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BACKWARD EULER FINITE DIFFERENCE APPROXIMATIONS OF A LOGARITHMIC HEAT EQUATION OVER A 2D RECTANGULAR DOMAIN

PANAGIOTIS PARASCHIS[†] AND GEORGIOS E. ZOURARIS[‡]

ABSTRACT. We formulate an initial and Dirichlet boundary value problem for a semilinear heat equation with logarithmic nonlinearity over a two dimensional rectangular domain. We approximate its solution by using for space discretization the standard second order finite difference scheme, and for time-stepping the Linearized Backward Euler method, or, the ε -Backward Euler method after applying a smooth ε -cutoff of the logarithmic term, where the small positive parameter ε acts as a discretization parameter (along with the time-step and the space mesh widths) and has no influence on the complexity of the method. We prove optimal order error estimates in the discrete $L_t^\infty(L_x^2)$ and the discrete $L_t^\infty(L_x^\infty)$ norm, where the constants are ε -free and no mesh conditions are imposed. Results from numerical experiments expose the efficiency of the numerical methods proposed. It is the first time in the literature where numerical methods for the approximation of the solution to the heat equation with logarithmic nonlinearity are applied and analysed.

May 6, 2021

1. INTRODUCTION

1.1. Formulation of the problem. Let $T > 0$, $D := [a_1, a_2] \times [b_1, b_2] \subset \mathbb{R}^2$, $Q := [0, T] \times D$ and $u : Q \rightarrow \mathbb{R}$ be the solution of the following initial and boundary value problem:

$$(1.1) \quad u_t = \Delta u + g(u) + f \quad \text{on } (0, T] \times \text{int}(D),$$

$$(1.2) \quad u(t, x) = 0 \quad \forall (t, x) \in (0, T] \times \partial D,$$

$$(1.3) \quad u(0, x) = u_0(x) \quad \forall x \in \text{int}(D),$$

where $f \in C(Q, \mathbb{R})$, $u_0 \in C(D, \mathbb{R})$ with

$$(1.4) \quad u_0|_{\partial D} = 0$$

and $g \in C(\mathbb{R}, \mathbb{R})$ is an odd function given by

$$(1.5) \quad g(s) = \begin{cases} 0, & s = 0, \\ s \log(|s|), & s \neq 0, \end{cases} \quad \forall s \in \mathbb{R},$$

which is not differentiable at $s = 0$ and it is not locally Lipschitz around $s = 0$. For theory related to this problem, we refer the reader to [5], [1] and [7]. Hereafter, we shall make the formal assumption that the problem above admits a unique solution which is sufficiently smooth for our purposes.

1.2. Formulation of the numerical methods.

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1.2.1. *Basic notation.* Let \mathbb{N} be the set of all positive integers. For given $N \in \mathbb{N}$, we introduce a uniform partition of the time interval $[0, T]$ with time-step $\tau := \frac{T}{N}$ and nodes $t_n := n\tau$ for $n = 0, \dots, N$. Also, for given $J_1, J_2 \in \mathbb{N}$, we define a uniform partition of $[a_1, a_2]$ with mesh-width $h_1 := \frac{a_2 - a_1}{J_1 + 1}$ and nodes $x_{1,i} := a_1 + i h_1$ for $i = 0, \dots, J_1 + 1$, along with a uniform partition of $[b_1, b_2]$ with mesh-width $h_2 := \frac{b_2 - b_1}{J_2 + 1}$ and nodes $x_{2,j} := b_1 + j h_2$ for $j = 0, \dots, J_2 + 1$. To simplify the notation, we set $\mathbb{I} := \{(i, j) : i = 0, \dots, J_1 + 1, j = 0, \dots, J_2 + 1\}$, $\mathbb{I}^\circ := \{(i, j) : i = 1, \dots, J_1, j = 1, \dots, J_2\}$ and $\partial\mathbb{I} := \mathbb{I} \setminus \mathbb{I}^\circ$. Then, we introduce the discrete matrix space

$$\mathbf{X}_H := \left\{ V = (V_\alpha)_{\alpha \in \mathbb{I}} \in \mathbb{R}^{(J_1+2) \times (J_2+2)} : V_\alpha = 0 \quad \forall \alpha \in \partial\mathbb{I} \right\},$$

a discrete Laplacian operator $\Delta_H : \mathbf{X}_H \rightarrow \mathbf{X}_H$ by

$$(\Delta_H V)_{(i,j)} := \frac{V_{(i-1,j)} - 2V_{(i,j)} + V_{(i+1,j)}}{h_1^2} + \frac{V_{(i,j-1)} - 2V_{(i,j)} + V_{(i,j+1)}}{h_2^2} \quad \forall (i, j) \in \mathbb{I}^\circ, \quad \forall V \in \mathbf{X}_H$$

and the operator $\mathsf{l}_H : C(\mathcal{D}) \rightarrow \mathbf{X}_H$ by $(\mathsf{l}_H^\circ[z])_{(i,j)} := z(x_{1,i}, x_{2,j})$ for all $(i, j) \in \mathbb{I}^\circ$ and $z \in C(\mathcal{D})$. Finally, we simplify the notation, by setting $u^n := \mathsf{l}_H[u(t_n, \cdot)]$ for $n = 0, \dots, N$, and by defining, for any $W \in \mathbf{X}_H$ and $\mathbf{g} \in C(\mathbb{R}, \mathbb{R})$, $\mathbf{g}(W) \in \mathbf{X}_H$ by $(\mathbf{g}(W))_\alpha := \mathbf{g}(W_\alpha)$ for all $\alpha \in \mathbb{I}^\circ$.

1.2.2. *A ε -cutoff function.* Let $\varepsilon \in (0, \frac{1}{2\varepsilon})$ and $p_\varepsilon \in \mathbb{P}^3[\varepsilon, 2\varepsilon]$ defined by

$$(1.6) \quad p_\varepsilon(s) := \left(\frac{s-\varepsilon}{\varepsilon}\right)^2 \left[\log(2\varepsilon) + \left(2 \log(2\varepsilon) - \frac{1}{2}\right) \left(\frac{2\varepsilon-s}{\varepsilon}\right) \right] \quad \forall s \in [\varepsilon, 2\varepsilon].$$

Then, we define an even function $\xi_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ by

$$(1.7) \quad \xi_\varepsilon(s) = \begin{cases} 0, & s \in [0, \varepsilon] \\ p_\varepsilon(s), & s \in (\varepsilon, 2\varepsilon], \\ \log(s), & s \in (2\varepsilon, +\infty), \end{cases} \quad \forall s \geq 0,$$

and an odd function $g_\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$ by

$$(1.8) \quad g_\varepsilon(s) := s \xi_\varepsilon(s) \quad \forall s \geq 0.$$

1.2.3. *The finite difference methods.* We describe below two finite differences methods approximating, at the time node t_n , the solution u to the problem (1.1)-(1.5) by a matrix $U^n \in \mathbf{X}_H$.

The first step is common for both methods:

Step A: Set

$$(1.9) \quad U^0 := \mathsf{l}_H[u_0] \in \mathbf{X}_H.$$

The ε -Backward Euler Finite Difference (ε BEFD) method is implicit, requiring, at every time step, the solution of a nonlinear system of algebraic equation and its structure is as follows:

(ε BEFD)-**Step B:** For $n = 0, \dots, N - 1$, find $U^{n+1} \in \mathbf{X}_H$ such that

$$(1.10) \quad \frac{U^{n+1} - U^n}{\tau} = \Delta_H(U^{n+1}) + g_\varepsilon(U^{n+1}) + \mathsf{l}_H[f(t_{n+1}, \cdot)].$$

The Linearized Backward Euler Finite Difference (LBEFD) method is linearly implicit, requiring, at every time step, the solution of a linear system of algebraic equations and its algorithm is given below:

(LBEFD)-**Step B:** For $n = 0, \dots, N - 1$, find $U^{n+1} \in \mathbf{X}_H$ such that

$$(1.11) \quad \frac{U^{n+1} - U^n}{\tau} = \Delta_H(U^{n+1}) + g(U^n) + \mathsf{l}_H[f(t_{n+1}, \cdot)].$$

Remark 1.1. *The numerical approximations of the (LBEFD) method are unconditionally well defined, since the nonlinear term has no contribution in the matrix of the linear system to be solved. The discussion on the well-posedness of the (ε BEFD) method is postponed until Section 3.2.*

1.3. Main results and related work. The numerical approximation of the solution u_{LgNLS} to the logarithmic nonlinear Schrödinger (LgNLS) equation has been considered, recently, in [3], where the authors first introduce an ε -regularization $g_{\varepsilon, \text{LgNLS}}(z) = z \log(\varepsilon + |z|)$ of the complex nonlinear term $g_{\text{LgNLS}}(z) = z \log(|z|)$ and then apply the linearly implicit finite difference method introduced in [6] for the cubic (NLS) equation. The convergence analysis adopts the viewpoint that the numerical method approximates the solution $u_{\varepsilon, \text{LgNLS}}$ to the $(\varepsilon\text{LgNLS})$ equation which follows after substituting g_{LgNLS} by $g_{\varepsilon, \text{LgNLS}}$. For that reason, its first outcome is an $O(\varepsilon)$ bound of the modelling error $u_{\text{LgNLS}} - u_{\varepsilon, \text{LgNLS}}$ in the $L_t^\infty(L_x^2)$ -norm (see Proposition 2.5 in [3]). Then, for the numerical method in the one dimensional case, a standard second order error estimate in the discrete $L_t^\infty(L_x^2)$ -norm is provided, under the, rather unrealistic, assumption that higher order derivatives of $u_{\varepsilon, \text{LgNLS}}$ are bounded by a constant independent of ε (see Theorem 3.1 in [3]). In the latter error estimate the exponential constant, coming as an outcome of the application of the discrete Gronwall argument, is of the form $\exp(O(T|\log(\varepsilon)|^2))$, and thus, convergence is achieved by assuming that the size of the time step and the space width is $O(\sqrt{\varepsilon} \exp(-O(T|\log(\varepsilon)|^2)))$. In [2], the (LgNLS) equation is regularized, again, by using $g_{\varepsilon, \text{LgNLS}}$ and its solution is approximated by a time-discrete Lie-Trotter splitting method, the convergence analysis of which arrives at an $O(|\log(\varepsilon)|\sqrt{\tau})$ error estimate in the $L_t^\infty(L_x^2)$ -norm. In [8], the authors regularize the term $g_{\text{LgKG}}(s) = s \log(s^2)$ of the logarithmic Klein-Gordon equation by $g_{\varepsilon, \text{LgKG}}(s) = s \log(\varepsilon + s^2)$ and, without addressing the estimation of the corresponding modelling error, obtain convergence results, similar to those in [3], for a linearized Crank-Nicolson method and for the standard Courant-Friedrichs-Lewy explicit method (see Theorems 3.1 and 3.2 in [8]). Our opinion is that the dependence of the Gronwall constant on ε in [3] is strongly related to the stability properties of the numerical method chosen. Also, we believe that the use of the solution to the fictional ε -regularized problem, as an intermediate in the convergence analysis of the numerical method, gives birth to constants that may depend singularly on ε , and thus introducing a unexisting competition of those constants with the discretization parameters.

