Positional Injectivity for Innocent Strategies

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Abstract

In asynchronous games, Melliès proved that innocent strategies are *positional*: their behaviour only depends on the position, not the temporal order used to reach it. This insightful result shaped our understanding of the link between dynamic (*i.e.* game) and static (*i.e.* relational) semantics.

In this paper, we investigate the positionality of innocent strategies in the traditional setting of Hyland-Ong-Nickau-Coquand pointer games. We show that though innocent strategies are not positional, total finite innocent strategies still enjoy a key consequence of positionality, namely *positional injectivity*: they are entirely determined by their positions. Unfortunately, this does not hold in general: we show a counter-example if finiteness and totality are lifted. For finite partial strategies we leave the problem open; we show however the partial result that two strategies with the same positions must have the same P-views of maximal length.

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1 Introduction

Game semantics presents higher-order computation interactively as an exchange of tokens in a two-player game between Player (the program under study), and Opponent (its execution environment) [15, 1]. Game semantics has had a strong theoretical impact on denotational semantics, achieving full abstraction results for languages for which other tools struggle.

At the heart of Hyland and Ong's celebrated model [15] are *innocent strategies*, matching *pure* programs. They matter conceptually and technically: many full abstraction results rely on innocent strategies and their definability properties. Accordingly, innocence is perhaps the most studied notion on the foundational side of game semantics, with questions including categorical reconstructions [13], alternative definitions [16, 14], non-deterministic [18, 6], concurrent [7], or quantitative [17, 4] extensions. In particular, our modern understanding of innocence is shaped by Melliès' homotopy-theoretic reformulation in asynchronous games [16]. In this paper, Melliès also introduced an important result: innocent strategies are *positional*.

Positionality is an elementary notion on games on graphs: a strategy is positional if its behaviour only depends on the current node – the "position" – and not the path leading there. In standard game semantics there is, at first sight, no clear notion of position: plays are primitive, and it is not clear what is the ambient graph. In contrast, asynchronous games and relatives (*e.g.* concurrent games) admit a transparent notion of position: two plays reach the same position if they feature the same moves, though not necessarily in the same order. In investigating positionality, Melliès' motivation was to bridge standard play-based game semantics with more static, *relational*-like semantics [2, 12]. Indeed, points of the *web* in relational semantics correspond to certain positions in game semantics. Positionality of innocent strategies entails that they are entirely defined by their positions (a property we shall call *positional injectivity*), so that collapsing game to relational semantics corresponds exactly to keeping only certain positions. See [8] for a recent account.

Now, traditional Hyland-Ong arena games are by no means disconnected from those developments: bridges with relational semantics were also investigated there, notably by Boudes [3]. There, points of the web match so-called *thick subtrees*, pomsets representing partial explorations of the arena with duplications. This provides *positions* for Hyland-Ong games. But then, are innocent strategies still positional? Though it came to us as a surprise, it is not hard to find a counter-example. So we focus on the key weakening of the question: are innocent strategies *positionally injective*? Our main result is positive, for *total finite* innocent strategies. We first link Hyland-Ong innocence with an alternative, causal formulation inspired from concurrent games [8], allowing a transparent link between a strategy and its positions. Drawing inspiration from the proof of injectivity of the relational model for MELL proof nets [10], we show how to track down duplications in certain well-engineered positions to recover a sufficient portion of the causal structure; and deduce positional injectivity. However, we show that in the general case (without *finiteness* and *totality*), positional injectivity fails. Finally, for finite (but not total) innocent strategies we show a partial result, namely that two strategies with the same positions have the same P-views of maximal length.

Tsukada and Ong [19] show an injective collapse from a category of innocent strategies onto the relational model. Their collapse is similar to ours, with an important distinction: they label moves in each play, coloring contiguous Opponent/Player pairs identically. Labels survive the collapse, allowing to read back causal links directly. This is possible because the web of atomic types is set to comprise countably many such labels – but then, the correspondence between positions and points of the web is lost. In contrast, our theorem requires us to prove injectivity directly, without such labeling.

