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# Fast Gossiping by Short Messages\*

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**Abstract.** Gossiping is the process of information diffusion in which each node of a network holds a packet that must be communicated to all other nodes in the network. We consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number  $p$  of packets at each round. In the first part of the paper we study the extremal case  $p = 1$  and we exactly determine the optimal number of communication rounds to perform gossiping for several classes of graphs, including Hamiltonian graphs and complete  $k$ -ary trees. For arbitrary graphs we give asymptotically matching upper and lower bounds. We also study the case of arbitrary  $p$  and we exactly determine the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, rings, and paths.

## 1 Introduction

Gossiping (also called total exchange or all-to-all communication) in distributed systems is the process of distribution of information known to each processor to every other processor of the system. This process of information dissemination is carried out by means of a sequence of message transmissions between adjacent nodes in the network.

Gossiping is a fundamental primitive in distributed memory multiprocessor system. There are a number of situations in multiprocessor computation, such as global processor synchronization, where gossiping occurs. Moreover, the gossiping problem is implicit in a large class of parallel computation problems, such as linear system solving, Discrete Fourier Transform, and sorting, where both input and output data are required to be distributed across the network [7]. Due to the interesting theoretical questions it poses and its numerous practical applications, gossiping has been widely studied under various communication models. Hedetniemi, Hedetniemi and Liestman [12] provide a survey of the area. Two

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more recent surveys paper collecting the latest results are [9, 14]. The reader can also profitably see the book [16].

The great majority of the previous work on gossiping has considered the case in which the packets known to a processor at any given time during the execution of the gossiping protocol can be freely concatenated and the resulting (longer) message can be transmitted in a constant amount of time, that is, it has been assumed that the time required to transmit a message is independent from its length. While this assumption is reasonable for short messages, it is clearly unrealistic in case the size of the messages becomes large. Notice that most of the gossiping protocols proposed in the literature require the transmission, in the last rounds of the execution of the protocol, of messages of size  $\Theta(n)$ , where  $n$  is the number of nodes in the network. Therefore, it would be interesting to have gossiping protocols that require only the transmission of bounded length messages between processors. In this paper we consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number  $p$  of packets at each round.

## 1.1 The Model

Consider a communication network modeled by a graph  $G = (V, E)$  where the node set  $V$  represents the set of processors of the network and  $E$  represents the set of the communication lines between processors.

Initially each node holds a packet that must be transmitted to any other node in the network by a sequence of *calls* between adjacent processors. During each call, communicating nodes can exchange up to  $p$  packets, where  $p$  is an *a priori* fixed integer. We assume that each processor can participate in at most one call at time. Therefore, we can see the gossiping process as a sequence of *rounds*: During each round a disjoint set of edges (matching) is selected and the nodes that are end vertices of these edges make a call. This communication model is usually referred to as *telephone model* [12] or *Full-Duplex 1-Port* ( $F_1$ ) [15]. We denote by  $g_{F_1}(p, G)$  the minimum possible number of rounds to complete the gossiping process in the network  $G$  subject to the above conditions. Another popular communication model is the *mail model* [12] or *Half-Duplex 1-Port* ( $H_1$ ) [15], in which in each round any node can either send a message to one of its neighbors or receive a message from it but not simultaneously. The problem of estimating  $g_{H_1}(p, G)$  has been considered in [4]. Analogous problems in bus networks have been considered in [10, 13]. Optimal bounds on  $g_{H_1}(1, G)$  when the edges of  $G$  are subject to random failures are given in [8]. Packet routing in interconnection networks in the  $F_1$  model has been considered in [1].

## 1.2 Results

We first study the extremal case in which gossiping is to be performed under the restriction that communicating nodes can exchange *exactly* one packet at each round. We provide several lower bounds on the gossiping time  $g_{F_1}(1, G)$  and we provide matching upper bounds for Hamiltonian graphs, complete trees,

and complete bipartite graphs. For general graphs we provide asymptotically tight upper and lower bounds. Subsequently, we study the case of arbitrary  $p$  and we compute exactly  $g_{F_1}(p, G)$  for complete graphs, hypercubes, rings and paths. Our result for hypercubes allows us to improve the corresponding result in the  $H_1$  model given in [4]. Due to the space limits, all proofs are omitted. We refer to the full version [5] for all omitted proofs and some additional results.

## 2 Gossiping by exchanging one packet at time

In this section we study  $g_{F_1}(1, G)$ , that is the minimum possible number of rounds to complete gossiping in a graph  $G$  under the condition that at each round communicating nodes can exchange *exactly one* packet. In order to avoid overburdening the notation, we will simply write  $g(G)$  to denote  $g_{F_1}(1, G)$ .

### 2.1 Lower bounds on $g(G)$

In this section we give some lower bounds on the time needed to complete the gossiping process.