In the paper at hands, our aim is to contribute in the understanding of the numerical approximation of semilinear evolution equations with logarithmic nonlinearity, which is challenging since the usual locally Lipschitz property is absent. We consider, as a model problem, the semilinear heat equation with logarithmic nonlinearity over a two dimensional rectangular domain described by (1.1)-(1.5). To discretize the problem in space, we use the standard second order finite difference method as a way to avoid, in the actual computations, the integration of the logarithmic term required by the finite element method. For time stepping, we propose the use of the Linearised Backward Euler method, or, of the Backward Euler method after substituting g by its ε -cutoff g_ε (see (1.8)), which is different than that used in the bibliography (cf., e.g. [3], [2], [8]). Thus, we arrive at the (LBEFD) method, or, the (εBEFD) method, respectively.

The convergence analysis is based on the introduction and the error estimation of a proper modified version of the proposed methods (see Sections 3.1 and 3.3), which follow by mollifying properly the non-linear term (cf. [9]). For the (LBEFD) method, we provide an error estimate in the discrete $L_t^\infty(L_x^2)$ and in the discrete $L_t^\infty(L_x^\infty)$ norm, i.e. there exists a constant $C_{\text{LBEFD}} > 0$, independent of τ , h_1 and h_2 , such that

$$\max_{0 \leq n \leq N} (\|U^n - u^n\|_{0, \text{H}} + |U^n - u^n|_{\infty, \text{H}}) \leq C_{\text{LBEFD}} [\tau |\log(\tau)| + h_1^2 + h_2^2].$$

For the (εBEFD) method, we derive, also, an error estimate in the discrete $L_t^\infty(L_x^2)$ and in the discrete $L_t^\infty(L_x^\infty)$ norm, i.e. there exists a constant $C_{\text{BEFD}} > 0$, independent of ε , τ , h_1 and h_2 , such that

$$\max_{0 \leq n \leq N} (\|U^n - u^n\|_{0, \text{H}} + |U^n - u^n|_{\infty, \text{H}}) \leq C_{\text{BEFD}} [\varepsilon |\log(\varepsilon)| + \tau + h_1^2 + h_2^2].$$

We would like to stress, that, to obtain the error estimates above, we do not impose coupling conditions on the discretization parameters τ , h_1 , h_2 and ε .

We close this section by giving a brief overview of the paper. In Section 2, we introduce additional notation, provide a series of auxiliary results and estimate the consistency error. Section 3 is

dedicated to the convergence analysis of the proposed method. Finally, we expose results from numerical experiments in Section 4.

2. PRELIMINARIES

2.1. Discrete norms. We provide X_H with the discrete $L^2(D)$ -inner product $(\cdot, \cdot)_{0,H}$ given by $(V, Z)_{0,H} := h_1 h_2 \sum_{\alpha \in \mathbb{I}^\circ} V_\alpha Z_\alpha$ for $V, Z \in X_H$, and we denote by $\|\cdot\|_{0,H}$ its induced norm, i.e. $\|v\|_{0,H} := [(V, V)_{0,H}]^{1/2}$ for $V \in X_H$. Also, we equip X_H with a discrete $L^\infty(D)$ -norm $|\cdot|_{\infty,H}$ defined by $|W|_{\infty,H} := \max_{\alpha \in \mathbb{I}^\circ} |W_\alpha|$ for $W \in X_H$ and with a discrete $H^1(D)$ -type norm $|\cdot|_{1,H}$ given by

$$|V|_{1,H} := \left[h_1 h_2 \sum_{j=1}^{J_2} \sum_{i=0}^{J_1} \left| \frac{V_{(i+1,j)} - V_{(i,j)}}{h_1} \right|^2 + h_1 h_2 \sum_{i=1}^{J_1} \sum_{j=0}^{J_2} \left| \frac{V_{(i,j+1)} - V_{(i,j)}}{h_2} \right|^2 \right]^{1/2} \quad \forall V \in X_H.$$

In the convergence analysis of the method, we will make use of the, easy to verify, discrete integration by part result

$$(2.1) \quad (\Delta_H V, V)_{0,H} = -|V|_{1,H}^2 \quad \forall V \in X_H$$

and of the discrete Poincaré-Friedrichs inequality

$$(2.2) \quad \|V\|_{0,H} \leq \frac{1}{2} \min\{a_2 - a_1, b_2 - b_1\} |V|_{1,H} \quad \forall V \in X_H.$$

2.2. A δ -mollifier. For $\delta > 0$, let $\mathbf{n}_\delta \in C^1(\mathbb{R}; \mathbb{R})$ (see, e.g. [9]) be an odd function defined by

$$(2.3) \quad \mathbf{n}_\delta(s) := \begin{cases} s, & \text{if } s \in [0, \delta], \\ q_\delta(s), & \text{if } s \in (\delta, 2\delta], \\ 2\delta, & \text{if } s > 2\delta, \end{cases} \quad \forall s \geq 0,$$

where $q_\delta \in \mathbb{P}^3[\delta, 2\delta]$ is a polynomial defined by

$$(2.4) \quad q_\delta(s) := s + \frac{(s-\delta)^2(2\delta-s)}{\delta^2} \quad \forall s \in [\delta, 2\delta].$$

In the lemma below, we present some basic properties of the function \mathbf{n}_δ .

Lemma 2.1. *For $\delta > 0$, it holds that*

$$(2.5) \quad \mathbf{n}'_\delta(s) \in \left[0, \frac{4}{3}\right] \quad \forall s \in \mathbb{R}$$

and

$$(2.6) \quad \max_{s \in \mathbb{R}} |\mathbf{n}_\delta(s)| = 2\delta.$$

Proof. Since \mathbf{n}_δ is an odd function, we conclude that \mathbf{n}'_δ is an even function, and thus it is sufficient to investigate the range of \mathbf{n}'_δ on $[0, +\infty)$. According to (2.3), we have

$$(2.7) \quad \mathbf{n}'_\delta(s) = 0 \quad \forall s \in [2\delta, +\infty) \quad \text{and} \quad \mathbf{n}'_\delta(s) = 1 \quad \forall s \in [0, \delta].$$

Also, from (2.3) and (2.4), we obtain

$$\mathbf{n}'_\delta(s) = q'_\delta(s) = \frac{\delta^2 + (s-\delta)(5\delta-3s)}{\delta^2} = \frac{-3s^2 + 8\delta s - 4\delta^2}{\delta^2} \quad \forall s \in [\delta, 2\delta].$$

Observing that $q'_\delta(\delta) = 1$, $q'_\delta(2\delta) = 0$ and $q''_\delta(s) \geq 0$ iff $s \leq \frac{4}{3}\delta$, we conclude that q'_δ is increasing on $[\delta, \frac{4\delta}{3}]$ and decreasing on $[\frac{4\delta}{3}, 2\delta]$. Since $\mathbf{n}'_\delta(\frac{4\delta}{3}) = \frac{4}{3}$, we, easily, arrive at

$$(2.8) \quad \mathbf{n}'_\delta(s) \in \left[0, \frac{4}{3}\right] \quad \forall s \in [\delta, 2\delta].$$

Thus, (2.5) follows as a simple outcome of (2.7) and (2.8).

Finally, (2.5) yields that \mathbf{n}_δ is increasing on \mathbb{R} , and hence $\mathbf{n}_\delta(s) \in [0, 2\delta]$ for $s \in [0, +\infty)$, from which (2.6), easily, follows. \square

Remark 2.1. *Obviously, it holds that $\mathbf{n}_\delta(u(t, x)) = u(t, x)$ for $(t, x) \in \mathbb{Q}$, when $\delta \geq \max_{\mathbb{Q}} |u|$.*

2.3. Properties of the ε -cutoff function. In the lemmas below, we present some useful properties of function g_ε defined by (1.8).

Lemma 2.2. *For $\varepsilon \in (0, \frac{1}{2e})$ and $c \geq e$, we have*

$$(2.9) \quad \max_{s \in \mathbb{R}} |g(s) - g_\varepsilon(s)| < 9\varepsilon |\log(2\varepsilon)|$$

and

$$(2.10) \quad \max_{s \in [-c, c]} |g_\varepsilon(s)| \leq c \log(c).$$

Proof. Taking into account that g and g_ε are odd functions, we use (1.8) and (1.7), to have

$$(2.11) \quad \begin{aligned} \max_{s \in \mathbb{R}} |g_\varepsilon(s) - g(s)| &= \max_{s \in [0, +\infty)} |g_\varepsilon(s) - g(s)| \\ &= \max \left\{ \max_{s \in [0, \varepsilon]} |g_\varepsilon(s) - g(s)|, \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s) - g(s)| \right\} \\ &= \max \left\{ \max_{s \in [0, \varepsilon]} |g(s)|, \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s) - g(s)| \right\} \end{aligned}$$

and

$$(2.12) \quad \max_{s \in [-c, c]} |g_\varepsilon(s)| = \max_{s \in [0, c]} |g_\varepsilon(s)| = \max \left\{ \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s)|, \max_{s \in [2\varepsilon, c]} |g(s)| \right\}.$$

