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Wei Xu

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STOCHASTIC VOLTERRA EQUATIONS FOR THE LOCAL TIMES OF SPECTRALLY POSITIVE STABLE PROCESSES

BY WEI XU

Department of Mathematics, Humboldt-Universität zu Berlin, xuwei@math.hu-berlin.de, xuwei.math@gmail.com

This paper is concerned with the macroevolution mechanism of local times of a spectrally positive stable process in the spatial direction. The main results state that conditioned on the finiteness of the first time at which the local time at zero exceeds a given value, the local times at positive half line are equal in distribution to the unique solution of a stochastic Volterra equation driven by a Poisson random measure whose intensity coincides with the Lévy measure. This helps us to provide not only a simple proof for the Hölder regularity, but also a uniform upper bound for all moments of the Hölder coefficient as well as a maximal inequality for the local times. Moreover, in collaboration with the stochastic Volterra equation, we extend the method of duality to establish an exponential-affine representation of the Laplace functional in terms of the unique solution of a nonlinear Volterra integral equation associated with the Laplace exponent of the stable process.

1. Introduction. Local times of Lévy processes not only have wide applications in various fields; see [6, 43], but they also have been studied in depth with abundant of interesting results obtained, e.g. various constructions (see [5, 17, 59]), Hilbert transform (see [8, 25]), Hölder regularity (see [4, 13, 26]) and so on. We refer to [9, 29, 62] for survey on local times and their applications. In particular, to understand thoroughly the dependence of Brownian local times in the space variable, Ray [60] and Knight [40] independently proved the well-known *Ray-Knight theorem* that links Brownian local times to Bessel processes. Later, the Ray-Knight theorem was generalized in [22, 64] to strongly symmetric Markov processes with finite 1-potential densities. For a general spectrally positive Lévy process, Le Gall and Le Jan [49, 50] considered the reflected processes of its time-reversed processes. Associated to the local times at 0, they introduced an exploration process to describe the genealogy of a *continuous-state branching process* (CB-process) and generalized an analogue of the Ray-Knight theorem for a functional of local times of the Lévy process; see also [20, 48] for details.

Because of the lack of Markovianity; see [21], local times (not their functionals) of general spectrally positive Lévy processes are quite untractable. Their microstructure and evolution mechanism have received considerable attention in recent years. Specifically, Lambert [44] connected a compound Poisson process with unit negative drift and killed upon hitting 0 to the jumping chronological contour processes of a splitting tree, and then showed that its local times are equal in distribution to a homogeneous, binary *Crump-Mode-Jagers branching process* (CMJ-process). For a general spectrally positive Lévy process, Lambert and Simatos [45] explored the genealogical structure of their local times preliminarily via an approximating sequence consisting of rescaled binary CMJ-processes. Later, a detailed genealogical interpretation was given in [47] by considering the corresponding totally ordered measured tree that satisfies the splitting property. Meanwhile, Forman et al. [26] established a locally uniform approximation for the local times of a driftless spectrally positive stable process by endowing each jump with a random graph. Up to now, the genealogical structure of local times of general spectrally positive Lévy processes seems to be fairly clear. However, their macroevolution mechanisms, by contrast, are still incomprehensible.

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The purpose of this work is to establish stochastic equations for the local times of spectrally positive stable processes and study their macroevolution mechanisms in the spatial direction. In contrast to the genealogical interpretations given in the aforementioned literature, stochastic equations have many advantages including

- They provide an intuitive description for the evolution of local times in the spatial direction as well as a detailed interpretation of their perturbations caused by each jump of stable processes.
- They allow us to study the local times by using tools and methods from the modern probability theory, e.g., stochastic integral inequalities, stochastic Fubini theorem and extreme value theory.
- They offer a kind of novel non-Markovian models and a convenient way of numerical analysis, which will benefit greatly the related fields, e.g., processor-sharing queues and stochastic volatility models.

1.1. Overview of main results. Let $\xi := \{\xi(t) : t \geq 0\}$ be a one-dimensional spectrally positive stable process with index $1 + \alpha \in (1, 2)$ and Laplace exponent

$$(1.1) \quad \Phi(\lambda) := b\lambda + c\lambda^{\alpha+1} = b\lambda + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y)\nu_\alpha(dy), \quad \lambda \geq 0,$$

where $b \geq 0$, $c > 0$ and $\nu_\alpha(dy)$, known as the *Lévy measure*, is a σ -finite measure on $(0, \infty)$ given by

$$(1.2) \quad \nu_\alpha(dy) := \frac{c\alpha(\alpha+1)}{\Gamma(1-\alpha)} \cdot y^{-\alpha-2} \cdot dy.$$

It is recurrent or drifts to $-\infty$ according as $b = 0$ or > 0 . Let W be the scale function of ξ and $\bar{\nu}_\alpha(x) := \nu_\alpha([x, \infty))$ the tail function of ν_α . Let $L_\xi := \{L_\xi(x, t) : x \in \mathbb{R}, t \geq 0\}$ be the local times of ξ , where $L_\xi(x, t)$ is usually interpreted as the amount of time that ξ spends at level x up to time t . Denote by $\tau_\xi^L(\zeta)$ the first time that the amount of local time accumulated at level 0 exceeds a given value $\zeta > 0$; more accurate definitions can be found in Section 2.1 and [9, 43].

Let L_ζ^ξ be the process $\{L_\xi(x, \tau_\xi^L(\zeta)) : x \geq 0\}$ conditioned on $\tau_\xi^L(\zeta) < \infty$. The first main result states that L_ζ^ξ is the *unique weak solution*¹ of the following *stochastic Volterra equation* (SVE)

$$(1.3) \quad L_\zeta^\xi(x) = \int_0^\infty \int_0^\zeta \nabla_y W(x) N_0(dy, dz) + \int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz), \quad x \geq 0,$$

where $\nabla_y W(x) := W(x) - W(x-y)$, $N_0(dy, dz)$ is a Poisson random measure (PRM) on $(0, \infty)^2$ with intensity $\bar{\nu}_\alpha(y)dydz$, $\tilde{N}_\alpha(ds, dy, dz)$ is a compensated PRM on $(0, \infty)^3$ with intensity $ds\nu_\alpha(dy)dz$ and independent of $N_0(dy, dz)$. The first stochastic integral in (1.3) represents the contribution of jumps up-crossing 0 to the local time at level x and the second stochastic integral, known as *stochastic Volterra integral* (SVI), can be interpreted as the perturbations caused by jumps with initial positions above 0. Since the convolution kernel delays the relaxation of its perturbations, the PRM $N_\alpha(ds, dy, dz)$ changes the local times continuously in the spatial variable. This stands in striking contrast to the jumps in *Itô's stochastic differential equations* (Itô's S-DEs) driven by PRM. Additionally, because of the joint impact of relative level $x-s$ and jump-size y on the convolution kernel, the SVE (1.3) cannot be written into the form of SVEs in [1, 2, 3, 57, 58].

Based on the SVE (1.3), in the second main result we use stochastic integral inequalities to provide a simple proof for the Hölder continuity of L_ζ^ξ and the finiteness of all moments of the Hölder coefficient given in [4, 13, 26]. As the novelty, we also establish a uniform upper bound for all moments of the Hölder coefficient and a maximal inequality for the local times in the spatial variable. With the crucial assistance from the SVE (1.3),

¹ A continuous process with distribution \mathcal{P} is called a *weak solution* of (1.3) if there exists a stochastic basis, a PRM $N_0(dy, dz)$ on $(0, \infty)^2$ with intensity $\bar{\nu}_\alpha(y)dydz$, a PRM $N_\alpha(ds, dy, dz)$ on $(0, \infty)^3$ independent of $N_0(dy, dz)$ with intensity $ds\nu_\alpha(dy)dz$ and a continuous process L_ζ^ξ with distribution \mathcal{P} such that (1.3) holds almost surely. We say the *weak uniqueness* holds if any two weak solutions are equal in distribution.

in the third main result we extend the method of duality developed in [3] to provide an explicit representation of the Laplace functional $\mathbf{E}[\exp\{-\lambda \cdot L_\zeta^\xi(x) - g * L_\zeta^\xi(x)\}]$ with $\lambda \geq 0$ and $g \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$. It states that the Laplace exponent can be written as an affine functional of the initial state, in terms of the unique solution of the *nonlinear Volterra integral equation* (nonlinear-VIE)

$$(1.4) \quad v_\lambda^g(x) = \lambda W'(x) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(x), \quad x > 0,$$

where W' is the derivative of W and \mathcal{V}_α is a nonlinear operator acting on a locally integrable function f by

$$(1.5) \quad \mathcal{V}_\alpha \circ f(x) := \int_0^\infty \left(\exp \left\{ - \int_{(x-y)^+}^x f(r) dr \right\} - 1 + \int_{(x-y)^+}^x f(r) dr \right) \nu_\alpha(dy), \quad x \geq 0.$$

Finally, we provide an alternative fractional integration and differential equation for the process L_ζ^ξ and its Laplace exponent. In contrast to the SVE (1.3), the alternative equation takes it a step further and extracts the impact of drift b on the local times from that of jumps. It also uncovers the remarkable similarity between L_ζ^ξ and CB-processes in the evolution mechanism, which, together with the genealogical interpretations in [26, 44, 47], tells that the SVE (1.3) defines a novel non-Markovian CB-process.

To illustrate the strength of these results, we use the SVE (1.3) to establish a stochastic equation for the heavy-traffic limit of rescaled queue-length processes of M/G/1 processor-sharing queues with unit service capacity, heavy-tailed service distribution and stopped upon becoming empty. It can be seen as a continuation of [45], where the weak convergence of rescaled queue-length processes was proved. In a sense, this helps to partially answer Problem 2 stated by Zwart in [67] about the heavy-traffic limit of heavy-tailed processor-sharing queues; readers may refer to the references of Zwart and his coauthors for details. enlightened by the self-exciting property observed in the SVE (1.3), in the forthcoming preprint [34] we use the evolution mechanism of local times of stable processes to model the sharp-raise clusters in rough volatilities and introduce a novel fractional stochastic volatility model with self-excited sharp-raises.

1.2. Methodologies. We start the construction of the SVE (1.3) from the result that the local times of nearly recurrent compound Poisson processes with unit negative drift, Pareto-distributed jumps are equal in distribution to a class of nearly critical binary CMJ-processes, which converge weakly to the process L_ζ^ξ after rescaling; see [44, 45]. Enlightened by the Hawkes representation of general branching particle systems established in [32, 65], we reconstruct the binary CMJ-processes as the intensity processes of nearly unstable *marked Hawkes point measures* (MHPs) by translating the birth time, life-length and survival state of each individual into the arrival time, random mark and kernel of an event respectively. Furthermore, we write each intensity process into a SVE driven by an infinite-dimensional martingale in which the integrand is a functional of the *resolvent function* related to the life-length distribution. Consequently, it suffices to prove the weak convergence of these SVEs after rescaling to the desired SVE (1.3). Unfortunately, the Pareto-distributed life-length gives raise to long-range dependence in the pre-limit SVEs, which derives a series of challenges and difficulties in the proof including

- Along with the inseparable impact of time and life-length on the convolution kernel, the infinite-dimensional driving noises not only lead to the failure of the approximation method and the integral-derivative method developed in [2, 3, 38], but also make it hard to seek an approximation for the pre-limit SVEs.
- The resolvent function fluctuates drastically and explodes around 0 after rescaling. This leads to the sharp swings in the cumulative impact of infinite short-lived events on the pre-limit SVEs and also makes the uniform control on the error processes challengeable.
- The resolvent function inherits long-range dependence from the life-length distribution. It prevents us from transforming the pre-limit SVEs into the form of Itô's SDEs and obtaining the weak convergence similarly as in [37, 65] by using the weak convergence results established in [41, 42] for Itô's SDEs.

To overcome the first two difficulties, we start by analyzing in depth the direct and indirect impact of each event on the pre-limit SVEs. Our analyses show that the cumulative direct impact of all events can be asymptotically ignored and a suitably rescaled version of their indirect impact asymptotically behaves as the backward difference of scale function. This motivates us to approximate the SVIs in the pre-limit SVEs by replacing the integrands with the backward difference of scale function. For the uniform control on the error processes, we first split them into several parts according to the source and then prove the finite-dimensional convergence of each part to 0 separately. Based on a deep analysis about the backward difference of scale function, we prove the C -tightness² of the approximating processes, which, together with the C -tightness result given in [45] for the local times of nearly recurrent compound Poisson processes, yields the tightness of error processes. To overcome the third difficulty, we establish a weak convergence result for SVIs with respect to infinite-dimensional martingales, whose tightness and finite-dimensional convergence are obtained from the foregoing tightness results and the weak convergence of the related Itô's stochastic integrals respectively. More precisely, for a given finite sequence of time points, we first introduce a sequence of Itô's stochastic integrals with respective to infinite-dimensional martingale satisfying that their finite-dimensional distributions at the given time points are equal to those of the corresponding SVIs, and then prove their weak convergence to a limit process whose finite-dimensional distribution at the given time points is equal to that of the desired limit SVI.

In the proof of existence and uniqueness of solutions of the nonlinear-VIE (1.4), the next two main difficulties stem from the nonlinear operator \mathcal{V}_α and the singularity of the function W' at the origin

- The interplay between the singularity of W' and \mathcal{V}_α makes the existence of local solutions of (1.4) around 0 quite difficult.
- Since \mathcal{V}_α is path-dependent and does not satisfy the Lipschitz condition, it is difficult to identify the non-explosion of local solutions and extend them into global solutions.

To bypass the first difficulty, we first prejudge the behavior of solutions near the origin with the help of an upper bound estimate of \mathcal{V}_α and the expansion given in [14] for solutions of fractional Riccati equations. In a specified closed set in some Lebesgue space, we then find a local solution of (1.4) successfully by using Banach's fixed point theorem. To overcome the second difficulty, associated with a fractional differential equation related to \mathcal{V}_α we first provide an upper bound estimate for a functional of each local solution, and then, along with the comparison principle for fractional differential equations, establish a uniform control on the local solutions.

1.3. Related Literature. Let us comment on the relationship between the present work and the existing literature. Firstly, based on the Markov property, Brownian local times were linked to Bessel processes via their transition semigroups in [40, 60] or their infinitesimal generators in [39, 52]. However, the lack of Markovianity of L_ζ^ξ makes it impossible to establish the SVE (1.4) similarly as in the preceding references. Even if it could be established successfully, the SVE (1.4) is beyond the scope of all existing literature [1, 2, 3, 57, 58] and the existence of its solutions seems to be quite difficult to be proved in the standard way. On the other hand, the present work establishes the well-posedness of the novel SVE (1.4). Secondly, the main results, as mentioned above, are obtained by establishing a weak convergence result for the corresponding long-range dependent MHPs. The first scaling limit theorem for Hawkes processes was established by Jaisson and Rosenbaum [37] in the study of the asymptotic behavior of Hawkes-based price-volatility models in the context of high-frequency trading. Their results state that under the short-memory condition, the rescaled intensity processes of nearly unstable Hawkes processes converge weakly to the well-known *CIR-model*. The analogous scaling limits were established for multivariate (marked) Hawkes processes in [23, 65] and a jump-diffusion limit was given in [33] for MHPs with exponential kernel. When the kernel is heavy-tailed, Jaisson and Rosenbaum [38] proved the weak convergence of the integral of rescaled intensity process to the integral of a fractional diffusion process,

²Readers may refer to Definition 3.25 in [36, p.351] for the definition of C -tightness.

see also [23, 63] for the multivariate case. Because of many difficulties deriving from long-range dependence, they left the weak convergence of rescaled intensity processes as an open problem. However, we stress that the weak convergence result in this work is established for the intensity processes of MHPs. As the final remark, we need to point out that different to the analogous version in [49, 50], the Ray-Knight theorem in this work is established for the local times rather than their functionals.

Organization of this paper. In Section 2, we first introduce general notation and properties of spectrally positive stable processes, and then formulate the main results. In Section 3, we introduce some elementary results and a SVE for the local times of a compound Poisson process with negative drift by linking them to a MHP. Section 4 is devoted to proving that L_ζ^ξ solves the SVE (1.3). Its Hölder continuity is proved in Section 5. In Section 6, we prove the exponential-affine transform formula as well as the existence and uniqueness of solutions of the nonlinear-VIE (1.4). The proof for the alternative representation of L_ζ^ξ are given in Section 7. Applications to processor-sharing queues are given in Section 8. Additional proofs and supporting results are presented in the Appendices.

Notation. For any $x \in \mathbb{R}$, let $x^+ := x \vee 0$, $x^- := x \wedge 0$ and $[x]$ be the integer part of x . For a Banach space \mathbb{V} with a norm $\|\cdot\|_{\mathbb{V}}$, let $D([0, \infty), \mathbb{V})$ be the space of all càdlàg \mathbb{V} -valued functions endowed with the Skorokhod topology and $C([0, \infty), \mathbb{V})$ the space of all continuous \mathbb{V} -valued functions endowed with the uniform topology. For any $\mathcal{T} \subset [0, \infty)$ and $p \in (0, \infty]$, let $L^p(\mathcal{T}; \mathbb{V})$ be the space of \mathbb{V} -valued measurable functions f on \mathcal{T} satisfying that $\|f\|_{L^p_{\mathbb{V}}} := \int_{\mathcal{T}} \|f(x)\|_{\mathbb{V}}^p dx < \infty$. We also write $\|f\|_{L^p_{\mathbb{V}}}$ for $\|f\|_{L^p_{[0, \mathcal{T}]}}$ and $\|f\|_{L^p}$ for $\|f\|_{L^p_{\infty}}$. We make the conventions that for $x, y \in \mathbb{R}$ with $y \geq x$,

$$\int_x^y = - \int_y^x = \int_{(x, y]}, \quad \int_{x^-}^{y^-} = \int_{[x, y)} \quad \text{and} \quad \int_x^\infty = \int_{(x, \infty)}.$$

Denote by $f * g$ the convolution of two functions f, g on \mathbb{R}_+ . Let Δ_h and ∇_h be the forward and backward difference operators with step size $h > 0$, i.e., $\Delta_h f(x) := f(x+h) - f(x)$ and $\nabla_h f(x) := f(x) - f(x-h)$. Let $\xrightarrow{\text{u.c.}}$, $\xrightarrow{\text{a.s.}}$, $\xrightarrow{\text{d}}$ and $\xrightarrow{\text{P}}$ be the uniform convergence on compacts, almost sure convergence, convergence in distribution and convergence in probability respectively. We also use $\stackrel{\text{a.s.}}{=}$, $\stackrel{\text{d}}{=}$ and $\stackrel{\text{P}}{=}$ to denote almost sure equality, equality in distribution and equality in probability respectively.

For a probability measure μ on \mathbb{R} , denote by \mathbf{P}_μ and \mathbf{E}_μ the law and expectation of the underlying process with initial state distributed as μ . When μ is a Dirac measure at point $x \in \mathbb{R}$, we write \mathbf{P}_x for \mathbf{P}_μ and \mathbf{E}_x for \mathbf{E}_μ . For simplicity, we also write \mathbf{P} for \mathbf{P}_0 and \mathbf{E} for \mathbf{E}_0 . For two σ -finite measures μ_1, μ_2 on \mathbb{R} , we say $\mu_1 \leq \mu_2$ if for any non-negative function f on \mathbb{R} ,

$$\int_{\mathbb{R}} f(x) \mu_1(dx) \leq \int_{\mathbb{R}} f(x) \mu_2(dx).$$

We use C to denote a positive constant whose value might change from line to line.

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2. Preliminaries and main results.

2.1. Spectrally positive stable processes. Suppose that the spectrally positive stable process ξ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses. For every $t \geq 0$, let $\mu_{\xi, t}(dy)$ be the *occupation measure* of ξ on the time interval $[0, t]$ given for every non-negative

and measurable function f on \mathbb{R} by

$$\int_0^t f(\xi(s)) ds \stackrel{\text{a.s.}}{=} \int_{\mathbb{R}} f(y) \mu_{\xi,t}(dy).$$

The measure $\mu_{\xi,t}$ is absolutely continuous with respect to the Lebesgue measure and the density, denoted by $\{L_{\xi}(x,t) : x \in \mathbb{R}\}$, is square integrable; see Theorem 1 in [9, p.126]. The quantity $L_{\xi}(x,t)$ is called the *local time* of ξ at level x and time t . The two-parameter process $L_{\xi} := \{L_{\xi}(x,t) : x \in \mathbb{R}, t \geq 0\}$ is jointly continuous and satisfies the *occupation density formula*

$$(2.1) \quad \int_0^t f(\xi(r)) dr \stackrel{\text{a.s.}}{=} \int_{\mathbb{R}} f(x) L_{\xi}(x,t) dx, \quad t \geq 0,$$

see Theorem 15 in [9, p.149]. Moreover, for any (\mathcal{F}_t) -stopping time τ , it is easy to identify that

$$(2.2) \quad \inf \{x \geq 0 : L_{\xi}(x,\tau) = 0\} \stackrel{\text{a.s.}}{=} \sup \{\xi(t) : t \in [0, \tau]\}.$$

The process $\{L_{\xi}(0,t) : t \geq 0\}$ is continuous and non-decreasing. This allows us to define the *inverse local time* $\tau_{\xi}^L := \{\tau_{\xi}^L(\zeta) : \zeta \geq 0\}$ at level 0 by $\tau_{\xi}^L(\zeta) = \infty$ if $\zeta > L_{\xi}(0, \infty)$ and

$$\tau_{\xi}^L(\zeta) := \inf \{s \geq 0 : L_{\xi}(0,s) \geq \zeta\}, \quad \text{if } \zeta \in [0, L_{\xi}(0, \infty)].$$

From Proposition 4 in [9, p.130], the process τ_{ξ}^L is a subordinator, killed at an independent exponential time if ξ is transient ($b > 0$), and its Laplace transform is of the form

$$(2.3) \quad \mathbf{E}[\exp\{-\lambda \cdot \tau_{\xi}^L(\zeta)\}] = \exp\{-\zeta/u^{\lambda}(0)\}, \quad \lambda > 0, \zeta \geq 0,$$

where $u^{\lambda} := \{u^{\lambda}(y) : y \in \mathbb{R}\}$ is the density of the λ -resolvent kernel of ξ . When $b = 0$, we have $L_{\xi}(0, \infty) \stackrel{\text{a.s.}}{=} \infty$ and $\tau_{\xi}^L(\zeta) < \infty$ a.s. When $b > 0$, the *potential density* u^0 is well-defined as the limit case $\lambda = 0$ for u^{λ} . In this case, we consider the limit case $\lambda \rightarrow 0+$ for (2.3) to get

$$\mathbf{P}(L_{\xi}(0, \infty) \geq \zeta) = 1 - \mathbf{P}(\tau_{\xi}^L(\zeta) = \infty) = \exp\{-\zeta/u^0(0)\},$$

which induces that $L_{\xi}(0, \infty)$ is exponentially distributed with mean $u^0(0)$ and $\mathbf{P}(\tau_{\xi}^L(\zeta) = \infty) > 0$ for any $\zeta > 0$. For $a, \theta > 0$, let $a \cdot \xi(\theta \cdot) := \{a\xi(\theta t) : t \geq 0\}$. The equality (2.1), along with the change of variables, implies the following two equivalences³

$$(2.4) \quad L_{a \cdot \xi(\theta \cdot)} \stackrel{\text{a.s.}}{=} \{(a\theta)^{-1} \cdot L_{\xi}(x/a, \theta t) : x \in \mathbb{R}, t \geq 0\} \quad \text{and} \quad \tau_{a \cdot \xi(\theta \cdot)}^L(\zeta) \stackrel{\text{a.s.}}{=} \theta^{-1} \cdot \tau_{\xi}^L(a\theta\zeta), \quad \zeta \geq 0.$$

Let $\{W(x) : x \in \mathbb{R}\}$ be the *scale function* of ξ , which is identically zero on $(-\infty, 0)$ and characterized on $[0, \infty)$ as a strictly increasing function whose Laplace transform is given by

$$(2.5) \quad \int_0^{\infty} e^{-\lambda x} W(x) dx = \frac{1}{\Phi(\lambda)}, \quad \lambda > 0.$$

The scale function W is continuous on \mathbb{R} and differentiable on $(0, \infty)$ with derivative denoted as W' ; see Theorem 8 in [9, p.194]. Applying the integration by parts to (2.5), we have

$$(2.6) \quad \int_0^{\infty} e^{-\lambda x} W'(x) dx = \int_0^{\infty} \lambda e^{-\lambda x} W(x) dx = \frac{1}{b + c\lambda^{\alpha}}, \quad \lambda > 0.$$

The Laplace transform of *Mittag-Leffler function*⁴ yields that W' has the representation

$$W'(x) = c^{-1} x^{\alpha-1} \cdot E_{\alpha, \alpha}(-b/c \cdot x^{\alpha}), \quad x > 0.$$

³Actually, these two equivalences hold for any Lévy process.

⁴The Mittag-Leffler function $E_{\alpha, \alpha}$ on \mathbb{R}_+ is defined by $E_{\alpha, \alpha}(x) := \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha(n+1))}$; see [31] for a precise definition and a survey of its properties, e.g.,

$$\int_0^{\infty} e^{-\lambda x} a x^{\alpha-1} E_{\alpha, \alpha}(-a \cdot x^{\alpha}) dx = \frac{a}{a + \lambda^{\alpha}}, \quad a, \lambda \geq 0.$$

The smoothness of $E_{\alpha,\alpha}$ induces that W is infinitely differentiable on $(0, \infty)$. When $b = 0$, we have $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ and

$$(2.7) \quad W(x) = \frac{x^\alpha}{c \cdot \Gamma(\alpha + 1)}, \quad W'(x) = \frac{x^{\alpha-1}}{c \cdot \Gamma(\alpha)}, \quad W''(x) = \frac{(\alpha - 1)x^{\alpha-2}}{c \cdot \Gamma(\alpha)}, \quad x > 0.$$

When $b > 0$, the function bW' is a *Mittag-Leffler density function* and $1 - bW(x) \rightarrow 0$ as $x \rightarrow \infty$. The properties of Mittag-Leffler distribution/density function; see [31, 54, 56], yield that the scale function W is Hölder continuous with index α . Moreover,

$$W(x) \sim \frac{x^\alpha}{c \cdot \Gamma(\alpha + 1)}, \quad W'(x) \sim \frac{x^{\alpha-1}}{c \cdot \Gamma(\alpha)}, \quad W''(x) \sim \frac{(\alpha - 1)x^{\alpha-2}}{c \cdot \Gamma(\alpha)}, \quad \text{as } x \rightarrow 0+,$$

and

$$W(x) \sim \frac{1}{b} - \frac{c \cdot x^{-\alpha}}{b^2 \cdot \Gamma(1 - \alpha)}, \quad W'(x) \sim \frac{c\alpha \cdot x^{-\alpha-1}}{b^2 \cdot \Gamma(1 - \alpha)}, \quad W''(x) \sim -\frac{c\alpha(\alpha + 1) \cdot x^{-\alpha-2}}{b^2 \cdot \Gamma(1 - \alpha)} \quad \text{as } x \rightarrow \infty.$$

A direct consequence of these asymptotic properties and (2.7) is that uniformly in $x > 0$,

$$(2.8) \quad W(x) \leq C \cdot x^\alpha, \quad |W'(x)| \leq C \cdot x^{\alpha-1} \quad \text{and} \quad |W''(x)| \leq C \cdot x^{\alpha-2}.$$

By the mean-value theorem, it is easy to identify that uniformly in $x, y > 0$,

$$(2.9) \quad |\nabla_y W(x)| = |\Delta_y W(x - y)| \leq C \cdot [x^\alpha \wedge ((x - y)^+)^{\alpha-1} y].$$

In addition to the scale function, we will also need a *Sonine pair* (K, L_K) on $(0, \infty)$ defined by

$$(2.10) \quad K(x) := \frac{x^{\alpha-1}}{c \cdot \Gamma(\alpha)} \quad \text{and} \quad L_K(x) := \frac{c \cdot x^{-\alpha}}{\Gamma(1 - \alpha)}, \quad x > 0,$$

which satisfies the *Sonine equation*, i.e.,

$$(2.11) \quad K * L_K = L_K * K \equiv 1.$$

In the theory of Volterra equations; see [30], the function L_K is also said to be the *resolvent of the first kind* related to K and vice versa. When $b > 0$, a simple calculation shows that the function bW' is the *resolvent of the second kind* corresponding to bK , which is usually introduced by means of the resolvent equation

$$(2.12) \quad bW' = bK - (bK) * (bW').$$

The function bK is usually referred as the *resolvent* associated to bW' . Convolving both sides of (2.12) by L_K and then dividing them by b , we have

$$(2.13) \quad L_K * W' = W' * L_K = 1 - bW.$$

Actually, this equality also holds when $b = 0$, since $W' = K$ in this case; see (2.7) and (2.10).

2.2. Main results. We now formulate the main results for the local times of ξ at the stopping time $\tau_\xi^L(\zeta)$ for a given value $\zeta > 0$. For convenience, we write L_ξ^ζ for the process $\{L_\xi(x, \tau_\xi^L(\zeta)) : x \geq 0\}$ under $\mathbf{P}(\cdot | \tau_\xi^L(\zeta) < \infty)$. Since $\tau_\xi^L(\zeta) < \infty$ a.s. when $b = 0$, this conditional probability law turns to be \mathbf{P} . When $b > 0$, the stopping time $\tau_\xi^L(L_\xi(0, \infty))$ is finite almost surely and equal to the last time that ξ hits 0. In this case, we are also interested in the process

$$(2.14) \quad L_\xi^\infty := \{L_\xi(x, \infty) : x \geq 0\} \stackrel{\text{a.s.}}{=} \{L_\xi(x, \tau_\xi^L(L_\xi(0, \infty))) : x \geq 0\},$$

under \mathbf{P} . Let ϱ be an exponential random variable with mean $u^0(0)$, independent of $N_\alpha(ds, dy, dz)$ and $N_0(dy, dz)$. Our first main theorem establishes SVEs for L_ξ^ζ and L_ξ^∞ .

THEOREM 2.1. *We have the following:*

- (1) For each $\zeta \geq 0$, the process L_ζ^ξ is a weak solution of (1.3).
- (2) If $b > 0$, the process L_∞^ξ is a weak solution of (1.3) with $\zeta = \varrho$.
- (3) The weak uniqueness of non-negative solutions holds for (1.3).

REMARK 2.2. By the change of variables and Proposition A.1 with $p = 2$, there exists a constant $C > 0$ such that for any $x \geq 0$,

$$\int_0^x ds \int_0^\infty |\nabla_y W(x-s)|^2 \nu_\alpha(dy) = \int_0^x ds \int_0^\infty |\nabla_y W(s)|^2 \nu_\alpha(dy) \leq C \cdot x^\alpha.$$

Taking expectations on both sides of (1.3) and then using Fubini's theorem along with (2.13), we have

$$\begin{aligned} \mathbf{E}[L_\zeta^\xi(x)] &= \mathbf{E}\left[\int_0^\infty \int_0^\zeta \nabla_y W(x) N_0(dy, dz)\right] \\ &= \zeta \int_0^\infty \nabla_y W(x) \bar{\nu}_\alpha(y) dy \\ (2.15) \quad &= \zeta \int_0^\infty \bar{\nu}_\alpha(y) dy \int_{(x-y)^+}^x W'(s) ds = \zeta \cdot W' * L_K(x) = \zeta(1 - bW(x)) \leq \zeta, \quad x \geq 0. \end{aligned}$$

The SVI in (1.3) has finite quadratic variation and is well defined as an Itô integral; see [35, p.59-63].