In Section 2 we introduce the setting and state our main result. In Section 3 we reformulate the problem via a *causal* presentation of game semantics. In Section 4 we present the proof of positional injectivity for total finite innocent strategies. In Section 5, we show some partial results beyond total finite strategies. Finally, in Section 6, we conclude. Detailed proofs are attached in appendix.

2 Innocent Strategies and Positions

2.1 Arenas and Constructions

We start this paper by giving a definition of *arenas*, which represent *types*.

▶ **Definition 1.** *An* ***arena*** *is* $A = \langle |A|, \leq_A, \lambda_A\rangle$ *where* $\langle |A|, \leq_A\rangle$ *is a partial order, and* $\lambda_A : |A| \to \{-,+\}$ *is a* ***polarity function****. Moreover, these data must satisfy:*

$$\begin{array}{rl} \text{finitary:} & \textit{for all } a \in |A|,\ [a]_A = \{a' \in |A| \mid a' \leq_A a\} \textit{ is finite,} \\ \text{forestial:} & \textit{for all } a_1, a_2 \leq_A a, \textit{ then } a_1 \leq_A a_2 \textit{ or } a_2 \leq_A a_1, \\ \text{alternating:} & \textit{for all } a_1 \rightarrowtriangle_A a_2, \textit{ then } \lambda_A(a_1) \neq \lambda_A(a_2), \\ \text{negative:} & \textit{for all } a \in \min(A) = \{a \in |A| \mid a \textit{ minimal}\},\ \lambda_A(a) = -, \end{array}$$

where $a_1 \rightarrowtriangle_A a_2$ *means* $a_1 <_A a_2$ *with no event strictly in between.*

Though our notations differ superficially, our arenas are similar to [15]. They present observable computational events (on a given type) along with their causal dependencies: positive moves are due to Player / the program, and negative moves to Opponent / the environment. We show in Figures 1 and 2, read from top to bottom, the representation of the

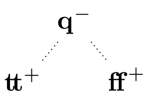


Figure 1 Arena **bool**

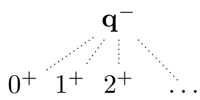


Figure 2 Arena **nat**

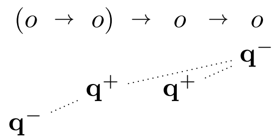


Figure 3 Arena $(o \Rightarrow o) \Rightarrow o \Rightarrow o$

datatypes **bool** and **nat** as arenas. Opponent initiates the execution with $\mathbf{q}^-$, annotated so as to indicate its polarity, and Player may respond any possible value, with a positive move.

We write 1 for the empty arena and o for the arena with exactly one (negative) move. More elaborate types involve matching constructions: the *product* and the *arrow*.

▶ **Definition 2.** *Consider A_1 and A_2 arenas. Then, we define $A_1 \parallel A_2$ as*

$$\begin{array}{rcl} |A_1 \parallel A_2| & = & (\{1\} \times |A_1|) \cup (\{2\} \times |A_2|) \\ (i,a) \leq_{A_1 \parallel A_2} (j,b) & \Leftrightarrow & i = j \quad \wedge \quad a \leq_{A_i} b \\ \lambda_{A_1 \parallel A_2}(i,a) & = & \lambda_{A_i}(a)\,, \end{array}$$

*called their **parallel composition** or **product**, and also written $A_1 \times A_2$.*

For any family $(A_i)_{i\in I}$ of arenas, this extends to $\prod_{i\in I} A_i$ in the obvious way. Any arena A decomposes (up to forest iso) as $A \cong \prod_{i\in I} A_i$ for some family $(A_i)_{i\in I}$ of arenas which are **well-opened**, *i.e.* with *exactly one* initial (*i.e. minimal*) move. We now define the *arrow*:

▶ **Definition 3.** *Consider A_1, A_2 arenas with A_2 well-opened. Then $A_1 \Rightarrow A_2$ has:*

$$\begin{array}{rcl} |A_1 \Rightarrow A_2| & = & (\{1\} \times |A_1|) \cup (\{2\} \times |A_2|) \\ (i,a) \leq_{A_1 \Rightarrow A_2} (j,b) & \Leftrightarrow & (i = j \wedge a \leq_{A_i} b) \vee (i = 2 \wedge a \in \min(A_2)) \\ \lambda_{A_1 \Rightarrow A_2}(i,a) & = & (-1)^i \cdot \lambda_{A_i}(a) \end{array}$$

This extends to all arenas with $A \Rightarrow \prod_{i\in I} B_i = \prod_{i\in I} A \Rightarrow B_i$ and $A \Rightarrow 1 = 1$.