Observing that g is non-positive on $[0, 1]$, $g(1) = 0$, g is strictly decreasing on $[0, \frac{1}{e}]$ and strictly increasing on $[\frac{1}{e}, +\infty)$, we use (1.6) to obtain

$$(2.13) \quad \max_{s \in [0, \varepsilon]} |g(s)| = \varepsilon |\log(\varepsilon)|,$$

$$(2.14) \quad \begin{aligned} \max_{s \in [2\varepsilon, c]} |g(s)| &= \max \left\{ \max_{s \in [2\varepsilon, e^{-1}]} |g(s)|, \max_{s \in [e^{-1}, c]} |g(s)| \right\} \\ &= \max \{ |g(e^{-1})|, c \log(c) \} \\ &= \max \left\{ \frac{1}{e}, c \log(c) \right\} \\ &= c \log(c), \end{aligned}$$

$$(2.15) \quad \begin{aligned} \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s)| &= \max_{s \in [\varepsilon, 2\varepsilon]} |s p_\varepsilon(s)| \\ &\leq 2\varepsilon \max_{s \in [\varepsilon, 2\varepsilon]} |p_\varepsilon(s)| \\ &\leq 2\varepsilon \left[3 |\log(2\varepsilon)| + \frac{1}{2} \right] \\ &\leq 6\varepsilon |\log(2\varepsilon)| + \varepsilon \end{aligned}$$

and

$$(2.16) \quad \begin{aligned} \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s) - g(s)| &\leq \max_{s \in [\varepsilon, 2\varepsilon]} |g_\varepsilon(s)| + \max_{s \in [\varepsilon, 2\varepsilon]} |g(s)| \\ &\leq 6\varepsilon |\log(2\varepsilon)| + \varepsilon + 2\varepsilon |\log(2\varepsilon)| \\ &\leq \varepsilon + 8\varepsilon |\log(2\varepsilon)|. \end{aligned}$$

Combining (2.11), (2.13), (2.16) and (2.12), (2.15), (2.14), we get

$$\begin{aligned} \max_{s \in \mathbb{R}} |g_\varepsilon(s) - g(s)| &\leq \max \{ \varepsilon |\log(\varepsilon)|, \varepsilon + 8\varepsilon |\log(2\varepsilon)| \} \\ &\leq \max \{ |g(\varepsilon)|, \varepsilon + 4 |g(2\varepsilon)| \} \\ &\leq \varepsilon + 4 |g(2\varepsilon)| \\ &< 9\varepsilon |\log(2\varepsilon)| \end{aligned}$$

and

$$\begin{aligned}
\max_{s \in [-c, c]} |g_\varepsilon(s)| &\leq \max \{6\varepsilon |\log(2\varepsilon)| + \varepsilon, c \log(c)\} \\
&\leq \max \left\{ 3|g(2\varepsilon)| + \frac{1}{2e}, c \log(c) \right\} \\
&\leq \max \left\{ 3|g(e^{-1})| + \frac{1}{2e}, c \log(c) \right\} \\
&\leq \max \left\{ \frac{7}{2e}, c \log(c) \right\} \\
&\leq c \log(c).
\end{aligned}$$

Thus, we arrive at (2.9) and (2.10). \square

Lemma 2.3. *Let $\varepsilon \in (0, \frac{1}{2e})$ and $c \geq e$. Then, we have*

$$(2.17) \quad g'_\varepsilon(s) \leq 1 \quad \forall s \in [-e^{-1}, e^{-1}],$$

$$(2.18) \quad \max_{|s| \in [e^{-1}, c]} |g'_\varepsilon(s)| \leq 2 \log(c),$$

$$(2.19) \quad \max_{s \in [-c, c]} |g'_\varepsilon(s)| \leq 8 \max \{ \log(c), |\log(2\varepsilon)| \}$$

and

$$(2.20) \quad g'_\varepsilon(s) \leq 2 \log(c) \quad \forall s \in [-c, c].$$

Proof. Since g_ε is an odd function, it follows that g'_ε is an even function, and thus it is sufficient to investigate the range of g'_ε on $[0, e^{-1}]$. According to (1.7) and (1.8), we have

$$(2.21) \quad \begin{aligned} g'_\varepsilon(s) &= 0 & \forall s \in [0, \varepsilon], \\ g'_\varepsilon(s) &= 1 + \log(s) \leq 0 & \forall s \in [2\varepsilon, e^{-1}]. \end{aligned}$$

Also, for $s \in [\varepsilon, 2\varepsilon]$, we have

$$(2.22) \quad g'_\varepsilon(s) = s p'_\varepsilon(s) + p_\varepsilon(s) = (s - \varepsilon) \varepsilon^{-3} [\log(2\varepsilon) q_{1,\varepsilon}(s) + q_{2,\varepsilon}(s)]$$

where $q_{1,\varepsilon}(s) := -8s^2 + 19\varepsilon s - 5\varepsilon^2$ and $q_{2,\varepsilon}(s) := 2s^2 - 4\varepsilon s + \varepsilon^2$. Observing that $q_{1,\varepsilon}(\varepsilon) = 6\varepsilon^2$, $q_{1,\varepsilon}(2\varepsilon) = \varepsilon^2$, and $q'_{1,\varepsilon}(s) \geq 0$ iff $s \leq \frac{19}{16}\varepsilon$, we conclude that $q_{1,\varepsilon}$ is increasing on $[\varepsilon, \frac{19}{16}\varepsilon]$ and decreasing on $[\frac{19}{16}\varepsilon, 2\varepsilon]$. Also, observing that $q_{2,\varepsilon}(\varepsilon) = -\varepsilon^2$, $q_{2,\varepsilon}(2\varepsilon) = \varepsilon^2$, and $q'_{2,\varepsilon}(s) \geq 0$ iff $s \geq \varepsilon$, we conclude that $q_{2,\varepsilon}$ is increasing on $[\varepsilon, 2\varepsilon]$. Thus, we obtain

$$(2.23) \quad q_{1,\varepsilon}(s) \in \left[0, q_{1,\varepsilon}\left(\frac{19\varepsilon}{16}\right)\right] \quad \forall s \in [\varepsilon, 2\varepsilon]$$

and

$$(2.24) \quad \max_{s \in [\varepsilon, 2\varepsilon]} |q_{2,\varepsilon}(s)| \leq \varepsilon^2 \quad \forall s \in [\varepsilon, 2\varepsilon].$$

Combining, (2.22), (2.23) and (2.24), we get

$$(2.25) \quad \begin{aligned} g'_\varepsilon(s) &\leq (s - \varepsilon) \varepsilon^{-3} q_{2,\varepsilon}(s) \\ &\leq (s - \varepsilon) \varepsilon^{-3} \varepsilon^2 \\ &\leq 1 \quad \forall s \in [\varepsilon, 2\varepsilon]. \end{aligned}$$

Thus, (2.17) follows, easily, from (2.21) and (2.25).

In addition, from (1.8), (1.7), (2.22), (2.23) and (2.24), we get

$$(2.26) \quad \begin{aligned} \max_{|s| \in [e^{-1}, c]} |g'_\varepsilon(s)| &= \max_{s \in [e^{-1}, c]} |g'(s)| = \max_{s \in [e^{-1}, c]} |1 + \log(s)| \\ &= \max_{s \in [e^{-1}, c]} (1 + \log(s)) \\ &= 1 + \log(c) \\ &\leq 2 \log(c), \end{aligned}$$

$$(2.27) \quad \max_{|s| \in [2\varepsilon, e^{-1}]} |g'_\varepsilon(s)| = \max_{s \in [2\varepsilon, e^{-1}]} |1 + \log(s)| = 1 + |\log(2\varepsilon)| \leq 2 |\log(2\varepsilon)|$$

and

$$(2.28) \quad \begin{aligned} \max_{|s| \in [\varepsilon, 2\varepsilon]} |g'_\varepsilon(s)| &= \max_{s \in [\varepsilon, 2\varepsilon]} |g'_\varepsilon(s)| \leq \varepsilon^{-2} [\varepsilon^2 + |\log(2\varepsilon)|] q_{1,\varepsilon} \left(\frac{19\varepsilon}{16} \right) \\ &\leq 1 + |\log(2\varepsilon)| \frac{201}{32} \\ &\leq |\log(2\varepsilon)| + |\log(2\varepsilon)| \frac{201}{32} \\ &\leq 8 |\log(2\varepsilon)|. \end{aligned}$$

Thus, (2.18) follows from (2.26). Also, we arrive at (2.19) by combining (2.21), (2.28), (2.27) and (2.26). Finally, (2.20) is a simple outcome of (2.17) and (2.18). \square

Lemma 2.4. *Let $\varepsilon \in (0, \frac{1}{2e})$ and $\mathbf{q} \in C^1(\mathbb{R}, \mathbb{R})$ with $\sup_{\mathbb{R}}(|\mathbf{q}| + |\mathbf{q}'|) < +\infty$ and $\mathbf{q}' \geq 0$ on \mathbb{R} . Then, it holds that*

$$(2.29) \quad (g_\varepsilon(\mathbf{q}(V)) - g_\varepsilon(\mathbf{q}(W)), V - W)_{0,\mathbb{H}} \leq C_s \|V - W\|_{0,\mathbb{H}}^2 \quad \forall V, W \in \mathbb{X}_{\mathbb{H}},$$

where $C_s = 2 \sup_{\mathbb{R}} |\mathbf{q}'| \log(c)$ with $c = \max\{e, \sup_{\mathbb{R}} |\mathbf{q}|\}$.

Proof. Let $\alpha \in \mathbb{I}^\circ$ and $V, W \in \mathbb{X}_{\mathbb{H}}$. For simplicity, we set $S_\alpha := [g_\varepsilon(\mathbf{q}(V_\alpha)) - g_\varepsilon(\mathbf{q}(W_\alpha))](V_\alpha - W_\alpha)$, $\beta_\alpha := s \mathbf{q}(V_\alpha) + (1-s) \mathbf{q}(W_\alpha)$ and $\gamma_\alpha := s V_\alpha + (1-s) W_\alpha$ for $s \in [0, 1]$. Applying the Taylor formula we get

$$(2.30) \quad \begin{aligned} S_\alpha &= \int_0^1 g'_\varepsilon(\beta_\alpha(s)) (\mathbf{q}(V_\alpha) - \mathbf{q}(W_\alpha)) (V_\alpha - W_\alpha) ds \\ &= \int_0^1 \left[\int_0^1 g'_\varepsilon(\beta_\alpha(s)) \mathbf{q}'(\gamma_\alpha(s')) (V_\alpha - W_\alpha)^2 ds' \right] ds. \end{aligned}$$

Let $c = \max\{e, \sup_{\mathbb{R}} |\mathbf{q}|\}$. It is easily seen that $\max_{s \in [0,1]} |\beta_\alpha(s)| \leq \sup_{\mathbb{R}} |\mathbf{q}| \leq c$. Thus, under the light of (2.20), we conclude that

$$(2.31) \quad g'_\varepsilon(\beta_\alpha(s)) \leq 2 \log(c) \quad \forall s \in [0, 1].$$