**Lemma 2.1** *For any graph  $G = (V, E)$ , with  $|V| = n$ , let  $\mu(G)$  be the size of a maximum matching in  $G$ , then  $g(G) \geq \left\lceil \frac{n(n-1)}{2\mu(G)} \right\rceil$ .*

**Lemma 2.2** *Let  $X \subset V$  be a vertex cutset of the graph  $G = (V, E)$  whose removal disconnects  $G$  into the connected components  $V_1, \dots, V_d$ , then  $g(G) \geq \left\lceil \sum_{i=1}^d \frac{\max\{|V_i|, n-|V_i|\}}{|M_X|} \right\rceil$ , where  $|M_X|$  is the size of a maximum matching  $M_X$  in  $G$  such that any edge in it has an endpoint in  $X$  and the other in  $V - X$ .*

**Corollary 2.1** *Let  $\alpha(G)$  be the independence number of  $G$ , then*

$$g(G) \geq \left\lceil \frac{\alpha(G)(n-1)}{n-\alpha(G)} \right\rceil. \quad (1)$$

Let  $T$  be a tree and  $v$  one of its nodes, we indicate the connected components into which the node set of  $T$  is splitted by the removal of  $v$  by  $V_1(v), \dots, V_{\deg(v)}(v)$ , ordered so that  $|V_1(v)| \geq \dots \geq |V_{\deg(v)}(v)|$ .

**Corollary 2.2** *Let  $T$  be a tree on  $n$  nodes of maximum degree  $\Delta = \max_{v \in V} \deg(v)$ , then  $g(T) \geq \max_{v: \deg(v)=\Delta} L(v)$ , where*

$$L(v) = \begin{cases} (\deg(v) - 1)n + 1 & \text{if } |V_1(v)| \leq n/2; \\ (\deg(v) - 2)n + 1 + 2|V_1(v)| & \text{if } |V_1(v)| > n/2. \end{cases}$$

## 2.2 Upper bounds

We will determine exactly  $g(G)$  for several classes of graphs, including Hamiltonian graphs and complete  $k$ -ary trees. We will also provide good upper bounds for general graphs. We first note that in any graph  $G = (V, E)$  the size of a maximum matching  $\mu(G)$  is at most  $\lfloor |V|/2 \rfloor$ . Therefore, from Lemma 2.1 we get that the gossiping time  $g(G)$  of *any* graph with  $n$  nodes is always lower bounded by

$$g(G) \geq \begin{cases} n - 1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases} \quad (2)$$

We can prove that this lower bound is attained by Hamiltonian graphs.

Let  $C_n = (V, E)$  denote the ring of length  $n$ ; we assume the vertex set be  $V = \{0, \dots, n - 1\}$  and the edge set be  $E = \{(v, w) : 1 = |v - w| \pmod{n}\}^2$ .

**Lemma 2.3** 
$$g(C_n) \leq \begin{cases} n - 1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** We shall discuss only the case  $n$  even. The case  $n$  odd is similar. For each integer  $t$  define the perfect matching in  $C_n$  given by

$$M_t = \begin{cases} \{(v, w) : v \text{ is even and } w = v + 1\} & \text{if } t \text{ is even} \\ \{(v, w) : v \text{ is odd and } w = v + 1 \pmod{n}\} & \text{if } t \text{ is odd;} \end{cases} \quad (3)$$

notice that  $M_t$  and  $M_{t+1}$  are disjoint for each  $t$ . The gossiping algorithm is shown in Figure 1. The easy proof of correctness is left to the reader.

### Gossiping-even( $C_n$ )

**Round  $t = 1$ :** each node  $v$  sends its own packet to the node  $w$  such that  $(v, w) \in M_1$ ;  
**Round  $t = 2$ :** each node  $v$  sends its own packet to the node  $w$  such that  $(v, w) \in M_2$ ;  
**Round  $t, 3 \leq t \leq n - 1$ :** For each node  $v$  let  $w$  be the node such that  $(v, w) \in M_t$ , node  $v$  sends a new packet to  $w$ , namely  $v$  sends the packet it has first got among those  $v$  has neither received from  $w$  nor sent to  $w$  in any previous round.

Figure 1: Gossiping Algorithm in  $C_n, n$  even.

From Lemma 2.3 and (2) we immediately get

**Theorem 2.1** *For any Hamiltonian graph  $G$  on  $n$  vertices we have*

$$g(G) = \begin{cases} n - 1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

<sup>2</sup> Here and in the rest of the paper with  $x = a \pmod{b}$  we denote the unique integer  $0 \leq x < b$  such that  $x = qb + a$ .

## 2.3 Trees

In this section we investigate the gossiping time in trees. We first give an upper bound on the gossiping time in *any* tree and afterwards we exactly compute the gossiping time of  $k$ -ary trees.

