(t)u_0$ , and then we adapt Lemma 3.2 to determine the full wave-front set.

**Lemma 4.1** *For any  $t \in \mathbb{R}$ ,  $U(t)u_0$  is smooth outside  $\{(0, y, 0) \in \mathbb{R}^3, y \in tF'(I)\}$ .*

*Proof.* As in Lemma 3.1, we prove successively that  $U(t)u_0$  is smooth outside  $x = 0$ ,  $y = 0$  and  $z = 0$ . Any derivative of  $U(t)u_0$  is of the form

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta \quad (14)$$

for some  $\alpha, \beta, \gamma \geq 0$ .

For  $x \neq 0$ , it follows from Lemma 2.1 (Point 2) that the integrand in (14) has a fast decay in  $\zeta$ . This proves that  $U(t)u_0$  is smooth outside  $x = 0$ .

If  $y \notin tF'(I)$ , we use the fact that the phase  $\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta$  is non critical with respect to  $\eta_1$  to get decay in  $\zeta$ . We set  $R_{\eta_1}H = D_{\eta_1}(Q^{-1}H)$  where  $Q = D_{\eta_1}(-i(\zeta^{1/3}(tF(\eta_1) - y\eta_1) - z\zeta)) = -\zeta^{1/3}(tF'(\eta_1) - y)$ . Note that  $Q \neq 0$  since  $y \notin tF'(I)$ . Doing  $N$  integration by parts, the above expression becomes

$$\iint_{\mathbb{R}^2} Y(\zeta)\zeta^\alpha R_{\eta_1}^N(\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta) e^{-i\zeta^{1/3}(tF(\eta_1)-y\eta_1)} e^{iz\zeta} d\eta_1 d\zeta. \quad (15)$$

We set  $H(x, \eta_1, \zeta) = \psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)\phi(\eta_1)\eta_1^\beta$ .

*Claim.* For any  $N$ , there exists  $C_N$  such that  $|R_{\eta_1}^N H(x, \eta_1, \zeta)| \leq C_N |\zeta|^{-N/3}$  for any  $\zeta \in \mathbb{R}$ , any  $\eta_1 \in I = \text{Support}(\phi)$  and any  $x \in \mathbb{R}$ .

Taking  $N$  sufficiently large, the claim implies that (15), and thus (14), converge, which proves the smoothness when  $y \notin tF'(I)$  thanks to the dominated convergence theorem.

*Proof of the claim.* We prove it first for  $N = 1$ . We have

$$R_{\eta_1} H = \frac{D_{\eta_1} H}{Q} - H \frac{D_{\eta_1} Q}{Q^2}. \quad (16)$$

Since  $H$  is bounded (thanks to Point 2 of Lemma 2.1) and  $|Q| \geq c|\zeta|^{1/3}$  with  $c > 0$  and  $|D_{\eta_1} Q| \leq C|\zeta|^{1/3}$  on the support of  $\phi$ , we have  $|H \frac{D_{\eta_1} Q}{Q^2}| \leq c|\zeta|^{-1/3}$ . For the first term in the right-hand side of (16), we only need to prove that  $D_{\eta_1} H$  is bounded. When  $D_{\eta_1}$  falls on  $\phi(\eta_1)$  or  $\eta_1^\beta$ , it is immediate. When  $D_{\eta_1}$  falls on  $\psi_{\eta_1}^{(\gamma)}(\zeta^{1/3}x)$ , we use Lemma 2.1 (Point 3) and also get the result. This ends the proof of the case  $N = 1$ . Now, we notice that our argument works not only for  $H$ , but for any function of the form  $\psi_{\eta_1}^{(\gamma')}(\zeta^{1/3}x)\phi^{(\delta)}(\eta_1)\eta_1^{\beta'}$  where  $\phi^{(\delta)}$  is any derivative of  $\phi$  and  $\beta', \gamma' \geq 0$ . Hence, applying the previous argument recursively, we obtain the claim for any  $N$ .

Finally, the case  $z \neq 0$  is checked in the same way as in the case  $t = 0$ , just shifting the phase by  $it\zeta^{1/3}F(\eta_1)$  in (11).