Now, we use (2.30), (2.31) and the assumption $\mathbf{q}' \geq 0$, to get

$$(2.32) \quad S_\alpha \leq 2 \log(c) \sup_{\mathbb{R}} |\mathbf{q}'| (V_\alpha - W_\alpha)^2.$$

Finally, under the light of (2.32), we have

$$(g_\varepsilon(\mathbf{q}(V)) - g_\varepsilon(\mathbf{q}(W)), V - W)_{0,\mathbb{H}} = h_1 h_2 \sum_{\alpha \in \mathbb{I}^\circ} S_\alpha \leq 2 \sup_{\mathbb{R}} |\mathbf{q}'| \log(c) \|V - W\|_{0,\mathbb{H}}^2,$$

which, obviously, yields (2.29). \square

2.4. Consistency Errors. Let $\varepsilon \in (0, \frac{1}{2e})$.

2.4.1. *Consistency error in time.* For $n = 0, \dots, N-1$, we define $\mathbf{r}_\varepsilon^n, \rho_\varepsilon^n \in \mathbb{X}_{\mathbb{H}}$ by

$$(2.33) \quad \frac{u^{n+1} - u^n}{\tau} = \mathbb{I}_{\mathbb{H}} [\Delta u(t_{n+1}, \cdot) + g_\varepsilon(u(t_{n+1}, \cdot)) + f(t_{n+1}, \cdot)] + \mathbf{r}_\varepsilon^n$$

and

$$(2.34) \quad \frac{u^{n+1} - u^n}{\tau} = \mathbb{I}_{\mathbb{H}} [\Delta u(t_{n+1}, \cdot) + g_\varepsilon(u(t_n, \cdot)) + f(t_{n+1}, \cdot)] + \rho_\varepsilon^n.$$

Using (1.1), (2.33) and (2.34), we obtain

$$\mathbf{r}_\varepsilon^n := \left[\frac{u^{n+1} - u^n}{\tau} - \mathbb{I}_{\mathbb{H}} [u_t(t_{n+1}, \cdot)] \right] - \mathbb{I}_{\mathbb{H}} [g_\varepsilon(u(t_{n+1}, \cdot)) - g(u(t_{n+1}, \cdot))]$$

and

$$\begin{aligned} \rho_\varepsilon^n &:= \left[\frac{u^{n+1} - u^n}{\tau} - \mathbb{I}_{\mathbb{H}} [u_t(t_{n+1}, \cdot)] \right] - \mathbb{I}_{\mathbb{H}} [g_\varepsilon(u(t_n, \cdot)) - g_\varepsilon(u(t_{n+1}, \cdot))] \\ &\quad - \mathbb{I}_{\mathbb{H}} [g_\varepsilon(u(t_{n+1}, \cdot)) - g(u(t_{n+1}, \cdot))] \end{aligned}$$

for $n = 0, \dots, N-1$. Applying the Taylor formula and using (2.9), we obtain

$$(2.35) \quad \max_{0 \leq n \leq N-1} |r_\varepsilon^n|_{\infty, \mathbb{H}} \leq C_{\text{TC}} \left[\tau \max_{\mathbb{Q}} |u_{tt}| + \varepsilon |\log(2\varepsilon)| \right].$$

Assuming that $\varepsilon \in \left(0, \frac{1}{2c_0}\right)$ with $c_0 := \max\{e, \max_{\mathbb{Q}} |u|\}$, and applying the Taylor formula, the mean value theorem, (2.9) and (2.19) (with $c = c_0$), it follows that

$$(2.36) \quad \max_{0 \leq n \leq N-1} |\rho_\varepsilon^n|_{\infty, \mathbb{H}} \leq C_{\text{LTC}} \left[\tau \max_{\mathbb{Q}} |u_{tt}| + \tau |\log(2\varepsilon)| \max_{\mathbb{Q}} |u_t| + \varepsilon |\log(2\varepsilon)| \right].$$

2.4.2. *Consistency error in space.* For $n = 0, \dots, N-1$, let $\mathfrak{s}_\varepsilon^n, \sigma_\varepsilon^n \in \mathbb{X}_{\mathbb{H}}$ be given by

$$(2.37) \quad \frac{u^{n+1} - u^n}{\tau} = \Delta_{\mathbb{H}}(u^{n+1}) + \mathbb{I}_{\mathbb{H}} [g_\varepsilon(u(t_{n+1}, \cdot)) + f(t_{n+1}, \cdot)] + \mathfrak{s}_\varepsilon^n$$

and

$$(2.38) \quad \frac{u^{n+1} - u^n}{\tau} = \Delta_{\mathbb{H}}(u^{n+1}) + \mathbb{I}_{\mathbb{H}} [g_\varepsilon(u(t_n, \cdot)) + f(t_{n+1}, \cdot)] + \sigma_\varepsilon^n.$$

Then, subtracting (2.37) from (2.33) and (2.38) from (2.34), we obtain

$$\mathfrak{s}_\varepsilon^n - r_\varepsilon^n = \rho_\varepsilon^n - \sigma_\varepsilon^n = \mathbb{I}_{\mathbb{H}} [\Delta u(t_{n+1}, \cdot)] - \Delta_{\mathbb{H}}(u^{n+1}), \quad n = 0, \dots, N-1.$$

After using the Taylor formula with respect to the space variables, we conclude

$$(2.39) \quad \max_{0 \leq n \leq N-1} |\mathfrak{s}_\varepsilon^n - r_\varepsilon^n|_{\infty, \mathbb{H}} = \max_{0 \leq n \leq N-1} |\sigma_\varepsilon^n - \rho_\varepsilon^n|_{\infty, \mathbb{H}} \leq C_{\text{sc}} \left(h_1^2 \max_{\mathbb{Q}} |\partial_{x_1}^4 u| + h_2^2 \max_{\mathbb{Q}} |\partial_{x_2}^4 u| \right).$$

3. CONVERGENCE ANALYSIS

3.1. **The modified (ε BEFD) method.** For $\delta > 0$ and $\varepsilon \in \left(0, \frac{1}{2e}\right)$, the modified (ε BEFD) method (see, e.g. [9]) constructs approximations $(V_\delta^n)_{n=0}^N \subset \mathbb{X}_{\mathbb{H}}$ of the solution u over the time nodes, by stepping as follows:

Step M1: Set

$$(3.1) \quad V_\delta^0 := U^0.$$

Step M2: For $n = 0, \dots, N-1$, find $V_\delta^{n+1} \in \mathbb{X}_{\mathbb{H}}$ such that

$$(3.2) \quad \frac{V_\delta^{n+1} - V_\delta^n}{\tau} = \Delta_{\mathbb{H}}(V_\delta^{n+1}) + g_\varepsilon(\mathfrak{n}_\delta(V_\delta^{n+1})) + \mathbb{I}_{\mathbb{H}} [f(t_{n+1}, \cdot)].$$

In the sequel, we investigate the existence of the modified (ε BEFD) approximations, using as a basic tool the following Brouwer-type fixed-point lemma (see [4]).

Lemma 3.1. *Let $(\mathbb{R}, \mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ be a real finite dimensional inner product space, $\|\cdot\|_{\mathcal{H}}$ be the associated norm, $\mu : \mathcal{H} \mapsto \mathcal{H}$ be a continuous operator, $\mathcal{S}_\varepsilon := \{z \in \mathcal{H} : \|z\|_{\mathcal{H}} = \varepsilon\}$ and $\mathcal{B}_\varepsilon := \{z \in \mathcal{H} : \|z\|_{\mathcal{H}} \leq \varepsilon\}$ for $\varepsilon > 0$. If there exists a positive constant $\beta > 0$ such that $(\mu(z), z)_{\mathcal{H}} > 0 \quad \forall z \in \mathcal{S}_\beta$, then there exists $w \in \mathcal{B}_\beta$ such that $\mu(w) = 0$.*

Proposition 3.1. *For $\delta > 0$ and $\varepsilon \in \left(0, \frac{1}{2e}\right)$, there exist $(V_\delta^n)_{n=1}^N \subset \mathbb{X}_{\mathbb{H}}$ satisfying (3.2).*

Proof. Let $Z \in \mathbb{X}_{\mathbb{H}}$ and $\mu : \mathbb{X}_{\mathbb{H}} \mapsto \mathbb{X}_{\mathbb{H}}$ be a continuous nonlinear operator defined by

$$\mu(V) := V - \tau \Delta_{\mathbb{H}}(V) - \tau g_\varepsilon(\mathfrak{n}_\delta(V)) + Z \quad \forall V \in \mathbb{X}_{\mathbb{H}}.$$

Let $|\mathbb{D}| := \text{area}(\mathbb{D})$, $\beta > 0$ and $V \in \mathbb{X}_{\mathbb{H}}$ with $\|V\|_{0, \mathbb{H}} = \beta$. Using (2.1), the Cauchy-Schwarz inequality and (2.6), we obtain

$$(3.3) \quad \begin{aligned} (\mu(V), V)_{0, \mathbb{H}} &= \|V\|_{0, \mathbb{H}}^2 + \tau |V|_{1, \mathbb{H}}^2 - \tau (g_\varepsilon(\mathfrak{n}_\delta(V)), V)_{0, \mathbb{H}} + (Z, V)_{0, \mathbb{H}} \\ &> \|V\|_{0, \mathbb{H}} [\|V\|_{0, \mathbb{H}} - \tau |\mathbb{D}|^{1/2} |g_\varepsilon(\mathfrak{n}_\delta(V))|_{\infty, \mathbb{H}} - \|Z\|_{0, \mathbb{H}}] \\ &\geq \beta \left[\beta - \tau |\mathbb{D}|^{1/2} \max_{|s| \in [0, 2\delta]} |g_\varepsilon(s)| - \|Z\|_{0, \mathbb{H}} \right]. \end{aligned}$$

Choosing $\beta = 1 + \tau |D|^{1/2} \max_{|s| \in [0, 2\delta]} |g_\varepsilon(s)| + \|Z\|_{0, \mathbb{H}}$, (3.3) yields $(\mu(V), V)_{0, \mathbb{H}} > 0$, which, under the light of Lemma 3.1, results the existence of a $W \in \mathbb{X}_{\mathbb{H}}$ such that $\|W\|_{0, \mathbb{H}} \leq \beta$ and $\mu(W) = 0$.