Consider a tree  $T = (V, E)$ . We recall that for each node  $v$  the set  $V_1(v)$  denotes the largest of the connected components into which  $T$  is splitted by the removal of  $v$ . Let  $\vartheta = \max |V_1(v)|$ , where the maximum is taken over all the internal nodes  $v$  having exactly  $\deg(v) - 1$  leaves as neighbors; notice that any other internal node  $u$  has  $|V_1(u)| \leq \vartheta - 1$ .

Call *pre-leaf* any node  $v$  such that  $|V_1(v)| = \vartheta$  and denote by  $\pi$  the maximum degree of a node in the subgraph consisting only of the edges  $(u, f)$  where  $f$  is either a leaf or a pre-leaf of  $T$ .

Finally, let  $\lambda$  be the maximum number of leaves connected to a same node and  $\Delta = \max_{v \in V} \deg(v)$ .

**Theorem 2.2** *For any tree  $T$  on  $n$  nodes  $g(T) \leq (\vartheta - 1)\Delta + \pi + (n - \vartheta - 1)\lambda$ .*

The gossiping algorithm in  $T$  is given in the Figure 2.

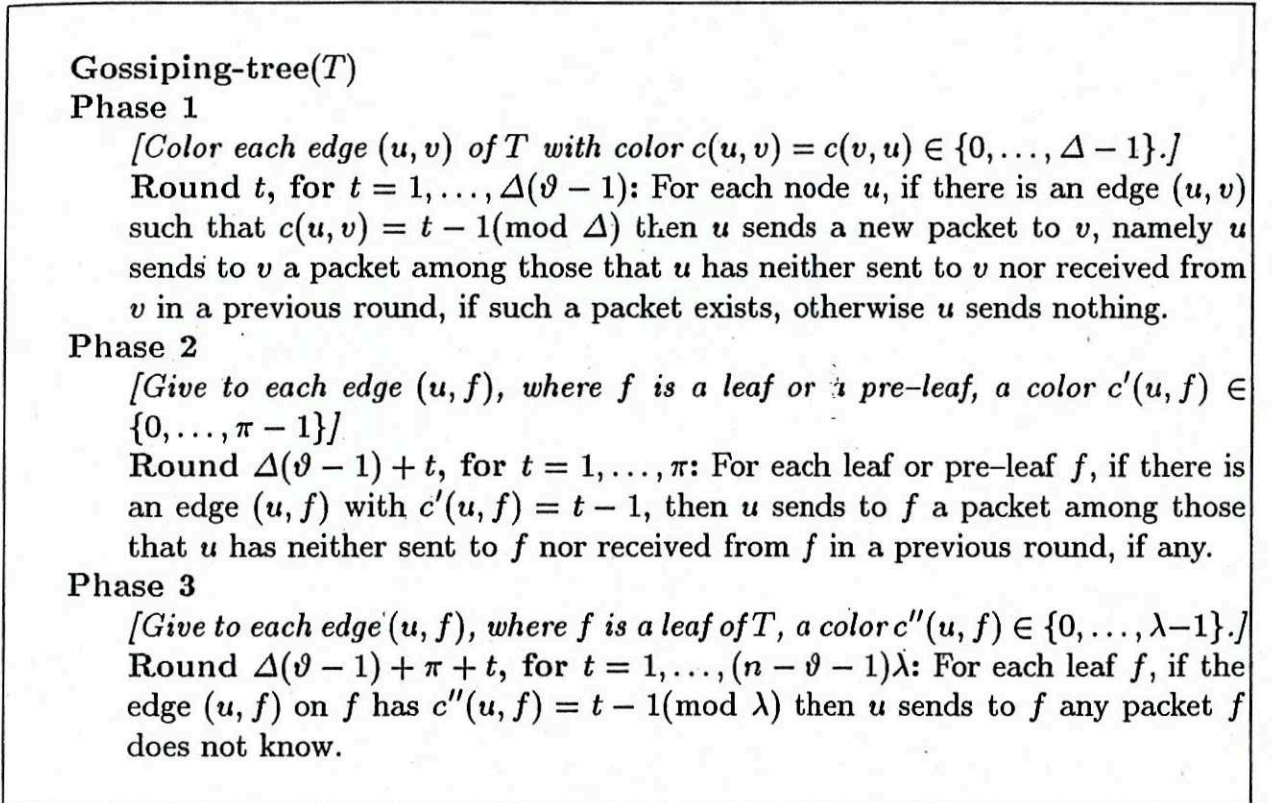


Figure 2: Gossiping Algorithm in a tree  $T$ .

Let  $\delta$  denote the minimum degree of an internal node in  $T$ . It is easy to see that we can upper bound  $\vartheta$  by  $n - \delta$ . Therefore, from Theorem 2.2 we have the following upper bound on  $g(T)$  that is expressed only in terms of degree properties of the nodes in  $T$ .