We will mostly use $A \Rightarrow B$ for B well-opened. Figure 3 displays $(o \Rightarrow o) \Rightarrow o \Rightarrow o$, matching the simple type $(o \to o) \to o \to o$ with atomic type o – the position of moves follows a correspondence between those and atoms of the type. These arena constructions describe call-by-name computation: once Opponent initiates computation with $\mathbf{q}^-$, two Player moves become available. Player may call the second argument (terminating computation) or evaluate the first argument, which in turn allows Opponent to call its argument.

2.2 Plays and Strategies

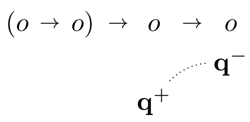
In Hyland-Ong games, players are allowed to *backtrack*, and resume the play from any earlier stage. This is made formal by the notion of *pointing strings*:

▶ **Definition 4.** *A **pointing string** over set Σ is a string $s \in \Sigma^*$, where each move may additionally come equipped with a **pointer** to an earlier move.*

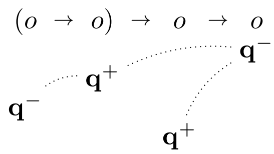
We often write $s = s_1 \dots s_n$ for pointing strings, leaving pointers implicit.

▶ **Definition 5.** *A **play** on arena A is a pointing string $s = s_1 \dots s_n$ over $|A|$ s.t.:*

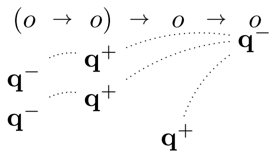
$$\begin{array}{rl} \text{rigid:} & \textit{If } s_i \textit{ points to } s_j\textit{, then } s_j \rightarrowtriangle_A s_i, \\ \text{alternating:} & \textit{for all } 1 \leq i < n,\ \lambda_A(s_i) \neq \lambda_A(s_{i+1}), \\ \text{legal:} & \textit{for all } 1 \leq i \leq n,\ \textit{either } s_i \in \min(A) \textit{ or } s_i \textit{ has a pointer.} \end{array}$$



■ **Figure 4** $\lambda f^{o\to o}.\, \lambda x^o.\, x$



■ **Figure 5** $\lambda f^{o\to o}.\, \lambda x^o.\, f\, x$



■ **Figure 6** $\lambda f^{o\to o}.\, \lambda x^o.\, f\,(f\, x)$

*A play is **well-opened** iff it has exactly one initial move. We write* $\mathsf{Plays}(A)$ *for the set of plays on* A*,* $\mathsf{Plays}^+(A)$ *for even-length plays, and* $\mathsf{Plays}_\bullet(A)$ *for well-opened plays.*

We write ε for the empty play, $\sqsubseteq$ for the prefix, and $\sqsubseteq^+$ if the smaller play has even length. Plays represent higher-order executions. Figures 4, 5 and 6 show plays on the arena of Figure 3; matching typical executions of the corresponding simply-typed λ-term. They are read from top to bottom, with pointers as dotted lines. As in Figure 3, the position of moves encodes their identity in the arena. Strategies, representing programs, are sets of plays:

▶ **Definition 6.** *A strategy* $\sigma : A$ *on arena* A *is a non-empty set* $\sigma \subseteq \mathsf{Plays}^+(A)$ *satisfying*

$$\begin{array}{rl} \text{prefix-closed:} & \forall s \in \sigma, \forall t \sqsubseteq^+ s, t \in \sigma\,, \\ \text{deterministic:} & \forall s \in \sigma,\ \ sab, sab' \in \sigma \implies sab = sab'\,. \end{array}$$

Implicit in the last clause is that sab and sab' also have the same pointers.