REMARK 2.3. By the exponential formula for PRMs; see [9, p.8], we have for any $\lambda \geq 0$,

$$\mathbf{E}\left[\exp\left\{-\lambda \int_0^\infty \int_0^\zeta \nabla_y W(x) N_0(dy, dz)\right\}\right] = \exp\left\{-\zeta \int_0^\infty (1 - e^{-\lambda \nabla_y W(x)}) \bar{\nu}_\alpha(y) dy\right\}.$$

From (2.15) and the fact that $\nabla_y W(x) \rightarrow 0$ uniformly in y as $x \rightarrow 0$, we have

$$\mathbf{E}\left[\exp\left\{-\lambda \int_0^\infty \int_0^\zeta \nabla_y W(x) N_0(dy, dz)\right\}\right] \sim \exp\left\{-\zeta \lambda \int_0^\infty \nabla_y W(x) \bar{\nu}_\alpha(y) dy\right\} \rightarrow e^{-\zeta \lambda}.$$

Thus the first term on the right side of (1.3) converges to ζ a.s. as $x \rightarrow 0+$. We make the convention that it is equal to ζ a.s. when $x = 0$, which is consistent with the fact that $L_\zeta^\xi(0) \stackrel{\text{a.s.}}{=} \zeta$.

REMARK 2.4. By (2.15), the SVE (1.3) can be written as

$$\begin{aligned} L_\zeta^\xi(x) &= \zeta(1 - bW(x)) + \int_0^\infty \int_0^\zeta \nabla_y W(x) \tilde{N}_0(dy, dz) \\ (2.16) \quad &+ \int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz), \quad x \geq 0, \end{aligned}$$

where $\tilde{N}_0(dy, dz) := N_0(dy, dz) - \bar{\nu}_\alpha(y) dy dz$. Here the first term on the right side of this equality represents the average local time at level x . The second term can be interpreted as the perturbations caused by jumps up-crossing 0; the third term can be translated into the perturbations caused by jumps with initial positions above 0 but below x . More precisely, the convolution kernel $\nabla_y W(x-s)$ describes the impact of a jump with initial position s and size y on the local time at level x . Notice that $\nabla_y W(x-s)$ increases when $x \in [s, s+y]$ and decreases as $x \rightarrow \infty$. It would be sensible to consider the jump size of each jump as its life-length/residual-life during which it perturbs the local times directly. This interpretation is consistent with the genealogical interpretations in [26, 47].

REMARK 2.5. Because of the delayed and smooth relaxation of its perturbations, the PRM $N_\alpha(ds, dy, dz)$ fails to make solutions of (1.3) jump. This phenomena cannot be observed in Ito's SDEs driven by PRM, since the PRM releases its perturbations instantaneously that give raise to jumps in the solutions. Consequently, the continuity of driving noises is a necessary condition for the continuity of solutions of Ito's SDEs; see [35, Chapter III-IV] and [59, Chapter II-V].

REMARK 2.6. *It is necessary to specify that the SVE (1.3) is beyond the scope of the existing literature, e.g. [1, 2, 3, 23, 38, 57, 58]. More precisely, all SVEs studied in these literature are driven by finite-dimensional semimartingale and always can be written as*

$$(2.17) \quad X(t) = H(t) + \int_0^t \mathbf{K}(t, s, X_s) dZ(t), \quad t \geq 0,$$

where H is a given function, \mathbf{K} is a $d \times k$ matrix-valued convolution kernel on $\mathbb{R}_+^2 \times \mathbb{R}$ and Z is a k -dimensional Itô's semimartingale whose differential characteristics are functions of X . Differently, the SVI in (1.3) is driven by an infinite-dimensional martingale; see [42] and Appendix C. Since the impact of time t on the convolution kernel $\nabla_y W(t)$ is tightly intertwined with that of mark y , one cannot write (1.3) into the form of (2.17). Consequently, it is difficult to prove the existence of solutions of (1.3) by using the approximation method used in [1, 3] or the martingale problem theory developed in [2].

REMARK 2.7. *Ito's SDEs with non-negative solutions have been widely studied in [7, 10, 15, 16, 27] under two key conditions: (i) when solutions hit 0, the diffusion vanishes and the drift turns to be non-negative; (ii) solutions cannot jump into the negative half-line. In particular, it is the strong Markovianity that turns the state 0 to be a tripper or a reflecting boundary, which results in the existence of non-negative solutions. However, the convolution kernel in (1.3) results in the lack of (strong) Markovianity of the solutions and makes the standard stopping time method fail to prove the existence of non-negative solutions. Fortunately, thanks to Theorem 2.1, the existence of non-negative solutions of (1.3) follows directly from the non-negativity of L_ζ^ξ .*

REMARK 2.8. *The point 0 is an absorbing state⁵ for the process L_ζ^ξ (and also L_∞^ξ), i.e., once it hits 0, it will stay at 0 forever. Indeed, the equivalence (2.2) shows that conditioned on $\tau_\xi^L(\zeta) < \infty$,*

$$\tau_0 := \inf \{x \geq 0 : L_\zeta^\xi(x) = 0\} < \infty, \text{ a.s.} \quad \text{and} \quad L_\zeta^\xi(\tau_0 + x) \stackrel{\text{a.s.}}{=} 0, \quad x \geq 0.$$

Usually, the lack of Markovianity makes it difficult to obtain this property from the SVE (1.3). Even for the SVE (2.17), the absorbing states and polarity are also unclear up to now.

The SVE (1.3) makes it possible to study the local times of ξ by using tools and methods from stochastic analysis, e.g., stochastic integral inequalities, stochastic Fubini theorem and martingale problem theory. To illustrate this, the next main theorem proves the Hölder continuity of L_ζ^ξ by using the Kolmogorov continuity theorem and also provides a uniform upper bound for all moments of the Hölder coefficients by using the Garsia-Rodemich-Rumsey inequality. For $\kappa \in (0, 1]$ and $x > 0$, the κ -Hölder coefficient of a Hölder continuous function f on $[0, x]$ is defined by

$$\|f\|_{C_x^{0, \kappa}} := \sup_{0 \leq y < z \leq x} \frac{|f(y) - f(z)|}{|y - z|^\kappa}.$$

THEOREM 2.9 (Hölder continuity). *For each $\zeta \geq 0$, we have the following:*

- (1) *The process L_ζ^ξ is Hölder-continuous of any order strictly less than $\alpha/2$.*
- (2) *For each $\kappa \in (0, \alpha/2)$ and $p \geq 0$, there exists a constant $C > 0$ such that for any $x \geq 0$,*

$$\mathbf{E}[\|L_\zeta^\xi\|_{C_x^{0, \kappa}}^p] \leq C \cdot (1 + x)^{p(\alpha - \kappa)}.$$

⁵Although the two terminologies *absorbing state* and *polarity* are initially introduced for Markov processes, it is sensible to use them to describe the analogous phenomena in other stochastic processes. Precisely, a state in a process is said to be an absorbing state if once it is entered, it is impossible to leave. A set is said to be a *polar set* for a process if it cannot be entered in finite time.

As a direct consequence of this theorem, we can establish a maximal inequality for L_ζ^ξ . In detail, for each $\kappa \in (0, \alpha/2)$ and $p \geq 0$, we have uniformly in $x \geq 0$,

$$\sup_{y \in [0, x]} |L_\zeta^\xi(y) - L_\zeta^\xi(0)|^p \leq \|L_\zeta^\xi\|_{C_x^{0, \kappa}}^p \cdot x^{p\kappa}, \quad a.s.$$

By the power mean inequality⁶ and Theorem 2.9,

$$\mathbf{E} \left[\sup_{y \in [0, x]} |L_\zeta^\xi(y)|^p \right] \leq C \cdot \zeta^p + C \cdot \mathbf{E} \left[\sup_{y \in [0, x]} |L_\zeta^\xi(y) - L_\zeta^\xi(0)|^p \right] \leq C \cdot \zeta^p + C \cdot \mathbf{E} [\|L_\zeta^\xi\|_{C_x^{0, \kappa}}^p] \cdot x^{p\kappa},$$

for some constant C depending only on p . This yields the following corollary immediately.

COROLLARY 2.10 (Maximal inequality). *For each $\zeta > 0$ and $p \geq 0$, there exists a constant $C > 0$ such that for any $x \geq 0$,*

$$\mathbf{E} \left[\sup_{y \in [0, x]} |L_\zeta^\xi(y)|^p \right] \leq C \cdot (1 + x)^{p\alpha}.$$

REMARK 2.11. *By applying the discretization technique to the occupation measures of stable Lévy processes, Boylan [13] proved the uniform Hölder regularity of their local times in both the time and spatial directions. Later, Barlow [4] constructed the explicit modulus of continuity and gave the optimal Hölder exponent. Recently, Forman et al. [26] proved the finiteness of all moments of the Hölder coefficient in the case of $b = 0$. The novelties of our results are in the uniform upper bound for all moments of the Hölder coefficient and the maximal inequality for the local times in the spatial direction.*

Because of the Markovianity and martingale property, Brownian local times are tractable and their Laplace transform can be written as an exponential affine function of the initial state, in terms of the solution of a Riccati equation. By contrast, the lack of Markovianity and martingale property makes the local times of stable processes quite intractable. As another example that illustrates the strength of the SVE (1.3), the third main theorem establishes an explicit representation of Laplace functionals of L_ζ^ξ . Let \mathcal{A}_∞ be the space of all functions f on $(0, \infty)$ satisfying that $\sup_{x \in (0, T]} x^{1-\alpha} |f(x)| < \infty$ for any $T > 0$. A continuous function v_λ^g on $(0, \infty)$ is said to be a \mathcal{A} -global solution of the nonlinear-VIE (1.4) if $v_\lambda^g \in \mathcal{A}_\infty$ and satisfies (1.4) on $(0, \infty)$.

THEOREM 2.12 (Laplace functionals). *For each $\lambda \geq 0$ and $g \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, we have for $x \geq 0$,*

$$(2.18) \quad \mathbf{E} \left[\exp \left\{ -\lambda \cdot L_\zeta^\xi(x) - g * L_\zeta^\xi(x) \right\} \right] = \exp \left\{ -\zeta \int_0^\infty \left(1 - \exp \left\{ -\int_{(x-y)^+}^x v_\lambda^g(s) ds \right\} \right) \bar{\nu}_\alpha(y) dy \right\},$$

where v_λ^g is the unique \mathcal{A} -global solution of (1.4).

REMARK 2.13. *By comparing (2.18) with (2.1) in [15] or (2.2) in [19], we see that the process L_ζ^ξ enjoys the analogue of the affine property of affine Markov processes. In addition, this property also has been observed in the rough Heston model that is defined by a SVE driven by Brownian motion; see [24]. Later, Abi Jaber et al. [1, 3] considered a class of SVEs of the form (2.17) with solutions being affine and also named them affine Volterra processes.*

By using the resolvent equations (2.12) and (2.13), the last main result in this work provides equivalent representations of the SVE (1.3) and the nonlinear-VIE (1.4), which help a lot to clarify the similarities and differences between the process L_ζ^ξ and CB-processes; we refer to [53] for a review on CB-processes.

⁶For any $p > 0, k \geq 2$ and $x_1, \dots, x_k \in \mathbb{R}$, by Jensen's inequality we have $|x_1 + \dots + x_k|^p \leq k^{(p-1)^+} (|x_1|^p + \dots + |x_k|^p)$.

THEOREM 2.14. *The SVE (1.3) is equivalent to*

$$(2.19) \quad \begin{aligned} L_\zeta^\xi(x) &= \zeta - b \cdot K * L_\zeta^\xi(x) + \int_0^\infty \int_0^\zeta \int_{(x-y)^+}^x K(r) dr \tilde{N}_0(dy, dz) \\ &+ \int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_{(x-s-y)^+}^{x-s} K(r) dr \tilde{N}_\alpha(ds, dy, dz), \quad x \geq 0, \end{aligned}$$

and the nonlinear-VIE (1.4) is equivalent to

$$(2.20) \quad v_\lambda^g(x) = \lambda K(x) + g * K(x) - (b \cdot v_\lambda^g + \mathcal{V}_\alpha \circ v_\lambda^g) * K(x), \quad x > 0.$$

By (2.10) and the Sonine equation (2.11), the nonlinear-VIE (2.20) can be written into a fractional differential equation. Indeed, denote by I_c^α and D_c^α the Riemann-Liouville fractional integral and derivative operators of order α modified by the constant $c > 0$ respectively. They act on a measurable function f according to

$$I_c^\alpha f(x) := \int_0^x \frac{(x-s)^{\alpha-1}}{c \cdot \Gamma(\alpha)} f(s) ds \quad \text{and} \quad D_c^\alpha f(x) := \frac{d}{dx} \int_0^x \frac{c(x-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s) ds, \quad x \geq 0.$$

Convolving both sides of (2.20) by L_K and then using (2.11), we have

$$L_K * v_\lambda^g(x) = \lambda + \int_0^x g(s) ds - \int_0^x (b \cdot v_\lambda^g + \mathcal{V}_\alpha \circ v_\lambda^g)(s) ds, \quad x \geq 0.$$

Notice that $I_c^\alpha f = K * f$ and $D_c^\alpha v_\lambda^g = \frac{d}{dx} (L_K * v_\lambda^g)$. A simple calculation yields the next corollary.

COROLLARY 2.15. *The nonlinear-VIE (1.4) is equivalent to*

$$(2.21) \quad D_c^\alpha v_\lambda^g = g - b \cdot v_\lambda^g - \mathcal{V}_\alpha \circ v_\lambda^g \quad \text{with} \quad I_c^{1-\alpha} v_\lambda^g(0) = \lambda.$$

REMARK 2.16. *By comparing (2.19) with the Itô's SDE for a CB-process⁷*

$$(2.22) \quad Y_\zeta(t) = \zeta - \int_0^t a Y_\zeta(s) ds + \int_0^t \int_0^{Y_\zeta(s)} \varsigma B_1(ds, dz) + \int_0^t \int_0^\infty \int_0^{Y_\zeta(s-)} y \tilde{N}_1(ds, dy, dz),$$

and then comparing (2.21) with the Riccati equation for the Laplace exponent of Y_ζ

$$\frac{\partial}{\partial t} v_\lambda = -a v_\lambda - \frac{\varsigma^2}{2} |v_\lambda|^2 - \int_0^\infty (e^{-y v_\lambda} - 1 + y v_\lambda) \nu_1(dy) \quad \text{with} \quad v_\lambda(0) = \lambda;$$

see [16, Section 3], we see that the processes L_ζ^ξ not only owns an evolution mechanism similar to that of Y_ζ , but also enjoys, formally at least, the striking analogue⁸ of the branching property. Moreover, it is same to Y_ζ that the point 0 is an absorbing state for L_ζ^ξ ; see Remark 2.8. In conclusion, the SVE (1.3) defines a fully novel non-Markov CB-process whose degree of Hölder regularity is less than that of Feller branching diffusion; see Theorem 2.9. Drawing from [24, 38], we may refer L_ζ^ξ as a rough CB-process.

REMARK 2.17. *The CB-process defined by (2.22) can be reconstructed as the scaling limit of discrete Galton-Watson branching processes. The life-length of individuals uniformly tends to 0 after rescaling and jumps in (2.22) result from the simultaneous births of infinite offsprings; readers may refer to [53] for details. In contrast, the variable y in (1.3) is positive and can be interpreted as the life-length of each individual during which it gives birth to its children randomly. This slows down the extinction of the population. In detail, the survival probability of the CB-process Y_ζ decreases to 0 at an exponential rate when $a > 0$, i.e., $\mathbf{P}(Y_\zeta(t) > 0) \sim C \cdot e^{-at}$ as $t \rightarrow \infty$. While, the survival probability of the rough CB-process L_ζ^ξ decreases to 0 at a power rate when $b > 0$; see the supplementary material [66].*

⁷Here $a \in \mathbb{R}$, $\varsigma \geq 0$, $B_1(ds, dz)$ is a Gaussian white noise on $(0, \infty)^2$ with intensity $ds dz$, $\tilde{N}_1(ds, dy, dz)$ is a PRM on $(0, \infty)^3$ with intensity $ds \nu_1(dy) dz$ and $\nu_1(dy)$ is a σ -finite measure on $(0, \infty)$ satisfying that $\int_0^\infty (1 \wedge y^2) \nu_1(dy) < \infty$.

⁸For $\zeta_1, \zeta_2 > 0$, assume $L_{\zeta_1}^\xi$ and $L_{\zeta_2}^\xi$ are two independent solutions of (1.3) with $\zeta = \zeta_1$ and $\zeta = \zeta_2$ respectively, we have $L_{\zeta_1}^\xi + L_{\zeta_2}^\xi$ is the unique weak non-negative solution of (1.3) with $\zeta = \zeta_1 + \zeta_2$.

3. Local times of compound Poisson processes. In this section, we first introduce some properties of local times of compound Poisson processes with negative drift and then establish a SVE for them. The proofs are elementary and will be merely sketched.

For two constants $\beta, \gamma > 0$ and a probability law $\nu(dx)$ on $(0, \infty)$ with finite mean m_ν , let $Y := \{Y(t) : t \geq 0\}$ be a compound Poisson process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with a drift $-\beta$, arrival rate γ and jump-size distribution ν . It is a Lévy process with bounded variation and Lévy measure $\gamma \cdot \nu(dx)$. Its Laplace exponent is of the form

$$\varphi(\lambda) := \beta\lambda + \gamma \int_0^\infty (e^{-\lambda x} - 1)\nu(dx) = \lambda \left(\beta - \gamma \int_0^\infty e^{-\lambda x} \bar{\nu}(x) dx \right), \quad \lambda \geq 0,$$

where $\bar{\nu}(x) := \nu([x, \infty))$ is the tail distribution of ν . The function φ is zero at zero and tends to ∞ at infinity. Moreover, it is infinitely differentiable and strictly convex on $(0, \infty)$. In particular, $\varphi'(0) = -\mathbf{E}[Y(1)] = \beta - \gamma \cdot m_\nu$ and hence φ is increasing on $[0, \infty)$ if $\varphi'(0) \geq 0$. The process Y drifts to $-\infty$, ∞ or is recurrent according as $\varphi'(0) > 0$, < 0 or $= 0$. Denote by τ_Y^+ the first passage time of Y in $[0, \infty)$, i.e., $\tau_Y^+ := \inf \{t > 0 : Y(t) \geq 0\}$. Actually, the process Y always moves from the negative half line into the positive half line by jumping, i.e., $Y(\tau_Y^+ -) < 0$ and $Y(\tau_Y^+) > 0$ a.s. The next proposition comes from Theorem 17(ii) in [9, p.204]. Let $\nu^*(dx)$ be the *size-biased distribution* of ν given by

$$\nu^*(dx) := \mathbf{1}_{\{x>0\}} \cdot \frac{\bar{\nu}(x)}{m_\nu} \cdot dx.$$

PROPOSITION 3.1. *If $\varphi'(0) \geq 0$, then $Y(\tau_Y^+)$ under $\mathbf{P}(\cdot | \tau_Y^+ < \infty)$ is distributed as ν^* .*

Let $L_Y := \{L_Y(x, t) : x \in \mathbb{R}, t \geq 0\}$ be the local times of Y satisfying the occupation density formula (2.1). The local time $L_Y(x, \infty)$ is infinite almost surely for some and hence all $x \in \mathbb{R}$ if and only if $\varphi'(0) = 0$. Denote by $\tau_Y^L := \{\tau_Y^L(\zeta) : \zeta \geq 0\}$ the right-inverse local time at level 0. When $\beta = 1$, the local time $L_Y(x, t)$ equals to the times that Y hits x in the time interval $(0, t]$, i.e.,

$$L_Y \stackrel{\text{a.s.}}{=} \#\{s \in (0, t] : Y(s) = x\} : x \in \mathbb{R}, t \geq 0\}$$

and τ_Y^L only jumps at positive integer points. The next proposition follows directly from (2.4).

PROPOSITION 3.2. *If $\beta > 0$, the process τ_Y^L only jumps at points $\{k/\beta : k = 1, 2, \dots\}$ and*

$$L_Y \stackrel{\text{a.s.}}{=} \{\beta^{-1} \cdot \#\{s \in (0, t] : Y(s) = x\} : x \in \mathbb{R}, t \geq 0\}.$$

We write $L_{k/\beta}^Y$ and $L_{k/\beta}^{Y,*}$ for the process $\{L_Y(x, \tau_Y^L(k/\beta)) : x \geq 0\}$ under $\mathbf{P}(\cdot | \tau_Y^L(k/\beta) < \infty)$ and $\mathbf{P}_{\nu^*}(\cdot | \tau_Y^L(k/\beta) < \infty)$ respectively. Notice that sample paths on $[0, \tau_Y^+]$ make no contribution to the local times $\{L_Y(x, \tau_Y^L(k/\beta)) : x \geq 0, k \geq 1\}$. The next proposition can be proved immediately by using Proposition 3.2, the strong Markov property and independent, stationary increments of Y .

PROPOSITION 3.3. *If $\varphi'(0) \geq 0$, then $L_{k/\beta}^Y \stackrel{d}{=} L_{k/\beta}^{Y,*} \stackrel{d}{=} \sum_{i=1}^k L_{1/\beta, i}^{Y,*}$ for any $k \geq 1$, where $\{L_{1/\beta, i}^{Y,*} : i = 1, 2, \dots\}$ is a sequence of i.i.d. copies of $L_{1/\beta}^{Y,*}$.*

3.1. Branching representation. Lambert [44] established a one-to-one correspondence between the local times of compound Poisson processes with drift -1 and homogeneous, binary CMJ-processes. More precisely, he observed that the jumping contour process of a homogeneous, binary CMJ-tree starting from one ancestor is a compound Poisson process with a drift -1 ; conversely, the local times on \mathbb{R}_+ of a compound Poisson process with drift -1 stopped upon hitting 0 are equal in distribution to a homogeneous, binary CMJ-process starting from one ancestor. In this section, we extend this connection slightly to the case of compound Poisson processes with arbitrary negative drift.

Recall the triple (β, γ, ν) with $\beta > 0$. We consider a binary CMJ-process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ defined by the following three properties:

- (P1) There are $k \in \mathbb{Z}_+$ ancestors at time 0 with residual life distributed as ν^* .
- (P2) Offsprings have a common life-length distribution ν .
- (P3) Each individual gives birth to its children according to a Poisson process with rate γ/β .

Denote by \mathcal{I}_k the collection of all individuals in the population. Each individual $x \in \mathcal{I}_k$ is endowed with a pair (σ_x, ℓ_x) that represents its birth time and life length. For convention, if x is an ancestor, we assume $\sigma_x = 0$ and ℓ_x equals to its residual life. For $t \geq 0$, let $\mathcal{I}_k(t) := \{x \in \mathcal{I}_k : 0 \leq t - \sigma_x < \ell_x\}$ and $\#\mathcal{I}_k(t)$ be the collection and number of all individuals alive at time t respectively.

LEMMA 3.4. *If $\varphi'(0) \geq 0$, then the process $L_{k/\beta}^Y$ is equal in distribution to $\{\beta^{-1} \cdot \#\mathcal{I}_k(\beta t) : t \geq 0\}$.*

PROOF. Let $Y_\beta := \{Y(t/\beta) : t \geq 0\}$, which is a compound Poisson process with a triple $(1, \gamma/\beta, \nu)$, local times L_{Y_β} and the right-inverse local time $\tau_{Y_\beta}^L$ at level 0. Let $L_k^{Y_\beta}$ be the process $\{L_{Y_\beta}(x, \tau_{Y_\beta}^L(k)) : x \geq 0\}$ under $\mathbf{P}(\cdot | \tau_{Y_\beta}^L(k) < \infty)$. Theorem 3.2 in [46] asserts that $L_k^{Y_\beta} \stackrel{d}{=} \#\mathcal{I}_k$. By (2.4) and Proposition 3.3, we have $L_{k/\beta}^Y \stackrel{d}{=} \beta^{-1} \cdot L_k^{Y_\beta}(\beta \cdot) \stackrel{d}{=} \beta^{-1} \cdot \#\mathcal{I}_k(\beta \cdot)$. \square

3.2. Stochastic Volterra representation. In this section, we establish a SVE for the process $L_{k/\beta}^Y$ by linking the preceding CMJ-process to a MHP; see Appendix B. It also can be obtained by using the connection between general branching processes and multivariate MHPs established in [65, Section 4].

As a preparation, we first clarify the genealogy of the CMJ-process. Denote by $\ell_{0,j}$ the residual life of the j -th ancestor at time 0, which is distributed as ν^* . We sort all offsprings by their birth times. Associated to the sequence $\{(\sigma_i, \ell_i) : i = 1, 2, \dots\}$ we define a (\mathcal{F}_t) -random point measure $N_\nu(ds, dy)$ on $(0, \infty)^2$ by

$$N_\nu(ds, dy) := \sum_{i=1}^{\infty} \mathbf{1}_{\{\sigma_i \in ds, \ell_i \in dy\}}.$$

By the branching mechanism, a child will be born at time t at the rate $\gamma/\beta \cdot \#\mathcal{I}_k(t-)$. Thus the random point measure $N_\nu(ds, dy)$ has a (\mathcal{F}_t) -intensity $\gamma/\beta \cdot \#\mathcal{I}_k(s-) \cdot ds \cdot \nu(dy)$. Notice that $\#\mathcal{I}_k$ has the representation

$$\#\mathcal{I}_k(t) = \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > t\}} + \sum_{\sigma_i \leq t} \mathbf{1}_{\{\ell_i > t - \sigma_i\}}, \quad t \geq 0.$$

Here the two sums on the right side of this equality represent the numbers of ancestors and offsprings alive at time t respectively. Similarly as in Appendix B, we can write the foregoing equation into

$$(3.1) \quad \#\mathcal{I}_k(t) = \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > t\}} + \int_0^t \int_0^\infty \mathbf{1}_{\{y > t-s\}} N_\nu(ds, dy), \quad t \geq 0.$$

Hence $N_\nu(ds, du)$ is a MHP on $(0, \infty)^2$. Before applying Proposition B.1 to establish a SVE for the process $L_{k/\beta}^Y$, we need to introduce two important quantities related to the triple (β, γ, ν) . Let R_ν be the *resolvent* associated to the function $\gamma/\beta \cdot \bar{\nu}$ defined by the unique solution of

$$(3.2) \quad R_\nu = \frac{\gamma}{\beta} \cdot \bar{\nu} + \frac{\gamma}{\beta} \cdot \bar{\nu} * R_\nu.$$

Actually, the function R_ν can be interpreted as the mean reproduction rate of an individual and its descents. In addition, we introduce a two-parameter function on \mathbb{R}_+^2

$$(3.3) \quad R(t, y) = \mathbf{1}_{\{y > t\}} + \int_0^t R_\nu(t-s) \cdot \mathbf{1}_{\{y > s\}} ds, \quad t, y \geq 0,$$

to describe the mean reproduction rate of an individual with life-length y and its descents; we refer to [65, Section 2] for more detailed explanations.

THEOREM 3.5. *If $\varphi'(0) \geq 0$, we have for $k \geq 1$, the process $L_{k/\beta}^Y$ is equal in distribution to the unique solution of the SVE*

$$(3.4) \quad \begin{aligned} Z_k(t) &= \frac{1}{\beta} \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > \beta t\}} + \int_0^t R_\Lambda(\beta(t-s)) \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > \beta s\}} ds \\ &+ \int_0^t \int_0^\infty \int_0^{Z_k(s^-)} \frac{1}{\beta} R(\beta(t-s), y) \tilde{N}(ds, dy, dz), \quad t \geq 0, \end{aligned}$$

where $\tilde{N}(ds, dy, dz)$ is a compensated PRM on $(0, \infty)^3$ with intensity $\beta\gamma \cdot ds\nu(dy)dz$.

PROOF. From (3.1) and Proposition B.1, the population process $\#\mathcal{I}_k(\cdot)$ is the unique solution of

$$\#\mathcal{I}_k(t) = \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > t\}} + \int_0^t R_\nu(t-s) \sum_{j=1}^k \mathbf{1}_{\{\ell_{0,j} > s\}} ds + \int_0^t \int_0^\infty \int_0^{\#\mathcal{I}_k(s^-)} R(t-s, y) \tilde{N}_\nu(ds, dy, dz),$$

where $\tilde{N}_\nu(ds, dy, dz)$ is compensated PRM on $(0, \infty)^3$ with intensity $\gamma/\beta \cdot ds\nu(dy)dz$. By the change of variables, the process $\beta^{-1} \cdot \#\mathcal{I}_k(\beta \cdot)$ is the unique solution to (3.4) with $\tilde{N}(ds, dy, dz) = \tilde{N}_\nu(\beta \cdot ds, \beta \cdot dy, \beta \cdot dz)$ and the desired result follows directly from Lemma 3.4. \square

4. Stochastic Volterra equation for L_ζ^ξ . In this section, we prove the first two claims in Theorem 2.1. By the self-similarity of stable processes and their local times; see (2.4), the next lemma shows that it suffices to prove them for a subclass of spectrally positive stable processes. It will help a lot to simplify our proofs.

LEMMA 4.1. *Let $c_0 > 0$. If Theorem 2.1(1) holds for any ξ with $(\alpha, b, c) \in (0, 1) \times [0, \infty) \times \{c_0\}$, then it holds for all ξ with $(\alpha, b, c) \in (0, 1) \times [0, \infty) \times (0, \infty)$.*

PROOF. For any ξ with parameters $(\alpha, b, c) \in (0, 1) \times [0, \infty) \times (0, \infty)$, its self-similarity induces that $a\xi$ with $a = (c_0/c)^{1/(1+\alpha)}$ is a spectrally positive stable process with parameters (α, ab, c_0) and Lévy measure $\nu_\alpha(a^{-1} \cdot dy) = c_0/c \cdot \nu_\alpha(dy)$. For $\zeta > 0$, by Theorem 2.1(1) the process $L_\zeta^{a\xi}$ is a weak solution to

$$L_\zeta^{a\xi}(x) = \int_0^\infty \int_0^\zeta \nabla_y W_a(x) N_{0,a}(dy, dz) + \int_0^x \int_0^\infty \int_0^{L_\zeta^{a\xi}(s)} \nabla_y W_a(x-s) \tilde{N}_a(ds, dy, dz), \quad x \geq 0,$$

where W_a is the scale function of $a\xi$, $N_{0,a}(dy, dz)$ is PRM on $(0, \infty)^2$ with intensity $c_0/c \cdot \bar{\nu}_\alpha(y)dydz$ and $\tilde{N}_a(ds, dy, dz)$ is a compensated PRM on $(0, \infty)^3$ with intensity $c_0/c \cdot ds\nu_\alpha(dy)dz$. By (2.5) as well as the continuity of W and W_a , we have $W(x) = aW_a(ax)$ for any $x \in \mathbb{R}$. Application of (2.4) gives that $L_\zeta^\xi \stackrel{\text{a.s.}}{=} aL_{\zeta/a}^{a\xi}(a \cdot)$. Taking these back into the preceding SVE and then using the change of variables,

$$L_\zeta^\xi(x) = \int_0^\infty \int_0^\zeta \nabla_y W(x) N_{0,a}(a \cdot dy, a^{-1} \cdot dz) + \int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(x-s) \tilde{N}_a(a \cdot ds, a \cdot dy, a^{-1} \cdot dz).$$

Notice that $\nu_\alpha(a \cdot dy) = a^{-\alpha-1}\nu_\alpha(dy)$; see (1.2). The PRM $N_{0,a}(a \cdot dy, a^{-1} \cdot dz)$ has intensity $\bar{\nu}_\alpha(y)dydz$ and the compensated PRM $\tilde{N}_a(a \cdot ds, a \cdot dy, a^{-1} \cdot dz)$ has intensity $ds\nu_\alpha(dy)dz$. Thus L_ζ^ξ is a weak solution of (1.3) and Theorem 2.1(1) holds for ξ . \square

4.1. Compound Poisson approximation. Based on Lemma 4.1, we start to prove Theorem 2.1(1) for any ξ with

$$\alpha \in (0, 1), \quad b \geq 0, \quad c \equiv \Gamma(1 - \alpha), \quad \nu_\alpha(dy) = \alpha(\alpha + 1)y^{-\alpha-2}dy \quad \text{and} \quad \bar{\nu}_\alpha(y) = \alpha y^{-\alpha-1}.$$

Let us consider a sequence of compound Poisson processes with negative drift and positive jumps, whose local times give a good approximation for the process L_ζ^ξ . Denote by Λ the Pareto-II distribution on \mathbb{R}_+ with location 0 and shape $\alpha + 1$, i.e.,

$$(4.1) \quad \Lambda(dx) := (\alpha + 1)(1 + x)^{-\alpha-2}dx \quad \text{with tail-distribution} \quad \bar{\Lambda}(x) = (1 + x)^{-\alpha-1}, \quad x \geq 0.$$

For $n \geq 1$, let $\{Y^{(n)}(t) : t \geq 0\}$ be a compound Poisson process with a drift -1 , arrival rate $\gamma_n \in (0, \alpha)$, jump-size distribution Λ , Laplace exponent $\varphi^{(n)}$ and local times $L_{Y^{(n)}}$. We now consider the behavior of $Y^{(n)}$ and $L_{Y^{(n)}}$ at a large time scale under the following condition.