We establish, now, the existence of the modified (ε BEFD) approximations by induction. First, we observe that V_δ^0 is well defined. Next, we assume that, for given $\kappa \in \{0, \dots, N-1\}$ there exists a modified approximation $V_\delta^\kappa \in \mathbb{X}_{\mathbb{H}}$. Then, we choose $Z = -V_\delta^\kappa - \tau \mathbb{I}_{\mathbb{H}} [f(t_{\kappa+1}, \cdot)]$, to obtain a root $V_\delta^{\kappa+1} \in \mathbb{X}_{\mathbb{H}}$ of the corresponding operator μ . \square

In the lemma below, assuming τ be small enough, we ensure the uniqueness of the modified (ε BEFD) approximations.

Proposition 3.2. *If $\delta > e$, $\varepsilon \in (0, \frac{1}{2e})$ and*

$$(3.4) \quad \tau \log(2\delta) < \frac{3}{8},$$

then the modified (ε BEFD) approximations $(V_\delta^n)_{n=1}^N \subset \mathbb{X}_{\mathbb{H}}$ satisfying (3.2) are unique.

Proof. Our assumptions and Proposition 3.1 yield the existence of $(V_\delta^n)_{n=1}^N \subset \mathbb{X}_{\mathbb{H}}$ satisfying (3.2). To ensure their uniqueness, we will employ an induction argument.

Obviously, V_δ^0 is unique. Now, let us assume that, for given $\kappa \in \{0, \dots, N-1\}$, the modified approximations $(V_\delta^\ell)_{\ell=0}^\kappa$ are unique, and that there exists $W \in \mathbb{X}_{\mathbb{H}}$ such that

$$(3.5) \quad \frac{W - V_\delta^\kappa}{\tau} = \Delta_{\mathbb{H}}(W) + g_\varepsilon(\mathbf{n}_\delta(W)) + \mathbb{I}_{\mathbb{H}} [f(t_{\kappa+1}, \cdot)].$$

Subtracting (3.5) from (3.2) (with $n = \kappa$), we obtain

$$(3.6) \quad V_\delta^{\kappa+1} - W = \tau \Delta_{\mathbb{H}}(V_\delta^{\kappa+1} - W) + \tau [g_\varepsilon(\mathbf{n}_\delta(V_\delta^{\kappa+1})) - g_\varepsilon(\mathbf{n}_\delta(W))].$$

Taking the $(\cdot, \cdot)_{0, \mathbb{H}}$ -inner product of both sides of (3.6) with $(V_\delta^{\kappa+1} - W)$ and using (2.1), we obtain

$$\|V_\delta^{\kappa+1} - W\|_{0, \mathbb{H}}^2 + \tau |V_\delta^{\kappa+1} - W|_{1, \mathbb{H}}^2 = \tau (g_\varepsilon(\mathbf{n}_\delta(V_\delta^{\kappa+1})) - g_\varepsilon(\mathbf{n}_\delta(W)), V_\delta^{\kappa+1} - W)_{0, \mathbb{H}},$$

which, after applying (2.29) (with $\mathbf{q} = \mathbf{n}_\delta$) and incorporating (2.5) and (2.6), yields

$$\|V_\delta^{\kappa+1} - W\|_{0, \mathbb{H}}^2 \leq \frac{8}{3} \tau \log(2\delta) \|V_\delta^{\kappa+1} - W\|_{0, \mathbb{H}}^2.$$

Thus, the latter inequality, along with (3.4), yield $\|V_\delta^{\kappa+1} - W\|_{0, \mathbb{H}} = 0$, or, equivalently, $W = V_\delta^{\kappa+1}$, which completes the induction. \square

Remark 3.1. *Let $\delta > e$ and $\varepsilon \in (0, \frac{1}{4\delta})$. Then, (3.2) can be written equivalently as*

$$V_\delta^{n+1} = \nu_n(V_\delta^{n+1}) \quad n = 0, \dots, N-1,$$

where $\nu_n : \mathbb{X}_{\mathbb{H}} \mapsto \mathbb{X}_{\mathbb{H}}$ is an operator given by

$$\nu_n(V) := (\mathfrak{I}_{\mathbb{H}} - \tau \Delta_{\mathbb{H}})^{-1} [V_\delta^n + \tau g_\varepsilon(\mathbf{n}_\delta(V)) + \tau \mathbb{I}_{\mathbb{H}} [f(t_{n+1}, \cdot)]] \quad \forall V \in \mathbb{X}_{\mathbb{H}}$$

and $\mathfrak{I}_{\mathbb{H}} : \mathbb{X}_{\mathbb{H}} \mapsto \mathbb{X}_{\mathbb{H}}$ stands for the identity operator. Observing that $\|(\mathfrak{I}_{\mathbb{H}} - \tau \Delta_{\mathbb{H}})^{-1} Z\|_{0, \mathbb{H}} \leq \|Z\|_{0, \mathbb{H}}$ for $Z \in \mathbb{X}_{\mathbb{H}}$, and using (2.6), (2.19) (with $c = 2\delta$), we obtain

$$(3.7) \quad \begin{aligned} \|\nu_n(V) - \nu_n(W)\|_{0, \mathbb{H}} &= \tau \|(\mathfrak{I}_{\mathbb{H}} - \tau \Delta_{\mathbb{H}})^{-1} [g_\varepsilon(\mathbf{n}_\delta(V)) - g_\varepsilon(\mathbf{n}_\delta(W))]\|_{0, \mathbb{H}} \\ &\leq \tau \|g_\varepsilon(\mathbf{n}_\delta(V)) - g_\varepsilon(\mathbf{n}_\delta(W))\|_{0, \mathbb{H}} \\ &\leq \tau \max_{|s| \in [0, 2\delta]} |g'_\varepsilon(s)| \|V - W\|_{0, \mathbb{H}} \\ &\leq 8\tau \max\{\log(2\delta), |\log(2\varepsilon)|\} \|V - W\|_{0, \mathbb{H}} \\ &\leq 8\tau |\log(2\varepsilon)| \|V - W\|_{0, \mathbb{H}} \quad \forall V, W \in \mathbb{X}_{\mathbb{H}}, \quad n = 0, \dots, N-1. \end{aligned}$$

Under the condition $8\tau |\log(2\varepsilon)| \leq \frac{1}{2}$, (3.7) yields that the operators $(\nu_n)_{n=0}^{N-1}$ become contractions on $\mathbb{X}_{\mathbb{H}}$, which results the uniqueness of the modified (ε BEFD) approximations and the convergence of the corresponding fixed point iteration method.

Now, we develop a convergence result for the modified (ε BEFD) method.

Theorem 3.3. Let $\delta_\star = 2 \max\{e, \max_{\mathbf{q}} |u|\}$, $\varepsilon \in (0, \frac{1}{2e})$ and $\tau \in (0, \frac{1}{11 \log(2\delta_\star)})$. Then, there exist constant $\mathfrak{C}_{\text{NL}, \delta_\star} > 0$, independent of ε , τ , h_1 and h_2 , such that:

$$(3.8) \quad \max_{0 \leq m \leq N} \|u^m - V_{\delta_\star}^m\|_{0, \text{H}} + \max_{0 \leq m \leq N} |u^m - V_{\delta_\star}^m|_{\infty, \text{H}} \leq \mathfrak{C}_{\text{NL}, \delta_\star} (\varepsilon |\log(2\varepsilon)| + \tau + h_1^2 + h_2^2).$$

Proof. To simplify the notation, we set $\mathbf{e}^m := u^m - V_{\delta_\star}^m$ for $m = 0, \dots, N$, and we use the symbol C to denote a generic non-negative constant that is independent of ε , τ , h_1 , h_2 and δ_\star , and may changes value from one line to the other. Also, we will use the symbol C_{δ_\star} to denote a generic non-negative constant that depends on δ_\star but is independent of ε , τ , h_1 and h_2 , and may changes value from one line to the other.

Under our assumptions, the modified (BEFD) approximations are well defined. Subtracting (3.2) from (2.37), we obtain the following error equations:

$$(3.9) \quad \mathbf{e}^{n+1} - \mathbf{e}^n = \tau \Delta_{\text{H}}(\mathbf{e}^{n+1}) + \tau [g_\varepsilon(\mathbf{n}_{\delta_\star}(u^{n+1})) - g_\varepsilon(\mathbf{n}_{\delta_\star}(V_{\delta_\star}^{n+1}))] + \tau \mathbf{s}_\varepsilon^n, \quad n = 0, \dots, N-1.$$

Discrete L^2 -error estimate. Take the $(\cdot, \cdot)_{0, \text{H}}$ -inner product of both sides of (3.9) with \mathbf{e}^{n+1} , and then use (2.1) to get

$$(3.10) \quad \|\mathbf{e}^{n+1}\|_{0, \text{H}}^2 - \|\mathbf{e}^n\|_{0, \text{H}}^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0, \text{H}}^2 + 2\tau |\mathbf{e}^{n+1}|_{1, \text{H}}^2 \leq \mathbf{K}_1^n + \mathbf{K}_2^n, \quad n = 0, \dots, N-1,$$

where

$$\begin{aligned} \mathbf{K}_1^n &:= 2\tau (\mathbf{s}_\varepsilon^n, \mathbf{e}^{n+1})_{0, \text{H}}, \\ \mathbf{K}_2^n &:= 2\tau (g_\varepsilon(\mathbf{n}_{\delta_\star}(u^{n+1})) - g_\varepsilon(\mathbf{n}_{\delta_\star}(V_{\delta_\star}^{n+1})), \mathbf{e}^{n+1})_{0, \text{H}}. \end{aligned}$$

Let $n \in \{0, \dots, N-1\}$. Using the Cauchy-Schwarz inequality, (2.35), (2.39), (2.2) and the arithmetic mean inequality, it follows that

$$(3.11) \quad \begin{aligned} \mathbf{K}_1^n &\leq 2\tau \|\mathbf{s}_\varepsilon^n\|_{0, \text{H}} \|\mathbf{e}^{n+1}\|_{0, \text{H}} \\ &\leq C\tau (\|\mathbf{s}_\varepsilon^n - \mathbf{r}_\varepsilon^n\|_{0, \text{H}} + \|\mathbf{r}_\varepsilon^n\|_{0, \text{H}}) |\mathbf{e}^{n+1}|_{1, \text{H}} \\ &\leq C\tau [\tau + \varepsilon |\log(2\varepsilon)| + h_1^2 + h_2^2]^2 + \tau |\mathbf{e}^{n+1}|_{1, \text{H}}^2. \end{aligned}$$