2.3 Visibility and Innocence

Innocence captures that the behaviour only depends on which program phrase currently has control. Intuitively, the "current program phrase" is captured by the *P-view*.

▶ **Definition 7.** *For any arena* A*, we set a partial function* $\ulcorner - \urcorner : \mathsf{Plays}(A) \rightharpoonup \mathsf{Plays}(A)$ *as:*

$$\begin{array}{rcll} \ulcorner si \urcorner & = & i & \text{if } i \in \min(A), \\ \ulcorner sn^- m^+ \urcorner & = & \ulcorner sn \urcorner m & \text{if the pointer of } m \text{ is in } \ulcorner sn \urcorner, \\ \ulcorner sn^+ t m^- \urcorner & = & \ulcorner sn \urcorner m & \text{if } m \text{ points to } n, \end{array}$$

undefined otherwise. In the last two cases, m *keeps its pointer in the resulting play.*

If defined, $\ulcorner s \urcorner$ *is the **P-view** of* s*. A play* $s \in \mathsf{Plays}(A)$ *is **visible** iff* $\forall t \sqsubseteq s$*,* $\ulcorner t \urcorner$ *is defined.*

We say that $s \in \mathsf{Plays}(A)$ is a **P-view** iff $\ulcorner s \urcorner = s$. A strategy $\sigma : A$ is **visible** iff any $s \in \sigma$ is visible. In that case, P-views are always well-defined, so that we may formulate:

▶ **Definition 8.** *A strategy* $\sigma : A$ *is **innocent** iff it is visible, and satisfies:*

$$\text{innocence:}\quad \textit{for all } sab, t \in \sigma,\ \textit{if } ta \in \mathsf{Plays}(A) \textit{ and } \ulcorner sa \urcorner = \ulcorner ta \urcorner,\ \textit{then } tab \in \sigma.$$

where, in tab*,* b *points "as in* sab*",* i.e. *so as to ensure that* $\ulcorner sab \urcorner = \ulcorner tab \urcorner$*.*

An innocent $\sigma : A$ is determined by $\ulcorner\!\ulcorner \sigma \urcorner\!\urcorner = \{\ulcorner s \urcorner \mid s \in \sigma\}$, its *P-view forest*. Figures 4, 5 and 6 present P-views, each inducing an innocent strategy *via* the P-view forest obtained by even-length prefix closure. Likewise, Figures 7 and 8 induce strategies for the so-called simply-typed "Kierstead terms" $\lambda f^{(o\to o)\to o}.\, f\,(\lambda x^o.\, f\,(\lambda y^o.\, x))$ and $\lambda f^{(o\to o)\to o}.\, f\,(\lambda x^o.\, f\,(\lambda y^o.\, y))$. P-views are well-opened, so innocent strategies are determined by their set $\sigma_\bullet$ of well-opened plays.

Innocent strategies form a cartesian closed category Inn with as objects arenas, and morphisms from A to B the innocent strategies $\sigma : A \Rightarrow B$. Composing $\sigma : A \Rightarrow B$ and $\tau : B \Rightarrow C$ involves a "parallel interaction plus hiding" mechanism, which we omit [15].