**Corollary 2.3** *For any tree  $T$  on  $n$  nodes  $g(T) \leq (n - \delta)\Delta + (\delta - 1)\lambda$ .*

Given a connected graph  $G = (V, E)$ , denote by  $\mathcal{T}$  the set of all spanning trees of  $G$  and for any vertex  $v \in V$  denote by  $\deg_T(v)$  the degree of  $v$  in  $T \in \mathcal{T}$ . Define  $d(G) = \min_{T \in \mathcal{T}} \max_{v \in V} \deg_T(v)$ . The following corollary is immediate.

**Corollary 2.4** For any connected graph  $G = (V, E)$  with  $n$  vertices

$$g(G) \leq (n - 1)d(G). \quad (4)$$

We point out that, although the problem of computing  $d(G)$  is *NP*-hard, there exists an efficient algorithm to compute a spanning tree of maximum degree at most  $d(G) + 1$  (see [11]). From Corollary 2.2 and Corollary 2.4 we have that for any tree with  $n$  nodes and maximum degree  $\Delta$  it holds  $n\Delta - n + 1 \leq g(T) \leq n\Delta - \Delta$ . Let us consider now the tree  $S_{n,\Delta}$  of Figure 3. If  $\Delta = n - 1$  then  $S_{n,n-1}$  is the star on  $n$  nodes and from Corollary 2.2 and Theorem 2.2 we have  $g(S_{n,n-1}) = (n - 1)^2$ . If  $\Delta > 2$  is constant with respect to  $n > 2\Delta$  then from Corollary 2.2 and Theorem 2.2 we get  $\Delta(n - 1) - (\Delta - 1) \leq g(S_{n,\Delta}) \leq \Delta(n - 1) - 2$ . It is not difficult to obtain a specific gossiping algorithm attaining the lower bound. Therefore, we have that for any  $n$  and  $\Delta$  there exists a graph  $G_{n,\Delta}$  with  $n$  vertices and maximum degree  $\Delta$  such that  $g(G_{n,\Delta}) = \Omega((n - 1)\Delta)$ , hence the bound (4) is asymptotically tight. In [8] it is conjectured that for any graph  $G$  it holds  $g_{H_1}(1, G) = \Omega(nd(G))$ . This conjecture, if true, together with Corollary 2.4 would imply the rather interesting fact that for any graph  $G$  it holds  $g(G) = \Theta(nd(G))$ .

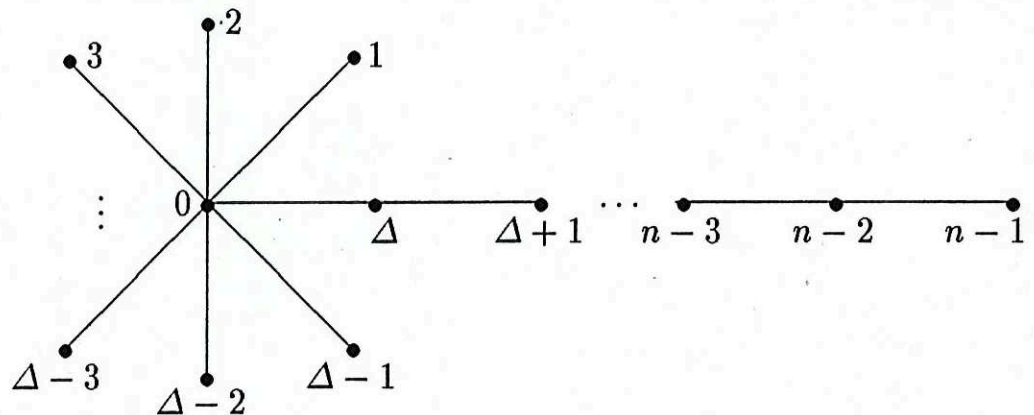


Figure 3: Tree  $S_{n,\Delta}$

We shall now exactly compute the gossiping time of  $k$ -ary trees, that is, rooted trees in which each internal node has exactly  $k$  sons. Let  $T_{k,n}$  denote any  $k$ -ary tree with  $n$  nodes.