CONDITION 4.2. Assume that $n^\alpha(1 - \gamma_n/\alpha) \rightarrow b$ as $n \rightarrow \infty$.

A routine computation, along with this condition, induces that the rescaled Laplace exponent $\bar{\varphi}^{(n)} := \{n^{1+\alpha}\varphi^{(n)}(\lambda/n) : \lambda \geq 0\}$ converges locally uniformly to $\bar{\varphi}$ on \mathbb{R}_+ as $n \rightarrow \infty$. By Corollary 4.3 in [36, p.440], the rescaled compound Poisson process $\xi^{(n)} := \{n^{-1} \cdot Y^{(n)}(n^{1+\alpha}t) : t \geq 0\}$ converges weakly to ξ in $D([0, \infty); \mathbb{R})$ as $n \rightarrow \infty$. Let $L_{\xi^{(n)}}$ be the local times of $\xi^{(n)}$ satisfying (2.1) and $\tau_{\xi^{(n)}}^L$ the right-inverse local time at level 0. Similarly, we also write $L_\zeta^{\xi^{(n)}}$ for the process $\{L_{\xi^{(n)}}(x, \tau_{\xi^{(n)}}^L(\zeta)) : x \geq 0\}$ under $\mathbf{P}(\cdot | \tau_{\xi^{(n)}}^L(\zeta) < \infty)$. The following convergence result comes from Theorem 2.4 in [45].

LEMMA 4.3. For $\zeta > 0$, we have $L_\zeta^{\xi^{(n)}} \rightarrow L_\zeta^\xi$ weakly in $D([0, \infty); \mathbb{R}_+)$ as $n \rightarrow \infty$.

For $n \geq 1$, let $Z_{[n^\alpha\zeta]}^{(n)}$ be the unique solution of (3.4) with $k = [n^\alpha\zeta]$, $\beta = 1$, $\gamma = \gamma_n$ and $\nu = \Lambda$. By (2.4) and Theorem 3.5,

$$(4.2) \quad L_\zeta^{\xi^{(n)}} \stackrel{\text{a.s.}}{=} \{n^{-\alpha} \cdot L_{[n^\alpha\zeta]}^{Y^{(n)}}(nx) : x \geq 0\} \stackrel{\text{d}}{=} \{n^{-\alpha} \cdot Z_{[n^\alpha\zeta]}^{(n)}(nt) : t \geq 0\} =: X_\zeta^{(n)}$$

and hence $X_\zeta^{(n)} \rightarrow L_\zeta^\xi$ weakly in $D([0, \infty); \mathbb{R}_+)$ as $n \rightarrow \infty$; see Lemma 4.3. Clearly, Theorem 2.1(1) can be proved by characterizing the cluster points of $\{X_\zeta^{(n)}\}_{n \geq 1}$ as weak solutions of (1.3).

Let $R_\Lambda^{(n)}$ and $R^{(n)}$ be the resolvent associated to the tail-distribution $\bar{\Lambda}$ defined as in (3.2)-(3.3) with $\beta = 1$, $\gamma = \gamma_n$ and $\nu = \Lambda$, i.e., $R_\Lambda^{(n)}(t) = R^{(n)}(t, y) = 0$ if $t \leq 0$ or $y \leq 0$ and for $t, y \geq 0$,

$$(4.3) \quad R_\Lambda^{(n)}(t) = \gamma_n \bar{\Lambda}(t) + \gamma_n \bar{\Lambda} * R_\Lambda^{(n)}(t),$$

$$(4.4) \quad R^{(n)}(t, y) = \mathbf{1}_{\{y > t\}} + \int_0^t R_\Lambda^{(n)}(t-s) \cdot \mathbf{1}_{\{y > s\}} ds.$$

By (4.2), Theorem 3.5 and the change of variables, the process $X_\zeta^{(n)}$ satisfies the SVE

$$X_\zeta^{(n)}(t) = \frac{1}{n^\alpha} \sum_{i=1}^{[n^\alpha\zeta]} \mathbf{1}_{\{\ell_{0,i} > nt\}} + \int_0^{nt} R_\Lambda^{(n)}(nt-s) \cdot \frac{1}{n^\alpha} \sum_{i=1}^{[n^\alpha\zeta]} \mathbf{1}_{\{\ell_{0,i} > s\}} ds$$

$$(4.5) \quad + \int_0^t \int_0^\infty \int_0^{X_\zeta^{(n)}(s-)} \frac{1}{n^\alpha} R^{(n)}(n(t-s), ny) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \quad t \geq 0,$$

where $\tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz)$ is a compensated PRM on $(0, \infty)^3$ with intensity $n^{1+\alpha} \gamma_n \cdot ds \cdot \Lambda(n \cdot dy) \cdot dz$. Here $\ell_{0,i}$ is distributed as the size-biased distribution of Λ

$$(4.6) \quad \Lambda^*(dx) := \alpha \bar{\Lambda}(x) dx = \alpha(1+x)^{-\alpha-1} dx \quad \text{with tail-distribution} \quad \bar{\Lambda}^*(x) := (1+x)^{-\alpha}, \quad x \geq 0.$$

4.2. Proofs for Theorem 2.1(1)-(2). In order to make the proofs much easier to be understood, we offer an intuitive description on how to derive the convergence of the SVE (4.5) to the SVE (1.3). The following auxiliary lemmas will be proved in Section 4.3.

We start the asymptotic analysis from the first two terms on the right side of (4.5). Denote by $X_{\zeta,0}^{(n)}(t)$ their sum. Let $M(\mathbb{R}_+)$ be the space of all σ -finite measures on \mathbb{R}_+ endowed with the topology of weak convergence and $M_p(\mathbb{R}_+) \subset M(\mathbb{R}_+)$ the space of all Radon point measures on \mathbb{R}_+ . Let $N_\zeta^{(n)}(dy)$ be a finite point measure on $(0, \infty)$ defined by

$$N_\zeta^{(n)}(dy) := \sum_{i=1}^{\lfloor n^\alpha \zeta \rfloor} \mathbf{1}_{\{\ell_{0,i}/n \in dy\}}.$$

From this and the change of variables, we can write $X_{\zeta,0}^{(n)}$ into

$$X_{\zeta,0}^{(n)}(t) = \int_0^\infty \left(\frac{\mathbf{1}_{\{y>t\}}}{n^\alpha} + \int_0^t n^{1-\alpha} R_\Lambda^{(n)}(n(t-s)) \cdot \mathbf{1}_{\{y>s\}} ds \right) N_\zeta^{(n)}(dy), \quad t \geq 0.$$

Obviously, the convergence of $X_{\zeta,0}^{(n)}$ relies on the asymptotic behavior of $\{n^{1-\alpha} R_\Lambda^{(n)}(nt) : t \geq 0\}$ and $N_\zeta^{(n)}$.

LEMMA 4.4. We have $\int_0^\cdot n^{1-\alpha} R_\Lambda^{(n)}(ns) ds \xrightarrow{u.c.} W$ as $n \rightarrow \infty$.

Notice that $n^\alpha \mathbf{P}(\ell_{0,i}/n \in dy) \rightarrow \alpha y^{-\alpha-1} dy = \bar{\nu}_\alpha(y) dy$ vaguely in $M(\mathbb{R}_+)$. The next proposition follows directly from the basic convergence theorem of empirical measures; see Theorem 5.3 in [61, p.138].

PROPOSITION 4.5. We have $N_\zeta^{(n)}(dy) \rightarrow N_\zeta(dy) := N_0(dy, [0, \zeta])$ weakly in $M_p(\mathbb{R}_+)$ as $n \rightarrow \infty$.

We start to consider the convergence of the SVI in (4.5). By (4.4),

$$n^{-\alpha} R^{(n)}(nt, ny) = n^{-\alpha} \mathbf{1}_{\{y>t\}} + \int_0^t n^{1-\alpha} R_\Lambda^{(n)}(n(t-s)) \mathbf{1}_{\{y>s\}} ds, \quad t, y \geq 0.$$

Clearly, the first term on the right side of this equality vanishes uniformly in $t, y \in \mathbb{R}$ as $n \rightarrow \infty$. Additionally, using the change of variable to the second term,

$$\int_0^t n^{1-\alpha} R_\Lambda^{(n)}(n(t-s)) \mathbf{1}_{\{y>s\}} ds = \int_{(t-y)^+}^t n^{1-\alpha} R_\Lambda^{(n)}(ns) ds.$$

By Lemma 4.4 and the fact that $W(x) = 0$ if $x < 0$, it can be well approximated by $W(t) - W(t-y) = \nabla_y W(t)$. Thus the SVI in (4.5) can be well approximated by

$$M^{(n)}(t) := \int_0^t \int_0^\infty \int_0^{X_\zeta^{(n)}(s-)} \nabla_y W(t-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz)$$

with the error denoted as $\varepsilon^{(n)}(t)$. By the preceding analysis and notation, we can write the SVE (4.5) into

$$(4.7) \quad X_\zeta^{(n)}(t) = \varepsilon^{(n)}(t) + X_{\zeta,0}^{(n)}(t) + M^{(n)}(t), \quad t \geq 0.$$

As the last preparation, the next two lemmas establish some convergence results for $\{M^{(n)}\}_{n \geq 1}$ and $\{\varepsilon^{(n)}\}_{n \geq 1}$.

LEMMA 4.6. *The sequence $\{M^{(n)}\}_{n \geq 1}$ is C -tight.*

LEMMA 4.7. *We have $\varepsilon^{(n)} + X_{\zeta,0}^{(n)} \rightarrow X_{\zeta,0}$ weakly in $D([0, \infty), \mathbb{R})$ as $n \rightarrow \infty$, where*

$$(4.8) \quad X_{\zeta,0}(t) := \int_0^\infty \nabla_y W(t) N_\zeta(dy) = \int_0^\infty \int_0^\zeta \nabla_y W(t) N_0(dy, dz), \quad t \geq 0.$$

PROOF FOR THEOREM 2.1(1). Corollary 3.33(b) in [36, p.353], together with Lemma 4.3, 4.6 and 4.7, yields the C -tightness of the sequence $\{(X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, M^{(n)})\}_{n \geq 1}$. Additionally, if

$$(4.9) \quad (X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, M^{(n)}) \rightarrow (X_\zeta, X_{\zeta,0}, M),$$

as $n \rightarrow \infty$ in the sense of finite-dimensional distributions, where

$$(4.10) \quad M(t) := \int_0^t \int_0^\infty \int_0^{X_\zeta(s)} \nabla_y W(t-s) \tilde{N}_\alpha(ds, dy, dz), \quad t \geq 0,$$

then Theorem 13.1 in [11, p.139] implies that $(X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, M^{(n)})$ converges weakly to $(X_\zeta, X_{\zeta,0}, M)$ in $D([0, \infty); \mathbb{R}^3)$. By Proposition 2.4 in [36, p.339] and the continuous mapping theorem,

$$\sup_{t \in [0, T]} |X_\zeta(t) - X_{\zeta,0}(t) - M(t)| \stackrel{d}{=} \lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |X_\zeta^{(n)}(t) - \varepsilon^{(n)}(t) - X_{\zeta,0}^{(n)}(t) - M^{(n)}(t)| \stackrel{\text{a.s.}}{=} 0,$$

for any $T \geq 0$. In conclusion, we have $X_\zeta \stackrel{\text{a.s.}}{=} X_{\zeta,0} + M$ and hence X_ζ is a weak solution of the SVE (1.3).

To finish this proof, it remains to prove the finite-dimensional convergence (4.9). It follows if for any $d \geq 1$ and $0 \leq t_1 < \dots < t_d$, the $3d$ -dimensional random variable

$$Z_d^{(n)} := (X_\zeta^{(n)}(t_1), \varepsilon^{(n)}(t_1) + X_{\zeta,0}^{(n)}(t_1), M^{(n)}(t_1), \dots, X_\zeta^{(n)}(t_d), \varepsilon^{(n)}(t_d) + X_{\zeta,0}^{(n)}(t_d), M^{(n)}(t_d))$$

converges in distribution to $Z_d := (X_\zeta(t_1), X_{\zeta,0}(t_1), M(t_1), \dots, X_\zeta(t_d), X_{\zeta,0}(t_d), M(t_d))$. For $i \in \{1, \dots, d\}$ and $t \geq 0$, let

$$(4.11) \quad M_i^{(n)}(t) := \int_0^t \int_0^\infty \int_0^{X_\zeta^{(n)}(s-)} \nabla_y W(t_i - s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz),$$

$$(4.12) \quad M_i(t) := \int_0^t \int_0^\infty \int_0^{X_\zeta(s)} \nabla_y W(t_i - s) \tilde{N}_\alpha(ds, dy, dz).$$

Similarly as in Remark 2.2, we can identify that both $M_i^{(n)}$ and M_i are well-defined and local martingales.

Notice that $M_i^{(n)}(t_i) \stackrel{\text{a.s.}}{=} M^{(n)}(t_i)$ and $M_i(t_i) \stackrel{\text{a.s.}}{=} M(t_i)$ for $i = 1, \dots, d$. Then

$$Z_d^{(n)} \stackrel{d}{=} (X_\zeta^{(n)}(t_1), \varepsilon^{(n)}(t_1) + X_{\zeta,0}^{(n)}(t_1), M_1^{(n)}(t_1), \dots, X_\zeta^{(n)}(t_d), \varepsilon^{(n)}(t_d) + X_{\zeta,0}^{(n)}(t_d), M_d^{(n)}(t_d)),$$

$$Z_d \stackrel{d}{=} (X_\zeta(t_1), X_{\zeta,0}(t_1), M_1(t_1), \dots, X_\zeta(t_d), X_{\zeta,0}(t_d), M_d(t_d)).$$

By the continuity of W and the fact that $N_\alpha(\{t\}, \mathbb{R}_+^2) \stackrel{\text{a.s.}}{=} 0$ for any $t > 0$, the set $\{t \geq 0 : \mathbf{P}(|Z_d(t) - Z_d(t-)| \neq 0) > 0\}$ is null. From Proposition 3.14 in [36, p.349], we have $Z_d^{(n)} \xrightarrow{d} Z_d$ if

$$(4.13) \quad (X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, M_1^{(n)}, \dots, M_d^{(n)}) \rightarrow (X_\zeta, X_{\zeta,0}, M_1, \dots, M_d), \quad \text{weakly in } D([0, \infty), \mathbb{R}^{2+d}).$$

We start to prove (4.13) by using the weak convergence results established in [42] for Itô's stochastic integrals driven by infinite-dimensional semimartingale; see Appendix C. Let $\hat{\nu}_\alpha(dy, dz) := \nu_\alpha(dy)dz$ be a σ -finite measure on \mathbb{R}_+^2 and $L^2(\hat{\nu}_\alpha)$ the collection of all functions on \mathbb{R}_+^2 that are square-integrable with respect to $\hat{\nu}_\alpha$.

For $n \geq 1$ and $t > 0$, let $\tilde{\mathbf{N}}^{(n)}(t) := \tilde{N}^{(n)}((0, nt], n \cdot dy, n^\alpha \cdot dz)$ and $\tilde{\mathbf{N}}_\alpha(t) := \tilde{N}_\alpha((0, t], dy, dz)$, which are two standard $L^2(\hat{\nu}_\alpha)^\#$ -martingales. We can write (4.11) and (4.12) into

$$(M_1^{(n)}(t), \dots, M_d^{(n)}(t)) = F(X_\zeta^{(n)}, -) \cdot \tilde{\mathbf{N}}^{(n)}(t) \quad \text{and} \quad (M_1(t), \dots, M_d(t)) = F(X_\zeta, -) \cdot \tilde{\mathbf{N}}_\alpha(t),$$

where the function $F : D([0, \infty); \mathbb{R}_+) \times \mathbb{R}_+ \mapsto (L^2(\hat{\nu}))^d$ is defined by

$$F(x, s) := (\nabla_y W(t_1 - s), \dots, \nabla_y W(t_d - s)) \cdot \mathbf{1}_{\{0 < z \leq x(s-)\}}.$$

From Condition 4.2 and Theorem 2.7 in [42], we have $\gamma_n n^{\alpha+1} \Lambda(n \cdot dy) \rightarrow \nu_\alpha(dy)$ vaguely and hence $\tilde{\mathbf{N}}^{(n)} \Rightarrow \tilde{\mathbf{N}}_\alpha$ as $n \rightarrow \infty$. Similarly as in the proof of Lemma 4.9 in [33], we can prove that the sequence of $L^2(\hat{\nu})^\#$ -martingales $\{\tilde{\mathbf{N}}^{(n)}\}_{n \geq 1}$ is uniformly tight. By Example 5.3 in [41], the function F satisfies Condition C.2 and also Condition C.1 in [42, p.248-249], which induces that

$$(X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, F(X_\zeta^{(n)}, -), \tilde{\mathbf{N}}^{(n)}) \Rightarrow (X_\zeta, X_{\zeta,0}, F(X_\zeta, -), \tilde{\mathbf{N}}_\alpha).$$

By Theorem 4.2 or 5.5 in [42], there exists a filtration such that $(X_\zeta, \tilde{\mathbf{N}}_\alpha)$ is adapted and

$$(X_\zeta^{(n)}, \varepsilon^{(n)} + X_{\zeta,0}^{(n)}, F(X_\zeta^{(n)}, -), \tilde{\mathbf{N}}^{(n)}, F(X_\zeta^{(n)}, -) \cdot \tilde{\mathbf{N}}^{(n)}) \Rightarrow (X_\zeta, X_{\zeta,0}, F(X_\zeta, -), \tilde{\mathbf{N}}_\alpha, F(X_\zeta, -) \cdot \tilde{\mathbf{N}}_\alpha),$$

as $n \rightarrow \infty$, which implies (4.13) immediately. The whole proof has been finished. \square

PROOF FOR THEOREM 2.1(2). From (2.3), the inverse local time τ_ξ^L is a subordinator killed at an independent exponential time ϱ . For any $\zeta > 0$, we have $\tau_\xi^L(\zeta) < \infty$ if and only if $\varrho > \zeta$. Moreover, the independent increments of ξ and the memorylessness of ϱ yield that conditioned on $\tau_\xi^L(\zeta) < \infty$, the two random elements $L_\xi(\cdot, \tau_\xi^L(\zeta))$ and ϱ are independent. Hence

$$(4.14) \quad \begin{aligned} \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(\zeta)) \in A, \varrho \in d\zeta | \tau_\xi^L(\zeta) < \infty) &= \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(\zeta)) \in A | \tau_\xi^L(\zeta) < \infty) \mathbf{P}(\varrho \in d\zeta | \tau_\xi^L(\zeta) < \infty) \\ &= \mathbf{P}(L_\xi^\xi \in A) \mathbf{P}(\varrho \in d\zeta | \varrho > \zeta). \end{aligned}$$

Notice that the event $\{L_\xi(0, \infty) \in d\zeta\}$ occurs if and only if $\{\tau_\xi^L(\zeta) < \infty, \varrho \in d\zeta\}$. By (2.14) and (4.14), we have for any Borel set A of $C([0, \infty), \mathbb{R}_+)$,

$$\begin{aligned} \mathbf{P}(L_\infty^\xi \in A) &= \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(L_\xi(0, \infty))) \in A) \\ &= \int_0^\infty \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(\zeta)) \in A, L_\xi(0, \infty) \in d\zeta) \\ &= \int_0^\infty \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(\zeta)) \in A, \tau_\xi^L(\zeta) < \infty, \varrho \in d\zeta) \\ &= \int_0^\infty \mathbf{P}(\tau_\xi^L(\zeta) < \infty) \mathbf{P}(L_\xi(\cdot, \tau_\xi^L(\zeta)) \in A, \varrho \in d\zeta | \tau_\xi^L(\zeta) < \infty) \\ &= \int_0^\infty \mathbf{P}(\varrho > \zeta) \mathbf{P}(L_\xi^\xi \in A) \mathbf{P}(\varrho \in d\zeta | \varrho > \zeta) = \int_0^\infty \mathbf{P}(L_\xi^\xi \in A) \mathbf{P}(\varrho \in d\zeta). \end{aligned}$$

Hence the process L_∞^ξ is a weak solution of (1.3) with $\zeta = \varrho$. \square

4.3. Proofs for auxiliary lemmas. In this section we provide the detailed proofs for the auxiliary lemmas in the last section. Denote by $\mathcal{L}_{\bar{\Lambda}}$, \mathcal{L}_{Λ^*} and $\mathcal{L}_{R_\Lambda^{(n)}}$ the Laplace transforms of $\bar{\Lambda}$, Λ^* and $R_\Lambda^{(n)}$ respectively, i.e.,

$$\mathcal{L}_{\bar{\Lambda}}(\lambda) := \int_0^\infty e^{-\lambda x} \bar{\Lambda}(x) dx, \quad \mathcal{L}_{\Lambda^*}(\lambda) := \int_0^\infty e^{-\lambda x} \Lambda^*(dx), \quad \mathcal{L}_{R_\Lambda^{(n)}}(\lambda) := \int_0^\infty e^{-\lambda x} R_\Lambda^{(n)}(x) dx, \quad \lambda \geq 0.$$

4.3.1. Upper bounds for resolvents. As a preparation, we first give some upper bound estimates for the resolvent $R_\Lambda^{(n)}$ and $R^{(n)}$, which will play an important role in the following proofs and analyses. Notice that $\bar{\Lambda}$ is a complete monotone function. Theorem 5.3.1 in [30, p.148] tells that $R_\Lambda^{(n)}$ is also completely monotone and $\|R_\Lambda^{(n)}\|_{L^\infty} < \infty$. The next proposition gives a uniform upper bound estimate for $\{R_\Lambda^{(n)}\}_{n \geq 1}$.

PROPOSITION 4.8. *There exists a constant $C > 0$ such that for any $t \geq 0$ and $n \geq 1$,*

$$(4.15) \quad R_\Lambda^{(n)}(t) \leq C \cdot (1+t)^{\alpha-1}.$$

PROOF. Let R_Λ be the resolvent associated to $\alpha\bar{\Lambda}$, i.e., $R_\Lambda = \alpha\bar{\Lambda} + \alpha\bar{\Lambda} * R_\Lambda$. It is bounded and completely monotone with $R_\Lambda(0) = \alpha$. It is easy to identify that $R_\Lambda^{(n)}$ and R_Λ have the representations

$$R_\Lambda^{(n)} = \sum_{k=1}^{\infty} (\gamma_n \cdot \bar{\Lambda})^{*k} \quad \text{and} \quad R_\Lambda = \sum_{k=1}^{\infty} (\alpha \cdot \bar{\Lambda})^{*k}.$$

Here f^{*k} is the k -th convolution of f . Since $\gamma_n \leq \alpha$, we have $R_\Lambda^{(n)}(t) \leq R_\Lambda(t)$ uniformly in $n \geq 1$ and $t \geq 0$. Let $\bar{R}_\Lambda := 1 - R_\Lambda/\alpha$ for $t \geq 0$, which is a distribution function on \mathbb{R}_+ . Using the integration by parts,

$$(4.16) \quad \int_0^\infty e^{-\lambda t} d\bar{R}_\Lambda(t) = \lambda \int_0^\infty e^{-\lambda t} \bar{R}_\Lambda(t) dt = \lambda \int_0^\infty e^{-\lambda t} [1 - R_\Lambda(t)/\alpha] dt = 1 - \frac{\lambda}{\alpha} \mathcal{L}_{R_\Lambda}(\lambda),$$

where \mathcal{L}_{R_Λ} denotes the Laplace transform of R_Λ . Taking Laplace transforms of both sides of $R_\Lambda = \alpha\bar{\Lambda} + \alpha\bar{\Lambda} * R_\Lambda$, we have $\mathcal{L}_{R_\Lambda}(\lambda) = \alpha\mathcal{L}_{\bar{\Lambda}}(\lambda)[1 + \mathcal{L}_{R_\Lambda}(\lambda)]$ and

$$\mathcal{L}_{R_\Lambda}(\lambda) = \frac{\alpha\mathcal{L}_{\bar{\Lambda}}(\lambda)}{1 - \alpha\mathcal{L}_{\bar{\Lambda}}(\lambda)}, \quad \lambda > 0.$$

It is obvious that the numerator goes to 1 as $\lambda \rightarrow 0+$. Moreover, a simple calculation with (4.1) induces that $\int_t^\infty \alpha\bar{\Lambda}(s) ds = (1+t)^{-\alpha}$ for $t \geq 0$. By Karamata's Tauberian theorem⁹; see Theorem 8.1.6 in [12, p.333], we have $1 - \alpha\mathcal{L}_{\bar{\Lambda}}(\lambda) \sim \Gamma(1-\alpha)\lambda^\alpha$ and $\mathcal{L}_{R_\Lambda}(\lambda) \sim \lambda^{-\alpha}/\Gamma(1-\alpha)$ as $\lambda \rightarrow 0+$. Taking these back into (4.16),

$$1 - \int_0^\infty e^{-\lambda t} d\bar{R}_\Lambda(t) \sim \frac{\lambda^{1-\alpha}}{\alpha\Gamma(1-\alpha)}.$$

Using Karamata's Tauberian theorem again, we have as $t \rightarrow \infty$,

$$1 - \bar{R}_\Lambda(t) \sim \frac{t^{\alpha-1}}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)} \quad \text{and hence} \quad R_\Lambda(t) = \alpha(1 - \bar{R}_\Lambda(t)) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)}.$$

Consequently, there exists a constant $C > 0$ such that $R_\Lambda^{(n)}(t) \leq R_\Lambda(t) \leq C(1+t)^{\alpha-1}$ for any $t \geq 0$. \square

PROPOSITION 4.9. *For $p > 1 + \alpha$, there exists a constant $C > 0$ such that for any $t \geq 0$ and $n \geq 1$,*

$$(4.17) \quad n^{\alpha+1} \int_0^t ds \int_0^\infty |n^{-\alpha} R^{(n)}(ns, ny)|^p \Lambda(n \cdot dy) \leq C \cdot (1+t)^{\alpha(p-1)}.$$

PROOF. By (4.4) and the power mean inequality; see footnote 5, the term on the left side of (4.17) can be bounded by the sum of the following four terms

$$I_1^{(n)}(t) := C \int_0^t ds \int_0^\infty |n^{-\alpha} \mathbf{1}_{\{y>s\}}|^p n^{\alpha+1} \Lambda(n \cdot dy),$$

⁹Let F be a probability distribution function on \mathbb{R}_+ with Laplace transform \mathcal{L}_F . If $(1 - F(tx))/(1 - F(t)) \rightarrow x^{-\alpha}$ or $(1 - \mathcal{L}_F(1/(t\lambda)))/(1 - \mathcal{L}_F(1/t)) \rightarrow x^{-\alpha}$ as $t \rightarrow \infty$ for some $\alpha \in (0, 1)$, then $1 - \mathcal{L}_F(1/t) \sim \Gamma(1-\alpha)(1 - F(t))$ as $t \rightarrow \infty$.

$$\begin{aligned}
I_2^{(n)}(t) &:= C \int_0^t \left| \int_0^s n^{1-\alpha} R_\Lambda^{(n)}(nr) dr \right|^p n^{\alpha+1} \bar{\Lambda}(ns) ds, \\
I_3^{(n)}(t) &:= C \int_0^t ds \int_0^{s/2} \left| \int_{s-y}^s n^{1-\alpha} R_\Lambda^{(n)}(nr) dr \right|^p n^{\alpha+1} \Lambda(n \cdot dy), \\
I_4^{(n)}(t) &:= C \int_0^t ds \int_{s/2}^s \left| \int_{s-y}^s n^{1-\alpha} R_\Lambda^{(n)}(nr) dr \right|^p n^{\alpha+1} \Lambda(n \cdot dy),
\end{aligned}$$

for some constant C depending only on p . Notice that the inner integral in $I_1^{(n)}(t)$ equals to $n^{(1-p)\alpha+1} \bar{\Lambda}(ns)$. By the change of variables and the fact that $\|\bar{\Lambda}\|_{L^1} < \infty$,

$$\sup_{t \geq 0} |I_1^{(n)}(t)| = C n^{(1-p)\alpha} \cdot \sup_{t \geq 0} \int_0^{nt} \bar{\Lambda}(s) ds = C \|\bar{\Lambda}\|_{L^1} \cdot n^{(1-p)\alpha},$$

which vanishes as $n \rightarrow \infty$, since $p > 1$. Integrating (4.15) induces that $\int_0^s R_\Lambda^{(n)}(r) dr \leq C(1+s)^\alpha$ uniformly in $n \geq 1$ and $s \geq 0$. Using the change of variables to $I_2^{(n)}$,

$$I_2^{(n)}(t) = \frac{C}{n^{(p-1)\alpha}} \int_0^{nt} \left| \int_0^s R_\Lambda^{(n)}(r) dr \right|^p \bar{\Lambda}(s) ds \leq \frac{C}{n^{(p-1)\alpha}} \int_0^{nt} (1+s)^{(p-1)\alpha-1} ds \leq C \cdot (1+t)^{(p-1)\alpha}.$$

Here the constant $C > 0$ is independent of n and t . We now turn to consider $I_3^{(n)}$. Using (4.15) again, we have $n^{1-\alpha} R_\Lambda^{(n)}(nr) \leq C \cdot r^{\alpha-1}$ uniformly in $n \geq 1$ and $r \geq 0$. Hence uniformly in $s, y \geq 0$

$$(4.18) \quad \int_{(s-y)^+}^s n^{1-\alpha} R_\Lambda^{(n)}(nr) dr \leq C \cdot (s^\alpha \wedge ((s-y)^+)^{\alpha-1} \cdot y).$$

Plugging this back into $I_3^{(n)}$ and then using the fact that $n^{\alpha+1} \Lambda(n \cdot dy) \leq \nu_\alpha(dy)$, we have uniformly in $n \geq 1$ and $t \geq 0$,

$$I_3^{(n)}(t) \leq C \int_0^t ds \int_0^{s/2} |s-y|^{p(\alpha-1)} \cdot y^{p-\alpha-2} dy \leq C \int_0^t s^{p(\alpha-1)} ds \int_0^{s/2} y^{p-\alpha-2} dy \leq C \cdot t^{(p-1)\alpha}.$$

Similarly, we also have $\int_{s-y}^s n^{1-\alpha} R_\Lambda^{(n)}(nr) dr \leq C \int_0^s r^{\alpha-1} dr \leq C s^\alpha$ uniformly in $s > y \geq 0$. Then

$$I_4^{(n)}(t) \leq C \int_0^t s^{p\alpha} ds \int_{s/2}^s y^{-\alpha-2} dy \leq C \int_0^t s^{(p-1)\alpha-1} ds \leq C \cdot t^{(p-1)\alpha},$$

uniformly in $n \geq 1$ and $t \geq 0$. The desired result follows by putting all estimates above together. \square

4.3.2. Moment estimates. In this section, we provide some uniform upper bounds for all moments of the sequence $\{X_\zeta^{(n)}\}_{n \geq 1}$ and the impact of each ancestor on the population system.