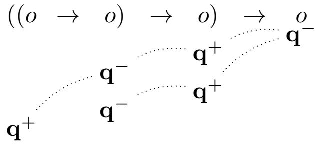


Figure 7 $K_x : ((o \to o) \to o) \to o$

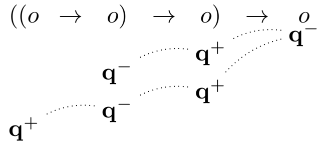


Figure 8 $K_y : ((o \to o) \to o) \to o$

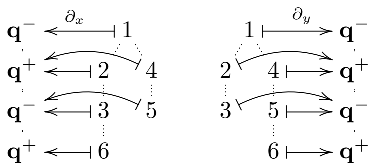


Figure 9 Deseq. K_x and K_y

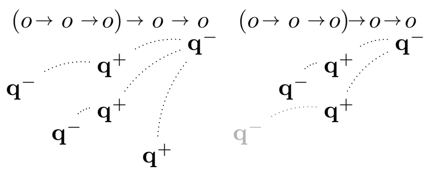


Figure 10 Non-positionality of innocence

2.4 Positions

Boudes' "thick subtrees" [3], called *positions* in this paper, are the central concept informing the link between innocent game semantics and relational semantics. They are simply desequentialized plays, or in other words prefixes of the arena with duplications.

To introduce positions, our first stop is the following notion of *configuration*.

▶ **Definition 9.** *A* ***configuration*** $x \in \mathscr{C}(A)$ *of arena* A *is a tuple* $x = \langle |x|, \leq_x, \partial_x \rangle$ *such that* $\langle |x|, \leq_x \rangle$ *is a finite tree, and* $\partial_x : |x| \to |A|$*, the* ***display map****, is a labeling function s.t.:*

minimality-respecting: *for all* $a \in |x|$*,* a *is* $\leq_x$*-minimal iff* $\partial_x(a)$ *is* $\leq_A$*-minimal,*
causality-preserving: *for all* $a_1, a_2 \in |x|$*, if* $a_1 \rightarrowtriangle_x a_2$ *then* $\partial_x(a_1) \rightarrowtriangle_A \partial_x(a_2)$*.*

We call **events** the elements of $|x|$. Note $\langle |x|, \leq_x \rangle$ has exactly one minimal event, which suffices as innocent strategies are determined by well-opened plays. Configurations include:

▶ **Definition 10.** *The* ***desequentialization*** $(\!|s|\!) \in \mathscr{C}(A)$ *of* $s = s_1 \dots s_n \in \mathsf{Plays}_\bullet(A)$ *has* $|(\!|s|\!)| = \{1, \dots, n\}$*,* $\partial_{(\!|s|\!)}(i) = s_i$*, and* $i \leq_{(\!|s|\!)} j$ *if there is a chain of pointers from* s_j *to* s_i *in* s*.*

We show in Figure 9 the desequentialization of the maximal P-views of K_x and K_y from Figures 7 and 8. Extracting $(\!|s|\!)$ is a first step, we must then forget the identity of its events:

▶ **Definition 11.** *A bijection* $\varphi : |x| \cong |y|$ *is an* ***isomorphism*** $\varphi : x \cong y$ *iff it is*

arena-preserving: *for all* $a \in |x|$*,* $\partial_y(\varphi(a)) = \partial_x(a)$*,*
causality-respecting: *for all* $a_1, a_2 \in |x|$*, we have* $a_1 \rightarrowtriangle_x a_2$ *iff* $\varphi(a_1) \rightarrowtriangle_y \varphi(a_2)$*.*

A ***position*** *of* A*, written* $\mathsf{x} \in (\!(A)\!)$*, is an isomorphism class of configurations.*

If $s \in \mathsf{Plays}_\bullet(A)$, the **position** $(\!(s)\!) \in (\!(A)\!)$ is the isomorphism class of $(\!|s|\!)$.

We pause to consider the *positionality of innocent strategies* as mentioned in the introduction. Though it will only play a very minor role, we define *positional* strategies:

▶ **Definition 12.** *Consider* $\sigma : A$ *a strategy on* A*. We set the condition:*

positional: $\forall sab, t \in \sigma,\ ta \in \mathsf{Plays}(A),\ (\!(sa)\!) = (\!(ta)\!) \implies \exists tab \in \sigma, (\!(sab)\!) = (\!(tab)\!)$.

Innocent strategies are not positional: Figure 10 displays (the two maximal P-views of) the innocent strategy for the λ-term $\lambda f^{o \to o \to o}.\, \lambda x^o.\, f\,(f \perp x)\,(f \perp \perp)$. On the right hand side, the last Opponent move is grayed out as an extension of a P-view triggering no response. After the fifth move the position is the same, contradicting positionality. In Melliès' asynchronous games [16], explicit copy indices help distinguish the two calls to f. The two plays no longer reach the same position, restoring positionality. But even in asynchronous games, if positions were quotiented by symmetry so as to match relational semantics, positionality would fail.