*Let us finish the proof of the inclusion  $WF(U(t)u_0) \subset \mathcal{G}_t$ .*

The Fourier transform of  $U(t)u_0$  is

$$\mathcal{F}(U(t)u_0)(\xi, \eta, \zeta) = Y(\zeta)\phi(\eta/\zeta^{1/3})\Psi_{\eta/\zeta^{1/3}}(\xi/\zeta^{1/3})e^{-it\sqrt{\alpha_1}(\eta, \zeta)}. \quad (17)$$

The change of phase with respect (12) has no influence on the properties of decay at infinity. Hence, the proof of Lemma 3.2 allows to conclude that  $WF(U(t)u_0) \subset \mathcal{G}_t$  for any  $t \in \mathbb{R}$ .

## 4.2 The inclusion $\mathcal{G}_t \subset WF(U(t)u_0)$

We fix  $t \in \mathbb{R}$  and we prove the non smoothness at  $(0, tF'(c), 0)$  for any  $c \in I$ . We can assume that  $F'(c)$  is not a critical value of  $F'$  since such points are dense in  $I$  because  $F$  is an analytic, non-affine, function (see Point 1 of Lemma 2.1 and Proposition 5.1). We want to show non-smoothness with respect to  $z$  at  $x = 0$ ,  $y = tF'(c)$  and  $z = 0$ . We set  $v(z) := (U(t)u_0)(0, tF'(c), z)$ . We will show that the Fourier transform of  $v$  is not fastly decaying.

Starting from (3), we get the explicit formula for the Fourier transform of  $v$ ,

$$\mathcal{F}v(\zeta) = Y(\zeta)\zeta^{1/2}K(\zeta)$$

where

$$K(\zeta) = \int_{\mathbb{R}} \phi(\eta_1)\psi_{\eta_1}(0)e^{-i\zeta^{1/3}t(F(\eta_1)-F'(c)\eta_1)}d\eta_1.$$

We denote by  $\eta_\alpha$  the critical points of the phase:  $F'(\eta_\alpha) = F'(c)$  and  $\eta_\alpha \in \text{Support}(\phi)$ . By assumption, we know that  $F''(\eta_\alpha) \neq 0$  for any  $\alpha$ . Applying the stationary phase theorem with respect to  $\eta_1$ , we obtain

$$K(\zeta) = \sum_{\alpha} e^{-i\zeta^{1/3}t(F(\eta_\alpha)-F'(c)\eta_\alpha)} \sum_{j \geq 1} a_j^\alpha(\zeta^{1/3}|t|)^{-j/2}$$

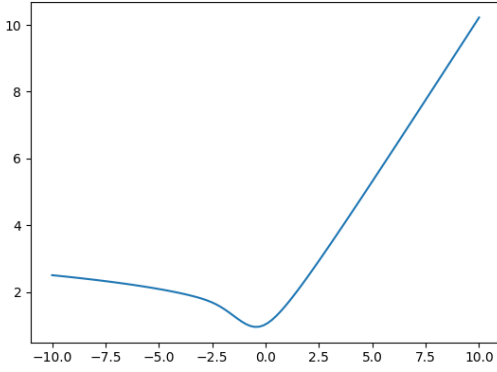
where

$$a_1^\alpha = \phi(\eta_\alpha)\psi_{\eta_\alpha}(0) \left( \frac{2\pi}{|F''(\eta_\alpha)|} \right)^{1/2} \exp(-i\frac{\pi}{4}\text{sgn}(F''(\eta_\alpha))) \neq 0$$

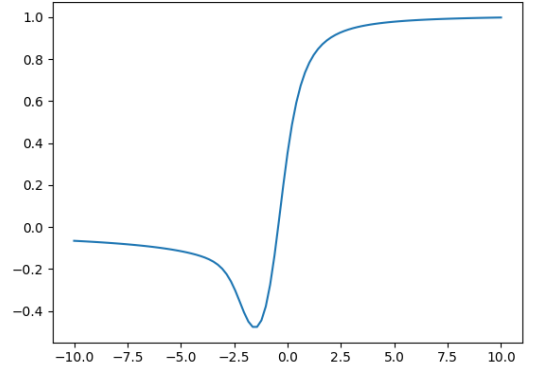
for any  $\alpha$ . Moreover, since  $\phi \geq 0$  and  $\psi_{\eta_\alpha}(0) > 0$  for any  $\alpha$ , the sum  $c = \sum_\alpha a_1^\alpha$  is a sum of the form  $a \exp(i\pi/4) + b \exp(-i\pi/4)$ , which cannot be 0. Hence,  $K(\zeta) \sim c(\zeta^{1/3}|t|)^{-1/2}$ , and  $\mathcal{F}v(\zeta)$  is not fastly decaying as  $\zeta \rightarrow +\infty$ . Applying Lemma A.2 to  $a = \mathcal{F}v$  which is a symbol in  $\zeta$ , this implies that  $v$  is not smooth at  $z = 0$ , thus  $U(t)u_0$  is not smooth at  $(0, tF'(c), 0)$ .