PROPOSITION 4.10. *For each $p \geq 0$, there exists a constant $C > 0$ such that for any $t \geq 0$ and $n \geq 1$,*

$$\mathbf{E} \left[\left| \int_0^t R_\Lambda^{(n)}(t-s) \cdot \mathbf{1}_{\{\ell_{0,1} > s\}} ds \right|^p \right] \leq C \cdot t^{\alpha \cdot (p-1)^+}.$$

PROOF. By Jensen's inequality, (4.6) and (4.15), there exists a constant $C > 0$ such that for any $n \geq 1$, $p \in [0, 1]$ and $t \geq 0$,

$$\mathbf{E} \left[\left| \int_0^t R_\Lambda^{(n)}(t-s) \cdot \mathbf{1}_{\{\ell_{0,1} > s\}} ds \right|^p \right] \leq \left(\int_0^t R_\Lambda^{(n)}(t-s) \mathbf{E}[\mathbf{1}_{\{\ell_{0,1} > s\}}] ds \right)^p \leq C \left(\int_0^t (t-s)^{\alpha-1} \frac{ds}{s^\alpha} \right)^p \leq C.$$

For $p > 1$, by Hölder's inequality¹⁰ and the previous upper bound,

$$\mathbf{E} \left[\left| \int_0^t R_\Lambda^{(n)}(t-s) \cdot \mathbf{1}_{\{\ell_{0,1} > s\}} ds \right|^p \right] \leq \|R_\Lambda^{(n)}\|_{L_t^1}^{p-1} \cdot \mathbf{E} \left[\int_0^t R_\Lambda^{(n)}(t-s) \cdot \mathbf{1}_{\{\ell_{0,1} > s\}} ds \right] \leq C \cdot \|R_\Lambda^{(n)}\|_{L_t^1}^{p-1}.$$

By (4.15), we have $\sup_{n \geq 1} \|R_\Lambda^{(n)}\|_{L_t^1} \leq C \cdot t^\alpha$ uniformly in $t \geq 0$ and the desired result follows. \square

PROPOSITION 4.11. *For each $p \geq 0$ and $T \geq 0$, we have $\sup_{n \geq 1} \sup_{t \in [0, T]} \mathbf{E}[|X_\zeta^{(n)}(t)|^p] < \infty$.*

PROOF. It suffices to prove the case of $p \in \mathbb{Z}_+$. Taking expectations on both sides of (4.5),

$$\mathbf{E}[X_\zeta^{(n)}(t)] \leq \zeta + n^{-\alpha} \sum_{k=1}^{[\zeta n^\alpha]} \mathbf{E} \left[\int_0^{nt} R_\Lambda^{(n)}(nt-s) \mathbf{1}_{\{\ell_{0,k} > s\}} ds \right].$$

From Proposition 4.10, we have $\mathbf{E}[X_\zeta^{(n)}(t)] \leq C$ uniformly in $t \geq 0$ and $n \geq 1$. By mathematical induction, it suffices to prove that for any $p \geq 1$, the desired $2p$ -order moment estimate holds under the assumption that $\sup_{n \geq 1} \sup_{t \in [0, T]} \mathbf{E}[|X_\zeta^{(n)}(t)|^p] < \infty$. By the power mean inequality; see footnote 5,

$$(4.19) \quad \begin{aligned} \mathbf{E}[|X_\zeta^{(n)}(t)|^{2p}] &\leq C \cdot \zeta^{2p} + C \mathbf{E} \left[\left| \int_0^{nt} R_\Lambda^{(n)}(nt-s) n^{-\alpha} \sum_{k=1}^{[\zeta n^\alpha]} \mathbf{1}_{\{\ell_{0,k} > s\}} ds \right|^{2p} \right] \\ &\quad + C \mathbf{E} \left[\left| \int_0^t \int_0^\infty \int_0^{X_\zeta^{(n)}(s-)} n^{-\alpha} R^{(n)}(n(t-s), ny) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz) \right|^{2p} \right], \end{aligned}$$

for some constant $C > 0$ independent of n and t . By (D.1), the last expectation can be bounded by

$$C \left| \int_0^t ds \int_0^\infty |n^{-\alpha} R^{(n)}(ns, ny)|^2 n^{\alpha+1} \Lambda(n \cdot dy) \right|^p + C \int_0^t ds \int_0^\infty |n^{-\alpha} R^{(n)}(ns, ny)|^{2p} n^{\alpha+1} \Lambda(n \cdot dy),$$

uniformly in $n \geq 1$ and $t \in [0, T]$. A simple calculation, together with (4.17), implies that these two terms can be bounded uniformly in $t \geq 0$ and $n \geq 1$ by $C \cdot (1+t)^{\alpha p}$ and $C \cdot (1+t)^{\alpha(2p-1)}$ respectively. We now consider the first expectation on the right side of (4.19). Notice that

$$\left| \sum_{k=1}^{[\zeta n^\alpha]} \int_0^{nt} R_\Lambda^{(n)}(nt-s) \mathbf{1}_{\{\ell_{0,k} > s\}} ds \right|^{2p} = \sum_{|\mathbf{k}^{(n)}|=2p} \prod_{i=1}^{[\zeta n^\alpha]} \left| \int_0^{nt} R_\Lambda^{(n)}(nt-s) \mathbf{1}_{\{\ell_{0,i} > s\}} ds \right|^{k_i}.$$

Here the sum on the right side of this inequality is over all $\mathbf{k}^{(n)} := (k_1, \dots, k_{[\zeta n^\alpha]}) \in \mathbb{N}^{[\zeta n^\alpha]}$ with $|\mathbf{k}^{(n)}| := \sum_{i=1}^{[\zeta n^\alpha]} k_i = 2p$. By Proposition 4.10, we have for some constant $C > 0$ depending only on p ,

$$\mathbf{E} \left[\left| \int_0^{nt} R_H^{(n)}(nt-s) n^{-\alpha} \sum_{k=1}^{[\zeta n^\alpha]} \mathbf{1}_{\{\ell_{0,k} > s\}} ds \right|^{2p} \right] \leq \frac{C}{n^{2p\alpha}} \cdot \sum_{|\mathbf{k}^{(n)}|=2p} (nt)^\alpha \sum_{i=1}^{[\zeta n^\alpha]} (k_i-1)^+.$$

Using the multinomial distribution and then the combination formula to the last sum,

$$\sum_{|\mathbf{k}^{(n)}|=2p} (nt)^\alpha \sum_{i=1}^{[\zeta n^\alpha]} (k_i-1)^+ = \sum_{j=1}^{[n^\alpha \zeta] \wedge (2p)} [n^\alpha \zeta] C_j \cdot (nt)^{\alpha(2p-j)},$$

¹⁰ For two integrable functions g, h on $[0, T]$ and $p > 1$, we have

$$\left| \int_0^T |g(s)h(s)| ds \right|^p = \left| \int_0^T |g(s)||h(s)|^{1/p} \cdot |h(s)|^{1-1/p} ds \right|^p \leq \int_0^T |g(s)|^p |h(s)| ds \cdot \left| \int_0^T |h(s)| ds \right|^{p-1}.$$

Here ${}_{[n^\alpha \zeta]}C_j := [n^\alpha \zeta]! / (j!([n^\alpha \zeta] - j)!) \leq \zeta^j \cdot n^{\alpha j} / j!$ for any $j = 1, \dots, [n^\alpha \zeta] \wedge (2p)$. Thus there exists a constant $C > 0$ such that for any $n \geq 1$ and $t \in [0, T]$,

$$\mathbf{E} \left[\left| \int_0^{nt} R_H^{(n)}(nt-s) n^{-\alpha} \sum_{k=1}^{[n^\alpha \zeta]} \mathbf{1}_{\{\ell_{0,k} > s\}} ds \right|^{2p} \right] \leq C \sum_{j=1}^{[n^\alpha \zeta] \wedge (2p)} \zeta^j \cdot \frac{t^{\alpha(2p-j)}}{j!} \leq C \cdot \zeta^{2p} \cdot (1+t)^{\alpha(2p-1)}.$$

The desired result follows by putting all estimates above together. \square

4.3.3. Proof for Lemma 4.4. The desired convergence is obtained by showing that the Laplace transform of the measure with density $n^{1-\alpha} R_\Lambda^{(n)}(n \cdot)$ converges toward the Laplace transform of the measure with density W' . Taking the Laplace transforms of both sides of (4.3), we have $\mathcal{L}_{R_\Lambda^{(n)}}(\lambda) = \gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda) [1 + \mathcal{L}_{R_\Lambda^{(n)}}(\lambda)]$ and hence $\mathcal{L}_{R_\Lambda^{(n)}}(\lambda) = \gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda) \cdot (1 - \gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda))^{-1}$. By the change of variables,

$$\begin{aligned} \int_0^\infty e^{-\lambda t} n^{1-\alpha} R_\Lambda^{(n)}(nt) dt &= \frac{\mathcal{L}_{R_\Lambda^{(n)}}(\lambda/n)}{n^\alpha} = \frac{\gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda/n)}{n^\alpha (1 - \gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda/n))} \\ &= \frac{\gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda/n)}{n^\alpha (1 - \frac{\gamma_n}{\alpha}) + \frac{\gamma_n}{\alpha} \cdot n^\alpha (1 - \mathcal{L}_{\Lambda^*}(\lambda/n))}. \end{aligned}$$

A simple calculation, along with Condition 4.2, shows that $\gamma_n \mathcal{L}_{\bar{\Lambda}}(\lambda/n) \rightarrow 1$ as $n \rightarrow \infty$. By (4.6) and Karamata's Tauberian theorem; see footnote 8, we have $n^\alpha (1 - \mathcal{L}_{\Lambda^*}(\lambda/n)) \rightarrow \Gamma(1 - \alpha) \lambda^\alpha$ as $n \rightarrow \infty$ and

$$\int_0^\infty e^{-\lambda t} n^{1-\alpha} R_\Lambda^{(n)}(nt) dt \rightarrow \frac{1}{b + c\lambda^\alpha}.$$

By (2.6), the function whose Laplace transform is equal to the last quantity is W' .

4.3.4. Proof for Lemma 4.6. As a direct consequence of the continuity of $\nabla_y W$, the process $M^{(n)}$ is also continuous. Let $p > 2/\alpha$. By the Kolmogorov tightness criterion; see Theorem 13.5 in [11], the sequence $\{M^{(n)}\}_{n \geq 1}$ is C -tight if for any $T > 0$, there exists a constant $C \geq 0$ such that for any $h \in [0, 1]$,

$$(4.20) \quad \sup_{n \geq 1} \sup_{t \in [0, T]} \mathbf{E} [|\Delta_h M^{(n)}(t)|^{2p}] \leq C \cdot h^2.$$

We start to prove (4.20) with the help of the technical results about W in Appendix A. We first split $\Delta_h M^{(n)}(t)$ into the following five parts:

$$\begin{aligned} M_1^{(n)}(t, h) &:= \int_t^{t+h} \int_0^{t+h-s} \int_0^{X_\zeta^{(n)}(s-)} \nabla_y W(t+h-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \\ M_2^{(n)}(t, h) &:= \int_t^{t+h} \int_{t+h-s}^\infty \int_0^{X_\zeta^{(n)}(s-)} W(t+h-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \\ M_3^{(n)}(t, h) &:= \int_0^t \int_0^{t-s} \int_0^{X_\zeta^{(n)}(s-)} \nabla_y \Delta_h W(t-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \\ M_4^{(n)}(t, h) &:= \int_0^t \int_{t-s}^{t+h-s} \int_0^{X_\zeta^{(n)}(s-)} \nabla_y \Delta_h W(t-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \\ M_5^{(n)}(t, h) &:= \int_0^t \int_{t+h-s}^\infty \int_0^{X_\zeta^{(n)}(s-)} \Delta_h W(t-s) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz). \end{aligned}$$

Applying (D.1), Proposition 4.11 and A.1 to $\mathbf{E}[|M_1^{(n)}(t, h)|^{2p}]$,

$$\mathbf{E}[|M_1^{(n)}(t, h)|^{2p}] \leq C \left| \int_t^{t+h} ds \int_0^{t+h-s} \frac{|\nabla_y W(t+h-s)|^2}{y^{\alpha+2}} dy \right|^p$$

$$\begin{aligned}
& + C \int_0^{t+h} ds \int_0^{t+h-s} \frac{|\nabla_y W(t+h-s)|^{2p}}{y^{\alpha+2}} dy \\
& \leq C \left| \int_0^t ds \int_0^\infty \frac{|\nabla_y W(s)|^2}{y^{\alpha+2}} dy \right|^p + C \int_0^h ds \int_0^\infty \frac{|\nabla_y W(s)|^{2p}}{y^{\alpha+2}} dy,
\end{aligned}$$

which can be bounded by $Ch^{p\alpha} \leq Ch^2$ uniformly in $n \geq 1$, $t \in [0, T]$ and $h \in [0, 1]$. With the help of Proposition A.2 and A.3, we also can prove the similar results for other terms. The inequality (4.20) follows by putting them together and the power mean inequality; see footnote 5. \square

4.3.5. Proof for Lemma 4.7. From (4.7), we have $\varepsilon^{(n)} + X_{\zeta,0}^{(n)} = X_{\zeta}^{(n)} - M^{(n)}$. Notice that $X_{\zeta}^{(n)} \rightarrow L_{\zeta}^{\xi}$ weakly in $D([0, \infty); \mathbb{R}_+)$; see Lemma 4.3 and (4.2). By the continuity of L_{ζ}^{ξ} , the sequence $\{X_{\zeta}^{(n)}\}_{n \geq 1}$ is C -tight. Together with this and Lemma 4.6, Corollary 3.33 in [36, p.353] yields the C -tightness of the sequence $\{\varepsilon^{(n)} + X_{\zeta,0}^{(n)}\}_{n \geq 1}$ immediately. Hence it remains to prove $\varepsilon^{(n)} + X_{\zeta,0}^{(n)} \rightarrow X_{\zeta,0}$ in the sense of finite-dimensional distributions, which follows directly from the next two propositions.

PROPOSITION 4.12. *We have $\varepsilon^{(n)} \rightarrow 0$ in the sense of finite-dimensional distributions as $n \rightarrow \infty$.*

PROOF. It suffices to prove $|\varepsilon^{(n)}(t)| \xrightarrow{P} 0$ as $n \rightarrow \infty$ for any $t \geq 0$. By (4.4), we split $\varepsilon^{(n)}(t)$ into the following two parts:

$$\begin{aligned}
\varepsilon_1^{(n)}(t) & := \int_0^t \int_0^\infty \int_0^{X_{\zeta}^{(n)}(s-)} n^{-\alpha} \cdot \mathbf{1}_{\{y>t-s\}} \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz), \\
\varepsilon_2^{(n)}(t) & := \int_0^t \int_0^\infty \int_0^{X_{\zeta}^{(n)}(s-)} \left(\int_{(t-s-y)^+}^{t-s} n^{1-\alpha} R_{\Lambda}^{(n)}(nr) dr - \nabla_y W(t-s) \right) \tilde{N}^{(n)}(n \cdot ds, n \cdot dy, n^\alpha \cdot dz).
\end{aligned}$$

Applying (D.1) and Proposition 4.11 to $\mathbf{E}[|\varepsilon_1^{(n)}(t)|^2]$, we have

$$\sup_{t \geq 0} \mathbf{E}[|\varepsilon_1^{(n)}(t)|^2] \leq C \sup_{t \geq 0} \int_0^t n^{1-\alpha} \bar{\Lambda}(n(t-s)) ds = C n^{-\alpha} \int_0^\infty \bar{\Lambda}(s) ds,$$

which goes to 0 as $n \rightarrow \infty$. Similarly, we also have

$$\sup_{t \in [0, T]} \mathbf{E}[|\varepsilon_2^{(n)}(t)|^2] \leq C \int_0^T ds \int_0^\infty \left| \int_{(s-y)^+}^s n^{1-\alpha} R_{\Lambda}^{(n)}(nr) dr - \nabla_y W(s) \right|^2 n^{\alpha+1} \Lambda(ndy).$$

For $\vartheta \in (\frac{1+\alpha}{2}, \frac{1}{1-\alpha} \wedge 1)$, the preceding integral can be bounded by the multiplication of the next two terms

$$\begin{aligned}
\varepsilon_{21}^{(n)}(T) & := \sup_{x \in [0, T]} \left| \int_0^x n^{1-\alpha} R_{\Lambda}^{(n)}(nr) dr - W(x) \right|^{2(1-\vartheta)}, \\
\varepsilon_{22}^{(n)}(T) & := \int_0^T ds \int_0^\infty \left| \int_{(s-y)^+}^s n^{1-\alpha} R_{\Lambda}^{(n)}(nr) dr - \nabla_y W(s) \right|^{2\vartheta} n^{\alpha+1} \Lambda(n \cdot dy).
\end{aligned}$$

From Lemma 4.4, we have $\varepsilon_{21}^{(n)}(T) \rightarrow 0$ as $n \rightarrow \infty$. It remains to prove that $\varepsilon_{22}^{(n)}(T)$ is bounded. By the power mean inequality and the fact that $n^{\alpha+1} \Lambda(n \cdot dy) \leq \nu_{\alpha}(dy)$,

$$(4.21) \quad \varepsilon_{22}^{(n)}(T) \leq C \int_0^T ds \int_0^\infty \frac{|\nabla_y W(s)|^{2\vartheta}}{y^{\alpha+2}} dy + C \int_0^T ds \int_0^\infty \left| \int_{(s-y)^+}^s n^{1-\alpha} R_{\Lambda}^{(n)}(nr) dr \right|^{2\vartheta} \frac{dy}{y^{\alpha+2}},$$

uniformly in $n \geq 1$. By Proposition A.1, the first integral on the right side of (4.21) can be bounded by $C \cdot T^{\alpha(2\vartheta-1)}$. Plugging (4.18) into the second integral on the right side of (4.21), it can be bounded by

$$C \int_0^T ds \int_s^\infty \frac{s^{2\vartheta\alpha}}{y^{\alpha+2}} dy + C \int_0^T ds \int_0^s (s-y)^{2\vartheta(\alpha-1)} y^{2\vartheta-\alpha-2} dy \leq C \cdot T^{\alpha(2\vartheta-1)}.$$

Hence $\varepsilon_{22}^{(n)}(T) < \infty$ and $\sup_{t \in [0, T]} \mathbf{E}[|\varepsilon_2^{(n)}(t)|^2] \rightarrow 0$ as $n \rightarrow \infty$. The desired result follows by putting these estimates together. \square

PROPOSITION 4.13. *We have $X_{\zeta, 0}^{(n)} \rightarrow X_{\zeta, 0}$ in the sense of finite-dimensional distributions as $n \rightarrow \infty$.*

PROOF. Notice that $X_{\zeta, 0}^{(n)}(0) = [n^\alpha \zeta]/n^\alpha$ is deterministic and converges to $\zeta = X_{\zeta, 0}(0)$ as $n \rightarrow \infty$. For any $T > 0$, $d \in \mathbb{Z}_+$, $0 < t_1 < \dots < t_d \leq T$ and $\lambda_1, \dots, \lambda_d \geq 0$, let

$$Y_d^{(n)}(\lambda, y) := \sum_{i=1}^d \lambda_i \cdot \left(\frac{\mathbf{1}_{\{y > t_i\}}}{n^\alpha} + \int_0^{t_i} n^{1-\alpha} R_\Lambda^{(n)}(n(t_i - s)) \cdot \mathbf{1}_{\{y > s\}} ds \right),$$

$$Y_d(\lambda, y) := \sum_{i=1}^d \lambda_i \int_0^{t_i} W'(t_i - s) \cdot \mathbf{1}_{\{y > s\}} ds = \sum_{i=1}^d \lambda_i \cdot \nabla_y W(t_i).$$

It suffices to prove that $\mathbf{E}[\exp\{-\int_0^\infty Y_d^{(n)}(\lambda, y) N_\zeta^{(n)}(dy)\}] \rightarrow \mathbf{E}[\exp\{-\int_0^\infty Y_d(\lambda, y) N_\zeta(dy)\}]$. By the definition of $N_\zeta^{(n)}$ we have

$$\begin{aligned} \mathbf{E}\left[\exp\left\{-\int_0^\infty Y_d^{(n)}(\lambda, y) N_\zeta^{(n)}(dy)\right\}\right] &= (\mathbf{E}[e^{-Y_d^{(n)}(\lambda, \ell_{0,1})}])^{[n^\alpha \zeta]} \\ &= \left(1 - \frac{1}{n^\alpha} \int_0^\infty (1 - e^{-Y_d^{(n)}(\lambda, y)}) \cdot n^\alpha \cdot \Lambda^*(n \cdot dy)\right)^{[n^\alpha \zeta]}, \end{aligned}$$

and this converges as $n \rightarrow \infty$ to

$$\exp\left\{-\zeta \int_0^\infty (1 - e^{-Y_d(\lambda, y)}) \bar{\nu}_\alpha(y) dy\right\} = \mathbf{E}\left[\exp\left\{-\int_0^\infty Y_d(\lambda, y) N_\zeta(dy)\right\}\right],$$

if and only if

$$(4.22) \quad \int_0^\infty (1 - e^{-Y_d^{(n)}(\lambda, y)}) \cdot n^\alpha \cdot \Lambda^*(n \cdot dy) - \int_0^\infty (1 - e^{-Y_d(\lambda, y)}) \bar{\nu}_\alpha(y) dy \rightarrow 0.$$

For any $\epsilon \in (0, t_1/2)$, we can write the preceding subtraction into the sum of the following four terms

$$\begin{aligned} \varepsilon_3^{(n)} &:= \int_0^\infty (1 - e^{-Y_d(\lambda, y)}) \cdot (n^\alpha \cdot \Lambda^*(n \cdot dy) - \bar{\nu}(y) dy), \\ \varepsilon_4^{(n)} &:= \int_0^\infty (e^{-Y_d(\lambda, y)} - e^{-Y_d^{(n)}(\lambda, y)}) \cdot n^\alpha \cdot \Lambda^*(n \cdot dy), \\ \varepsilon_5^{(n)} &:= \int_0^\epsilon (1 - e^{-Y_d^{(n)}(\lambda, y)}) \cdot n^\alpha \cdot \Lambda^*(n \cdot dy), \\ \varepsilon_6^{(n)} &:= \int_0^\epsilon (1 - e^{-Y_d(\lambda, y)}) \cdot n^\alpha \cdot \Lambda^*(n \cdot dy). \end{aligned}$$

The vague convergence of $n^\alpha \cdot \Lambda^*(n \cdot dy)$ to $\bar{\nu}_\alpha(y) dy$ implies that $|\varepsilon_3^{(n)}| \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 4.4, we have $Y_d^{(n)}(\lambda, y) \rightarrow Y_d(\lambda, y)$ uniformly in $y > 0$, which yields that as $n \rightarrow \infty$,

$$|\varepsilon_4^{(n)}| \leq \sup_{y > 0} |e^{-Y_d(\lambda, y)} - e^{-Y_d^{(n)}(\lambda, y)}| \cdot n^\alpha \bar{\Lambda}^*(n\epsilon) \leq \alpha \cdot \epsilon^{-\alpha} \cdot \sup_{y > 0} |e^{-Y_d(\lambda, y)} - e^{-Y_d^{(n)}(\lambda, y)}| \rightarrow 0.$$

Notice that $Y_d^{(n)}(\lambda, y) = \sum_{i=1}^d \lambda_i \int_0^y n^{1-\alpha} R_\Lambda^{(n)}(n(t_i - s)) ds$ and $Y_d(\lambda, y) = \sum_{i=1}^d \lambda_i \cdot \nabla_y W(t_i)$ for any $y \in (0, \epsilon)$. By the change of variables and (4.18),

$$\sup_{n \geq 1} |1 - e^{-Y_d^{(n)}(\lambda, y)}| \leq \sup_{n \geq 1} |Y_d^{(n)}(\lambda, y)| = \sup_{n \geq 1} \sum_{i=1}^d \lambda_i \int_{t_i - y}^{t_i} n^{1-\alpha} R_\Lambda^{(n)}(ns) ds \leq C \cdot |t_1|^{\alpha-1} \cdot y.$$

Plugging this back into $\varepsilon_5^{(n)}$ and then using the fact that $n^\alpha \cdot \Lambda^*(n \cdot dy) \leq \bar{\nu}(y)dy$,

$$\sup_{n \geq 1} |\varepsilon_5^{(n)}| \leq C \int_0^\varepsilon y \bar{\nu}(y) dy \leq C \int_0^\varepsilon y^{-\alpha} dy \leq C \varepsilon^{1-\alpha},$$

which goes to 0 as $\varepsilon \rightarrow 0+$. Similarly, we also can prove that $\sup_{n \geq 1} |\varepsilon_6^{(n)}| \rightarrow 0$ as $\varepsilon \rightarrow 0+$. The convergence (4.22) follows directly by putting these estimates together. \square

5. Hölder continuity. In this section, we prove the Hölder continuity of L_ζ^ξ ; see Theorem 2.9. For simplicity on exposition, we also denote by $X_{\zeta,0}$ and M the two terms on the right side of (1.3) respectively without ambiguity, i.e., the SVE (1.3) can be written into $L_\zeta^\xi(x) = X_{\zeta,0}(x) + M(x)$ for any $x \geq 0$.

LEMMA 5.1. *For each $p \geq 0$ and $\zeta > 0$, there exists a constant $C > 0$ such that for any $x \geq 0$,*

$$(5.1) \quad \mathbf{E}[|X_{\zeta,0}(x)|^p] \leq C \cdot (1+x)^{\alpha(p-1)^+}.$$

PROOF. By (2.16), we have $X_{\zeta,0}(x) \geq 0$ a.s. and

$$(5.2) \quad X_{\zeta,0}(x) = \zeta(1 - bW(x)) + \int_0^\infty \int_0^\zeta \nabla_y W(x) \tilde{N}_0(dy, dz), \quad x \geq 0.$$

When $p \leq 1$, by Jensen's inequality we have $\mathbf{E}[|X_{\zeta,0}(x)|^p] \leq \mathbf{E}[X_{\zeta,0}(x)]^p \leq \zeta^p$. For $p > 1$, By the power mean inequality and Theorem D.1, there exists a constant $C > 0$ such that for any $x > 0$,

$$(5.3) \quad \mathbf{E}[|X_{\zeta,0}(x)|^p] \leq C\zeta^p + C \left| \int_0^\infty |\nabla_y W(x)|^2 \bar{\nu}_\alpha(y) dy \right|^{p/2} + C \int_0^\infty |\nabla_y W(x)|^p \bar{\nu}_\alpha(y) dy.$$

By (2.8) and (2.15),

$$\int_0^\infty |\nabla_y W(x)|^p \bar{\nu}_\alpha(y) dy \leq \int_0^\infty \nabla_y W(x) \bar{\nu}_\alpha(y) dy \cdot |W(x)|^{p-1} \leq Cx^{(p-1)\alpha}.$$

The desired result follows by plugging this back into (5.3). \square

LEMMA 5.2. *For each $p \geq 1$, there exists a constant $C > 0$ such that for any $x_1, x_2 \geq 0$,*

$$\mathbf{E}[|X_{\zeta,0}(x_2) - X_{\zeta,0}(x_1)|^{2p}] \leq C \cdot (|x_2 - x_1|^{2p\alpha} + |x_2 - x_1|^{p\alpha}).$$

PROOF. By the power mean inequality and (5.2), there exists a constant $C > 0$ such that for any $x_1, x_2 \geq 0$, the expectation $\mathbf{E}[|X_{\zeta,0}(x_2) - X_{\zeta,0}(x_1)|^{2p}]$ can be bounded by

$$C|W(x_2) - W(x_1)|^{2p} + C\mathbf{E}\left[\left|\int_0^\infty \int_0^\zeta (\nabla_y W(x_2) - \nabla_y W(x_1)) \tilde{N}_0(dy, dz)\right|^{2p}\right].$$

By the uniform α -Hölder continuity of W on \mathbb{R}_+ ; see Section 2.1, the first term can be bounded by $C \cdot |x_2 - x_1|^{2p\alpha}$ uniformly in $x_1, x_2 \geq 0$. Applying (D.1) to the second term, it can be bounded by

$$(5.4) \quad C \left| \int_0^\infty |\nabla_y W(x_2) - \nabla_y W(x_1)|^2 \bar{\nu}(y) dy \right|^p + C \int_0^\infty |\nabla_y W(x_2) - \nabla_y W(x_1)|^{2p} \bar{\nu}(y) dy,$$

for some constant $C > 0$ independent of x_1, x_2 . Using the uniform α -Hölder continuity of W on $(0, \infty)$ again, we have $|\nabla_y W(x_2) - \nabla_y W(x_1)| \leq C(|x_2 - x_1|^\alpha \wedge y^\alpha)$ uniformly in $x_1, x_2, y \geq 0$. Plugging this into (5.4), it can be bounded by

$$C \left| \int_0^\infty (|x_2 - x_1|^\alpha \wedge y^\alpha)^2 \bar{\nu}(y) dy \right|^p + C \int_0^\infty (|x_2 - x_1|^\alpha \wedge y^\alpha)^{2p} \bar{\nu}(y) dy \leq C \cdot |x_2 - x_1|^{p\alpha},$$

with $C > 0$ independent of x_1, x_2 . The desired result follows by putting these estimates together. \square

LEMMA 5.3. *For each $p \geq 0$ and $\zeta > 0$, there exists a constant $C > 0$ such that for any $x \geq 0$,*

$$(5.5) \quad \mathbf{E}[|L_\zeta^\xi(x)|^p] \leq C \cdot (1+x)^{p\alpha}.$$

PROOF. Here we just prove this lemma with $p = 2^k$ and $k \in \mathbb{N}$. The general case can be proved in the same way. When $k = 0$, by (2.15) we have $\mathbf{E}[L_\zeta^\xi(x)] \leq \zeta$. For $k \geq 1$, by mathematical induction it suffices to prove (5.5) for $p = 2^k$ under the assumption that it holds for $p = 2^i$ with $i = 0, 1, \dots, k-1$. Applying the Burkholder-Davis-Gundy inequality and the power mean inequality to $\mathbf{E}[|M(x)|^{2^k}]$,

$$(5.6) \quad \begin{aligned} \mathbf{E}[|M(x)|^{2^k}] &\leq C \cdot \mathbf{E}\left[\left|\int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} |\nabla_y W(x-s)|^2 N_\alpha(ds, dy, dz)\right|^{2^{k-1}}\right] \\ &\leq C \cdot \mathbf{E}\left[\left|\int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} |\nabla_y W(x-s)|^2 \tilde{N}_\alpha(ds, dy, dz)\right|^{2^{k-1}}\right] \\ &\quad + C \cdot \mathbf{E}\left[\left|\int_0^x ds \int_0^\infty L_\zeta^\xi(s) |\nabla_y W(x-s)|^2 \nu_\alpha(dy)\right|^{2^{k-1}}\right] \\ &\leq C \cdot \sum_{i=1}^k \mathbf{E}\left[\left|\int_0^x ds \int_0^\infty L_\zeta^\xi(s) |\nabla_y W(x-s)|^{2^i} \nu_\alpha(dy)\right|^{2^{k-i}}\right], \end{aligned}$$

with $C > 0$ depending only on k . Applying Hölder's inequality to each term in the last sum; see footnote 9,

$$\begin{aligned} &\mathbf{E}\left[\left|\int_0^x ds \int_0^\infty L_\zeta^\xi(s) |\nabla_y W(x-s)|^{2^i} \nu_\alpha(dy)\right|^{2^{k-i}}\right] \\ &\leq \int_0^x ds \int_0^\infty \mathbf{E}[|L_\zeta^\xi(s)|^{2^{k-i}}] |\nabla_y W(x-s)|^{2^i} \nu_\alpha(dy) \\ &\quad \times \left|\int_0^x ds \int_0^\infty |\nabla_z W(x-s)|^{2^i} \nu_\alpha(dz)\right|^{2^{k-i}-1} \\ &\leq \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|^{2^{k-i}}] \cdot \left|\int_0^x ds \int_0^\infty |\nabla_y W(s)|^{2^i} \nu_\alpha(dy)\right|^{2^{k-i}} \\ &\leq C \cdot (1+x)^{2^{k-i}\alpha} \cdot \left|\int_0^x ds \int_0^\infty |\nabla_y W(s)|^{2^i} \nu_\alpha(dy)\right|^{2^{k-i}}. \end{aligned}$$