Also, (2.5), (2.6) and (2.29) (with $\mathbf{q} = \mathbf{n}_{\delta_\star}$) yield

$$(3.12) \quad \begin{aligned} \mathbf{K}_2^n &\leq \frac{16}{3} \tau \log(2\delta_\star) \|\mathbf{e}^{n+1}\|_{0, \text{H}}^2 \\ &\leq \frac{32}{3} \tau \log(2\delta_\star) (\|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0, \text{H}}^2 + \|\mathbf{e}^n\|_{0, \text{H}}^2) \\ &\leq \frac{32}{3} \tau \log(2\delta_\star) \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0, \text{H}}^2 + C_{\delta_\star} \tau \|\mathbf{e}^n\|_{0, \text{H}}^2 \\ &\leq \frac{32}{33} \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0, \text{H}}^2 + C_{\delta_\star} \tau \|\mathbf{e}^n\|_{0, \text{H}}^2. \end{aligned}$$

Combining, (3.10), (3.11) and (3.12), we conclude that

$$(3.13) \quad \|\mathbf{e}^{n+1}\|_{0, \text{H}}^2 \leq (1 + C_{\delta_\star} \tau) \|\mathbf{e}^n\|_{0, \text{H}}^2 + C\tau (\tau + \varepsilon |\log(2\varepsilon)| + h_1^2 + h_2^2)^2, \quad n = 0, \dots, N-1.$$

Since $\mathbf{e}^0 = 0$, after employing a standard Gronwall argument on (3.13), we arrive at

$$(3.14) \quad \max_{0 \leq n \leq N} \|\mathbf{e}^n\|_{0, \text{H}} \leq C_{\delta_\star} (\tau + \varepsilon |\log(2\varepsilon)| + h_1^2 + h_2^2).$$

Discrete L^∞ -error estimate. Let $n \in \{0, \dots, N-1\}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{I}^\circ$ such that $|\mathbf{e}_\alpha^{n+1}| = |\mathbf{e}^{n+1}|_{\infty, \text{H}}$. Multiplying both sides of (3.9) with \mathbf{e}_α^{n+1} , we obtain

$$\begin{aligned} \left[1 + \frac{2\tau}{h_1^2} + \frac{2\tau}{h_2^2}\right] (\mathbf{e}_\alpha^{n+1})^2 &= \mathbf{e}_\alpha^n \mathbf{e}_\alpha^{n+1} + \frac{\tau}{h_1^2} [\mathbf{e}_{(\alpha_1-1, \alpha_2)}^{n+1} + \mathbf{e}_{(\alpha_1+1, \alpha_2)}^{n+1}] \mathbf{e}_\alpha^{n+1} + \frac{\tau}{h_2^2} [\mathbf{e}_{(\alpha_1, \alpha_2-1)}^{n+1} + \mathbf{e}_{(\alpha_1, \alpha_2+1)}^{n+1}] \mathbf{e}_\alpha^{n+1} \\ &\quad + \tau [g_\varepsilon(\mathbf{n}_{\delta_\star}(u_\alpha^{n+1})) - g_\varepsilon(\mathbf{n}_{\delta_\star}((V_{\delta_\star}^{n+1})_\alpha))] \mathbf{e}_\alpha^{n+1} \\ &\quad + \tau [((\mathbf{s}_\varepsilon^n)_\alpha - (\mathbf{r}_\varepsilon^n)_\alpha) + (\mathbf{r}_\varepsilon^n)_\alpha] \mathbf{e}_\alpha^{n+1} \end{aligned}$$

which, along with (2.35) and (2.39), yields that

$$(3.15) \quad \begin{aligned} |\mathbf{e}_\alpha^{n+1}|^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2 &\leq |\mathbf{e}_\alpha^n|^2 + C\tau \left[\varepsilon |\log(\varepsilon)| + \tau + h_1^2 + h_2^2 \right] |\mathbf{e}_\alpha^{n+1}| \\ &\quad + 2\tau \left[g_\varepsilon(\mathbf{n}_{\delta_*}(u_\alpha^{n+1})) - g_\varepsilon(\mathbf{n}_{\delta_*}((V_{\delta_*}^{n+1})_\alpha)) \right] \mathbf{e}_\alpha^{n+1}. \end{aligned}$$

Applying (2.32) (with $\mathbf{q} = \mathbf{n}_{\delta_*}$) along with (2.6) and (2.5), we conclude that

$$(3.16) \quad \begin{aligned} 2\tau \left[g_\tau(\mathbf{n}_{\delta_*}(u_\alpha^{n+1})) - g_\tau(\mathbf{n}_{\delta_*}((V_{\delta_*}^{n+1})_\alpha)) \right] \mathbf{e}_\alpha^{n+1} &\leq \frac{16}{3} \tau \log(2\delta_*) (\mathbf{e}_\alpha^{n+1})^2 \\ &\leq \frac{32}{3} \tau \log(2\delta_*) \left[(\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n)^2 + (\mathbf{e}_\alpha^n)^2 \right] \\ &\leq C_{\delta_*} \tau (\mathbf{e}_\alpha^n)^2 + \frac{32}{3} \tau \log(2\delta_*) (\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n)^2 \\ &\leq C_{\delta_*} \tau (\mathbf{e}_\alpha^n)^2 + \frac{32}{33} (\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n)^2. \end{aligned}$$

Using (3.15) and (3.16), we have

$$|\mathbf{e}^{n+1}|_{\infty, \mathbb{H}}^2 \leq (1 + C_{\delta_*}) |\mathbf{e}^n|_{\infty, \mathbb{H}}^2 + C_{\delta_*} \tau \left(\varepsilon |\log(\varepsilon)| + \tau + h_1^2 + h_2^2 \right) |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}}, \quad n = 0, \dots, N-1,$$

which, easily, yields

$$(3.17) \quad |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}} \leq (1 + C_{\delta_*}) |\mathbf{e}^n|_{\infty, \mathbb{H}} + C_{\delta_*} \tau \left(\varepsilon |\log(\varepsilon)| + \tau + h_1^2 + h_2^2 \right), \quad n = 0, \dots, N-1,$$

Finally, we apply a standard Gronwall argument on (3.17), to arrive at

$$(3.18) \quad \max_{0 \leq n \leq N} |\mathbf{e}^n|_{\infty, \mathbb{H}} \leq C_{\delta_*} \left(\varepsilon |\log(2\varepsilon)| + \tau + h_1^2 + h_2^2 \right).$$

□

Thus, (3.8), easily, follows from (3.14) and (3.18).

3.2. Convergence of the (ε BEFD) method.

Theorem 3.4. *Let $\delta_* = 2 \max\{e, \max_{\mathbb{Q}} |u|\}$, $\varepsilon \in (0, \frac{1}{2e})$, $\tau \in (0, \frac{1}{11 \log(2\delta_*)})$ and C_{NL, δ_*} be the constant specified in Theorem 3.3. If*

$$(3.19) \quad C_{\text{NL}, \delta_*} \left(\varepsilon |\log(2\varepsilon)| + \tau + h_1^2 + h_2^2 \right) \leq \frac{\delta_*}{2},$$

then, the matrices $(U^m)_{m=0}^N \subset \mathbb{X}_{\mathbb{H}}$, given by $U^m := V_{\delta_}^m$ for $m = 0, \dots, N$, are (ε BEFD) approximations and satisfy*

$$(3.20) \quad \max_{0 \leq m \leq N} (\|u^m - U^m\|_{0, \mathbb{H}} + |u^m - U^m|_{\infty, \mathbb{H}}) \leq C_{\text{NL}, \delta_*} \left(\varepsilon |\log(2\varepsilon)| + \tau + h_1^2 + h_2^2 \right).$$

Proof. Using that $\delta_* \geq 2 \max_{\mathbb{Q}} |u|$, along with (3.8) and (3.19), we obtain

$$\begin{aligned} |V_{\delta_*}^n|_{\infty, \mathbb{H}} &\leq |u^n - V_{\delta_*}^n|_{\infty, \mathbb{H}} + |u^n|_{\infty, \mathbb{H}} \\ &\leq C_{\text{NL}, \delta_*} \left[\tau |\log(2\tau)| + h_1^2 + h_2^2 \right] + \max_{\mathbb{Q}} |u| \\ &\leq \delta_*, \quad n = 1, \dots, N, \end{aligned}$$

which, along with (2.3), yields $\mathbf{n}_{\delta_*}(V_{\delta_*}^n) = V_{\delta_*}^n$ for $n = 1, \dots, N$. Thus, for $\delta = \delta_*$, the modified (ε BEFD) approximations defined are (ε BEFD) approximations and the error estimate (3.20) follow as a natural outcome of (3.8). □

Remark 3.2. *Let $\delta_* = 2 \max\{e, \max_{\mathbb{Q}} |u|\}$, $\varepsilon \in (0, \frac{1}{2e})$ and $\tau \in (0, \frac{1}{11 \log(2\delta_*)})$. Then, every set $(U^m)_{m=0}^N \subset \mathbb{X}_{\mathbb{H}}$ of (ε BEFD) approximations satisfying $\max_{0 \leq m \leq N} |U^m|_{\infty, \mathbb{H}} \leq \delta_*$, is, also, a set of modified (ε BEFD) approximations and according to Proposition 3.2, is unique. However, we are not able to provide a general uniqueness result for the (ε BEFD) approximations.*

3.3. The modified (LBEFD) method. For $\delta > 0$, the modified (LBEFD) approximations $(W_\delta^n)_{n=0}^N \subset \mathbf{X}_H$ of the solution u are specified by the algorithm below:

Step ML1: Set

$$(3.21) \quad W_\delta^0 := U^0.$$

Step ML2: For $n = 0, \dots, N-1$, find $W_\delta^{n+1} \in \mathbf{X}_H$ such that

$$(3.22) \quad \frac{W_\delta^{n+1} - W_\delta^n}{\tau} = \Delta_H(W_\delta^{n+1}) + g(\mathbf{n}_\delta(W_\delta^n)) + I_H[f(t_{n+1}, \cdot)].$$

Next, we develop a convergence result for the modified (LBEFD) approximations.

Theorem 3.5. *Let $\delta_\star = 2 \max\{e, \max_Q |u|\}$ and $\tau \in (0, \frac{1}{4\delta_\star})$. Then, there exists constant $\mathfrak{C}_{L, \delta_\star}$, independent of τ, h_1 and h_2 , such that*

$$(3.23) \quad \max_{0 \leq m \leq N} \|u^m - W_{\delta_\star}^m\|_{0,H} + \max_{0 \leq m \leq N} |u^m - W_{\delta_\star}^m|_{\infty,H} \leq \mathfrak{C}_{L, \delta_\star} [\tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Proof. We simplify the notation, by setting $e^m := u^m - W_{\delta_\star}^m$ for $m = 0, \dots, N$, and by adopting the notation convection of the proof of Theorem 3.3.