## 5 The function $F_k(\mu) = \sqrt{\lambda_k(\mu)}$

In this Section, we illustrate Theorem 1.2 with some plots and asymptotics of the functions  $F_k$  defined by  $\mu \rightarrow \sqrt{\lambda_k(\mu)}$ . As shown by Theorem 1.2 (and Remark 1.8), the speeds of the propagation of singularities along the singular curve are determined by the derivative  $F'_k(\mu)$ . Below, we plot  $F = F_1$  and  $F'$  for  $\mu \in (-10, 10)^5$ .



(a) Plot of  $F(\mu)$  for  $\mu \in (-10, 10)$



(b) Plot of  $F'(\mu)$  for  $\mu \in (-10, 10)$

Recall that the  $F_k$ 's are analytic (see Point 1 of Lemma 2.1). We state a more precise version of Proposition 1.3:

**Proposition 5.1** *For any  $k \in \mathbb{N} \setminus \{0\}$ , there holds  $F'_k(\mu) \rightarrow 1^-$  as  $\mu \rightarrow +\infty$ ,  $F'_k(\mu) \rightarrow 0^-$  as  $\mu \rightarrow -\infty$ , and  $F'_k$  is minimal for some value  $\mu_k^* < 0$ . There exists  $a_k \in (-1, 0)$  such that the range of  $F'_k$  is  $[a_k, 1)$ .*

Proposition 5.1 will be a consequence of the following result:

**Proposition 5.2** *Denote by  $\lambda_k(\mu)$  the  $k$ -th eigenvalue of  $H_\mu = -d_x^2 + (\mu + x^2)^2$ . Then, for  $k \in \mathbb{N} \setminus \{0\}$ , as  $\mu \rightarrow +\infty$ ,*

$$\lambda_k(\mu) = \mu^2 + \sqrt{2}(2k-1)\sqrt{\mu} + \sum_{\ell=2}^{\infty} b_{\ell,k}\mu^{2-3\ell/2} \quad (18)$$

and

$$\frac{d}{d\mu}\sqrt{\lambda_k(\mu)} = 1 - \frac{2k-1}{2\sqrt{2}}\mu^{-3/2} + o(\mu^{-3/2}) \quad (19)$$

<sup>5</sup>We thank Julien Guillod for his help in making the first numerical experiments.

These derivatives are  $> 0$  and converge to 1.

As  $\mu \rightarrow -\infty$ , for  $k \in \mathbb{N} \setminus \{0\}$ ,

$$\lambda_{2k-1}(\mu) = 2(2k-1)\sqrt{-\mu} + \sum_{\ell=2}^{\infty} c_{\ell,k}(-\mu)^{2-3\ell/2} \quad (20)$$

$$\lambda_{2k}(\mu) = \lambda_{2k-1} + o(\mu^{-\infty}) \quad (21)$$

and

$$\frac{d}{d\mu} \sqrt{\lambda_{2k-1}(\mu)} = -\frac{\sqrt{2(2k-1)}}{4}(-\mu)^{-3/4} + o((-\mu)^{3/4}) \quad (22)$$

and the same for  $\frac{d}{d\mu} \sqrt{\lambda_{2k}(\mu)}$ . These derivative are  $< 0$  and converge to 0.

*Proof of Proposition 5.2.* For  $\mu > 0$ , we consider the operator  $T_\mu : \psi \mapsto \psi(\cdot/\mu^{1/4})$ . Then  $H_\mu = T_\mu^{-1}G_\mu T_\mu$  where  $G_\mu = \mu^2 + \mu^{1/2}(-d_x^2 + 2x^2 + x^4/\mu^{3/2})$ . The eigenvalues of  $-d_x^2 + 2x^2 + hx^4$  for  $h \rightarrow 0$  can be computed with the usual perturbation theory (see [RS78, Chapter XII.3]), and this yields (18) with  $h = \mu^{-3/2}$ . Moreover the formal expansion can be differentiated with respect to  $\mu$ , hence we get (19).