By Proposition A.1 with $p = 2^i$, the right hand of the last inequality can be bounded by $C \cdot (1+x)^{p\alpha}$ uniformly in $x \geq 0$. Taking these estimates back into (5.6), we have $\mathbf{E}[|M(x)|^p] \leq C \cdot (1+x)^{p\alpha}$ uniformly in $x \geq 0$. The desired result follows from this, (5.1) and the power mean inequality. \square

LEMMA 5.4. *For each $p \geq 1$, there exists a constant $C > 0$ such that for any $x \geq 0$ and $x_1, x_2 \in [0, x]$,*

$$\mathbf{E}[|M(x_1) - M(x_2)|^{2p}] \leq C \cdot (1+x)^{p\alpha} \cdot |x_2 - x_1|^{p\alpha}.$$

PROOF. Without loss of generality, we assume $0 \leq x_1 < x_2 \leq x$. Similarly as in the proof of Lemma 4.6, by the power mean inequality we have $\mathbf{E}[|M(x_1) - M(x_2)|^{2p}] \leq 5^{2p} \cdot \sum_{i=1}^5 \mathbf{E}[|M_i(x_1, x_2)|^{2p}]$ with

$$\begin{aligned} M_1(x_1, x_2) &:= \int_{x_1}^{x_2} \int_0^{x_2-s} \int_0^{L_\zeta^\xi(s)} \nabla_y W(x_2-s) \tilde{N}_\alpha(ds, dy, dz), \\ M_2(x_1, x_2) &:= \int_{x_1}^{x_2} \int_{x_2-s}^\infty \int_0^{L_\zeta^\xi(s)} W(x_2-s) \tilde{N}_\alpha(ds, dy, dz), \\ M_3(x_1, x_2) &:= \int_0^{x_1} \int_{x_1-s}^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y \Delta_{x_2-x_1} W(x_1-s) \tilde{N}_\alpha(ds, dy, dz), \\ M_4(x_1, x_2) &:= \int_0^{x_1} \int_{x_1-s}^{x_2-s} \int_0^{L_\zeta^\xi(s)} \nabla_y \Delta_{x_2-x_1} W(x_1-s) \tilde{N}_\alpha(ds, dy, dz), \end{aligned}$$

$$M_5(x_1, x_2) := \int_0^{x_1} \int_{x_2-s}^{\infty} \int_0^{L_\zeta^\xi(s)} \Delta_{x_2-x_1} W(x_1-s) \tilde{N}_\alpha(ds, dy, dz).$$

Applying (D.1) to $\mathbf{E}[|M_1(x_1, x_2)|^{2p}]$ and then using the change of variables, there exists a constant $C > 0$ depending only on p such that

$$\begin{aligned} \mathbf{E}[|M_1(x_1, x_2)|^{2p}] &\leq C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|^p] \cdot \left| \int_{x_1}^{x_2} \int_0^{x_2-s} |\nabla_y W(x_2-s)|^2 \nu_\alpha(dy) ds \right|^p \\ &\quad + C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|] \cdot \int_{x_1}^{x_2} \int_0^{x_2-s} |\nabla_y W(x_2-s)|^{2p} \nu_\alpha(dy) ds \\ &= C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|^p] \cdot \left| \int_0^{x_2-x_1} ds \int_0^s |\nabla_y W(s)|^2 \nu_\alpha(dy) \right|^p \\ &\quad + C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|] \cdot \int_0^{x_2-x_1} ds \int_0^s |\nabla_y W(s)|^{2p} \nu_\alpha(dy). \end{aligned}$$

By Proposition A.1 and Lemma 5.3, the foregoing quantities can be bounded by

$$C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|^p] \cdot |x_2 - x_1|^{p\alpha} + C \sup_{t \in [0, x]} \mathbf{E}[|L_\zeta^\xi(t)|] \cdot |x_2 - x_1|^{\alpha(2p-1)} \leq C \cdot (1+x)^{p\alpha} \cdot |x_2 - x_1|^{p\alpha},$$

uniformly in $x \geq 0$ and $x_1, x_2 \in [0, x]$. Similarly, for $i \in \{2, 3, 4, 5\}$, by Proposition A.2 and A.3 we also have $\mathbf{E}[M_i(x_1, x_2)|^{2p}] \leq C \cdot (1+x)^{p\alpha} \cdot |x_2 - x_1|^{p\alpha}$ uniformly in $x \geq 0$ and $x_1, x_2 \in [0, x]$. The desired inequality follows by putting these estimates together. \square

PROOF FOR THEOREM 2.9. By the Kolmogorov continuity theorem along with Lemma 5.2 and 5.4, the two processes $X_{\zeta,0}$ and M are locally Hölder continuous of any order strictly less than $\alpha/2$. Then claim (1) holds. For the second claim, it suffices to consider the case of $p > 1$. By the Garsia-Rodemich-Rumsey inequality; see Lemma 1.1 in [28] with $\psi(u) = |u|^p$ and $p(u) = |u|^{q+1/p}$ for $q > 1/p$, there exists a constant $C_{p,q} > 0$ such that for any $x_2 > x_1 \geq 0$,

$$|M(x_2) - M(x_1)|^p \leq C_{p,q} \cdot |x_2 - x_1|^{pq-1} \int_{x_1}^{x_2} dv \int_{x_1}^{x_2} \frac{|M(u) - M(v)|^p}{|u-v|^{pq+1}} du, \quad a.s.$$

In particular, choosing $p > (\alpha/2 - \kappa)^{-1}$ and $q = 1/p + \kappa$ we have

$$\|M\|_{C_x^{0,\kappa}}^p = \sup_{0 \leq x_1 < x_2 \leq x} \frac{|M(x_2) - M(x_1)|^p}{|x_2 - x_1|^{p\kappa}} \leq C_{p,q} \int_0^x dv \int_0^x \frac{|M(u) - M(v)|^p}{|u-v|^{p\kappa+2}} du, \quad a.s.$$

From Lemma 5.4, there exists a constant $C > 0$ such that for any $x \geq 0$,

$$\mathbf{E}[\|M\|_{C_x^{0,\kappa}}^p] \leq C(1+x)^{p\alpha/2} \int_0^x dv \int_0^x |u-v|^{p\alpha/2-p\kappa-2} du \leq C(1+x)^{p(\alpha-\kappa)}.$$

Similarly, by Lemma 5.2 we also have $\mathbf{E}[\|X_{\zeta,0}\|_{C_x^{0,\kappa}}^p] \leq C(1+x)^{p(\alpha-\kappa)}$ uniformly in $x \geq 0$. Then claim (2) follows from these two upper bound estimates. \square

6. Laplace functionals and weak uniqueness. In this section, we firstly prove the affine representation of the Laplace functionals of L_ζ^ξ in two steps: (i) on some finite interval, local solutions of (1.4) exist uniquely and (2.18) holds; (ii) this finite interval can be extended successfully to the whole positive real line. Then we prove the weak uniqueness holds for the SVE (1.3). As a preparation, we introduce several function spaces that will be used in the following proofs. For $T, J > 0$,

- $\mathcal{A}_{T,J}$: the space of functions f on $(0, \infty)$ satisfying $\sup_{t \in (0, T]} t^{1-\alpha} |f(t)| \leq J$;

- $\mathcal{B}_{T,J}$: the space of functions f on \mathbb{R}_+ satisfying $\|f\|_{L_T^\infty} \leq J$.

Notice that $\mathcal{A}_{T,J_1} \subset \mathcal{A}_{T,J_2}$ for any $J_2 \geq J_1 > 0$. Let $\mathcal{A}_T := \cup_{J>0} \mathcal{A}_{T,J}$, which collects all functions on $(0, \infty)$ satisfying that $\sup_{x \in (0, T]} x^{1-\alpha} |f(x)| < \infty$. It is obvious that \mathcal{A}_T is decreasing in T and $\mathcal{A}_\infty = \cap_{T>0} \mathcal{A}_T$.

6.1. Nonlinear Volterra equation. We say a pair $(v_\lambda^g, T) \in \mathcal{A}_T \times (0, \infty)$ is a \mathcal{A} -local solution of (1.4) if v_λ^g is continuous and satisfies (1.4) on $(0, T]$. Moreover, for a pair $(v_\lambda^g, T_\lambda^g)$ with $T_\lambda^g \in (0, \infty]$ and a function v_λ^g on $[0, T_\lambda^g)$, we say it is a \mathcal{A} -noncontinuable solution¹¹ of (1.4) if for any $T \in (0, T_\lambda^g)$, the pair (v_λ^g, T) is a \mathcal{A} -local solution of (1.4) and $\limsup_{x \rightarrow T_\lambda^g-} |v_\lambda^g(x)| = \infty$ if $T_\lambda^g < \infty$. In particular, if $T_\lambda^g = \infty$ the function v_λ^g turns to be a \mathcal{A} -global solution of (1.4). In this section, we prove the existence and uniqueness of \mathcal{A} -noncontinuable solutions of (1.4) with the help of the following technical estimates for the nonlinear operator \mathcal{V}_α . The next useful inequality can be proved immediately by using the mean-value theorem

$$(6.1) \quad |(e^{-x} - 1 + x) - (e^{-z} - 1 + z)| \leq (e^{|x| \vee |z|}) \cdot (|x| \vee |z|) \cdot |x - z|, \quad x, z \in \mathbb{R}.$$

PROPOSITION 6.1. *There exists a constant $C > 0$ such that for any $T, J > 0$, $f \in \mathcal{A}_{T,J}$ and $t \in (0, T]$,*

$$(6.2) \quad \sup_{y \geq 0} \left| \int_{(t-y)^+}^t f(r) dr \right| \leq \frac{J}{\alpha} \cdot t^\alpha \quad \text{and} \quad \int_0^\infty \left| \int_{(t-y)^+}^t f(r) dr \right|^2 \nu_\alpha(dy) \leq C \cdot J^2 \cdot t^{\alpha-1}.$$

PROOF. The first desired inequality follows directly from a simple calculation. For the second one, we have

$$(6.3) \quad \int_0^\infty \left| \int_{(t-y)^+}^t f(r) dr \right|^2 \nu_\alpha(dy) \leq \|f\|_{L_t^1}^2 \cdot \bar{\nu}_\alpha(t/2) + \int_0^{t/2} \|f\|_{L_{[t-y, t]}^1}^2 \nu_\alpha(dy).$$

It is easy to see that $\|f\|_{L_t^1}^2 \cdot \bar{\nu}_\alpha(t/2) \leq C \cdot J^2 \cdot t^{\alpha-1}$ uniformly in $t \in (0, T]$ and $T > 0$. Since $\|f\|_{L_{[t-y, t]}^1} \leq J \cdot (t-y)^{\alpha-1} \cdot y \leq 2J \cdot t^{\alpha-1} \cdot y$ for any $0 < y < t/2 < t \leq T$, we have

$$(6.4) \quad \int_0^{t/2} \|f\|_{L_{[t-y, t]}^1}^2 \nu_\alpha(dy) \leq 4J^2 t^{2\alpha-2} \int_0^{t/2} y^2 \nu_\alpha(dy) \leq C \cdot J^2 \cdot t^{\alpha-1},$$

with $C > 0$ depending only on α and c . The second desired inequality follows. \square

PROPOSITION 6.2. *There exists a constant $C > 0$ such that for any $T, J > 0$, $f \in \mathcal{A}_{T,J}$ and $t \in (0, T]$,*

$$|\mathcal{V}_\alpha \circ f(t)| \leq C \cdot J^2 e^{\frac{J}{\alpha} t^\alpha} \cdot t^{\alpha-1} \quad \text{and} \quad |(\mathcal{V}_\alpha \circ f) * W'(t)| \leq C \cdot J^2 e^{\frac{J}{\alpha} t^\alpha} \cdot t^{2\alpha-1}.$$

PROOF. By the first inequality in (6.2) and (6.1) with $z = 0$,

$$\left| \exp \left\{ - \int_{(t-y)^+}^t f(r) dr \right\} - 1 + \int_{(t-y)^+}^t f(r) dr \right| \leq e^{\frac{J}{\alpha} t^\alpha} \cdot \left| \int_{(t-y)^+}^t f(r) dr \right|^2.$$

Plugging this back into (1.5) and then using the second inequality in (6.2), we have

$$|\mathcal{V}_\alpha \circ f(t)| \leq e^{\frac{J}{\alpha} t^\alpha} \int_0^\infty \left| \int_{(t-y)^+}^t f(r) dr \right|^2 \nu_\alpha(dy) \leq C \cdot J^2 e^{\frac{J}{\alpha} t^\alpha} \cdot t^{\alpha-1}.$$

From this and (2.8), we have uniformly in $T, J \geq 0$, $f \in \mathcal{A}_{T,J}$ and $t \in [0, T]$,

$$|(\mathcal{V}_\alpha \circ f) * W'(t)| \leq C \cdot J^2 \cdot e^{\frac{J}{\alpha} t^\alpha} \cdot \int_0^t s^{\alpha-1} (t-s)^{\alpha-1} ds \leq C \cdot J^2 \cdot e^{\frac{J}{\alpha} t^\alpha} \cdot t^{2\alpha-1}.$$

\square

¹¹The terminology ‘‘noncontinuable solution’’ comes from the theory of Volterra equations; see Chapter 12 in [30].

PROPOSITION 6.3. For $\theta \in (1, \frac{1}{1-\alpha})$, there exists a constant $C > 0$ such that for any $T, J > 0$ and $f_1, f_2 \in \mathcal{A}_{T,J}$,

$$\|(\mathcal{V}_\alpha \circ f_1 - \mathcal{V}_\alpha \circ f_2) * W'\|_{L_T^\theta} \leq C \cdot JT^\alpha e^{\frac{J}{\alpha} T^\alpha} \cdot \|f_1 - f_2\|_{L_T^\theta}.$$

PROOF. Let $\bar{f} := f_1 - f_2$. By (1.5), (6.1) and the first inequality in (6.2),

$$(6.5) \quad |\mathcal{V}_\alpha \circ f_1(t) - \mathcal{V}_\alpha \circ f_2(t)| \leq e^{\frac{J}{\alpha} t^\alpha} \int_0^\infty (\|f_1\|_{L_{[(t-y)^+, t]}^1} \vee \|f_2\|_{L_{[(t-y)^+, t]}^1}) \|\bar{f}\|_{L_{[(t-y)^+, t]}^1} \nu_\alpha(dy)$$

and hence $|(\mathcal{V}_\alpha \circ f_1 - \mathcal{V}_\alpha \circ f_2) * W'(t)| \leq e^{\frac{J}{\alpha} t^\alpha} (I_1(t) + I_2(t))$ for any $t \in [0, T]$, where

$$\begin{aligned} I_1(t) &:= \int_0^t W'(t-s) ds \int_{s/2}^\infty (\|f_1\|_{L_{[(s-y)^+, s]}^1} \vee \|f_2\|_{L_{[(s-y)^+, s]}^1}) \|\bar{f}\|_{L_{[(s-y)^+, s]}^1} \nu_\alpha(dy), \\ I_2(t) &:= \int_0^t W'(t-s) ds \int_0^{s/2} (\|f_1\|_{L_{[s-y, s]}^1} \vee \|f_2\|_{L_{[s-y, s]}^1}) \|\bar{f}\|_{L_{[s-y, s]}^1} \nu_\alpha(dy). \end{aligned}$$

By Minkowski's inequality, we have $\|(\mathcal{V}_\alpha \circ f_1 - \mathcal{V}_\alpha \circ f_2) * W'\|_{L_T^\theta} \leq e^{\frac{J}{\alpha} T^\alpha} (\|I_1\|_{L_T^\theta} + \|I_2\|_{L_T^\theta})$. Notice that

$$I_1(t) \leq \int_0^t W'(t-s) \cdot (\|f_1\|_{L_s^1} \vee \|f_2\|_{L_s^1}) \cdot \|\bar{f}\|_{L_s^1} \cdot \bar{\nu}_\alpha(s/2) ds.$$

By Hölder's inequality, we have $\|\bar{f}\|_{L_s^1} \leq \|\bar{f}\|_{L_s^\theta} \cdot s^{1-1/\theta}$. By (2.8) and the first inequality in (6.2), there exists a constant $C > 0$ such that for any $T, J > 0$ and $t \in (0, T]$,

$$I_1(t) \leq C \cdot J \cdot \int_0^t (t-s)^{\alpha-1} s^{-1/\theta} \|\bar{f}\|_{L_s^\theta} ds \leq C \cdot J \cdot t^{\alpha-1/\theta} \cdot \|\bar{f}\|_{L_t^\theta} \quad \text{and} \quad \|I_1\|_{L_T^\theta} \leq C \cdot J \cdot T^\alpha \cdot \|\bar{f}\|_{L_T^\theta}.$$

Noting that $\|f_1\|_{L_{[s-y, s]}^1} \vee \|f_2\|_{L_{[s-y, s]}^1} \leq J \cdot (s-y)^{\alpha-1} \cdot y$ for any $y \in (0, s/2)$, we have

$$\begin{aligned} \int_0^{s/2} (\|f_1\|_{L_{[s-y, s]}^1} \vee \|f_2\|_{L_{[s-y, s]}^1}) \|\bar{f}\|_{L_{[s-y, s]}^1} \nu_\alpha(dy) &\leq C \cdot J \int_0^{s/2} (s-y)^{\alpha-1} y^{-\alpha-1} \|\bar{f}\|_{L_{[s-y, s]}^1} dy \\ &= C \cdot J \int_0^{s/2} (s-y)^{\alpha-1} y^{-\alpha-1} \int_0^y |\bar{f}(s-r)| dr dy \\ &= C \cdot J \int_0^{s/2} |\bar{f}(s-y)| dy \int_y^{s/2} (s-r)^{\alpha-1} r^{-\alpha-1} dr \\ &\leq C \cdot J \cdot s^{\alpha-1} \int_0^{s/2} y^{-\alpha} |\bar{f}(s-y)| dy \\ &\leq C \cdot J \cdot s^{\alpha-1} \int_0^s (s-y)^{-\alpha} |\bar{f}(y)| dy, \end{aligned}$$

for some constant C independent of s, J, T and \bar{f} . Here the two equalities follow from the change of variables and Fubini's theorem respectively. Plugging this into $I_2(t)$ and then using (2.8),

$$\begin{aligned} I_2(t) &\leq C \cdot J \cdot \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \int_0^s (s-y)^{-\alpha} |\bar{f}(y)| dy \\ &= C \cdot J \cdot \int_0^t |\bar{f}(s)| ds \int_0^{t-s} (t-s-y)^{\alpha-1} (y+s)^{\alpha-1} y^{-\alpha} dy. \end{aligned}$$

This equality comes from Fubini's theorem and the change of variables. Let $\eta \in (0 \vee \frac{1-\theta\alpha}{\theta-\theta\alpha}, 1)$. Since $(y+s)^{\alpha-1} \leq s^{(1-\eta)(\alpha-1)} \cdot y^{\eta(\alpha-1)}$ for any $y, s > 0$, we have

$$\begin{aligned} I_2(t) &\leq C \cdot J \cdot \int_0^t s^{(1-\eta)(\alpha-1)} |\bar{f}(s)| ds \int_0^{t-s} (t-s-y)^{\alpha-1} y^{\eta(\alpha-1)-\alpha} dy \\ &\leq C \cdot J \cdot \int_0^t (t-s)^{\eta(\alpha-1)} s^{(1-\eta)(\alpha-1)} |\bar{f}(s)| ds. \end{aligned}$$

By Young's convolution inequality and then Hölder's inequality,

$$\begin{aligned} \|I_2\|_{L_T^\theta} &\leq C \cdot J \cdot \left(\int_0^T t^{\theta\eta(\alpha-1)} dt \right)^{1/\theta} \cdot \int_0^T s^{(1-\eta)(\alpha-1)} |\bar{f}(s)| ds \\ &\leq C \cdot J \cdot T^{\eta(\alpha-1)+1/\theta} \cdot \left(\int_0^T s^{(1-\eta)(\alpha-1)\frac{\theta}{\theta-1}} ds \right)^{\frac{\theta-1}{\theta}} \|\bar{f}\|_{L_T^\theta} \leq C \cdot J \cdot T^\alpha \cdot \|\bar{f}\|_{L_T^\theta} \end{aligned}$$

with $C > 0$ independent of T, J and \bar{f} . The desired result follows by putting these estimates together. \square

The next two propositions can be proved similarly and their detailed proofs are omitted.

PROPOSITION 6.4. *There exists a constant $C > 0$ such that for any $T, J > 0$, $f \in \mathcal{B}_{T,J}$ and $t \in (0, T]$,*

$$|\mathcal{V}_\alpha \circ f(t)| \leq C \cdot J^2 e^{Jt} \cdot t^{1-\alpha} \quad \text{and} \quad |(\mathcal{V}_\alpha \circ f) * W'(t)| \leq C \cdot J^2 e^{Jt} \cdot t.$$

PROPOSITION 6.5. *There exists a constant $C > 0$ such that for any $T, J > 0$ and $f_1, f_2 \in \mathcal{B}_{T,J}$,*

$$\|(\mathcal{V}_\alpha \circ f_1 - \mathcal{V}_\alpha \circ f_2) * W'\|_{L_T^1} \leq C \cdot JT e^{JT} \cdot \|f_1 - f_2\|_{L_T^1}.$$

LEMMA 6.6. *For each $\lambda \geq 0$ and $g \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, the nonlinear Volterre equation (1.4) has a unique \mathcal{A} -noncontinuable solution.*

PROOF. By using Banach's fixed point theorem, we prove this lemma in the following three steps.

Step 1. We first prove the existence of \mathcal{A} -local solutions near 0. We consider the map \mathcal{R}_0 that acts on a locally integrable function f on \mathbb{R}_+ according to $\mathcal{R}_0 \circ f := \lambda W' + (g - \mathcal{V}_\alpha \circ f) * W'$. Recall the constant $\theta \in (1, 1/(1-\alpha))$. By (2.8), Proposition 6.2 and 6.3, there exists a constant $C_0 > 0$ such that for any $T, J > 0$, $f_1, f_2 \in \mathcal{A}_{T,J}$ and $t \in (0, T]$,

$$|\mathcal{R}_0 \circ f_1(t)| \leq C_0 (|\lambda| + \|g\|_{L^\infty} + J^2 \cdot T^\alpha e^{\frac{J}{\alpha} T^\alpha}) \cdot t^{\alpha-1}$$

and

$$\|\mathcal{R}_0 \circ f_1 - \mathcal{R}_0 \circ f_2\|_{L_T^\theta} = \|(\mathcal{V}_\alpha \circ f_1 - \mathcal{V}_\alpha \circ f_2) * W'\|_{L_T^\theta} \leq C_0 J \cdot T^\alpha e^{\frac{J}{\alpha} T^\alpha} \|f_1 - f_2\|_{L_T^\theta}.$$

Choosing $J_0 > 2C(|\lambda| + \|g\|_{L^\infty})$ and $T_0 \in (0, 1)$ such that $T_0^\alpha \cdot C_0 \cdot J_0 e^{J_0/\alpha} < 1/2$, we have for any $f_1, f_2 \in \mathcal{A}_{T_0, K_0}$ and $t \in (0, T_0]$,

$$|\mathcal{R}_0 \circ f_1(t)| \leq J_0 \cdot t^{\alpha-1} \quad \text{and} \quad \|\mathcal{R}_0 \circ f_1 - \mathcal{R}_0 \circ f_2\|_{L_{T_0}^\theta} < \|f_1 - f_2\|_{L_{T_0}^\theta}.$$

Thus \mathcal{R}_0 is a contractive map from \mathcal{A}_{T_0, K_0} to itself. It can be easily identify that \mathcal{A}_{T_0, J_0} is a closed, bounded and convex subset in $L^\theta((0, T_0]; \mathbb{R})$. By Banach's fixed point theorem, there exists a unique function $v_0 \in \mathcal{A}_{T_0, J_0}$ satisfying (1.4), i.e., $v_0 = \mathcal{R}_0 \circ v_0$ almost everywhere on $(0, T_0]$. By the properties of convolution, the function $v_\lambda^g := \mathcal{R}_0 \circ v_0$ is continuous and equal to v_0 almost everywhere. Hence $\mathcal{R}_0 \circ v_\lambda^g = \mathcal{R}_0 \circ v_0 = v_\lambda^g$ pointwisely on $(0, T_0]$ and (v_λ^g, T_0) is a \mathcal{A} -local solution of (1.4).

Step 2. We now extend the preceding \mathcal{A} -local solution into a \mathcal{A} -noncontinuable solution. Denote by \mathcal{T} the collection of all $T > 0$ such that (1.4) has a \mathcal{A} -local solution on $(0, T]$. We assert that \mathcal{T} is an open interval containing $(0, T_0]$. Indeed, for any $t_0 \in \mathcal{T}$ and some $j_0 > 0$, assume that $v_\lambda^g \in \mathcal{A}_{t_0, j_0}$ is a \mathcal{A} -local solution of (1.4). For $t \geq 0$, let

$$(6.6) \quad H_1(t) := \lambda W'(t_0 + t) + W' * g(t_0 + t) - \int_0^{t_0} \mathcal{V}_\alpha \circ v_\lambda^g(s) W'(t_0 + t - s) ds.$$

By (2.8), we have $|\lambda W'(t_0 + t)| \leq C \cdot t_0^{\alpha-1}$ and $|W' * g(t_0 + t)| \leq C \cdot \|g\|_{L^\infty} \cdot (t_0 + t)^\alpha$ uniformly in $t \geq 0$. From Proposition 6.2, we also have $|\mathcal{V}_\alpha \circ v_\lambda^g(s)| \leq C \cdot s^{\alpha-1}$ uniformly in $s \in (0, t_0]$ and hence $\int_0^{t_0} \mathcal{V}_\alpha \circ v_\lambda^g(s) W'(t_0 + t - s) ds$

$t - s)ds \leq C$ uniformly in $t \geq 0$. Putting these estimates together, there exists a constant $C_{H_1} > 0$ such that $|H_1(t)| \leq C_{H_1}$ for any $t \in [0, 1]$. We consider the map \mathcal{R}_1 acting on functions $f \in L^\infty(\mathbb{R}_+; \mathbb{R})$ by $\mathcal{R}_1 \circ f := H_1 - W' * (\mathcal{V}_\alpha \circ f)$. Recall $\mathcal{B}_{T,J}$ for $T, J > 0$. From Proposition 6.4 and 6.5, there exists a constant $C_1 > 0$ such that for any $T \in [0, 1]$, $J > 0$, $f_1, f_2 \in \mathcal{B}_{T,J}$ and $t \in [0, T]$,

$$|\mathcal{R}_1 \circ f_1(t)| \leq C_1(C_{H_1} + J^2 e^{J \cdot T} \cdot T) \quad \text{and} \quad \|\mathcal{R}_1 \circ f_1 - \mathcal{R}_1 \circ f_2\|_{L^1_T} \leq C_1 \cdot JT \cdot e^{2JT} \cdot \|f_1 - f_2\|_{L^1_T}.$$

Choosing $J_1 > 2C_1 \cdot C_{H_1}$ and $T_1 \in [0, 1]$ such that $T_1 \cdot C_1 J_1 e^{2J_1} < 1/2$, we have for any $f_1, f_2 \in \mathcal{B}_{T_1, J_1}$,

$$\sup_{t \in [0, T_1]} |\mathcal{R}_1 \circ f_1(t)| \leq J_1 \quad \text{and} \quad \|\mathcal{R}_1 \circ f_1 - \mathcal{R}_1 \circ f_2\|_{L^1_{T_1}} < \|f_1 - f_2\|_{L^1_{T_1}}.$$

Thus \mathcal{R}_1 is a contractive map from \mathcal{B}_{T_1, J_1} to itself. Notice that \mathcal{B}_{T_1, J_1} is a closed, bounded and convex subset of $L^1([0, T_1]; \mathbb{R})$. Applying Banach's fixed point theorem again, there exists a unique function $v_1 \in \mathcal{B}_{T_1, J_1}$ satisfying that $v_1 = \mathcal{R}_1 \circ v_1$ almost everywhere. For $t \in [0, T_1]$, let $v_\lambda^g(t_0 + t) := \mathcal{R}_1 \circ v_1(t)$. One can verify that $(v_\lambda^g, t_0 + T_1)$ is a \mathcal{A} -local solution of (1.4) and hence t_0 is an interior point of \mathcal{T} .

Let $T_\lambda^g := \sup \mathcal{T}$ and v_λ^g be a continuous function on $(0, T_\lambda^g)$ satisfying that (v_λ^g, T) is a \mathcal{A} -local solution of (1.4) for any $T \in (0, T_\lambda^g)$. To assert that $(v_\lambda^g, T_\lambda^g)$ is a \mathcal{A} -noncontinuable solution, it remains to identify that $\limsup_{t \rightarrow T_\lambda^g -} |v_\lambda^g(t)| = \infty$ if $T_\lambda^g < \infty$. Actually, if $\sup_{t \in [t_0, T_\lambda^g -)} |v_\lambda^g(t)| \leq J_2$ for some constant $J_2 > 0$, then $\sup_{t \in (0, T_\lambda^g)} t^{1-\alpha} |v_\lambda^g(t)| < \infty$. Let H_2 be a function on \mathbb{R}_+ defined as in (6.6) with t_0 replaced by T_λ^g . Let \mathcal{R}_2 be a map acting on functions $f \in L^\infty(\mathbb{R}_+; \mathbb{R})$ by $\mathcal{R}_2 \circ f := H_2 + W' * (\mathcal{V}_\alpha \circ f)$. Similarly as in the previous paragraph, there exist constants $C_{H_2} > 0$, $T_2 \in [0, 1]$ and $J_2 > 0$ such that $|H_2(t)| \leq C_{H_2}$ for any $t \in [0, 1]$ and \mathcal{R}_2 is a contractive map from \mathcal{B}_{T_2, J_2} to itself. By Banach's fixed point theorem again, there exists a unique function $v_2 \in \mathcal{B}_{T_2, J_2}$ satisfying that $v_2 = \mathcal{R}_2 \circ v_2$ almost everywhere. For $t \in [0, T_2]$, let $v_\lambda^g(T_\lambda^g + t) := \mathcal{R}_2 \circ v_2(t)$. Then $(v_\lambda^g, T_\lambda^g + T_2)$ is a \mathcal{A} -local solution of (1.4) and $T_\lambda^g + T_2 \in \mathcal{T}$, which contradicts the assumption that $T_\lambda^g = \sup \mathcal{T}$. Consequently, $(v_\lambda^g, T_\lambda^g)$ is a \mathcal{A} -noncontinuable solution of (1.4).