Let us first notice that for  $n = k + 1$  the tree  $T_{k,n}$  is the star  $S_{k+1,k}$ . Consider then a tree  $T_{k,n}$  with  $n \geq 2k + 1$  nodes. Let  $u$  be a node of  $T_{k,n}$  whose sons are all leaves, by Corollary 2.2 we get

$$g(T_{k,n}) \geq \max_v L(v) \geq L(u) = \begin{cases} kn + 1 & \text{if } n = 2k + 1 \\ (k + 1)(n - 1) - k & \text{if } n \geq 3k + 1. \end{cases} \quad (5)$$

We show now that (5) holds with equality. Applying Theorem 2.2 to  $T_{k,n}$  we get that

$$g(T_{k,n}) \leq (\vartheta - 1)\Delta + \pi + (n - \vartheta - 1)\lambda = (\vartheta - 1)(k + 1) + \pi + (n - \vartheta - 1)k. \quad (6)$$

Unless exactly  $k - 1$  sons of the root are leaves (cf. the tree in Figure 4)  $T_{k,n}$  has  $\vartheta = n - k - 1$  and  $\pi \leq \Delta = k + 1$ , that by (6) and (5) gives

$$g(T_{k,n}) = (n - k - 2)(k + 1) + k + 1 + k^2 = (k + 1)(n - 1) - k.$$

Consider now the remaining case when  $T_{k,n}$  is the tree of Figure 4. The only pre-leaf is the root, and  $\vartheta = n - k$ . If  $n \geq 3k + 1$  we have  $\pi = k$  and from (6) we get

$$g(T_{k,n}) \leq (n - k - 1)(k + 1) + k + (k - 1)k = (n - 1)(k + 1) - k;$$

if  $n = 2k + 1$  we have  $\pi = \Delta = k + 1$  and  $g(T_{k,2k+1}) \leq kn + 1$ .

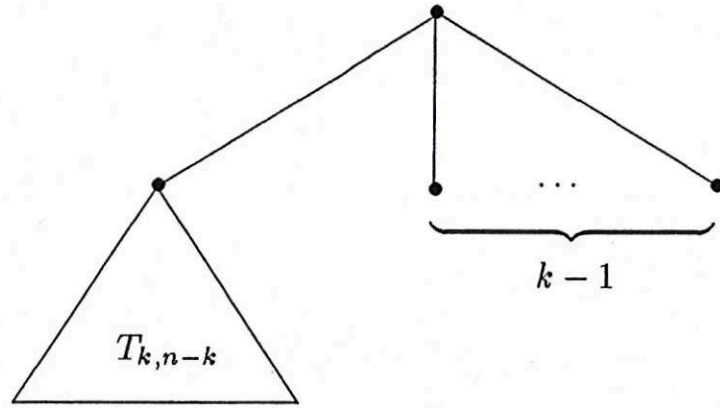


Figure 4

Therefore, we have proved the following result.

**Theorem 2.3** For any  $k$ -ary tree on  $n$  nodes  $T_{k,n}$  it holds that

$$g(T_{k,n}) = \begin{cases} k^2 & \text{if } n = k + 1 \\ 2k^2 + k + 1 & \text{if } n = 2k + 1 \\ (k + 1)(n - 1) - k & \text{if } n \geq 3k + 1. \end{cases}$$

The particular case  $k = 1$  of above theorem deserves to be explicitly stated.

**Corollary 2.5** For the path on  $n$  nodes  $P_n$  we have  $g(P_n) = \begin{cases} 1 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 2n - 3 & \text{if } n \geq 4. \end{cases}$

## 2.4 Complete bipartite graphs

Let  $K_{r,s} = (V(K_{r,s}), E(K_{r,s}))$  be the complete bipartite graph on the node set  $V(K_{r,s}) = \{a_0, \dots, a_{r-1}\} \cup \{b_0, \dots, b_{s-1}\}$ , with  $\{a_1, \dots, a_{r-1}\} \cap \{b_0, \dots, b_{s-1}\} = \emptyset$ ,  $r \geq s$ , and edge set  $E(K_{r,s}) = \{a_0, \dots, a_{r-1}\} \times \{b_0, \dots, b_{s-1}\}$ . In the next theorem we determine the gossiping time of  $K_{r,s}$ .

**Theorem 2.4** For each  $r$  and  $s$ ,  $r \geq s \geq 1$ , it holds  $g(K_{r,s}) = \lceil (r + s - 1)r/s \rceil$ .



**Proof.** The lower bound  $g(K_{r,s}) \geq \lceil (r+s-1)r/s \rceil$  is an immediate consequence of Corollary 2.1 since the complete bipartite graph has  $\alpha(K_{r,s}) = r$ .

In order to give a gossiping algorithm in  $K_{r,s}$  requiring  $\lceil (r+s-1)r/s \rceil$  communication rounds, we define the matchings

$$M_j = \{(b_i, a_{i+j \pmod{r}}) : 0 \leq i \leq s-1\},$$

for  $j = 0, \dots, r-1$ . The algorithm is shown in Figure 5.