We turn to the weaker *positional injectivity.* If $\sigma : A$, its **positions** are those reached by well-opened plays, *i.e.* $(\!|\sigma|\!) = \{(\!|s|\!) \mid s \in \sigma_\bullet\} \subseteq (\!|A|\!)$. We may finally ask our main question:

▶ **Question 1** (Positional Injectivity). *If σ, τ are innocent and $(\!|\sigma|\!) = (\!|\tau|\!)$, do we have $\sigma = \tau$?*

2.5 Links with the Relational Model

To fully appreciate this question, it is informative to consider the link with the relational model. We start with the following observation concerning positions on the arrow arena.

▶ **Fact 1.** *Consider A and B arenas, and write $\mathcal{M}_f(X)$ for the **finite multisets** on X. Then, we have a bijection $(\!|A \Rightarrow B|\!) \cong \mathcal{M}_f((\!|A|\!)) \times (\!|B|\!)$.*

Recall [12] that the relational model forms a cartesian closed category $\mathsf{Rel}_!$ having *sets* as objects; and as morphisms from A to B the *relations* $R \subseteq \mathcal{M}_f(A) \times B$. Considering simple types generated from o and the arrow $A \to B$, and setting the relational interpretation of o as $[\![o]\!]_{\mathsf{Rel}_!} = \{\mathbf{q}\}$, then for any type A, there is a bijection $r_A : (\!|[\![A]\!]_{\mathsf{Inn}}|\!) \cong [\![A]\!]_{\mathsf{Rel}_!}$.

▶ **Theorem 13.** *This extends to a functor $(\!| - |\!) : \mathsf{Inn} \to \mathsf{Rel}_!$, which preserves the interpretation: for any term $M : A$ of the simply-typed λ-calculus, $r_A((\!|[\![M]\!]_{\mathsf{Inn}}|\!)) = [\![M]\!]_{\mathsf{Rel}_!}$.*

This *relational collapse* of innocent strategies has been studied extensively [3, 16, 19, 4, 9]. The inclusion $\subseteq$ is easy; the difficulty in proving $\supseteq$ is that game-semantic interaction is temporal: positions arising relationally might, in principle, fail to appear game-semantically because reproducing them yields a deadlock. For innocent strategies this does not happen: this may be proved through connections with syntax [3, 19] or semantically [4, 9].

In [19], Tsukada and Ong prove a similar collapse injective. This seems to answer Question 1 positively – but this is not so simple. The interpretation in $\mathsf{Rel}_!$ is parametrized by a set X for the ground type o. In [19], X is required to be countably infinite: this way one allocates one tag for each pair of chronologically contiguous O/P moves, encoding the *causal / axiom links.* In contrast, for Question 1 we are forced to interpret o with a singleton set $\{\mathbf{q}\}$, or lose the correspondence between points of the web and positions. We must reconstruct strategies directly from their desequentializations, with no help from labeling or coloring.

2.6 Main result

At first this seems desperate. In [19], an innocent strategy may already be reconstructed from the desequentialization of its P-views. But here, the two plays of Figures 7 and 8 yield the configurations of Figure 9, which are isomorphic – so give the same position. Nevertheless K_x and K_y *can* be distinguished, via their behaviour under replication. In both plays of Figure 11, we replay the move to which the deepest $\mathbf{q}^+$ points. This brings K_x and K_y to react differently, obtaining plays whose positions separate $(\!|K_x|\!)$ and $(\!|K_y|\!)$. So, by observing the behaviour of a strategy under replication, we can infer some temporal information.

Most of the paper will be devoted to turning this idea into a proof. However, we have only been able to prove the result with the following additional restrictions on strategies.