Step 3. We prove the uniqueness. Assume that $(v_\lambda^g, T_\lambda^g)$ and $(\hat{v}_\lambda^g, \hat{T}_\lambda^g)$ are two \mathcal{A} -noncontinuable solutions of (1.4) with $T_\lambda^g \leq \hat{T}_\lambda^g$. Similarly as in Step 1, there exist two constants $T_0 \in (0, 1)$ and $J_0 > 0$ such that $v_\lambda^g, \hat{v}_\lambda^g \in \mathcal{A}_{T_0, J_0}$ and the map \mathcal{R}_0 is contractive from \mathcal{A}_{T_0, J_0} to itself. Then Banach's fixed point theorem induces that $\|v_\lambda^g - \hat{v}_\lambda^g\|_{L^1_{T_0}} = 0$. Their continuity yields that $v_\lambda^g = \hat{v}_\lambda^g$ on $(0, T_0]$. Similarly as in Step 2, let $t_0 := \inf\{t > 0 : v_\lambda^g(t) \neq \hat{v}_\lambda^g(t)\}$, H_1 be the function defined by (6.6), $v_{\lambda,1}^g(t) = v_\lambda^g(t_0 + t)$ and $\hat{v}_{\lambda,1}^g(t) = \hat{v}_\lambda^g(t_0 + t)$ for $t \in [0, T_\lambda^g - t_0)$. We also can find some constants $T_1 \in (0, 1) \cap [0, T_\lambda^g - t_0)$ and $J_1 > 0$ such that $v_{\lambda,1}^g, \hat{v}_{\lambda,1}^g \in \mathcal{B}_{T_1, J_1}$ and the map \mathcal{R}_1 is contractive from \mathcal{B}_{T_1, J_1} to itself. Again, Banach's fixed point theorem induces that $\|v_{\lambda,1}^g - \hat{v}_{\lambda,1}^g\|_{L^1_{T_1}} = 0$. Their continuity yields that $v_\lambda^g = \hat{v}_\lambda^g$ on $(0, t_0 + T_1]$, which contradicts the definition of t_0 . Hence the uniqueness holds. \square

6.2. Laplace functionals. For convenience, we assume that the process L_ζ^ξ , the PRM $\tilde{N}_\alpha(ds, dy, dz)$ are defined on a filtrated probability basis $(\Omega, \mathcal{G}, \mathcal{G}_r, \mathbf{P})$ satisfying the general hypotheses and $N_0(dy, dz)$ is \mathcal{G}_0 -measurable¹². For each $\lambda \geq 0$ and $g \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, we assume that (v_λ^g, T) is a \mathcal{A} -local solution of (1.4), i.e., $T > 0$ and $v_\lambda^g \in \mathcal{A}_{T,J}$ for some $J > 0$. For $x, r \geq 0$, conditioned on \mathcal{G}_r we take expectations on both sides of (1.3) and get

$$(6.7) \quad \mathbf{E}[L_\zeta^\xi(x) | \mathcal{G}_r] = \int_0^\infty \int_0^\zeta \nabla_y W(x) N_0(dy, dz) + \int_0^{x \wedge r} \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz).$$

PROPOSITION 6.7. *For any $x \in [0, T]$, the following hold:*

¹²Otherwise, by the definition of weak solutions of (1.3); see footnote 1, a realization of L_L^ξ , $N_0(dy, dz)$ and $\tilde{N}_\alpha(ds, dy, dz)$ can be found on some filtrated probability basis such that (1.3) holds.

- (1) The random variable $Y_x(x) := \lambda L_\zeta^\xi(x) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * L_\zeta^\xi(x)$ is integrable, i.e., $\mathbf{E}[|Y_x(x)|] < \infty$.
- (2) The Doob martingale $\{Y_x(t) := \mathbf{E}[Y_x(x)|\mathcal{G}_t] : t \in [0, x]\}$ has the representation

$$Y_x(t) = \int_0^\infty \int_0^\zeta \int_{(x-y)^+}^x v_\lambda^g(s) ds N_0(dy, dz) + \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \tilde{N}_\alpha(ds, dy, dz).$$

PROOF. By Proposition 6.2, we have $\|\mathcal{V}_\alpha \circ v_\lambda^g\|_{L_x^1} \leq C \cdot x^\alpha$. Moreover, by (2.15) we have $\mathbf{E}[|Y_x(x)|] \leq \zeta \cdot (\lambda + \|g\|_{L_x^1} + \|\mathcal{V}_\alpha \circ v_\lambda^g\|_{L_x^1}) < \infty$ and

$$(6.8) \quad Y_x(0) = \int_0^\infty \int_0^\zeta \lambda \nabla_y W(x) N_0(dy, dz) + \int_0^\infty \int_0^\zeta \nabla_y W(\cdot) N_0(dy, dz) * (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x).$$

By using Proposition 6.2 and (2.15) again, we have for any $\epsilon > 0$,

$$\left| \int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-s) \int_0^\epsilon \nabla_y W(s) \bar{\nu}_\alpha(y) dy ds \right| \leq (\|g\|_{L_x^1} + \|\mathcal{V}_\alpha \circ v_\lambda^g\|_{L_x^1}) \sup_{s \in [0, x]} \int_0^\epsilon \nabla_y W(s) \bar{\nu}_\alpha(y) dy,$$

which is finite and goes to 0 as $\epsilon \rightarrow 0+$. Moreover, by Fubini's theorem,

$$\begin{aligned} \int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-s) \nabla_y W(s) ds &= \int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-s) \int_{(s-y)^+}^s W'(r) dr ds \\ &= \int_{(x-y)^+}^x (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(s) ds, \end{aligned}$$

which can be bounded by $C(1 \wedge y)$ uniformly in $y > 0$; see Proposition 6.2. Hence for any $\epsilon > 0$,

$$\int_0^\epsilon \left| \int_{(x-y)^+}^x (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(s) ds \right| \bar{\nu}_\alpha(y) dy \leq C \cdot \epsilon^\alpha,$$

which vanishes as $\epsilon \rightarrow 0+$. By these and the stochastic Fubini theorem; see Theorem D.2, the stochastic integral

$$\int_0^\infty \int_0^\zeta \int_{(x-y)^+}^x (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(s) ds N_0(dy, dz)$$

is well defined and equal almost surely to the second stochastic integral on the right side of (6.8). By (1.4),

$$\begin{aligned} &\lambda \nabla_y W(x) + \int_{(x-y)^+}^x (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(s) ds \\ &= \int_{(x-y)^+}^x (\lambda W'(s) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(s)) ds = \int_{(x-y)^+}^x v_\lambda^g(s) ds. \end{aligned}$$

Plugging this back into (6.8), we have

$$(6.9) \quad Y_x(0) = \int_0^\infty \int_0^\zeta \int_{(x-y)^+}^x v_\lambda^g(s) ds N_0(dy, dz).$$

By (6.7), we have for $t \in [0, x]$,

$$(6.10) \quad \begin{aligned} Y_x(t) &= \int_0^\infty \int_0^\zeta \int_{(x-y)^+}^x v_\lambda^g(s) ds N_0(dy, dz) + \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \lambda \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz) \\ &\quad + \int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-r) dr \int_0^{r \wedge t} \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(r-s) \tilde{N}_\alpha(ds, dy, dz). \end{aligned}$$

Proposition 6.2, together with the assumption that $v_\lambda^g \in \mathcal{A}_T$, implies that $g - \mathcal{V}_\alpha \circ v_\lambda^g \in \mathcal{A}_T$. Then there exists a constant $C > 0$ such that $|g(t) - \mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq C \cdot K(t)$ for any $t \in (0, T]$. By Proposition A.4,

$$\int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-r) dr \left| \int_0^r ds \int_0^\epsilon |\nabla_y W(r-s)|^2 \nu_\alpha(dy) \right|^{1/2}$$

$$(6.11) \leq C \int_0^x K(r) dr \cdot \left| \int_0^x ds \int_0^\epsilon |\nabla_y W(s)|^2 \nu_\alpha(dy) \right|^{1/2} \leq C \cdot x^\alpha \cdot \left| \int_0^x ds \int_0^\epsilon |\nabla_y W(s)|^2 \nu_\alpha(dy) \right|^{1/2},$$

which goes to 0 as $\epsilon \rightarrow 0+$. Moreover, by the change of variables,

$$\int_0^x (g - \mathcal{V}_\alpha \circ v_\lambda^g)(x-r) \cdot \mathbf{1}_{\{0 \leq s < r \wedge t\}} \cdot \nabla_y W(r-s) dr = (g - \mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(x-s) \cdot \mathbf{1}_{\{0 \leq s < t\}}.$$

By Proposition A.5, we have for $\epsilon > 0$,

$$\int_0^x ds \int_0^\epsilon |(g - \mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(x-s)|^2 \nu_\alpha(dy) \leq C \int_0^x ds \int_0^\epsilon \frac{|K * \nabla_y W(x-s)|^2}{y^{\alpha+2}} dy,$$

which is finite and goes to 0 as $\epsilon \rightarrow 0+$. From this, (6.11) and the stochastic Fubini theorem; see Theorem D.2, the stochastic integral

$$\int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} (g - \mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz)$$

is well defined and equal almost surely to the last stochastic integral on the right side of (6.10). Moreover, by Fubini's theorem we have for any $s \in [0, T]$ and $y > 0$,

$$\lambda \nabla_y W(s) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(s) = \int_{(s-y)^+}^s (\lambda W'(r) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * W'(r)) dr = \int_{(s-y)^+}^s v_\lambda^g(r) dr.$$

Consequently, the sum of the last two stochastic integrals on the right side of (6.10) can be replaced by

$$\int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \tilde{N}_\alpha(ds, dy, dz)$$

and claim (2) holds. \square

Associated to v_λ^g , we define a stochastic process $Z_x := \{Z_x(t) : t \in [0, x]\}$ by

$$Z_x(t) := \mathbf{E}[\lambda L_\zeta^\xi(x) + g * L_\zeta^\xi(x) | \mathcal{G}_t] - \int_t^x \mathcal{V}_\alpha \circ v_\lambda^g(x-s) \mathbf{E}[L_\zeta^\xi(s) | \mathcal{G}_t] ds.$$

By Proposition 6.7(2) and (6.9), the process Z_x also has the following representation

$$\begin{aligned} Z_x(t) &= Y_x(t) + \int_0^t \mathcal{V}_\alpha \circ v_\lambda^g(x-s) L_\zeta^\xi(s) ds = Y_x(0) + \int_0^t \mathcal{V}_\alpha \circ v_\lambda^g(x-s) L_\zeta^\xi(s) ds \\ &\quad + \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \tilde{N}_\alpha(ds, dy, dz). \end{aligned}$$

Thus Z_x is a (\mathcal{G}_t) -semimartingale. Applying Itô's formula to $\exp\{-Z_x(t)\}$ and then using (1.4),

$$(6.12) \quad e^{-Z_x(t)} = e^{-Y_x(0)} + M_x(t), \quad t \in [0, x],$$

where $M_x := \{M_x(t) : t \in [0, x]\}$ is a (\mathcal{G}_r) -local martingale and

$$(6.13) \quad M_x(t) := \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} e^{-Z_x(s)} \left(\exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} - 1 \right) \tilde{N}_\alpha(ds, dy, dz).$$

In the next lemma, we prove the martingality of $e^{-Z_x} := \{e^{-Z_x(t)} : t \in [0, x]\}$ and the equality (2.18) by using the method developed in [1, Lemma 6.3] and [3, Lemma 7.3].

LEMMA 6.8. *For each $x \in [0, T]$, the local-martingale e^{-Z_x} is a true (\mathcal{G}_r) -martingale and (2.18) holds.*

PROOF. For each $t \geq 0$, define

$$U_x(t) := \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \left(\exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} - 1 \right) \tilde{N}_\alpha(ds, dy, dz),$$

which is a uniformly square integrable (\mathcal{G}_r) -martingale, i.e., by the Burkholder-Davis-Gundy inequality, (2.15), Proposition 6.1 and the change of variables,

$$\begin{aligned} \sup_{t \geq 0} \mathbf{E}[|U_x(t)|^2] &\leq \sup_{t \geq 0} \int_0^t \mathbf{E}[L_\zeta^\xi(s)] ds \int_0^\infty \left(\exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} - 1 \right)^2 \nu_\alpha(dy) \\ &\leq C \int_0^x ds \int_0^\infty \left| \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right|^2 \nu_\alpha(dy) = C \int_0^x s^{\alpha-1} ds \leq Cx^\alpha. \end{aligned}$$

Denote by $\mathcal{E}_{U_x} := \{\mathcal{E}_{U_x}(t) : t \geq 0\}$ the Doléan-Dade exponential of U_x . By Itô's formula,

$$\mathcal{E}_{U_x}(t) = \exp \left\{ - \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \tilde{N}_\alpha(ds, dy, dz) - \int_0^t \nu_\alpha \circ v_\lambda^g(x-s) L_\zeta^\xi(s) ds \right\}.$$

Notice that \mathcal{E}_{U_x} is a non-negative local martingale and hence a supermartingale. By Fatou's lemma, we have $\mathbf{E}[\mathcal{E}_{U_x}(t)] \leq 1$ and hence it suffices to identify that $\mathbf{E}[\mathcal{E}_{U_x}(t)] = 1$ for any $t \geq 0$. For each $t_0 \geq 0$ and $n \geq 1$, let $\tau_n := \inf\{t \geq 0 : L_\zeta^\xi(t) \geq n\} \wedge t_0$ and $\mathcal{E}_{U_x}^n(\cdot) := \mathcal{E}_{U_x}(\tau_n \wedge \cdot)$. By the inequality $|1 + (z-1)e^z| \leq z^2 e^z$ for any $z \in \mathbb{R}$ and Proposition 6.1, there exists a constant $C > 0$ such that for any $t \geq 0$,

$$\begin{aligned} &\int_0^t \mathbf{1}_{\{s \leq \tau_n\}} L_\zeta^\xi(s) ds \int_0^\infty \left| 1 - \left(1 + \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right) \exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} \right| \nu_\alpha(dy) \\ &\leq n \int_0^x ds \int_0^\infty \left(\int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right)^2 \exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} \nu_\alpha(dy) \leq C \cdot \int_0^x s^{\alpha-1} ds \leq Cx^\alpha. \end{aligned}$$

By Theorem IV.3 in [51] with $y(s, z) = \mathbf{1}_{\{s \leq \tau_n\}} \cdot (\exp\{-\int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr\} - 1)$, the process $\mathcal{E}_{U_x}^n$ is a martingale for each $n \geq 1$. Thus

$$1 = \mathbf{E}[\mathcal{E}_{U_x}^n(t_0)] = \mathbf{E}[\mathcal{E}_{U_x}^n(t_0); \tau_n = t_0] + \mathbf{E}[\mathcal{E}_{U_x}^n(t_0); \tau_n < t_0] = \mathbf{E}[\mathcal{E}_{U_x}(t_0); \tau_n = t_0] + \mathbf{E}[\mathcal{E}_{U_x}^n(t_0); \tau_n < t_0].$$

By the monotone convergence theorem and the fact that $\tau_n \xrightarrow{\text{a.s.}} t_0$ as $n \rightarrow \infty$, we have $\mathbf{E}[\mathcal{E}_{U_x}(t_0); \tau_n = t_0] \rightarrow \mathbf{E}[\mathcal{E}_{U_x}(t_0)]$. Thus it suffices to prove that $\mathbf{E}[\mathcal{E}_{U_x}^n(t_0); \tau_n < t_0] \rightarrow 0$ as $n \rightarrow \infty$. Associate with the martingale $\mathcal{E}_{U_x}^n$, we define a probability law \mathbf{Q}_x^n on $(\Omega, \mathcal{G}, \mathcal{G}_r)$ by

$$\frac{d\mathbf{Q}_x^n}{d\mathbf{P}} = \mathcal{E}_{U_x}^n(\tau_n).$$

Since $\mathcal{E}_{U_x}^n(0) \stackrel{\text{a.s.}}{=} 1$, the PRM $N_0(dy, dz)$ is \mathcal{G}_0 -measurable and has the same distribution under \mathbf{P} and \mathbf{Q}_x^n . By Girsanov's theorem for random measure; see Theorem 3.17 in [36, p.170], the PRM $N_\alpha(ds, dy, dz)$ is a random point measure under \mathbf{Q}_x^n with intensity

$$\mathbf{1}_{\{s \leq \tau_n\}} \cdot \exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} ds \nu_\alpha(dy) dz,$$

and the SVE (1.3) is equal in distribution to the following SVE under \mathbf{Q}_x^n ,

$$L_\zeta^{\xi, n}(t) = X_{\zeta, 0}(t) + B^n(t) + M^n(t), \quad t \geq 0,$$

where $X_{\zeta, 0}$ is defined as in (4.8), M^n is defined as in (4.10) with X_ζ replaced by $L_\zeta^{\xi, n}$ and

$$B^n(t) := \int_0^t \mathbf{1}_{\{s \leq \tau_n\}} L_{\zeta, n}^\xi(s) ds \int_0^\infty \nabla_y W(t-s) \left(\exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} - 1 \right) \nu_\alpha(dy).$$

Noting that $\nabla_y W(t-s) = 0$ when $s \geq t$ and $\int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr = 0$ when $s \geq x$, we have for any $t \geq 0$,

$$B^n(t) = \int_0^x \mathbf{1}_{\{s \leq \tau_n\}} L_{\zeta,n}^\xi(s) ds \int_0^\infty \nabla_y W(t-s) \left(\exp \left\{ - \int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr \right\} - 1 \right) \nu_\alpha(dy).$$

By the arithmetic-geometric mean inequality and Proposition 6.1, the preceding inner integral can be bounded by $C \cdot ((x-s)^+ \cdot (t-s)^+)^{(\alpha-1)/2}$ uniformly in $t, s, y > 0$. The definition of τ_n implies that

$$(6.14) \quad \sup_{t \geq 0} |B^n(t)| \leq C \cdot n \cdot \int_0^x (x-s)^{\alpha-1} ds \leq C \cdot n \cdot x^\alpha, \quad a.s.$$

Proposition 6.1 implies that $\exp\{-\int_{(x-s-y)^+}^{x-s} v_\lambda^g(r) dr\}$ is uniformly bounded in $s, y \geq 0$. Similarly as in the proofs of Lemma 5.1-5.4, there exists a constant $C > 0$ such that for any $t > 0$ and $t_1, t_2 \in [0, t]$,

$$\mathbf{E}^{\mathbf{Q}_x^n} [|X_{\zeta,0}(t)|^p + |L_{\zeta}^{\xi,n}(t)|^p] < C \cdot (1+t)^{p\alpha}$$

and

$$\mathbf{E}^{\mathbf{Q}_x^n} [|X_{\zeta,0}(t_1) - X_{\zeta,0}(t_2)|^p + |M^n(t_1) - M^n(t_2)|^p] \leq C \cdot (1+t)^{p\alpha} \cdot |t_1 - t_2|^{p\alpha}.$$

Here $\mathbf{E}^{\mathbf{Q}_x^n}$ is the expectation under \mathbf{Q}_x^n . Together with these estimates, an argument similar to that in the proof of Theorem 2.9 implies that for any $\kappa \in (0, \alpha/2)$, both $X_{\zeta,0}$ and M^n are locally κ -Hölder continuous under \mathbf{Q}_x^n and the Hölder coefficient has finite moments of all orders. Like the argument before Corollary 2.10, we have $\mathbf{E}^{\mathbf{Q}_x^n} [\sup_{s \in [0,t]} |X_{\zeta,0}(s)|^p] + \mathbf{E}^{\mathbf{Q}_x^n} [\sup_{s \in [0,t]} |M^n(s)|^p] < \infty$ for any $t, p \geq 0$, which, together with (6.14), implies that $\mathbf{E}^{\mathbf{Q}_x^n} [\sup_{s \in [0,t]} |L_{\zeta}^{\xi,n}(s)|^p] < \infty$. By the definition of τ_n and Chebyshev's inequality,

$$\mathbf{E}[\mathcal{E}_{U_x}^n(\tau_n); \tau_n < t] = \mathbf{Q}_x^n(\tau_n < t) = \mathbf{Q}_x^n\left(\sup_{s \in [0,t]} L_{\zeta}^{\xi,n}(s) \geq n\right) \leq \frac{1}{n} \mathbf{E}^{\mathbf{Q}_x^n} \left[\sup_{s \in [0,t]} L_{\zeta}^{\xi,n}(s) \right],$$

which vanishes as $n \rightarrow \infty$. Hence $\mathbf{E}[\mathcal{E}_{U_x}(t_0)] = 1$ and \mathcal{E}_{U_x} is a (\mathcal{G}_r) -martingale under \mathbf{P} . By Proposition 6.1,

$$\int_0^\infty \left| 1 - \exp \left\{ - \int_{(x-y)^+}^x v_\lambda^g(r) dr \right\} \right| \bar{\nu}_\alpha(y) dy \leq C \int_0^\infty \left| \int_{(x-y)^+}^x v_\lambda^g(r) dr \right| \bar{\nu}_\alpha(y) dy.$$

Similarly as in the proof for Proposition 6.1, we can prove that the last quantity is finite. From this and the exponential formula for PRMs; see [9, p.8],

$$\mathbf{E}[\exp\{-Y_x(0)\}] = \exp \left\{ -\zeta \int_0^\infty \left(1 - \exp \left\{ - \int_{(x-y)^+}^x v_\lambda^g(s) ds \right\} \right) \bar{\nu}_\alpha(y) dy \right\} < \infty.$$

Notice that $e^{-Z_x} = e^{-Y_x(0)} \mathcal{E}_{U_x}$ on $[0, x]$. The standard conditional expectation argument yields that the local-martingale e^{-Z_x} is a true (\mathcal{G}_r) -martingale under \mathbf{P} . The equality (2.18) can be obtained immediately from the facts that $Z_x(x) = \lambda L_{\zeta}^\xi(x) + g * L_{\zeta}^\xi(x) \geq 0$ and $\mathbf{E}[e^{-Z_x(x)}] = \mathbf{E}[e^{-Y_x(0)}]$. \square

6.3. Proof for Theorem 2.12. By Lemma 6.6 and 6.8, associated to the unique \mathcal{A} -noncontinuable solution $(v_\lambda^g, T_\lambda^g)$ of (1.4) we see that Theorem 2.12 holds for any $x \in [0, T_\lambda^g)$. Thus it suffices to prove that $(v_\lambda^g, T_\lambda^g)$ is a \mathcal{A} -global solution, i.e., $T_\lambda^g = \infty$. In the sequel, we assume for contradiction that $T_\lambda^g < \infty$, which implies that $\limsup_{t \rightarrow T_\lambda^g-} |v_\lambda^g(t)| = \infty$. With the help of the following propositions, we prove that $|v_\lambda^g(t)| \leq C \cdot t^{\alpha-1}$ uniformly in $(0, T_\lambda^g)$; see Lemma 6.14, which leads to a contradiction to the preceding assumption,

PROPOSITION 6.9. *If $T_\lambda^g < \infty$, there exists a constant $C > 0$ such that for any $t \in [0, T_\lambda^g]$ and $y > 0$,*

$$(6.15) \quad \left| \int_{(t-y)^+}^t v_\lambda^g(r) dr \right| \leq C \quad \text{and} \quad |\mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq C \int_0^\infty \left| \int_{(t-y)^+}^t v_\lambda^g(r) dr \right|^2 \nu_\alpha(dy).$$

PROOF. Notice that $e^{-x} - 1 + x \geq 0$ for any $x \in \mathbb{R}$. By (1.4), we have $v_\lambda^g(t) \leq \lambda W'(t) + g * W'(t) \leq Ct^{\alpha-1}$ and hence $\int_{(t-y)^+}^t v_\lambda^g(r) dr \leq C$ uniformly in $t \in [0, T_\lambda^g]$ and $y > 0$. We now prove $\int_{(t-y)^+}^t v_\lambda^g(r) dr \geq -C$ uniformly in $t \in [0, T_\lambda^g]$ and $y > 0$. If not, the fact that $v_\lambda^g \in \mathcal{A}_T$ for any $T \in (0, T_\lambda^g)$ yields that for any $y > 0$,

$$\int_{(T_\lambda^g - y)^+}^{T_\lambda^g} v_\lambda^g(r) dr = -\infty.$$

Moreover, by the inequality $1 - e^{-x} \leq x$ for any $x \in \mathbb{R}$, Lemma 6.8 and (2.18), we have for any $t \in (0, T_\lambda^g)$,

$$0 \leq \int_0^\infty \left(1 - \exp \left\{ - \int_{(t-y)^+}^t v_\lambda^g(s) ds \right\}\right) \bar{\nu}_\alpha(y) dy \leq \int_0^\infty \bar{\nu}_\alpha(y) dy \int_{(t-y)^+}^t v_\lambda^g(s) ds.$$

The continuity of v_λ^g on $(0, T_\lambda^g)$ implies that the last integral tends to $-\infty$ as t increases to T_λ^g , which leads to a contradiction and hence the first desired inequality holds. The second one can be proved similarly as in the proof of Proposition 6.2. \square

Recall the constant $\theta \in (1, \frac{1}{1-\alpha})$ defined in Proposition 6.3. We need the next two constants in the follows

$$(6.16) \quad \eta \in \left(\left(\frac{1}{\theta} - \frac{1+\alpha^2}{2} \right)^+, \frac{1}{2\theta} \right) \quad \text{and} \quad \ell \in \left(\frac{1+\alpha}{2}, \frac{\eta+1-1/\theta}{1-\alpha} \wedge 1 \right).$$

The first inequality in Proposition 6.2, together with $0 < \theta\eta < 1/2$ and $\theta(\alpha-1) + 1 > 0$, implies that the following function is well defined on $[0, T_\lambda^g)$,

$$H(t) := \int_0^t s^{-\theta\eta} \cdot |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)|^\theta ds.$$

PROPOSITION 6.10. *If $T_\lambda^g < \infty$, there exists a constant $C > 0$ such that for any $t \in (0, T_\lambda^g)$,*

$$(6.17) \quad |\mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq Ct^{\alpha-1} + C|H(t)|^{2/\theta} \cdot t^{\alpha+2\eta+1-2/\theta} + Ct^{2(\alpha+\eta-\ell+1-1/\theta)} \int_0^t \frac{|H(t-s)|^{2/\theta}}{s^{\alpha+2-2\ell}} ds.$$

PROOF. For $y > 0$, integrating both sides of (1.4) over $((t-y)^+, t]$ and then using Fubini's theorem,

$$\int_{(t-y)^+}^t v_\lambda^g(s) ds = \lambda \cdot \nabla_y W(t) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(t).$$

Plugging this into the second inequality in (6.15) and then using the Cauchy-Schwarz inequality, we have $|\mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq C[J_1(t) + J_2(t) + J_3(t)]$ uniformly in $t \in (0, T_\lambda^g)$, where

$$J_1(t) := \lambda^2 \int_0^\infty |\nabla_y W(t)|^2 \nu_\alpha(dy), \quad J_2(t) := \int_0^\infty |g * \nabla_y W(t)|^2 \nu_\alpha(dy),$$

$$J_3(t) := \int_0^\infty |(\mathcal{V}_\alpha \circ v_\lambda^g) * \nabla_y W(t)|^2 \nu_\alpha(dy).$$

Similarly as in (6.3)-(6.4), we have $J_1(t) \leq C \cdot t^{\alpha-1}$ uniformly in $t \in (0, T_\lambda^g)$. By Hölder's inequality, we have $|g * \nabla_y W(t)| \leq \|g\|_{L^\infty} \cdot \|\nabla_y W\|_{L_t^1} \leq \|g\|_{L^\infty} \cdot t^{1/2} \cdot \|\nabla_y W\|_{L_t^2}$. Plugging this into $J_2(t)$ and then using Proposition A.1 with $p = 2$ as well as Fubini's theorem, we have uniformly in $t \in [0, T_\lambda^g)$,

$$J_2(t) \leq C \cdot t \cdot \int_0^\infty \|\nabla_y W\|_{L_t^2}^2 \nu_\alpha(dy) = C \cdot t \cdot \int_0^t dr \int_0^\infty |\nabla_y W(r)|^2 \nu_\alpha(dy) \leq C \cdot t^{\alpha+1}.$$

We now turn to analyze $J_3(t)$. Splitting the interval of integration and then using the Cauchy-Schwarz inequality, we have $J_3(t) \leq J_{31}(t) + J_{32}(t)$ with $J_{31}(t) := \|\mathcal{V}_\alpha \circ v_\lambda^g\| * W(t)\|^2 \cdot \bar{\nu}_\alpha(t)$ and

$$J_{32}(t) := 2 \int_0^t \left| \int_{t-y}^t \mathcal{V}_\alpha \circ v_\lambda^g(s) W(t-s) ds \right|^2 \nu_\alpha(dy),$$

$$J_{32}(t) := 2 \int_0^t \left| \int_0^{t-y} \mathcal{V}_\alpha \circ v_\lambda^g(s) \nabla_y W(t-s) ds \right|^2 \nu_\alpha(dy).$$

By Hölder's inequality and (2.8),

$$\begin{aligned} |\mathcal{V}_\alpha \circ v_\lambda^g * W(t)| &= \int_0^t s^{-\eta} |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)| \cdot s^\eta W(s) ds \\ &\leq |H(t)|^{1/\theta} \cdot \left(\int_0^t |s^\eta W(s)|^{\theta/(\theta-1)} ds \right)^{1-1/\theta} \leq C \cdot |H(t)|^{1/\theta} \cdot t^{\alpha+\eta+1-1/\theta} \end{aligned}$$

and hence $J_{31}(t) \leq C \cdot |H(t)|^{2/\theta} \cdot t^{\alpha+2\eta+1-2/\theta}$ uniformly in $t \in (0, T_\lambda^g)$. Similarly, we also have

$$\begin{aligned} \int_{t-y}^t |\mathcal{V}_\alpha \circ v_\lambda^g(s)| W(t-s) ds &= \int_0^y s^{-\eta} |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)| \cdot s^\eta W(s) ds \\ &\leq \left(\int_0^y s^{-\eta\theta} |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)|^\theta ds \right)^{1/\theta} \cdot \left(\int_0^y |s^\eta W(s)|^{\theta/(\theta-1)} ds \right)^{1-1/\theta}, \end{aligned}$$

which can be bounded by $C \cdot |H(t)|^{1/\theta} \cdot y^{\alpha+\eta+1-1/\theta}$. Taking this back into $J_{32}(t)$, we have $J_{32}(t) \leq C |H(t)|^{2/\theta} \cdot t^{\alpha+2\eta+1-1/\theta}$ uniformly in $t \in [0, T_\lambda^g]$. For $J_{33}(t)$, we first consider its inner integral. Like the preceding argument, by Hölder's inequality we have

$$(6.18) \quad \begin{aligned} \int_0^{t-y} |\mathcal{V}_\alpha \circ v_\lambda^g(s) \nabla_x W(t-s)| ds &= \int_y^t (s-y)^{-\eta} |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)| \cdot (s-y)^\eta \nabla_y W(s) ds \\ &\leq \left(\int_y^t (s-y)^{-\eta\theta} |\mathcal{V}_\alpha \circ v_\lambda^g(t-s)|^\theta ds \right)^{1/\theta} \cdot \left(\int_y^t |(s-y)^\eta \nabla_y W(s)|^{\frac{\theta}{\theta-1}} ds \right)^{1-1/\theta}. \end{aligned}$$

By the change of variables, the first term on the right side of this inequality equals to $|H(t-y)|^{1/\theta}$. Recall the constant ℓ in (6.16). By (2.9), we have $|\nabla_y W(s)| \leq C \cdot s^{\alpha(1-\ell)} \cdot (s-y)^{\ell(\alpha-1)} \cdot y^\ell$ uniformly in $s > y > 0$. Plugging this into the second term on the right side of the inequality in (6.18), it can be bounded by

$$C t^{\alpha(1-\ell)} \cdot \left(\int_y^t (s-y)^{(\eta+(\alpha-1)\ell)\frac{\theta}{\theta-1}} ds \right)^{\frac{\theta-1}{\theta}} \cdot y^\ell \leq C \cdot t^{\alpha+\eta-\ell+1-1/\theta} \cdot y^\ell.$$

uniformly in $t \in [0, T_\lambda^g]$ and $y \in (0, t)$. Taking these two estimates back into (6.18) and then $J_{33}(t)$, we have

$$J_{33}(t) \leq C \cdot t^{2(\alpha+\eta-\ell+1-1/\theta)} \cdot \int_0^t |H(t-y)|^{2/\theta} \cdot y^{2\ell-\alpha-2} dy.$$

Here the constant $C > 0$ is independent of t . Then (6.17) follows by putting all estimates above together. \square

PROPOSITION 6.11. *If $T_\lambda^g < \infty$, there exists a constant $C_* > 0$ such that for any $t \in (0, T_\lambda^g)$,*

$$(6.19) \quad H(t) \leq C_* t^{-\eta\theta} + C_* \int_0^t (t-s)^{-\eta\theta} \cdot |H(s)|^2 dt.$$

PROOF. Raising both sides of the inequality (6.17) to the θ power and then using the power mean inequality, we have for some constant $C > 0$ independent of t ,

$$|\mathcal{V}_\alpha \circ v_\lambda^g(t)|^\theta \leq C t^{\theta(\alpha-1)} + C |H(t)|^2 \cdot t^{\theta(\alpha+2\eta+1)-2} + C t^{2\theta(\alpha+\eta-\ell+1)-2} \left(\int_0^t \frac{|H(t-s)|^{2/\theta}}{s^{\alpha+2-2\ell}} ds \right)^\theta.$$