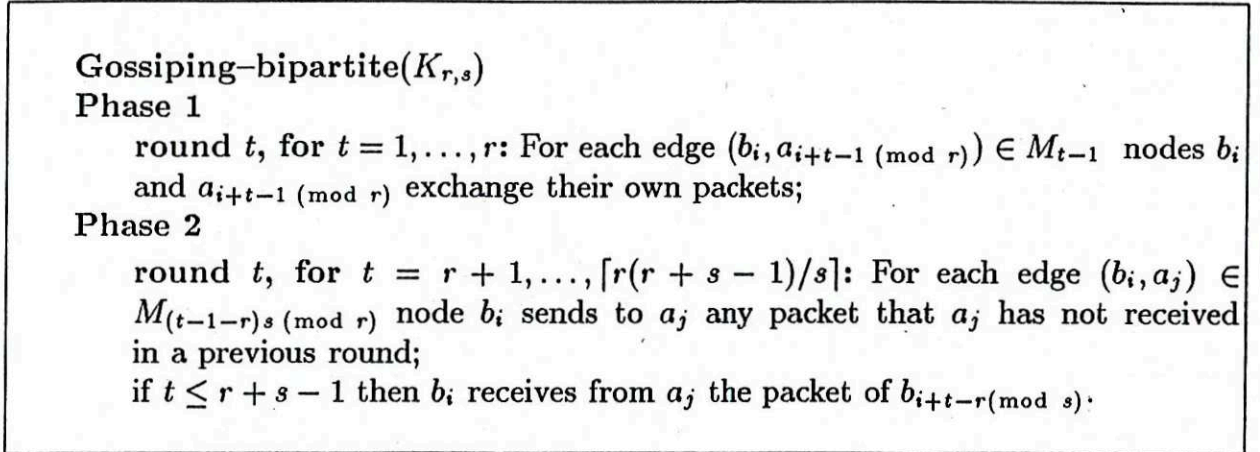


Figure 5: Gossiping Algorithm in  $K_{r,s}$ .

## 2.5 Generalized Petersen Graphs

In Section 2.2 we have seen that Hamiltonian graphs have the minimum possible gossiping time among all graphs with  $n$  nodes. A natural question to ask is to see if there are non-Hamiltonian graphs on  $n$  vertices with gossiping time equal to  $n$  if  $n$  is odd and  $n-1$  if  $n$  is even. A quick check shows that this is not the case for rectangular grids  $G_{t,s}$  with both  $t$  and  $s$  odd<sup>3</sup>. In fact, we know that  $\alpha(G_{t,s}) = \lceil \frac{s \cdot t}{2} \rceil$  and from Corollary 2.1 we get  $g(G_{t,s}) \geq s \cdot t + 1$ . Moreover, it is also easy to check that the gossiping time of the Petersen graph on 10 vertices is at least 10. Therefore, one could be tempted to conjecture that the gossiping time  $g(G)$  of a graph  $G$  is equal to the minimum possible only if  $G$  is Hamiltonian. This conjecture, although nice sounding, would be wrong as the following classes of graphs, including the Generalized Petersen Graphs, shows.

Let  $P_{k,\pi}$  be the graph consisting of two cycles of size  $k$  connected by a perfect matching in the following way: given a permutation  $\pi$  of  $\{0, \dots, k-1\}$  the graph  $P_{k,\pi} = (V(P_{k,\pi}), E(P_{k,\pi}))$  has vertex set  $V(P_{k,\pi}) = \{a_0, \dots, a_{k-1}\} \cup \{b_0, \dots, b_{k-1}\}$  and edge set

$$E(P_{k,\pi}) = \{(a_i, a_{i+1 \pmod{k}}) : 0 \leq i < k\} \cup \{(b_i, b_{i+1 \pmod{k}}) : 0 \leq i < k\} \\ \cup \{(a_i, b_{\pi(i)}) : 0 \leq i < k\}.$$

<sup>3</sup> It is well known that all rectangular grids  $G_{t,s}$  are Hamiltonian but for values of  $t$  and  $s$  both odd.

The Petersen Graph has  $k = 5$  and  $\pi(i) = 3i \pmod{5}$ , for  $i = 0, 1, 2, 3, 4$ ; Generalized Petersen Graphs (GPG) have  $k$  odd and  $\pi(s \cdot i \pmod{k}) = i$ ,  $i = 0, \dots, k - 1$ , for a fixed integer  $s$ . From Lemma 2.1 we know that  $g(P_{k,\pi}) \geq |V(P_{k,\pi})| - 1 = 2k - 1$ . We can prove the following theorem.

**Theorem 2.5** *For any  $k$  and  $\pi$  such that  $P_{k,\pi}$  is 3-edge-colorable, we have  $g(P_{k,\pi}) = 2k - 1$ .*

Notice that each cubic GPG, other than the Petersen graph itself, is 3-edge-colorable. Moreover, the class of 3-edge-colorable  $P_{k,\pi}$ 's includes the family of non Hamiltonian GPGs with  $k = 5 \pmod{6}$  and  $s = 2$  (see [2] and references therein quoted).