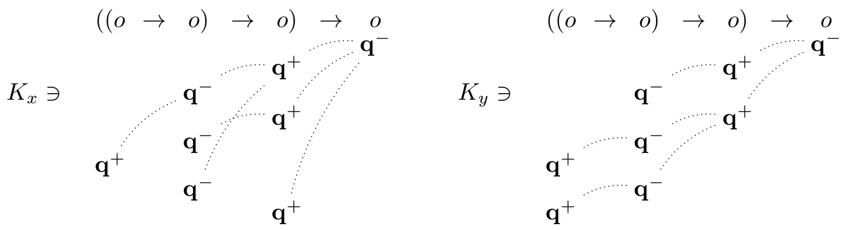


Figure 11 Plays yielding positions distinguishing K_x and K_y

▶ **Definition 14.** *For A an arena, we define conditions on innocent strategies $\sigma : A$ as:*

total: *for all $s \in \sigma$, if $sa \in \mathsf{Plays}(A)$ then there exists b such that $sab \in \sigma$,*
finite: *the set $\ulcorner\sigma\urcorner = \{\ulcorner s\urcorner \mid s \in \sigma\}$ is finite.*

Total finite strategies are already well-known: on arenas interpreting simple types they exactly correspond to β-normal η-long normal forms of simply-typed λ-terms.

We now state our main result, **positional injectivity**:

▶ **Theorem 15.** *For any $\sigma, \tau : A$ innocent total finite, $\sigma = \tau$ iff $\llparenthesis \sigma \rrparenthesis = \llparenthesis \tau \rrparenthesis$.*

As observed in Section 2.1, all arenas decompose as $A = \prod_{i \in I} A_i$ with A_i well-opened. As $\times$ is a cartesian product in Inn, strategies $\sigma : A$ also decompose as $\sigma = \langle \sigma_i \mid i \in I \rangle$ with $\sigma_i : A_i$ for all $i \in I$. From innocence it follows that $\llparenthesis \langle \sigma_i \mid i \in I \rangle \rrparenthesis \cong \sum_{i \in I} \llparenthesis \sigma_i \rrparenthesis$, so it suffices to prove Theorem 15 for A well-opened. From now on, we consider all arenas well-opened.

3 Causal Presentation

Besides the behaviour of strategies under replication, plays also include the order, irrelevant for our purposes, in which branches are explored by Opponent. To isolate the effect of replication, we introduce a *causal* version of strategies inspired from *concurrent games* [5].

3.1 Augmentations

This formulation rests on the notion of *augmentations*. Intuitively those correspond to expanded trees of P-views, which enrich configurations with causal wiring from the strategy.

▶ **Definition 16.** *An **augmentation** on arena A is a tuple $\mathscr{q} = \langle |\mathscr{q}|, \leq_{\llparenthesis \mathscr{q} \rrparenthesis}, \leq_{\mathscr{q}}, \partial_{\mathscr{q}} \rangle$, where $\llparenthesis \mathscr{q} \rrparenthesis = \langle |\mathscr{q}|, \leq_{\llparenthesis \mathscr{q} \rrparenthesis}, \partial_{\mathscr{q}} \rangle \in \mathscr{C}(A)$, and $\langle |\mathscr{q}|, \leq_{\mathscr{q}} \rangle$ is a tree satisfying:*

rule-abiding: *for all $a_1, a_2 \in |\mathscr{q}|$, if $a_1 \leq_{\llparenthesis \mathscr{q} \rrparenthesis} a_2$, then $a_1 \leq_{\mathscr{q}} a_2$,*
courteous: *for all $a_1 \rightarrowtriangle_{\mathscr{q}} a_2$, if $\lambda(a_1) = +$ or $\lambda(a_2) = -$, then $a_1 \rightarrowtriangle_{\llparenthesis \mathscr{q} \rrparenthesis} a_2$,*
deterministic: *for all $a^- \rightarrowtriangle_{\mathscr{q}} a_1^+$ and $a^- \rightarrowtriangle_{\mathscr{q}} a_2^+$, then $a_1 = a_2$,*