Convolving both sides of this inequality by the power function $s^{-\theta\eta}$,

$$(6.20) \quad \begin{aligned} H(t) &\leq C \int_0^t (t-s)^{-\eta\theta} \cdot s^{\theta(\alpha-1)} ds + C \int_0^t (t-s)^{-\eta\theta} \cdot |H(s)|^2 \cdot s^{\theta(\alpha+2\eta+1)-2} ds \\ &\quad + C \int_0^t (t-s)^{-\eta\theta} \cdot s^{2\theta(\alpha+\eta-\ell+1)-2} \left(\int_0^s \frac{|H(s-r)|^{2/\theta}}{r^{\alpha+2-2\ell}} dr \right)^\theta ds. \end{aligned}$$

Notice that $\eta\theta < 1/2$, $\theta(\alpha - 1) + 1 > 0$ and $\theta(\alpha + 2\eta + 1) - 2 > 0$. A simple calculation shows that uniformly in $t \in (0, T_\lambda^g)$, the first term on the right side of the above inequality can be bounded by $C \cdot t^{-\eta\theta}$ and the second term can be bounded by

$$(6.21) \quad C \cdot t^{\theta(\alpha+2\eta+1)-2} \int_0^t (t-s)^{-\eta\theta} \cdot |H(s)|^2 ds.$$

Using Hölder's inequality and the fact that $2\ell - \alpha - 1 > 0$, we have

$$\begin{aligned} \left(\int_0^s \frac{|H(s-r)|^{2/\theta}}{r^{\alpha+2-2\ell}} dr \right)^\theta &= \left(\int_0^s \frac{|H(s-r)|^{2/\theta}}{r^{(\alpha+2-2\ell)/\theta}} \cdot r^{(1-1/\theta)(2\ell-\alpha-2)} dr \right)^\theta \\ &\leq \left(\int_0^s z^{2\ell-\alpha-2} dz \right)^{\theta-1} \int_0^s \frac{|H(s-r)|^2}{r^{\alpha+2-2\ell}} dr \\ &= \frac{s^{(\theta-1)(2\ell-\alpha-1)}}{|2\ell-\alpha-1|^{\theta-1}} \int_0^s \frac{|H(s-r)|^2}{r^{\alpha+2-2\ell}} dr. \end{aligned}$$

Plugging this into the third term on the right side of (6.20), it can be bounded uniformly in $t \in (0, T_\lambda^g)$ by

$$\begin{aligned} &C \int_0^t (t-s)^{-\eta\theta} \cdot s^{2\theta(\alpha+\eta-\ell+1)-2+(\theta-1)(2\ell-\alpha-1)} \cdot \int_0^s \frac{|H(s-r)|^2}{r^{\alpha+2-2\ell}} dr ds \\ &\leq C t^{\theta(\alpha+2\eta+1)-2-(2\ell-\alpha-1)} \cdot \int_0^t (t-s)^{-\eta\theta} \cdot \int_0^s \frac{|H(s-r)|^2}{r^{\alpha+2-2\ell}} dr ds \\ &= C \cdot t^{\theta(\alpha+2\eta+1)-2-(2\ell-\alpha-1)} \cdot \int_0^t |H(s)|^2 \int_0^{t-s} \frac{(t-s-r)^{-\eta\theta}}{r^{\alpha+2-2\ell}} dr ds. \end{aligned}$$

Here the inequality comes from the fact that $2\theta(\alpha + \eta - \ell + 1) - 2 > 0$ as well as $(\theta - 1)(2\ell - \alpha - 1) > 0$, and the equality comes from Fubini's theorem. Noting that $2\ell - \alpha - 1 > 0$ and $\eta\theta < 1/2$, a simple calculation induces that uniformly in $t \geq s > 0$,

$$\int_0^{t-s} \frac{(t-s-r)^{-\eta\theta}}{r^{\alpha+2-2\ell}} dr \leq C \cdot (t-s)^{-\eta\theta+2\ell-\alpha-1} \leq C \cdot t^{2\ell-\alpha-1} \cdot (t-s)^{-\eta\theta}$$

Consequently, the third term on the right side of (6.20) can be bounded uniformly in $t \in (0, T_\lambda^g)$ by

$$C \cdot t^{\theta(\alpha+2\eta+1)-2} \cdot \int_0^t (t-s)^{-\eta\theta} |H(s)|^2 ds,$$

which can be merged with (6.21). Then (6.19) follows by putting these estimates together. \square

PROPOSITION 6.12. *For any $C^* > 0$, there exists a unique continuous and non-negative solution of*

$$(6.22) \quad \psi(t) = C^* t^{-\eta\theta} + C^* \int_0^t (t-s)^{-\eta\theta} \cdot |\psi(s)|^2 ds, \quad t > 0.$$

Moreover, for any $T > 0$, there exists a constant $C > 0$ such that $\psi(t) \leq C \cdot t^{-\eta\theta}$ for any $t \in (0, T]$.

PROOF. By Theorem 6.1(ii) in [3] and $0 < \eta\theta < 1/2$, there exists a unique solution $\tilde{\psi} \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}_+)$ of (6.22). Let $\psi(t) := C^* t^{-\eta\theta} + C^* \int_0^t (t-s)^{-\eta\theta} \cdot |\tilde{\psi}(s)|^2 ds$ for $t > 0$. By the properties of convolution, it is easy to identify that ψ is continuous on $(0, \infty)$ and equal to $\tilde{\psi}$ almost everywhere. Thus ψ is the unique continuous and non-negative solution of (6.22). By Theorem 2.a in [14], there exist two constants $C_\psi, r_\psi > 0$ such that $\psi(t) \leq C_\psi t^{-\eta\theta}$ uniformly on $t \in (0, r_\psi]$. For $T > r_\psi$, the continuity of ψ yields that $\psi(t) \leq C \cdot t^{-\eta\theta}$ for any $t \in (0, T]$ and some $C > 0$. \square

PROPOSITION 6.13. *If $T_\lambda^g < \infty$, there exists a constant $C > 0$ such that $H(t) \leq C \cdot t^{-\eta\theta}$ for any $t \in (0, T_\lambda^g)$.*

PROOF. Choosing the two constants $C^* > C_* > 0$ such that the inequality (6.19) turns to be strict. It is easy to identify that both $t^{\eta\theta} H(t)$ and $t^{\eta\theta} \psi(t)$ are continuous on $[0, T_\lambda^g]$. By Theorem 2.1 in [18]¹³, the function H can be uniformly bounded by ψ on $(0, T_\lambda^g)$ and the desired result follows directly from Proposition 6.12. \square

LEMMA 6.14. *If $T_\lambda^g < \infty$, there exists a constant $C > 0$ such that $|v_\lambda^g(t)| \leq C \cdot t^{\alpha-1}$ for any $t \in (0, T_\lambda^g)$.*

PROOF. Plugging Proposition 6.13 into (6.17), we have

$$|\mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq C t^{\alpha-1} + C t^{\alpha+1-2/\theta} + C t^{2(\alpha+\eta-\ell+1-1/\theta)} \int_0^t \frac{(t-s)^{-2\eta}}{s^{\alpha+2-2\ell}} ds.$$

Notice that $\alpha+1-2/\theta \in (\alpha-1, 1-\alpha)$, $\eta < 1/2$ and $\alpha+2-2\ell < 1$, we have $|\mathcal{V}_\alpha \circ v_\lambda^g(t)| \leq C \cdot t^{\alpha-1}$ uniformly in $t \in [0, T_\lambda^g]$. Taking this back into (1.4), we can get the desired result immediately. \square

6.4. Proof for Theorem 2.1(3). Assume that $L_{\zeta,1}^\xi$ and $L_{\zeta,2}^\xi$ are two weak solutions of (1.3). For any $x, z \geq 0$ and $g \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, let v_0^g be the unique \mathcal{A} -global solution of (1.4) with $\lambda = 0$. By Theorem 2.12,

$$\mathbf{E}[e^{-z \cdot g * L_{\zeta,1}^\xi(x)}] = \exp \left\{ -\zeta \int_0^\infty \left(1 - \exp \left\{ - \int_{(x-y)^+}^x v_0^g(s) ds \right\} \right) \bar{\nu}_\alpha(y) dy \right\} = \mathbf{E}[e^{-z \cdot g * L_{\zeta,2}^\xi(x)}].$$

The one-to-one correspondence between non-negative random variables and their Laplace transforms yields that the two non-negative random variables $g * L_{\zeta,1}^\xi(x)$ and $g * L_{\zeta,2}^\xi(x)$ are equal in distribution. Hence the two solutions $L_{\zeta,1}^\xi$ and $L_{\zeta,2}^\xi$ have the same probability law on $L^1([0, x]; \mathbb{R}_+)$ and also on $C([0, x]; \mathbb{R}_+)$. By the arbitrariness of x , the weak uniqueness of non-negative solutions holds for (1.3).

7. Fractional integral representations. In this section we prove the two equivalences in Theorem 2.14. When $b = 0$, they follow directly from (2.7). We now prove them with $b > 0$. The equivalence between (1.4) and (2.20) follows from the resolvent equation (2.12). Indeed, by Theorem 4.6¹⁴ in [30, p.48] and (2.12) we have v_λ^g solves (2.20) if and only if

$$\begin{aligned} v_\lambda^g &= \lambda K + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * K - bW' * (\lambda K + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * K) \\ &= \lambda(K - bW' * K) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * (K - bW' * K). \end{aligned}$$

Multiplying both sides by b and then using (2.12) again, we have

$$bv_\lambda^g = \lambda(bK - bW' * bK) + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * (bK - bW' * bK) = \lambda bW' + (g - \mathcal{V}_\alpha \circ v_\lambda^g) * bW',$$

which is equivalent to (1.4).

The equivalence between (1.3) and (2.19) can be proved in the same way. For convenience, we assume $K(x) = 0$ if $x \leq 0$. Notice that $\int_{s-y}^s K(r) dr = \int_0^s \nabla_y K(r) dr$ for any $s, y \geq 0$. By Theorem 4.6 in [30, p.48] and (2.12) again, the process L_ζ^ξ is a solution of (2.19) if and only if it solves

$$\begin{aligned} L_\zeta^\xi(x) &= \zeta - \zeta \cdot bW(x) + \int_0^\infty \int_0^\zeta \int_0^x \nabla_y K(r) dr \tilde{N}_0(dy, dz) \\ &\quad - \int_0^x bW'(x-t) dt \int_0^\infty \int_0^\zeta \int_0^t \nabla_y K(r) dr \tilde{N}_0(dy, dz) \end{aligned}$$

¹³For $T > 0$ and $\rho \in (0, 1)$, let f_1, f_2 be two functions on $(0, T]$ satisfying that $t^\rho f_i(t) \in C([0, T]; \mathbb{R})$ with $i = 1, 2$. If $f_1(t) < C_1 t^{-\rho} + C_1 \int_0^t (t-s)^{-\rho} f_1(s) ds$ and $f_2(t) = C_2 t^{-\rho} + C_2 \int_0^t (t-s)^{-\rho} f_2(s) ds$ with $C_1 < C_2$, then $f_1 < f_2$ on $(0, T]$.

¹⁴For two function $f, k \in L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R})$, we have $x = f + k * x$ if and only if $x = f + R_k * f$, where R_k is the unique solution of $R_k = k + k * R_k$.

$$(7.1) \quad \begin{aligned} & + \int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_0^{x-s} \nabla_y K(r) dr \tilde{N}_\alpha(ds, dy, dz) \\ & - \int_0^x bW'(x-t) dt \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(s)} \int_0^{t-s} \nabla_y K(r) dr \tilde{N}_\alpha(ds, dy, dz). \end{aligned}$$

By the change of variables and Proposition A.4,

$$(7.2) \quad \begin{aligned} & \int_0^x bW'(x-t) \left| \int_0^t ds \int_0^\epsilon \left| \int_0^{t-s} \nabla_y K(r) dr \right|^2 \nu_\alpha(dy) \right|^{1/2} dt \\ & \leq C \cdot W(x) \cdot \left| \int_0^x ds \int_0^\epsilon \frac{\left| \int_0^s \nabla_y K(r) dr \right|^2}{y^{\alpha+2}} dy \right|^{1/2}, \end{aligned}$$

which goes to 0 as $\epsilon \rightarrow 0+$. By the change of variables and Fubini's theorem,

$$\begin{aligned} \int_0^x bW'(x-t) \int_{t-s-y}^{t-s} K(r) dr \cdot \mathbf{1}_{\{0 \leq s < t\}} dt & = \int_0^{x-s} bW'(x-s-t) \int_0^s \nabla_y K(r) dr dt \\ & = \int_0^{x-s} \nabla_y K(t) \int_t^{x-s} bW'(x-s-r) dr dt \\ & = \int_0^{x-s} \nabla_y K(t) bW(x-s-t) dr dt = bW * \nabla_y K(x-s). \end{aligned}$$

By the change of variables and Proposition A.6, we have for $\epsilon > 0$,

$$\int_0^x ds \int_0^\epsilon |bW * \nabla_y K(x-s)|^2 \nu_\alpha(dy) \leq C \int_0^x ds \int_0^\epsilon \frac{|W * \nabla_y K(s)|^2}{y^{\alpha+2}} dy,$$

which is finite and goes to 0 as $\epsilon \rightarrow 0+$. From this, (7.2) and the stochastic Fubini theorem; see Theorem D.2, the stochastic integral

$$\int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} bW * \nabla_y K(x-s) \tilde{N}_\alpha(ds, dy, dz)$$

is well defined and equal almost surely to the last stochastic integral on the right side of (7.1). Moreover, by Fubini's theorem and (2.13),

$$\begin{aligned} \int_0^{x-s} \nabla_y K(r) dr - bW * \nabla_y K(x-s) & = (1 - bW) * \nabla_y K(x-s) \\ & = W' * L_K * \nabla_y K(x-s) = \nabla_y W(x-s). \end{aligned}$$

Thus the subtraction of the last two terms on the right side of (7.1) is equal almost surely to

$$\int_0^x \int_0^\infty \int_0^{L_\zeta^\xi(s)} \nabla_y W(x-s) \tilde{N}_\alpha(ds, dy, dz).$$

Similarly, the following stochastic integral

$$\int_0^\infty \int_0^\zeta \int_0^x bW'(x-t) \int_0^t \nabla_y K(r) dr dt \tilde{N}_0(dy, dz)$$

is also well defined and equal almost surely to the second stochastic integral on the right side of (7.1). Moreover,

$$\int_0^x \nabla_y K(r) dr - \int_0^x bW'(x-t) \int_0^t \nabla_y K(r) dr dt = \int_0^x \nabla_y W'(r) dy = \nabla_y W(x).$$

The subtraction of the first two integrals on the right side of (7.1) is equal almost surely to

$$\int_0^\infty \int_0^\zeta \nabla_y W(x) \tilde{N}_0(dy, dz).$$

Putting these results together, we see that (7.1) turns into (2.16) and hence (2.19) is equivalent to (1.3). \square

8. Application to M/G/1 processor-sharing queues. As the continuation of [45], we use the preceding results to establish a SVE for the heavy-traffic limit of heavy-tailed M/G/1 processor sharing queues. Recall the sequence $\{\gamma_n\}_{n \geq 1}$ and the Pareto distribution Λ defined in Section 4.1. In the n -th processor-sharing queue, the arrival of customers to the system is described by a Poisson process with rate $\gamma_n > 0$ and the amount of processing time that each customer requires from the server is distributed as Λ . Additionally, there are z_n initial customers in the system at time 0, whose residual service times are independent and identically distributed with common distribution Λ^* . Here we are interested in the heavy-traffic limit of the queue-length process before the queue becoming empty. More precisely, let $q^{(n)} := \{q^{(n)}(t) : t \geq 0\}$ be the queue-length process and $\tau^{(n)}$ the first time that the queue becomes empty, i.e. $\tau^{(n)} := \inf\{t > 0 : q^{(n)}(t) = 0\}$. We write $Q^{(n)}$ for the rescaled queue-length process $\{n^{-\alpha/(1+\alpha)} \cdot q^{(n)}(nt) : t \in [0, \tau^{(n)}]\}$ under $\mathbf{P}(\cdot | \tau^{(n)} < \infty)$.

Denote by \mathcal{E} the set of all positive excursions with finite length. For each $f \in \mathcal{E}$, let e_f be the right end point of f and

$$(8.1) \quad \mathcal{I}_f(t) := \int_0^t f(s) ds, \quad t \geq 0.$$

It is obvious that \mathcal{I}_f is a continuous and non-decreasing function on \mathbb{R}_+ , which allows us to define its right-inverse function \mathcal{I}_f^{-1} by $\mathcal{I}_f^{-1}(t) = e_f$ if $t > \mathcal{I}_f(\infty)$ and $\mathcal{I}_f^{-1}(t) := \inf\{s \geq 0 : \mathcal{I}_f(s) \geq t\}$ if $t \in [0, \mathcal{I}_f(\infty)]$. Let \mathcal{L} be the Lamperti transformation on \mathcal{E} , which is a map acting on an excursion $f \in \mathcal{E}$ by $\mathcal{L} \circ f(t) := f(\mathcal{I}_f^{-1}(t))$ for $t \geq 0$. The next corollary is a direct consequence of Theorem 6.5 in [45] and Theorem 2.14.

COROLLARY 8.1. *If Condition 4.2 holds and $z_n/n^{\alpha/(1+\alpha)} \rightarrow \zeta > 0$ as $n \rightarrow \infty$, we have $Q^{(n)} \rightarrow Q_\zeta$ weakly in $D([0, \infty), \mathbb{R}_+)$, where the limit process $Q_\zeta \in \mathcal{E}$ is the unique weak solution of*

$$(8.2) \quad \begin{aligned} Q_\zeta(t) &= \zeta - b \int_0^t \frac{(\mathcal{I}_{Q_\zeta}^{-1}(t) - \mathcal{I}_{Q_\zeta}^{-1}(s))^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} ds + \int_0^\infty \int_0^\zeta \left(\int_{(\mathcal{I}_{Q_\zeta}^{-1}(t)-y)^+}^{\mathcal{I}_{Q_\zeta}^{-1}(t)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_{Q,0}(dy, dz) \\ &+ \int_0^t \int_0^\infty \left(\int_{(\mathcal{I}_{Q_\zeta}^{-1}(t)-\mathcal{I}_{Q_\zeta}^{-1}(s)-y)^+}^{\mathcal{I}_{Q_\zeta}^{-1}(t)-\mathcal{I}_{Q_\zeta}^{-1}(s)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_Q(ds, dy), \quad t \geq 0, \end{aligned}$$

where $\nu_\alpha(dy)$ is given by (1.2) with $c = \Gamma(1-\alpha)$, $\tilde{N}_{Q,0}(dy, dz)$ and $\tilde{N}_Q(ds, dy)$ are two compensated PRMs on $(0, \infty)^2$ with intensity $\bar{\nu}_\alpha(y) dy dz$ and $d\nu_\alpha(dy)$, respectively.

PROOF. Let L_ζ^ξ be the unique weak solution of (2.19) with $c = \Gamma(1-\alpha)$. From Theorem 6.5 in [45], we have $Q^{(n)} \rightarrow \mathcal{L} \circ L_\zeta^\xi \in \mathcal{E}$ weakly in $D([0, \infty), \mathbb{R}_+)$. By (2.19) and the change of variables,

$$\begin{aligned} \mathcal{L} \circ L_\zeta^\xi(t) &= \zeta - b \int_0^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)} \frac{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t) - s)^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} L_\zeta^\xi(s) ds \\ &+ \int_0^\infty \int_0^\zeta \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_0(dy, dz) \\ &+ \int_0^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)} \int_0^\infty \int_0^{L_\zeta^\xi(s)} \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-s-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-s} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_\alpha(ds, dy, dz) \\ &= \zeta - b \int_0^t \frac{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t) - \mathcal{I}_{L_\zeta^\xi}^{-1}(s))^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} L_\zeta^\xi(\mathcal{I}_{L_\zeta^\xi}^{-1}(s)) d\mathcal{I}_{L_\zeta^\xi}^{-1}(s) \\ &+ \int_0^\infty \int_0^\zeta \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_0(dy, dz) \\ &+ \int_0^t \int_0^\infty \int_0^{L_\zeta^\xi(\mathcal{I}_{L_\zeta^\xi}^{-1}(s))} \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{L_\zeta^\xi}^{-1}(s)-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{L_\zeta^\xi}^{-1}(s)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_\alpha(d\mathcal{I}_{L_\zeta^\xi}^{-1}(s), dy, dz) \end{aligned}$$

$$\begin{aligned}
&= \zeta - b \int_0^t \frac{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t) - \mathcal{I}_{L_\zeta^\xi}^{-1}(s))^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} \mathcal{L} \circ L_\zeta^\xi(s) d\mathcal{I}_{L_\zeta^\xi}^{-1}(s) \\
&\quad + \int_0^\infty \int_0^\zeta \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_0(dy, dz) \\
(8.3) \quad &\quad + \int_0^t \int_0^\infty \left(\int_{(\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{L_\zeta^\xi}^{-1}(s)-y)^+}^{\mathcal{I}_{L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{L_\zeta^\xi}^{-1}(s)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_\alpha(d\mathcal{I}_{L_\zeta^\xi}^{-1}(s), dy, (0, \mathcal{L} \circ L_\zeta^\xi(s))).
\end{aligned}$$

For any $s \in [0, \mathcal{I}_{L_\zeta^\xi}(\infty)]$, by (8.1) we have

$$s = \mathcal{I}_{L_\zeta^\xi}(\mathcal{I}_{L_\zeta^\xi}^{-1}(s)) = \int_0^{\mathcal{I}_{L_\zeta^\xi}^{-1}(s)} L_\zeta^\xi(r) dr = \int_0^s L_\zeta^\xi(\mathcal{I}_{L_\zeta^\xi}^{-1}(r)) d\mathcal{I}_{L_\zeta^\xi}^{-1}(r) = \int_0^s \mathcal{L} \circ L_\zeta^\xi(r) d\mathcal{I}_{L_\zeta^\xi}^{-1}(r).$$

Differentiating both sides of this equality and then moving $\mathcal{L} \circ L_\zeta^\xi$ to the left side of the first equality,

$$\frac{ds}{\mathcal{L} \circ L_\zeta^\xi(s)} = d\mathcal{I}_{L_\zeta^\xi}^{-1}(s).$$

Integrating both side of this equality over $[0, t]$, we have

$$\mathcal{I}_{L_\zeta^\xi}^{-1}(t) = \int_0^t \frac{ds}{\mathcal{L} \circ L_\zeta^\xi(s)} = \mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t).$$

Plugging these back into the terms on the right side of the last equality in (8.3) yields that

$$\begin{aligned}
\mathcal{L} \circ L_\zeta^\xi(t) &= \zeta - b \int_0^t \frac{(\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t) - \mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(s))^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} ds \\
&\quad + \int_0^\infty \int_0^\zeta \left(\int_{(\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t)-y)^+}^{\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_{Q,0}(dy, dz) \\
&\quad + \int_0^t \int_0^\infty \left(\int_{(\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(s)-y)^+}^{\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(t)-\mathcal{I}_{\mathcal{L} \circ L_\zeta^\xi}^{-1}(s)} \frac{r^{\alpha-1} dr}{\Gamma(\alpha)\Gamma(1-\alpha)} \right) \tilde{N}_Q(ds, dy),
\end{aligned}$$

where $\tilde{N}_{Q,0}(dy, dz) := \tilde{N}_0(dy, dz)$ and $\tilde{N}_Q(ds, dy) := \tilde{N}_\alpha(d\mathcal{I}_{L_\zeta^\xi}^{-1}(s), dy, (0, \mathcal{L} \circ L_\zeta^\xi(s)))$. It is easy to identify that $\tilde{N}_Q(ds, dy)$ is a compensated PRM on $(0, \infty)^2$ with intensity $\mathcal{L} \circ L_\zeta^\xi(s) d\mathcal{I}_{L_\zeta^\xi}^{-1}(s) \nu_\alpha(dy) = ds \nu_\alpha(dy)$.

Consequently, the limit process $\mathcal{L} \circ L_\zeta^\xi$ is a weak solution of (8.2). The weak uniqueness of solutions of (8.2) follows directly from Theorem 2.1. \square

APPENDIX A: SOME TECHNICAL RESULTS ABOUT THE SCALE FUNCTION

PROPOSITION A.1. *For $p > 1 + \alpha$, there exists a constant $C > 0$ such that for any $h \geq 0$,*

$$\int_0^\infty \frac{|\Delta_h W(s)|^p}{(s+h)^{\alpha+1}} ds + \int_0^h ds \int_0^\infty \frac{|\nabla_y W(s)|^p}{y^{\alpha+2}} dy \leq Ch^{(p-1)\alpha}.$$

PROOF. From (2.9), there exists a constant $C > 0$ such that for any $h \geq 0$,

$$\int_0^\infty \frac{|\Delta_h W(s)|^p}{(s+h)^{\alpha+1}} ds \leq C \int_0^h (s+h)^{p\alpha-\alpha-1} ds + C \int_h^\infty \frac{h^p s^{p(\alpha-1)}}{(s+h)^{\alpha+1}} ds \leq C \cdot h^{(p-1)\alpha}.$$

Similarly, there exists a constant $C > 0$ such that for any $h \geq 0$,

$$\begin{aligned} \int_0^h ds \int_0^s |\nabla_y W(s)|^p y^{-\alpha-2} dy &\leq C \int_0^h ds \int_0^{s/2} \frac{(s-y)^{p(\alpha-1)}}{y^{\alpha+2-p}} dy + C \int_0^h ds \int_{s/2}^\infty \frac{s^{p\alpha}}{y^{\alpha+2}} dy \\ &\leq C \int_0^h s^{p(\alpha-1)} ds \int_0^{s/2} y^{p-\alpha-2} dy + C \int_0^h s^{p\alpha-\alpha-1} ds \leq C \cdot h^{(p-1)\alpha}. \end{aligned}$$

□

PROPOSITION A.2. For $p \geq 2$, there exists a constant $C > 0$ such that for any $h \geq 0$,

$$(A.1) \quad \int_0^\infty ds \int_0^s \frac{|\nabla_y \Delta_h W(s)|^p}{y^{\alpha+2}} dy \leq Ch^{(p-1)\alpha}.$$

PROOF. We first split the double integral in (A.1) into the following four parts:

$$\begin{aligned} J_1(h) &:= \int_0^{4h} ds \int_0^s \frac{|\nabla_y \Delta_h W(s)|^p}{y^{\alpha+2}} dy, & J_2(h) &:= \int_{4h}^\infty ds \int_{s-h}^s \frac{|\nabla_y \Delta_h W(s)|^p}{y^{\alpha+2}} dy, \\ J_3(h) &:= \int_{4h}^\infty ds \int_{s/2}^{s-h} \frac{|\nabla_y \Delta_h W(s)|^p}{y^{\alpha+2}} dy, & J_4(h) &:= \int_{4h}^\infty ds \int_0^{s/2} \frac{|\nabla_y \Delta_h W(s)|^p}{y^{\alpha+2}} dy. \end{aligned}$$

The power mean inequality, along with the equality $\nabla_y \Delta_h W(s) = \nabla_y W(s+h) - \nabla_y W(s)$, implies that uniformly in $h \geq 0$,

$$J_1(h) \leq C \int_0^{4h} ds \int_0^s \frac{|\nabla_y W(s+h)|^p}{y^{\alpha+2}} dy + C \int_0^{4h} ds \int_0^s \frac{|\nabla_y W(s)|^p}{y^{\alpha+2}} dy.$$

By Proposition A.1, the second term on the right side of this inequality can be bounded by $Ch^{(p-1)\alpha}$ uniformly in $h \geq 0$. Applying the change of variables to the first term and then using Proposition A.1 again, it can be bounded uniformly in $h \geq 0$ by

$$C \int_h^{5h} ds \int_0^{s-h} \frac{|\nabla_y W(s)|^p}{y^{\alpha+2}} dy \leq C \int_0^{5h} ds \int_0^s \frac{|\nabla_y W(s)|^p}{y^{\alpha+2}} dy \leq Ch^{(p-1)\alpha}.$$

Hence $J_1(h) \leq C \cdot h^{(p-1)\alpha}$ uniformly in $h \geq 0$. Similarly, notice that $\nabla_y \Delta_h W(s) = \Delta_h \nabla_y W(s) = \Delta_h W(s) - \Delta_h W(s-y)$, by the power mean inequality we have uniformly in $h \geq 0$,

$$J_2(h) \leq C \cdot \int_{4h}^\infty ds \int_{s-h}^s \frac{|\Delta_h W(s)|^p}{y^{\alpha+2}} dy + C \cdot \int_{4h}^\infty ds \int_0^h \frac{|\Delta_h W(y)|^p}{(s-y)^{\alpha+2}} dy.$$

Here the change of variables is also used to get the second integral. From (2.9) we have $|\Delta_h W(s)| \leq Cs^{\alpha-1}h$ for any $s \geq h \geq 0$ and $|\Delta_h W(y)| \leq Ch^\alpha$ for any $y \in (0, h]$. Thus

$$\begin{aligned} J_2(h) &\leq Ch^p \int_{4h}^\infty s^{p(\alpha-1)} ds \int_{s-h}^s \frac{dy}{y^{\alpha+2}} + Ch^{p\alpha+1} \int_{4h}^\infty \frac{ds}{(s-h)^{\alpha+2}} \\ &\leq Ch^{p+1} \int_{4h}^\infty \frac{s^{p(\alpha-1)}}{(s-h)^{\alpha+2}} ds + Ch^{p\alpha+1} \int_{3h}^\infty s^{-\alpha-2} ds \leq Ch^{(p-1)\alpha}. \end{aligned}$$

We now turn to consider $J_3(h)$ and $J_4(h)$. By (2.8), we have uniformly in $h \geq 0$ and $s \geq y > 0$,

$$|\nabla_y \Delta_h W(s)| = \left| \int_0^y d\tilde{y} \int_0^h W''(s+\tilde{h}-\tilde{y}) d\tilde{h} \right| \leq C \int_0^y d\tilde{y} \int_0^h |s+\tilde{h}-\tilde{y}|^{\alpha-2} d\tilde{h} \leq Ch \cdot \frac{|s-y| \wedge y}{(s-y)^{2-\alpha}}.$$

Plugging this into $J_3(h)$ and $J_4(h)$ yields that uniformly in $h \geq 0$,

$$J_3(h) \leq Ch^p \int_{4h}^\infty ds \int_{s/2}^{s-h} \frac{(s-y)^{p(\alpha-1)}}{y^{\alpha+2}} dy \leq Ch^{p\alpha} \int_{4h}^\infty s^{-\alpha-1} ds \leq Ch^{(p-1)\alpha}$$

and

$$J_4(h) \leq Ch^p \int_{4h}^{\infty} |s/2|^{p(\alpha-2)} ds \int_0^{s/2} y^{p-\alpha-2} dy \leq Ch^{(p-1)\alpha}.$$

The desired result follows immediately by putting these estimates together. \square

PROPOSITION A.3. *For $p \geq 2$, there exists a constant $C > 0$ such that for any $h \geq 0$,*

$$\int_0^{\infty} ds \int_s^{s+h} \frac{|\nabla_y W(s+h) - W(s)|^p}{y^{\alpha+2}} dy \leq Ch^{(p-1)\alpha}.$$

PROOF. We split the preceding double integral into the following three parts:

$$\begin{aligned} J_1(h) &:= \int_0^h ds \int_s^h \frac{|\nabla_y W(s+h) - W(s)|^p}{y^{\alpha+2}} dy, \\ J_2(h) &:= \int_0^h ds \int_h^{s+h} \frac{|\nabla_y W(s+h) - W(s)|^p}{y^{\alpha+2}} dy, \\ J_3(h) &:= \int_h^{\infty} ds \int_s^{s+h} \frac{|\nabla_y W(s+h) - W(s)|^p}{y^{\alpha+2}} dy. \end{aligned}$$