Subtract (3.22) from (2.38) (with $\varepsilon = \tau$), to get the corresponding error equations:

$$(3.24) \quad e^{n+1} - e^n = \tau \Delta_H(e^{n+1}) + \tau [g_\tau(\mathbf{n}_{\delta_\star}(u^n)) - g(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n))] + \tau \sigma_\tau^n, \quad n = 0, \dots, N-1.$$

Discrete L^2 -error estimate. Take the $(\cdot, \cdot)_{0,H}$ -inner product of both sides of (3.24) with e^{n+1} and then use (2.1) to arrive at

$$(3.25) \quad \|e^{n+1}\|_{0,H}^2 - \|e^n\|_{0,H}^2 + \|e^{n+1} - e^n\|_{0,H}^2 + 2\tau |e^{n+1}|_{1,H}^2 \leq \sum_{\ell=1}^3 L_\ell^n, \quad n = 0, \dots, N-1,$$

where

$$\begin{aligned} L_1^n &:= 2\tau (\sigma_\tau^n, e^{n+1})_{0,H}, \\ L_2^n &:= 2\tau (g_\tau(\mathbf{n}_{\delta_\star}(u^n)) - g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n)), e^{n+1})_{0,H}, \\ L_3^n &:= 2\tau (g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n)) - g(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n)), e^{n+1})_{0,H}. \end{aligned}$$

Let $n \in \{0, \dots, N-1\}$. First, we use the Cauchy-Schwarz inequality, (2.36), (2.39), (2.2) and the arithmetic mean inequality, to have

$$(3.26) \quad \begin{aligned} L_1^n &\leq 2\tau \|\sigma_\tau^n\|_{0,H} \|e^{n+1}\|_{0,H} \\ &\leq 2\tau (\|\sigma_\tau^n - \rho_\tau^n\|_{0,H} + \|\rho_\tau^n\|_{0,H}) \|e^{n+1}\|_{0,H} \\ &\leq C\tau [h_1^2 + h_2^2 + \tau |\log(2\tau)|] |e^{n+1}|_{1,H} \\ &\leq C\tau [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 + \frac{\tau}{2} |e^{n+1}|_{1,H}^2. \end{aligned}$$

Also, we use the Cauchy-Schwarz inequality, (2.6), (2.5), (2.29) (with $\mathbf{q} = \mathbf{n}_{\delta_\star}$), the arithmetic mean inequality and (2.10) (with $c = 2\delta_\star$), to get

$$(3.27) \quad \begin{aligned} L_2^n &= 2\tau (g_\tau(\mathbf{n}_{\delta_\star}(u^n)) - g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n)), e^{n+1} - e^n)_{0,H} \\ &\quad + 2\tau (g_\tau(\mathbf{n}_{\delta_\star}(u^n)) - g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n)), e^n)_{0,H} \\ &\leq 2\tau \|g_\tau(\mathbf{n}_{\delta_\star}(u^n)) - g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n))\|_{0,H} \|e^{n+1} - e^n\|_{0,H} + \frac{16}{3} \tau \log(2\delta_\star) \|e^n\|_{0,H}^2 \\ &\leq \tau^2 [\|g_\tau(\mathbf{n}_{\delta_\star}(u^n))\|_{0,H} + \|g_\tau(\mathbf{n}_{\delta_\star}(W_{\delta_\star}^n))\|_{0,H}]^2 + \|e^{n+1} - e^n\|_{0,H}^2 + C_{\delta_\star} \tau \|e^n\|_{0,H}^2 \\ &\leq C\tau^2 [4\delta_\star \log(2\delta_\star)]^2 + \|e^{n+1} - e^n\|_{0,H}^2 + C_{\delta_\star} \tau \|e^n\|_{0,H}^2 \\ &\leq C_{\delta_\star} (\tau^2 + \tau \|e^n\|_{0,H}^2) + \|e^{n+1} - e^n\|_{0,H}^2. \end{aligned}$$

Finally, we use the Cauchy-Schwarz inequality, (2.2), (2.9) and the arithmetic mean inequality, to obtain

$$\begin{aligned}
(3.28) \quad \mathbb{L}_3^n &\leq 2\tau \|g_\tau(\mathbf{n}_{\delta_*}(W_{\delta_*}^n)) - g(\mathbf{n}_{\delta_*}(W_{\delta_*}^n))\|_{0,\mathbb{H}} \|\mathbf{e}^{n+1}\|_{0,\mathbb{H}} \\
&\leq C\tau^2 |\log(2\tau)| |\mathbf{e}^{n+1}|_{1,\mathbb{H}} \\
&\leq C\tau [\tau |\log(2\tau)|]^2 + \frac{\tau}{2} |\mathbf{e}^{n+1}|_{1,\mathbb{H}}^2.
\end{aligned}$$

Combining, (3.25), (3.26), (3.27) and (3.28), we conclude that

$$\|\mathbf{e}^{n+1}\|_{0,\mathbb{H}}^2 \leq (1 + C_{\delta_*}\tau) \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau [\tau^{1/2} + \tau |\log(2\tau)| + h_1^2 + h_2^2]^2, \quad n = 0, \dots, N-1.$$

Apply a standard discrete Gronwall argument and use that $\mathbf{e}^0 = 0$, to get

$$(3.29) \quad \max_{0 \leq m \leq N} \|\mathbf{e}^m\|_{0,\mathbb{H}} \leq C_{\delta_*} [\tau^{1/2} + \tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Now, use (3.27), the arithmetic mean inequality, (2.6), (2.5), (2.19) (with $c = 2\delta_*$) and (3.29), to reestimate \mathbb{L}_2^n as follows

$$\begin{aligned}
(3.30) \quad \mathbb{L}_2^n &\leq \tau^2 \|g_\tau(\mathbf{n}_{\delta_*}(u^n)) - g_\tau(\mathbf{n}_{\delta_*}(W_{\delta_*}^n))\|_{0,\mathbb{H}}^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 \\
&\leq \tau^2 \max_{|s| \in [0, 2\delta_*]} |g'_\tau(s)|^2 \max_{s \in \mathbb{R}} |\mathbf{n}'_{\delta_*}|^2 \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 \\
&\leq C\tau^2 (\max\{\log(2\delta_*), |\log(2\tau)|\})^2 \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 \\
&\leq C_{\delta_*}\tau^2 |\log(2\tau)|^2 [\tau^{1/2} + \tau |\log(2\tau)| + h_1^2 + h_2^2]^2 \\
&\quad + \|\mathbf{e}^{n+1} - \mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau \|\mathbf{e}^n\|_{0,\mathbb{H}}^2, \quad n = 0, \dots, N-1.
\end{aligned}$$

Combining, (3.25), (3.26), (3.28) and (3.30), and observing that $\sqrt{2\tau} \in (0, e)$, we get

$$\begin{aligned}
\|\mathbf{e}^{n+1}\|_{0,\mathbb{H}}^2 &\leq (1 + C_{\delta_*}\tau) \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau [1 + \tau^{1/2} |\log(2\tau)|]^2 [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 \\
&\leq (1 + C_{\delta_*}\tau) \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau [1 + |g(\sqrt{2\tau})|]^2 [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 \\
&\leq (1 + C_{\delta_*}\tau) \|\mathbf{e}^n\|_{0,\mathbb{H}}^2 + C_{\delta_*}\tau [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2, \quad n = 0, \dots, N-1.
\end{aligned}$$

Now, we apply, again, a standard discrete Gronwall argument to conclude

$$(3.31) \quad \max_{0 \leq m \leq N} \|\mathbf{e}^m\|_{0,\mathbb{H}} \leq C_{\delta_*} [\tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Discrete L^∞ -error estimate. Let $n \in \{0, \dots, N-1\}$ and $\alpha \in \mathbb{I}^\circ$ such that $|\mathbf{e}_\alpha^{n+1}| = |\mathbf{e}^{n+1}|_{\infty,\mathbb{H}}$. Proceeding as in the proof of Theorem 3.3, we obtain

$$(3.32) \quad |\mathbf{e}_\alpha^{n+1}|^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2 \leq |\mathbf{e}_\alpha^n|^2 + \Lambda_\alpha^{1,n} + \Lambda_\alpha^{2,n} + \Lambda_\alpha^{3,n} + \Lambda_\alpha^{4,n},$$

where

$$\begin{aligned}
\Lambda_\alpha^{1,n} &:= 2\tau [|(\sigma_\tau^n)_\alpha - (\rho_\tau^n)_\alpha| + |(\rho_\tau^n)_\alpha|] |\mathbf{e}_\alpha^{n+1}|, \\
\Lambda_\alpha^{2,n} &:= 2\tau [g_\tau(\mathbf{n}_{\delta_*}(u_\alpha^n)) - g_\tau(\mathbf{n}_{\delta_*}((V_{\delta_*}^n)_\alpha))] (\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n), \\
\Lambda_\alpha^{3,n} &:= 2\tau [g_\tau(\mathbf{n}_{\delta_*}(u_\alpha^n)) - g_\tau(\mathbf{n}_{\delta_*}((V_{\delta_*}^n)_\alpha))] \mathbf{e}_\alpha^n, \\
\Lambda_\alpha^{4,n} &:= 2\tau [g_\tau(\mathbf{n}_{\delta_*}((V_{\delta_*}^n)_\alpha)) - g(\mathbf{n}_{\delta_*}((V_{\delta_*}^n)_\alpha))] \mathbf{e}_\alpha^{n+1}.
\end{aligned}$$

Using (2.36), (2.39), (2.9) and the arithmetic mean inequality, we have

$$\begin{aligned}
(3.33) \quad \Lambda_\alpha^{1,n} &\leq C\tau [\tau |\log(2\tau)| + h_1^2 + h_2^2] |\mathbf{e}^{n+1}|_{\infty,\mathbb{H}} \\
&\leq C\tau [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 + \frac{\tau}{4} |\mathbf{e}^{n+1}|_{\infty,\mathbb{H}}^2
\end{aligned}$$

and

$$(3.34) \quad \begin{aligned} \Lambda_\alpha^{4,n} &\leq 18 \tau [\tau |\log(2\tau)|] |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}} \\ &\leq C \tau [\tau |\log(2\tau)|]^2 + \frac{\tau}{4} |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}}^2. \end{aligned}$$