### 3 Gossiping by exchanging more than one packet at time

In this section we shall study the minimum number of time units  $g_{F_1}(p, G)$  necessary to perform gossiping in a graph  $G$ , under the restriction that at each time instant communicating nodes can exchange up to  $p$  packets,  $p$  fixed but arbitrary otherwise. We assume that  $p$  is smaller than the number of nodes of the graph  $G$ , otherwise the problem is equivalent to the classical one. Again, for ease of notation, we shall write  $g(p, G)$  to denote  $g_{F_1}(p, G)$ .

#### 3.1 Lower Bounds

First of all we shall present a simple lower bound on  $g(p, G)$  based on elementary counting arguments. Nonetheless, we shall prove in the sequel that the obtained lower bound is tight for complete graphs with an even number of nodes and for hypercubes. In order to derive the lower bound, let us define  $I(p, t)$  as the maximum number of packets a vertex can have possibly received after  $t$  communication rounds in *any* graph. Since at each round  $i$ , with  $1 \leq i \leq t$ , any vertex can receive at most  $\min\{p, 2^{i-1}\}$  packets, it follows that  $I(p, t) = 1 + \sum_{i=1}^t \min\{p, 2^{i-1}\}$ , or, equivalently

$$I(p, t) = 1 + \sum_{i=1}^{\lceil \log p \rceil} 2^{i-1} + p(t - \lceil \log p \rceil) = 2^{\lceil \log p \rceil} + p(t - \lceil \log p \rceil) \quad (7)$$

for any  $t \geq \lceil \log p \rceil$ . Therefore, for any graph  $G = (V, E)$ , the gossiping time  $g(p, G)$  is always lower bounded by the smallest integer  $t^*$  for which  $I(p, t^*) \geq |V|$ . Since  $t^*$  is obviously greater or equal to  $\lceil \log |V| \rceil \geq \lceil \log p \rceil$ , we can use (7) and obtain  $g(p, G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p} (|V| - 2^{\lceil \log p \rceil}) \right\rceil$ . Moreover, notice that if the number of nodes in the graph is odd then at each round there is a node that does not receive any message. This implies that after any round  $t$  there exists a node who can have possibly received at most  $I(p, t - 1)$  packets. Therefore,  $g(p, G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p} (|V| - 2^{\lceil \log p \rceil}) \right\rceil + 1$ . The above arguments give the following lemma.

**Lemma 3.1** For any graph  $G = (V, E)$ ,  $|V| = n$ , and integer  $p$  such that  $2^{\lceil \log p \rceil} \leq n$  we have

$$g(p, G) \geq \begin{cases} \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil & \text{if } n \text{ is even,} \\ \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil + 1 & \text{if } n \text{ is odd.} \end{cases}$$

### 3.2 Rings and Paths

Let  $g(\infty, G)$  denote the gossiping time of the graph  $G$  in absence of any restriction on the size of the messages. It is obvious that for each  $p$  it holds  $g(p, G) \geq g(\infty, G)$ , it is possible to see that equality holds for any  $p \geq 2$  when  $G$  is either the ring  $C_n$  or the path  $P_n$  on  $n$  nodes. It is well known that [14]

$$g(\infty, P_n) = 2 \left\lceil \frac{n}{2} \right\rceil - 1 \quad \text{and} \quad g(\infty, C_n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

We just point out that it is easy to see that the algorithms attaining  $g(\infty, C_n)$  and  $g(\infty, P_n)$  do not need to send more than 2 packets at time. Therefore the following results hold.

**Theorem 3.1** For each  $n \geq 3$  and  $p \geq 2$  it holds

$$g(p, C_n) = g(2, C_n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 3.2** For each  $n \geq 2$  and  $p \geq 2$  it holds  $g(p, P_n) = g(2, P_n) = 2 \left\lceil \frac{n}{2} \right\rceil - 1$ .

### 3.3 Complete graphs

Let  $K_n$  be the complete graph on  $n$  nodes. We recall that  $g(\infty, K_n)$  is equal to  $\lceil \log n \rceil$  if  $n$  is even, and  $\lceil \log n \rceil + 1$  if  $n$  is odd.

**Theorem 3.3** For each even integer  $n$  and integer  $p$  such that  $2^{\lceil \log p \rceil} \leq n$  it holds  $g(p, K_n) = \lceil \log p \rceil + \left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil$ .

**Theorem 3.4** For each odd integer  $N$  and integer  $p$  such that  $2^{\lceil \log p \rceil} \leq N + 1$  it holds  $\lceil \log p \rceil + \left\lceil \frac{N - 2^{\lceil \log p \rceil}}{p} \right\rceil + 1 \leq g(p, K_N) \leq \lceil \log p \rceil + \left\lceil \frac{N + 1 - 2^{\lceil \log p \rceil}}{p} \right\rceil + 2$ .