For  $\mu = -\mu_0 < 0$ , we see that the transformation  $x \mapsto \mu_0^{1/4}(x \mp \mu_0^{1/2})$  conjugates  $H_\mu$  to the operator  $\mu_0^{1/2}(-d_x^2 + 4x^2 \pm 4\mu_0^{-3/4}x^3 + \mu_0^{-3/2}x^4)$ . Using again perturbation theory and the separation into pairs of eigenvalues in double wells (see [HS84]), we get (20) and (21), and (22) follows.

*Proof of Proposition 5.1.* The convergences at  $\pm\infty$  are proved by Proposition 5.2. This behaviour at  $\pm\infty$  implies the existence of  $\mu_k^*$  such that  $F_k'(\mu_k^*) = a_k$  is minimal. We denote by  $\psi_\mu^k$  the normalized eigenfunction corresponding to  $\lambda_k(\mu)$ . Taking the first derivative (with value in the domain  $D(H_0)$ ) with respect to  $\mu$  of the eigenfunction equation  $(H_\mu - \lambda_k(\mu))\psi_\mu^k = 0$ , and then integrating against  $\psi_\mu^k$ , we obtain  $\lambda_k'(\mu) = \int_{\mathbb{R}} (\mu + x^2)\psi_\mu^k(x)^2 dx$ . Thus,

$$F_k'(\mu) = \frac{1}{\sqrt{\lambda_k(\mu)}} \int_{\mathbb{R}} (\mu + x^2)\psi_\mu^k(x)^2 dx$$

which is positive for  $\mu \geq 0$ , hence  $\mu_k^* < 0$ .

It remains to show that  $|F_k'(\mu)| < 1$  for every  $\mu$ : by the Cauchy-Schwarz inequality, we get

$$F_k'(\mu)^2 \leq \frac{1}{\lambda_k(\mu)} \int_{\mathbb{R}} (\mu + x^2)^2 \psi_\mu^k(x)^2 dx \int_{\mathbb{R}} \psi_\mu^k(x)^2 dx$$

and, from the quadratic form associated to  $H_\mu$ ,

$$\int_{\mathbb{R}} (\mu + x^2)^2 \psi_\mu^k(x)^2 dx < \lambda_k(\mu),$$

which concludes the proof.

# Appendix

## A Fourier transform of symbols

**Definition A.1** A smooth function  $a : \mathbb{R}^d \rightarrow \mathbb{C}$  is called a symbol of degree  $\leq m$  if there exists  $0 < \rho \leq 1$  so that the partial derivatives of  $a$  satisfy

$$\forall \alpha \in \mathbb{N}^d, \quad |D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m - \rho|\alpha|}.$$

The space of symbols is an algebra for the pointwise product. If  $a$  is a real valued symbol of degree  $m < 1$  and  $\rho > m$ ,  $e^{ia}$  is a symbol of degree 0 (with a different  $\rho$ ).

We will need the

**Lemma A.2** If  $a$  is a symbol, the Fourier transform  $\mathcal{F}a$  of  $a$  is smooth outside  $x = 0$  and all derivatives of  $\mathcal{F}a$  decay fastly at infinity. If moreover  $a$  does not belong to the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$ , then  $\mathcal{F}a$  is non smooth at  $x = 0$ .

*Proof.* For  $x \neq 0$  and for any  $\alpha, \beta \in \mathbb{N}^d$ , we have

$$(\mathcal{F}a)^{(\beta)}(x) = C_\beta \int_{\mathbb{R}^d} \xi^\beta a(\xi) e^{-ix\xi} d\xi = \frac{c_\beta^\alpha}{x^\alpha} \int_{\mathbb{R}^d} D_\xi^\alpha (\xi^\beta a(\xi)) e^{-ix\xi} d\xi. \quad (23)$$

The multi-index  $\beta \in \mathbb{N}^d$  being fixed, this last integral converges for  $|\alpha|$  sufficiently large since  $a$  is a symbol. By the dominated convergence theorem, this implies that  $\mathcal{F}a$  is smooth outside  $x = 0$ . Moreover, (23) also implies that all derivatives of  $\mathcal{F}a$  decay fastly at infinity.

Finally, if  $\mathcal{F}a$  were smooth at 0, then  $\mathcal{F}a$  would be in the Schwartz space as well as  $a$ .

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