By the power mean inequality, we have uniformly in $h \geq 0$,

$$J_1(h) \leq C \int_0^h ds \int_s^h \frac{|\nabla_y W(s+h)|^p}{y^{\alpha+2}} dy + C \int_0^h ds \int_s^h \frac{|W(s)|^p}{y^{\alpha+2}} dy.$$

Since $W(s) \leq Cs^\alpha$ uniformly in $s \geq 0$; see (2.8), the second term on the right side of this inequality can be bounded by $C \int_0^h s^{p\alpha-\alpha-1} ds \leq C \cdot h^{(p-1)\alpha}$ uniformly in $h \in [0, 1]$. For the first term, choosing a positive constant θ satisfying that $1 + \alpha < p\theta < (1 - \alpha)^{-1} \wedge p$, by (2.9) we have

$$\nabla_y W(s+h) = |\nabla_y W(s+h)|^{1-\theta} \cdot |\nabla_y W(s+h)|^\theta \leq C(s+h)^{(1-\theta)\alpha} (s+h-y)^{\theta(\alpha-1)} y^\theta,$$

uniformly in $h \geq s \geq 0$ and $y \in [h, s+h]$. Then

$$\begin{aligned} \int_0^h ds \int_s^h \frac{|\nabla_y W(s+h)|^p}{y^{\alpha+2}} dy &\leq C \int_0^h ds \int_s^h (s+h)^{(1-\theta)p\alpha} (s+h-y)^{p\theta(\alpha-1)} y^{p\theta-\alpha-2} dy \\ &\leq C \cdot h^{(1-\theta)p\alpha} \int_0^h ds \int_s^h (s+h-y)^{p\theta(\alpha-1)} y^{p\theta-\alpha-2} dy \\ &\leq C \cdot h^{(1-\theta)p\alpha} \int_0^h s^{p\theta-\alpha-2} ds \int_0^s (y+h-s)^{p\theta(\alpha-1)} dy \\ &\leq C \cdot h^{(1-\theta)p\alpha+p\theta(\alpha-1)+1} \int_0^h s^{p\theta-\alpha-2} ds \leq Ch^{(p-1)\alpha}. \end{aligned}$$

Hence $J_1(h) \leq C \cdot h^{(p-1)\alpha}$ uniformly in $h \geq 0$. We turn to consider $J_2(h)$. Notice that $\nabla_y W(s+h) - W(s) = \Delta_h W(s) - W(s+h-y)$ for any $y \in [h, s+h]$. By the power mean inequality,

$$J_2(h) \leq C \int_0^h ds \int_h^{s+h} \frac{|\Delta_h W(s)|^p}{y^{\alpha+2}} dy + C \int_0^h ds \int_h^{s+h} \frac{|W(s+h-y)|^p}{y^{\alpha+2}} dy,$$

uniformly in $h \geq 0$. Since $|\Delta_h W(s)| \leq C(s+h)^\alpha$ and $W(s+h-y) \leq C(s+h-y)^\alpha$ uniformly in $s, h \geq 0$ and $y \in [0, s+h]$, there exists a constant $C > 0$ such that for any $h \geq 0$,

$$J_2(h) \leq C \int_0^h ds \int_h^{s+h} \frac{(s+h)^{p\alpha}}{y^{\alpha+2}} dy + C \int_0^h ds \int_h^{s+h} \frac{(s+h-y)^{p\alpha}}{y^{\alpha+2}} dy.$$

The first double integral on the right side of this inequality can be bounded by $Ch^{-\alpha-1} \int_0^h (s+h)^{p\alpha} ds \leq Ch^{(p-1)\alpha}$. Using the change of variables and then Fubini's theorem to the second double integral,

$$\int_0^h ds \int_h^{s+h} \frac{(s+h-y)^{p\alpha}}{y^{\alpha+2}} dy = \int_0^h ds \int_0^s \frac{(s-y)^{p\alpha}}{(y+h)^{\alpha+2}} dy = \int_0^h ds \int_s^h \frac{(y-s)^{p\alpha}}{(s+h)^{\alpha+2}} dy \leq C \cdot h^{(p-1)\alpha}$$

and hence $J_2(h) \leq C \cdot h^{(p-1)\alpha}$ uniformly in $h \geq 0$. Similarly, we also have

$$\begin{aligned} J_3(h) &\leq \int_h^\infty ds \int_s^{s+h} \frac{|\Delta_h W(s)|^p}{y^{\alpha+2}} dy + \int_h^\infty ds \int_s^{s+h} \frac{|W(s+h-y)|^p}{y^{\alpha+2}} dy \\ &\leq Ch^p \int_h^\infty ds \int_s^{s+h} \frac{s^{p(\alpha-1)}}{y^{\alpha+2}} dy + C \int_h^\infty ds \int_s^{s+h} \frac{(s+h-y)^{p\alpha}}{y^{\alpha+2}} dy. \end{aligned}$$

The first term on the right side of the second inequality can be bounded by $C \cdot h^p \int_h^\infty s^{p(\alpha-1)-\alpha-1} ds \leq C \cdot h^{(p-1)\alpha}$ uniformly in $h \geq 0$. By the change of variables, the second term equals to

$$C \int_h^\infty ds \int_0^h \frac{(h-y)^{p\alpha}}{(y+s)^{\alpha+2}} dy \leq C \int_h^\infty ds \int_0^h \frac{y^{p\alpha}}{s^{\alpha+2}} dy \leq C \cdot h^{(p-1)\alpha}$$

and $J_3(h) \leq Ch^{(p-1)\alpha}$ uniformly in $h \geq 0$. The desired result follows by putting these estimates together. \square

PROPOSITION A.4. For any $x \geq 0$, we have as $\epsilon \rightarrow 0+$,

$$\int_0^x ds \int_0^\epsilon \frac{|\nabla_y W(s)|^2}{y^{\alpha+2}} dy + \int_0^x ds \int_0^\epsilon \frac{|\int_0^s \nabla_y K(r) dr|^2}{y^{\alpha+2}} dy \rightarrow 0.$$

PROOF. Here we just prove the convergence of the first integral to 0. The second one can be prove in the same way. For convenience, we assume $x \geq \epsilon \geq 0$. By Fubini's theorem and the fact that $W(s-y) = 0$ for $y \geq s$, we can split the targeted integral into two parts

$$I_1(\epsilon) := \int_0^\epsilon dy \int_0^y \frac{|W(s)|^2}{y^{\alpha+2}} ds \quad \text{and} \quad I_2(\epsilon) := \int_0^\epsilon dy \int_y^x \frac{|\nabla_y W(s)|^2}{y^{\alpha+2}} ds.$$

By (2.8), we have $\int_0^y |W(s)|^2 ds \leq C \cdot y^{2\alpha+1}$ uniformly in $y \geq 0$ and hence $I_1(\epsilon) \leq C \cdot \epsilon^\alpha \rightarrow 0$ as $\epsilon \rightarrow 0+$. Let $\vartheta \in (\alpha+1, (1-\alpha)^{-1} \wedge 2)$. Using (2.9), we have uniformly in $s \geq y \geq 0$,

$$|\nabla_y W(s)|^2 = |\nabla_y W(s)|^\vartheta \cdot |\nabla_y W(s)|^{2-\vartheta} \leq C \cdot y^\vartheta (s-y)^{\vartheta(\alpha-1)} \cdot s^{(2-\vartheta)\alpha}.$$

Plugging this into $I_2(\epsilon)$ implies that

$$I_2(\epsilon) \leq C \cdot \int_0^\epsilon dy \int_y^x \frac{(s-y)^{\vartheta(\alpha-1)}}{y^{\alpha+2-\vartheta}} \cdot s^{(2-\vartheta)\alpha} ds \leq C \cdot x^{(2-\vartheta)\alpha} \int_0^\epsilon \frac{(x-y)^{\vartheta(\alpha-1)+1}}{y^{\alpha+2-\vartheta}} dy \leq C \cdot \epsilon^{\vartheta-\alpha-1},$$

which goes to 0 as $\epsilon \rightarrow 0+$. The desired result follows by putting these estimates together. \square

PROPOSITION A.5. For any $x \geq 0$, we have as $\epsilon \rightarrow 0+$,

$$\int_0^x ds \int_0^\epsilon \frac{|K * \nabla_y W(s)|^2}{y^{\alpha+2}} dy \rightarrow 0.$$

PROOF. We still assume $x \geq \epsilon \geq 0$ and then split the targeted double integral into the following two parts

$$I_3(\epsilon) := \int_0^\epsilon dy \int_0^y \frac{|K * W(s)|^2}{y^{\alpha+2}} ds \quad \text{and} \quad I_4(\epsilon) := \int_0^\epsilon dy \int_y^x \frac{|K * \nabla_y W(s)|^2}{y^{\alpha+2}} ds.$$

By (2.8), we have uniformly in $y \geq 0$,

$$\int_0^y |K * W(s)|^2 ds \leq C \cdot \int_0^y \left| \int_0^s (s-r)^{\alpha-1} r^\alpha dr \right|^2 ds \leq C \cdot \int_0^y s^{4\alpha} ds \leq C \cdot y^{4\alpha+1}.$$

Taking this into $I_3(\epsilon)$, we have $I_3(\epsilon) \leq C \cdot \epsilon^{3\alpha} \rightarrow 0$ as $\epsilon \rightarrow 0+$. By the Cauchy-Schwarz inequality, we have $I_4(\epsilon) \leq C [I_{41}(\epsilon) + I_{42}(\epsilon)]$ uniformly in $x \geq \epsilon \geq 0$, where

$$\begin{aligned} I_{41}(\epsilon) &:= \int_0^\epsilon \frac{dy}{y^{\alpha+2}} \int_y^x \left| \int_0^y (s-r)^{\alpha-1} W(r) dr \right|^2 ds, \\ I_{42}(\epsilon) &:= \int_0^\epsilon \frac{dy}{y^{\alpha+2}} \int_y^x \left| \int_0^s (s-r)^{\alpha-1} \nabla_y W(r) dr \right|^2 ds. \end{aligned}$$

By (2.8) and the fact that $(s-r)^{\alpha-1} \leq (y-r)^{\frac{\alpha-1}{2}} \cdot (s-y)^{\frac{\alpha-1}{2}}$ for $s > y > r > 0$, there exists a constant $C > 0$ such that for any $x \geq y \geq 0$,

$$\int_y^x \left| \int_0^y (s-r)^{\alpha-1} W(r) dr \right|^2 ds \leq C \cdot \int_y^x \left| \int_0^y (y-r)^{\frac{\alpha-1}{2}} r^\alpha dr \right|^2 (s-y)^{\alpha-1} ds \leq C \cdot x^\alpha \cdot y^{3\alpha+1}.$$

Plugging this back into $I_{41}(\epsilon)$, we have $I_{41}(\epsilon) \leq C \cdot \epsilon^{2\alpha} \rightarrow 0$ as $\epsilon \rightarrow 0+$. Let $\vartheta \in (\frac{\alpha+1}{2}, \frac{\alpha+1/2}{1-\alpha} \wedge 1)$. By (2.9), we have $|\nabla_y W(r)| \leq C \cdot (r-y)^{(\alpha-1)\vartheta} y^\vartheta \cdot x^{\alpha(1-\vartheta)}$ uniformly in $0 \leq y \leq r \leq x$ and hence

$$\begin{aligned} \int_y^x \left| \int_0^s (s-r)^{\alpha-1} \nabla_y W(r) dr \right|^2 ds &\leq C \cdot y^{2\vartheta} \int_y^x \left| \int_0^s (s-r)^{\alpha-1} (r-y)^{(\alpha-1)\vartheta} dr \right|^2 ds \\ &\leq C \cdot y^{2\vartheta} \int_y^x (s-y)^{2\alpha+2(\alpha-1)\vartheta} ds \leq C \cdot y^{2\vartheta}. \end{aligned}$$

Taking this back into $I_{42}(\epsilon)$, we have $I_{42}(\epsilon) \leq C \cdot \epsilon^{2\theta-\alpha-1} \rightarrow 0$ as $\epsilon \rightarrow 0+$. The desired result follows by putting all results above together. \square

PROPOSITION A.6. *For any $x \geq 0$, we have as $\epsilon \rightarrow 0+$,*

$$\int_0^x ds \int_0^\epsilon \frac{|W * \nabla_y K(s)|^2}{y^{\alpha+2}} dy \rightarrow 0.$$

PROOF. Notice that $|W * \nabla_y K(s)| \leq C \cdot x^\alpha \cdot \int_0^s \nabla_y K(r) dr$ uniformly in $x \geq s \geq 0$. Thus

$$\int_0^x ds \int_0^\epsilon \frac{|W * \nabla_y K(s)|^2}{y^{\alpha+2}} dy \leq C \cdot \int_0^x ds \int_0^\epsilon \frac{\left| \int_0^s \nabla_y K(r) dr \right|^2}{y^{\alpha+2}} dy,$$

which goes to 0 as $\epsilon \rightarrow 0+$; see Proposition A.4. \square

APPENDIX B: MARKED HAWKES POINT MEASURES

Let \mathbb{U} be a Lusin topological space endowed with the Borel σ -algebra \mathcal{U} . Let $\{\sigma_k : k = 1, 2, \dots\}$ be a sequence of increasing, (\mathcal{F}_t) -adapted random times and $\{\eta_k : k = 1, 2, \dots\}$ be a sequence of i.i.d. \mathbb{U} -valued random variables with distribution $\nu_H(du)$. We assume that η_k is independent of $\{\sigma_j : j = 1, \dots, k\}$ for any $k \geq 0$. In terms of these two sequences we define the (\mathcal{F}_t) -random point measure

$$N_H(ds, du) := \sum_{k=1}^{\infty} \mathbf{1}_{\{\sigma_k \in ds, \eta_k \in du\}}$$

on $(0, \infty) \times \mathbb{U}$. We say $N_H(ds, du)$ is a *marked Hawkes point measure* (MHP) on $(0, \infty) \times \mathbb{U}$ if it has a (\mathcal{F}_t) -intensity $Z(s-)d\nu_H(du)$ with the intensity process $Z := \{Z(t) : t \geq 0\}$ given by

$$Z(t) = \mu(t) + \sum_{k=1}^{N_H(t)} \phi(t - \sigma_k, \eta_k), \quad t \geq 0,$$

for some *kernel* $\phi : \mathbb{R}_+ \times \mathbb{U} \rightarrow [0, \infty)$ and some \mathcal{F}_0 -measurable, non-negative functional-valued random variable $\{\mu(t) : t \geq 0\}$. We usually interpret $\phi(\cdot, u)$ and μ as the impacts of an event with mark u and all events prior to time 0 on the arrival of future events respectively. Following the argument in [35, p.93]; see also Section 2 in [32], on an extension of the original probability space we can define a time-homogeneous PRM $N(ds, du, dz)$ on $(0, \infty) \times \mathbb{U} \times \mathbb{R}_+$ with intensity $d\nu_H(du)dz$ such that

$$N_H(ds, du) = \int_0^{Z(s-)} N(ds, du, dz)$$

and hence the intensity process at time t can be rewritten into

$$Z(t) = \mu(t) + \int_0^t \int_{\mathbb{U}} \int_0^{Z(s-)} \phi(t - s, u) N(ds, du, dz).$$

Denote by $\phi_H := \{\phi_H(t) : t \geq 0\}$ the mean impacts of an event on the arrival of future events with $\phi_H(t) := \int_{\mathbb{U}} \phi(t, u) \nu_H(du)$. We assume ϕ_H is locally integrable. Let $R_H := \{R_H(t) : t \geq 0\}$ be the *resolvent* of ϕ_H defined as the unique solution to $R_H = \phi_H + \phi_H * R_H$. It is usual to interpret R_H as the mean impacts of an event and its triggered events on the arrivals of future events. In addition, we introduce a two-parameter function $R(t, u) = \phi(t, u) + R_H * \phi(t, u)$ on $\mathbb{R}_+ \times \mathbb{U}$ to describe the mean impacts of an event with mark u on the arrivals of future events. An argument similar to the one used in Section 2 in [32] induces the following proposition immediately.

PROPOSITION B.1. *The intensity process Z satisfies the following SVE*

$$Z(t) = \mu(t) + \int_0^t R_H(t - s) \mu(s) ds + \int_0^t \int_{\mathbb{U}} \int_0^{Z(s-)} R(t - s, u) \tilde{N}(ds, du, dz), \quad t \geq 0$$

where $\tilde{N}(ds, du, dz) := N(ds, du, dz) - d\nu_H(du)dz$.

APPENDIX C: STOCHASTIC INTEGRALS WITH RESPECT TO $\mathbb{H}^\#$ -SEMIMARTINGALE

In this section we give a brief introduction to the stochastic integrals with respect to infinite-dimensional semimartingales; readers may refer to [42] for more details. Let \mathbb{H} be a separable Banach space endowed with a norm $\|\cdot\|_{\mathbb{H}}$. We now give the definition of $\mathbb{H}^\#$ -semimartingales.

DEFINITION C.1. *We say Y is a (\mathcal{F}_t) -adapted $\mathbb{H}^\#$ -semimartingale, if it is a stochastic process indexed by $\mathbb{H} \times \mathbb{R}_+$ such that*

- For each $f \in \mathbb{H}$, $Y(f) := \{Y(f, t) : t \geq 0\}$ is a càdlàg (\mathcal{F}_t) -semimartingale with $Y(f, 0) \stackrel{\text{a.s.}}{=} 0$;
- For each $t \geq 0$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and $f_1, \dots, f_m \in \mathbb{H}$, $Y(\sum_{k=1}^m \alpha_k f_k, t) \stackrel{\text{a.s.}}{=} \sum_{k=1}^m \alpha_k Y(f_k, t)$.

Let \mathbb{H}_0 be a dense subset of \mathbb{H} and \mathcal{S}_0 the collection of \mathbb{H} -valued stochastic processes of the form

$$X(t) := \sum_{k=1}^m \xi_k(t) \varphi_k \quad \text{with} \quad \xi_k(t) := \sum_{i=0}^{\infty} \eta_i^k \cdot \mathbf{1}_{[\tau_i^k, \tau_{i+1}^k)}(t),$$

where $m \geq 1$, $\varphi_1, \dots, \varphi_m \in \mathbb{H}_0$, $\{\tau_i^k\}_{i \geq 0}$ is a sequence of non-decreasing (\mathcal{F}_t) -stopping times and $\eta_i^k \in \mathbb{R}^d$ is $\mathcal{F}_{\tau_i^k}$ -measurable. For any $X \in \mathcal{S}_0$, define

$$X_- \cdot Y(t) = \sum_{k=1}^m \int_0^t \xi_k(s-) dY(\varphi_k, t), \quad t \geq 0.$$

DEFINITION C.2. *The $\mathbb{H}^\#$ -semimartingale Y is standard if*

$$(C.1) \quad \mathcal{H}_t := \left\{ \sup_{s \leq t} |X_- \cdot Y(s)| : X \in \mathcal{S}_0, \sup_{s \leq t} \|X(s)\|_{\mathbb{H}} \leq 1 \right\}$$

is stochastically bounded for each $t \geq 0$.

For any \mathbb{H} -valued càdlàg process X and standard $\mathbb{H}^\#$ -semimartingale Y , we can find a sequence $\{X^\epsilon\}_{\epsilon > 0} \subset \mathcal{S}_0$ such that as $\epsilon \rightarrow 0$,

$$\sup_{t \in [0, T]} \|X^\epsilon(t) - X(t)\|_{\mathbb{H}} \xrightarrow{\text{a.s.}} 0 \quad \text{and} \quad X_- \cdot Y := \lim_{\epsilon \rightarrow 0^+} X_-^\epsilon \cdot Y$$

exists a.s. in the sense that $\sup_{t \in [0, T]} |X_- \cdot Y(t) - X_-^\epsilon \cdot Y(t)| \xrightarrow{\mathbb{P}} 0$. Moreover, the limit process $X_- \cdot Y$ is càdlàg, independent of $\{X^\epsilon\}_{\epsilon > 0}$ and called the *stochastic integral* of X with respect to Y . For any (\mathcal{F}_t) -stopping time σ , we have $X_- \cdot Y(t \wedge \sigma) = X_-^\sigma \cdot Y$ with $X_-^\sigma(t) := X_-(t) \mathbf{1}_{[0, \sigma)}(t)$ for $t \geq 0$.

DEFINITION C.3. *A sequence of $\mathbb{H}^\#$ -semimartingales $\{Y_n\}_{n \geq 1}$ is uniformly tight if $\{\mathcal{H}_{n,t}\}_{n \geq 1}$ is uniformly stochastically bounded for any $t \geq 0$, where $\mathcal{H}_{n,t}$ is defined by (C.1) with Y replaced by Y_n . We say Y_n converges weakly to Y and write $Y_n \Rightarrow Y$ if for any $m \geq 1$ and $f_1, \dots, f_m \in \mathbb{H}$,*

$$(Y_n(f_1), \dots, Y_n(f_m)) \xrightarrow{d} (Y(f_1), \dots, Y(f_m)) \quad \text{in } D([0, \infty), \mathbb{R}^m).$$

In addition, we also write $(X_n, Y_n) \Rightarrow (X, Y)$ if

$$(X_n, Y_n(f_1), \dots, Y_n(f_m)) \xrightarrow{d} (X, Y(f_1), \dots, Y(f_m)) \quad \text{in } D([0, \infty), \mathbb{H} \times \mathbb{R}^m).$$

APPENDIX D: STOCHASTIC INTEGRALS WITH RESPECT TO POISSON RANDOM MEASURE

Let $\tilde{N}_1(ds, dy, dz)$ be a compensated (\mathcal{F}_t) -PRM on $(0, \infty)^3$ with intensity $ds\nu_1(dy)dz$, where $\nu_1(dy)$ is a σ -finite measure on \mathbb{R}_+ such that $\nu_1(x, \infty) < \infty$ for any $x > 0$. Let $\{X(t) : t \geq 0\}$ be a (\mathcal{F}_t) -predictable and non-negative process.

THEOREM D.1 (Maximal inequality). *For $p \geq 1$ and $T > 0$, let f be a measurable function on \mathbb{R}_+^2 satisfying $\int_0^T ds \int_0^\infty |f(s, y)|^{2p} \nu_1(dy) < \infty$. If $\sup_{t \in [0, T]} \mathbf{E}[|X(t)|^p] < \infty$, then there exists a constant $C > 0$ depending only on p such that*

$$(D.1) \quad \mathbf{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_0^\infty \int_0^{X(s)} f(s, y) \tilde{N}_1(ds, dy, dz) \right|^{2p} \right] \leq C \sup_{t \in [0, T]} \mathbf{E}[|X(t)|^p] \cdot \left| \int_0^T \int_0^\infty |f(s, y)|^2 \nu_1(dy) ds \right|^p \\ + C \sup_{t \in [0, T]} \mathbf{E}[|X(t)|] \cdot \int_0^T \int_0^\infty |f(s, y)|^{2p} \nu_1(dy) ds.$$

PROOF. By the maximal inequality established in [55, Theorem 1, p.297] for purely discontinuous martingales, there exists a constant $C > 0$ depending only on p such that

$$\mathbf{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \int_0^\infty \int_0^{X(s)} f(s, y) \tilde{N}_1(ds, dy, dz) \right|^{2p} \right]$$

$$\begin{aligned}
&\leq C \cdot \mathbf{E} \left[\left| \int_0^T X(s) ds \int_0^\infty |f(s, y)|^2 \nu_1(dy) \right|^p + \int_0^T X(s) ds \int_0^\infty |f(s, y)|^{2p} \nu_1(dy) \right] \\
&\leq C \cdot \mathbf{E} \left[\left| \int_0^T X(s) ds \int_0^\infty |f(s, y)|^2 \nu_1(dy) \right|^p \right] + C \sup_{t \in [0, T]} \mathbf{E}[|X(t)|] \cdot \int_0^T ds \int_0^\infty |f(s, y)|^{2p} \nu_1(dy).
\end{aligned}$$

By Hölder's inequality; see footnote 9, the first expectation on the right side of the second inequality can be bounded by

$$\int_0^T \mathbf{E}[|X(r)|^p] \int_0^\infty |f(r, y)|^2 \nu_1(dy) dr \cdot \left| \int_0^T \int_0^\infty |f(s, z)|^2 \nu_1(dz) ds \right|^{p-1},$$

which can be bounded by $\sup_{t \in [0, T]} \mathbf{E}[|X(t)|^p] \cdot \left| \int_0^T \int_0^\infty |f(s, y)|^2 \nu_1(dy) ds \right|^p$. The desired result holds. \square

THEOREM D.2 (Stochastic Fubini theorem). *Let $(\mathbb{V}, \mathcal{V}, m)$ be a measurable space. For $T \geq 0$, let f be a measurable function on \mathbb{V} and g, h two measurable functions on $\mathbb{V} \times [0, T] \times \mathbb{R}_+$ satisfying that as $\epsilon \rightarrow 0+$,*

$$\int_{\mathbb{V}} |f(v)| m(dv) \left| \int_0^T ds \int_0^\epsilon |g(v, s, y)|^2 \nu_1(dy) \right|^{1/2} + \int_0^T ds \int_0^\epsilon \left| \int_{\mathbb{V}} f(v) g(v, s, y) m(dv) \right|^2 \nu_1(dy) \rightarrow 0$$

and

$$\int_{\mathbb{V}} |f(v)| m(dv) \int_0^T ds \int_0^\epsilon |h(v, s, y)| \nu_1(dy) + \int_0^T ds \int_0^\epsilon \left| \int_{\mathbb{V}} f(v) h(v, s, y) m(dv) \right| \nu_1(dy) \rightarrow 0.$$

If $\|X\|_{L_T^\infty} < \infty$ a.s., we have

$$\begin{aligned}
&\int_{\mathbb{V}} f(v) m(dv) \int_0^T \int_0^\infty \int_0^{X(s)} g(v, s, y) \tilde{N}_1(ds, dy, dz) \\
&\stackrel{\text{a.s.}}{=} \int_0^T \int_0^\infty \int_0^{X(s)} \int_{\mathbb{V}} f(v) g(v, s, y) m(dv) \tilde{N}_1(ds, dy, dz)
\end{aligned}
\tag{D.2}$$

and

$$\begin{aligned}
&\int_{\mathbb{V}} f(v) m(dv) \int_0^T \int_0^\infty \int_0^{X(s)} h(v, s, y) N_1(ds, dy, dz) \\
&\stackrel{\text{a.s.}}{=} \int_0^T \int_0^\infty \int_0^{X(s)} \int_{\mathbb{V}} f(v) h(v, s, y) m(dv) N_1(ds, dy, dz).
\end{aligned}
\tag{D.3}$$

PROOF. Here we just prove the first the equality (D.2) and the second one can be proved in the same way. It is easy to identify that the two integrals in (D.2) are well-defined. We now show they are equal almost surely. For any $\epsilon \in (0, 1]$, by the assumption that $\nu_1(\epsilon, \infty) < \infty$ and Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{V}} f(v) m(dv) \int_0^T \int_0^\infty \int_0^{X(s)} g(v, s, y) \tilde{N}_1(ds, dy, dz) \\
&= \int_{\mathbb{V}} f(v) m(dv) \int_0^T \int_0^\infty \int_0^{X(s)} g(v, s, y) N_1(ds, dy, dz) \\
&\quad - \int_{\mathbb{V}} f(v) m(dv) \int_0^T \int_\epsilon^\infty X(s) g(v, s, y) ds \nu(dy) \\
&= \int_0^T \int_\epsilon^\infty \int_0^{X(s)} \int_{\mathbb{V}} f(v) g(v, s, y) m(dv) N_1(ds, dy, dz) \\
&\quad - \int_0^T \int_\epsilon^\infty X(s) \int_{\mathbb{V}} f(v) g(v, s, y) m(dv) ds \nu(dy) \\
&= \int_0^T \int_\epsilon^\infty \int_0^{X(s)} \int_{\mathbb{V}} f(v) g(v, s, y) m(dv) \tilde{N}_1(ds, dy, dz).
\end{aligned}$$

Here the two stochastic integrals with respect to $N_1(ds, dy, dz)$ on the right side of the first and second equality are finite sums. Thus the difference between the two integrals in (D.2) can be bounded by $|A_1(\epsilon)| + |A_2(\epsilon)|$ uniformly in $\epsilon \in (0, 1]$ with

$$\begin{aligned} A_1(\epsilon) &:= \int_{\mathbb{V}} f(v)m(dv) \int_0^T \int_0^\epsilon \int_0^{X(s)} g(v, s, y) \tilde{N}_1(ds, dy, dz), \\ A_2(\epsilon) &:= \int_0^T \int_0^\epsilon \int_0^{X(s)} \int_{\mathbb{V}} f(v)g(v, s, y)m(dv) \tilde{N}_1(ds, dy, dz). \end{aligned}$$

It suffices to prove that $|A_1(\epsilon)| + |A_2(\epsilon)| \xrightarrow{P} 0$ as $\epsilon \rightarrow 0+$. For any $\eta > 0$ and $J > 0$, we have $\mathbf{P}(|A_1(\epsilon)| \geq \eta) \leq \mathbf{P}(|A_1(\epsilon)| \geq \eta, \|X\|_{L_T^\infty} \leq J) + \mathbf{P}(\|X\|_{L_T^\infty} > J)$. Since $\|X\|_{L_T^\infty} < \infty$ a.s., then $\mathbf{P}(\|X\|_{L_T^\infty} > J) \rightarrow 0$ as $J \rightarrow \infty$. Notice that

$$\mathbf{P}(|A_1(\epsilon)| \geq \eta, \|X\|_{L_T^\infty} \leq J) \leq \mathbf{P}\left(\left|\int_{\mathbb{V}} f(v)m(dv) \int_0^T \int_0^\epsilon \int_0^{X(s) \wedge J} g(v, s, y) \tilde{N}_1(ds, dy, dz)\right| \geq \eta\right).$$

By Chebyshev's inequality and then Fubini's theorem,

$$\mathbf{P}(|A_1(\epsilon)| \geq \eta, \|X\|_{L_T^\infty} \leq J) \leq \frac{1}{\eta} \int_{\mathbb{V}} |f(v)|m(dv) \mathbf{E}\left[\left|\int_0^T \int_0^\epsilon \int_0^{X(s) \wedge J} g(v, s, y) \tilde{N}_1(ds, dy, dz)\right|\right].$$

By (D.1) with $p = 1/2$ and then Jensen's inequality, the last expectation can be bounded by

$$\begin{aligned} &\mathbf{E}\left[\left|\int_0^T \int_0^\epsilon \int_0^{X(s) \wedge J} |g(v, s, y)|^2 N_1(ds, dy, dz)\right|^{1/2}\right] \\ &\leq \left|\mathbf{E}\left[\int_0^T \int_0^\epsilon \int_0^{X(s) \wedge J} |g(v, s, y)|^2 N_1(ds, dy, dz)\right]\right|^{1/2} \leq \left|J \int_0^T ds \int_0^\epsilon |g(v, s, y)|^2 \nu_1(dy)\right|^{1/2} \end{aligned}$$

and hence

$$\mathbf{P}(|A_1(\epsilon)| \geq \eta, \|X\|_{L_T^\infty} \leq J) \leq \frac{\sqrt{J}}{\eta} \int_{\mathbb{V}} |f(v)| \cdot \left|\int_0^T ds \int_0^\epsilon |g(v, s, y)|^2 \nu_1(dy)\right|^{1/2} m(dv),$$

which goes to 0 as $\epsilon \rightarrow 0+$. Putting these estimates together, we have $\mathbf{P}(|A_1(\epsilon)| \geq \eta) \rightarrow 0$ and hence $|A_1(\epsilon)| \xrightarrow{P} 0$ as $\epsilon \rightarrow 0+$. Similarly, we also can prove that $|A_2(\epsilon)| \xrightarrow{P} 0$ as $\epsilon \rightarrow 0+$. \square

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