Applying (2.32) (with $\mathbf{q} = \mathbf{n}_\delta$) along with (2.5), (2.6), we conclude that

$$(3.35) \quad \begin{aligned} \Lambda_\alpha^{3,n} &\leq \frac{16}{3} \tau \log(2\delta_\star) (\mathbf{e}_\alpha^n)^2 \\ &\leq C_{\delta_\star} \tau |\mathbf{e}_\alpha^n|_{\infty, \mathbb{H}}^2. \end{aligned}$$

Also, (2.10) (with $c = 2\delta_\star$) and the arithmetic mean inequality yields

$$(3.36) \quad \begin{aligned} \Lambda_\alpha^{2,n} &\leq 2\tau [|g_\tau(\mathbf{n}_{\delta_\star}(u_\alpha^n))| + |g_\tau(\mathbf{n}_{\delta_\star}((V_{\delta_\star}^n)_\alpha))|] |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n| \\ &\leq 2\tau [4\delta_\star \log(2\delta_\star)] |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n| \\ &\leq C_{\delta_\star} \tau^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2. \end{aligned}$$

Using (3.32), (3.33), (3.34), (3.35) and (3.36), we arrive at

$$(1 - \frac{\tau}{2}) |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}}^2 \leq (1 + C_{\delta_\star} \tau) |\mathbf{e}^n|_{\infty, \mathbb{H}}^2 + C_{\delta_\star} \tau [\tau^{1/2} + \tau |\log(2\tau)| + h_1^2 + h_2^2]^2$$

for $n = 0, \dots, N-1$. Since $\tau < 1$ and $\mathbf{e}^0 = 0$, a standard Gronwall argument yields

$$(3.37) \quad \max_{0 \leq m \leq N} |\mathbf{e}^m|_{\infty, \mathbb{H}} \leq C_{\delta_\star} [\tau^{1/2} + \tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Let us reestimate $\Lambda_\alpha^{2,n}$, by using (2.6), (2.19) (with $c = 2\delta_\star$), (2.5), (3.37) and the arithmetic mean inequality, as follows

$$(3.38) \quad \begin{aligned} \Lambda_\alpha^{2,n} &\leq 2\tau \max_{|s| \in [0, 2\delta_\star]} |g'_\tau(s)| \max_{s \in \mathbb{R}} |\mathbf{n}'_{\delta_\star}(s)| |\mathbf{e}_\alpha^n| |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n| \\ &\leq C \tau |\log(2\tau)| |\mathbf{e}_\alpha^n| |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n| \\ &\leq C \tau^2 |\log(2\tau)|^2 |\mathbf{e}_\alpha^n|_{\infty, \mathbb{H}}^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2 \\ &\leq C_{\delta_\star} \tau [1 + \tau^{1/2} |\log(2\tau)|]^2 [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2 \\ &\leq C_{\delta_\star} \tau [1 + |g(\sqrt{2\tau})|]^2 [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2 \\ &\leq C_{\delta_\star} \tau [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2 + |\mathbf{e}_\alpha^{n+1} - \mathbf{e}_\alpha^n|^2. \end{aligned}$$

Now, combining (3.32), (3.33), (3.34), (3.35) and (3.38), we get

$$(1 - \frac{\tau}{2}) |\mathbf{e}^{n+1}|_{\infty, \mathbb{H}}^2 \leq (1 + C_{\delta_\star} \tau) |\mathbf{e}^n|_{\infty, \mathbb{H}}^2 + C_{\delta_\star} \tau [\tau |\log(2\tau)| + h_1^2 + h_2^2]^2$$

for $n = 0, \dots, N-1$, which, after using a standard Gronwall argument, yields

$$(3.39) \quad \max_{0 \leq m \leq N} |\mathbf{e}^m|_{\infty, \mathbb{H}} \leq C_{\delta_\star} [\tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Thus, (3.23), easily, follows from (3.31) and (3.39). \square

3.4. Convergence of the (LBEFD) method.

Theorem 3.6. *Let $\delta_\star = 2 \max\{e, \max_{\mathbf{q}} |u|\}$, $\tau \in (0, \frac{1}{4\delta_\star})$ and $\mathfrak{C}_{\mathbb{L}, \delta_\star}$ be the positive constant specified in Theorem 3.5. If*

$$\mathfrak{C}_{\mathbb{L}, \delta_\star} [\tau |\log(2\tau)| + h_1^2 + h_2^2] \leq \frac{\delta_\star}{2},$$

then the approximations $(U^m)_{m=0}^N$ of the (LBEFD) method satisfy

$$U^m = W_{\delta_\star}^m, \quad m = 0, \dots, N,$$

and

$$\max_{0 \leq m \leq N} \|u^m - U^m\|_{0, \mathbb{H}} + \max_{0 \leq m \leq N} |u^m - U^m|_{\infty, \mathbb{H}} \leq \mathfrak{C}_{\mathbb{L}, \delta_\star} [\tau |\log(2\tau)| + h_1^2 + h_2^2].$$

Proof. It is similar to the proof of Theorem 3.4, and thus it is omitted. \square

4. NUMERICAL RESULTS

The proposed numerical methods has been implemented in `python` programs. The (LBEFD) code solves the linear systems of algebraic equations by applying the usual Conjugate Gradient method. The (ε BEFD) code solves the nonlinear systems of algebraic equations by applying the iterative Newton's method along with the GMRES method for inverting the corresponding Jacobian matrices by calling the subroutine `gmres` of the library `scipy.sparse.linalg`.

When the exact solution to the problem is known, we test the performance of our finite difference methods by computing the error in the discrete $L_t^\infty(L_x^2)$ -norm $\mathbf{E}^0(N, J_1, J_2) := \max_{0 \leq n \leq N} \|U^n - u^n\|_{0,H}$ and in the discrete $L_t^\infty(L_x^\infty)$ -norm $\mathbf{E}^\infty(N, J_1, J_2) := \max_{0 \leq n \leq N} |U^n - u^n|_{\infty,H}$. Then, after choosing $\nu \in \mathbb{N}$, function $\mathbf{f} : (0, +\infty) \mapsto (0, +\infty)^3$ and $(N, J_1, J_2) = \mathbf{f}(\nu)$, we compute the experimental order of convergence with respect to ν , corresponding to given values ν_1 and ν_2 of ν , by using the formula:

$$\log [\mathbf{E}(\mathbf{f}(\nu_1)) / \mathbf{E}(\mathbf{f}(\nu_2))] / \log(\nu_2/\nu_1),$$

where $\mathbf{E} = \mathbf{E}^0$ or \mathbf{E}^∞ .

4.1. Example 1. Let $T = 1$, $D = [0, 1] \times [0, 1]$, $N = \mathbf{f}_1(\nu) = \nu$, $J_1 = \mathbf{f}_2(\nu) = \sqrt{\nu}$, $J_2 = \mathbf{f}_3(\nu) = \sqrt{\nu}$, $\varepsilon = \tau^2$, $\nu \in \{200, 400, 800, 1600, 3200\}$ and load f such that the function

$$u(t, x) = \frac{1}{2} \exp(2 + \sin(2\pi t)) \sin(2\pi x_1) \sin(2\pi x_2)$$

to be the exact solution to the problem (1.1)-(1.5). The errors we computed are shown on Table 1, and confirm a first order experimental order of convergence with respect to ν , for both methods and norms.

(LBEFD) method					(ε BEFD) method with $\varepsilon = \tau^2$				
Example 1					Example 1				
ν	$\mathbf{E}^0(\mathbf{f}(\nu))$	Rate	$\mathbf{E}^\infty(\mathbf{f}(\nu))$	Rate	ν	$\mathbf{E}^0(\mathbf{f}(\nu))$	Rate	$\mathbf{E}^\infty(\mathbf{f}(\nu))$	Rate
200	7.033(-2)	—	1.393(-1)	—	200	7.009(-2)	—	1.388(-1)	—
400	3.583(-2)	0.97	7.138(-2)	0.96	400	3.569(-2)	0.97	7.107(-2)	0.96
800	1.883(-2)	0.92	3.761(-2)	0.92	800	1.876(-2)	0.92	3.745(-2)	0.92
1600	9.413(-3)	1.00	1.882(-2)	0.99	1600	9.376(-3)	1.00	1.874(-2)	0.99
3200	4.880(-3)	0.94	9.769(-3)	0.94	3200	4.862(-3)	0.94	9.729(-3)	0.94

TABLE 1.

4.2. Example 2. Let $T = 1$, $D = [0, 1] \times [0, 1]$, $N = \mathbf{f}_1(\nu) = \nu$, $J_1 = \mathbf{f}_2(\nu) = \sqrt{\nu}$, $J_2 = \mathbf{f}_3(\nu) = \sqrt{\nu}$, $\varepsilon = \tau^2$, $\nu \in \{200, 400, 800, 1600, 3200\}$ and load f such that the function

$$u(t, x) = 100 e^t (x_1^5 + x_2^5) \prod_{i=1}^4 (x_1 - a_i)(x_2 - a_i),$$

with $a_1 = 0$, $a_2 = 1$, $a_3 = 0.5$ and $a_4 = 0.25$, to be the exact solution to the problem (1.1)-(1.5), with non zero initial value. Computing again the numerical approximation errors we conclude a first order experimental order of convergence, for both methods as it is shown on Table 2.

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(LBEFD) method					(εBEFD) method with $\varepsilon = \tau^2$				
Example 2					Example 2				
ν	$E^0(\mathfrak{f}(\nu))$	Rate	$E^\infty(\mathfrak{f}(\nu))$	Rate	ν	$E^0(\mathfrak{f}(\nu))$	Rate	$E^\infty(\mathfrak{f}(\nu))$	Rate
200	3.435(-3)	—	1.208(-2)	—	200	3.437(-3)	—	1.209(-2)	—
400	1.766(-3)	0.95	6.336(-3)	0.93	400	1.768(-3)	0.95	6.340(-3)	0.93
800	9.299(-4)	0.92	3.406(-3)	0.89	800	9.306(-4)	0.92	3.408(-3)	0.89
1600	4.659(-4)	0.99	1.708(-3)	0.99	1600	4.663(-4)	0.99	1.709(-3)	0.99
3200	2.412(-4)	0.94	8.830(-4)	0.95	3200	2.414(-4)	0.94	8.836(-4)	0.95

TABLE 2.

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