For  $N$  odd, we believe that the true value of  $g(p, K_N)$  is  $\lceil \log p \rceil + \left\lceil \frac{N - 2^{\lceil \log p \rceil}}{p} \right\rceil + 1$ ; we can verify this equality for small values of  $N$  and  $p$ . In case  $p = 2$ , Theorem 3.1 and Lemma 3.1 tell us that  $g(2, K_N) = (N + 3)/2 = g(2, C_N)$ , for each odd  $N \geq 3$ . Moreover, we can prove that

**Theorem 3.5** If  $p$  is a multiple of 4 then  $g(p, K_N) = \lceil \log p \rceil + \left\lceil \frac{N - 2^{\lceil \log p \rceil}}{p} \right\rceil + 1$ .

### 3.4 Hypercube

Let  $H_d$  be the  $d$ -dimensional hypercube with  $2^d$  nodes. We can prove

**Theorem 3.6** For each integer  $p < 2^d$  it holds

$$g(p, H_d) = \lceil \log p \rceil + \left\lceil \frac{1}{p} (2^d - 2^{\lceil \log p \rceil}) \right\rceil.$$

**Remark 3.1** It is worth pointing out that the obvious inequality  $g_{H_1}(p, G) \leq 2g_{F_1}(p, G)$  and above theorem allow us to improve the upper bound on  $g_{H_1}(p, H_d)$  given by Theorem 4 of [4] for all values of  $p$  not power of two. Indeed, the authors of [4] have  $g_{H_1}(p, H_d) \leq 2d + 2^{d+1}/p - 2/p$  while from Theorem 3.6 and above inequality we get

**Theorem 3.7** For each integer  $p < 2^d$  we have

$$g_{H_1}(p, H_d) \leq 2\lceil \log p \rceil + 2 \left\lceil \frac{1}{p} (2^d - 2^{\lceil \log p \rceil}) \right\rceil.$$

## 4 Concluding Remarks and Open Problems

We have considered the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number  $p$  of packets at each round. In the extremal case  $p = 1$  we have given optimal algorithms to perform gossiping in several classes of graphs, including Hamiltonian graphs, paths, complete  $k$ -ary trees, and complete bipartite graphs. For arbitrary graphs we gave asymptotically matching upper and lower bounds.

In the case of arbitrary  $p$  we have determined the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, rings, paths and complete bipartite graphs  $K_{r,r}$ . Several open problems remain in the area. We list the most important of them here.

- It would be interesting to determine the computational complexity of computing  $g_{F_1}(1, G)$  ( $g_{F_1}(p, G)$ ) for general graphs, it is very likely that it is NP-hard. (We know that computing  $g_{F_1}(\infty, G)$  is NP-hard, see [15]).
- We have left open the problem of determining the gossiping time  $g_{F_1}(p, G_{t,s})$ , and more generally  $g_{F_1}(p, G_{t,s})$ , of rectangular grids  $G_{t,s}$  with both  $t$  and  $s$  odd. We know from Corollary 2.1 that  $g_{F_1}(1, G_{t,s}) \geq st + 1$ . Does equality holds? We can prove that  $g_{F_1}(1, G_{3,3}) = 10$ . A general upper bound on  $g_{F_1}(1, G_{t,s})$  can be obtained by observing that  $G_{t,s} = P_t \times P_s$ , where  $P_t$  and  $P_s$  are the paths on  $t$  and  $s$  nodes, respectively, and  $\times$  denotes the cartesian graph product. Now, given two graphs  $G = (V, E)$  and  $H = (W, F)$  it is easy to see that  $g_{F_1}(1, G \times H) \leq \min\{g_{F_1}(1, G) + |V|g_{F_1}(1, H), g_{F_1}(1, H) + |W|g_{F_1}(1, G)\}$  that, together with Corollary 2.5, immediately gives  $g_{F_1}(1, G_{t,s}) \leq 2ts - 3 - \max\{t, s\}$ .
- We know from (2) that for any graph  $G$  with  $n$  vertices one has  $g_{F_1}(1, G) \geq n$  if  $n$  is odd,  $g_{F_1}(1, G) \geq n - 1$  if  $n$  is even and from Theorem 2.1 we get that the equality holds for Hamiltonian graphs. It would be interesting to characterize

the class of graphs for which this lower bound is tight. We know from the results of Section 2.5 that this class is larger than the class of the Hamiltonian graphs.

• Finally, we mention that in [6] we have analyzed the minimum total number of calls necessary to perform gossiping under the restriction that communicating nodes can exchange up to  $p$  packets during each call. In the full version of the present paper [5] we will also present results concerning the construction of sparse interconnection networks with gossiping time equal to that of the complete graph.

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