*we then write $\mathscr{q} \in \mathsf{Aug}(A)$, and call $\llparenthesis \mathscr{q} \rrparenthesis \in \mathscr{C}(A)$ the **desequentialization** of $\mathscr{q}$.*

Events of $|\mathscr{q}|$ inherit a polarity with $\lambda(a) = \lambda_A(\partial_{\mathscr{q}}(a))$. By *rule-abiding* and *courteous*, $\langle |\mathscr{q}|, \leq_{\mathscr{q}} \rangle$ and $\langle |\mathscr{q}|, \leq_{\llparenthesis \mathscr{q} \rrparenthesis} \rangle$ have the same minimal event $\mathsf{init}(\mathscr{q})$, called the **initial** event. If $a \in |\mathscr{q}|$ is not initial, there is a unique $a' \in |\mathscr{q}|$ such that $a' \rightarrowtriangle_{\mathscr{q}} a$, written $a' = \mathsf{pred}(a)$ and called the **predecessor** of a. Likewise, a non-initial $a \in |\mathscr{q}|$ also has a unique $a'' \in |\mathscr{q}|$

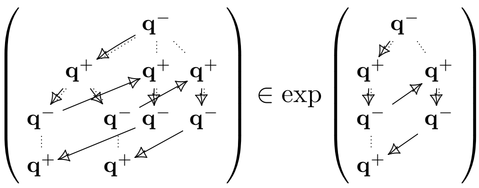


Figure 12 Causal K_x and its expansion

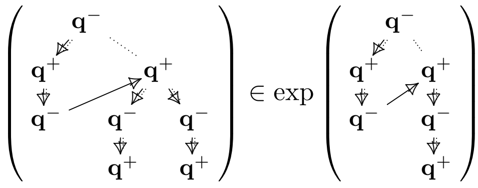


Figure 13 Causal K_y and its expansion

such that $a'' \rightarrowtriangle_{(\!|\mathscr{q}|\!)} a$, written $a'' = \mathsf{just}(a)$ and called the **justifier** of a. By *courteous* and as immediate causality alternates in A (and hence in $(\!|\mathscr{q}|\!)$), both $\mathsf{pred}(a)$ and $\mathsf{just}(a)$ have polarity opposite to a. They may not coincide, however from *courteous* they do for a negative.

Figures 12 and 13 show augmentations – though the corresponding definitions remain to be seen, those are the causal expansions of K_x and K_y matching the plays of Section 2.6. In such diagrams, immediate causality from the configuration appears as dotted lines, whereas that coming from the augmentation itself appears as $\rightarrowtriangle$. We set a few auxiliary conditions:

▶ **Definition 17.** *Let $\mathscr{q} \in \mathsf{Aug}(A)$ be an augmentation. We set the conditions:*

receptive:	*for all $a \in |\mathscr{q}|$, if $\partial_{\mathscr{q}}(a) \rightarrowtriangle_A b^-$, there is $a \rightarrowtriangle_{\mathscr{q}} b'$ such that $\partial_{\mathscr{q}}(b') = b$,*
$+$-covered:	*for all $a \in |\mathscr{q}|$ maximal in $\mathscr{q}$, we have $\lambda(a) = +$,*
$-$-linear:	*for all $a \rightarrowtriangle_{\mathscr{q}} a_1^-$, $a \rightarrowtriangle_{\mathscr{q}} a_2^-$, if $\partial_{\mathscr{q}}(a_1) = \partial_{\mathscr{q}}(a_2)$ then $a_1 = a_2$.*

We say that $\mathscr{q} \in \mathsf{Aug}(A)$ is **total** iff it is receptive and $+$-covered. We will also refer to receptive $-$-linear augmentations as **causal strategies**.

3.2 From Strategies to Causal Strategies

We may easily represent an innocent strategy as a causal strategy:

▶ **Proposition 18.** *For $\sigma : A$ finite innocent on A well-opened, we set components*

$$|\hat{\sigma}| = \{\ulcorner s \urcorner \mid s \in \sigma \wedge s \neq \varepsilon\} \cup \{\ulcorner sa \urcorner \mid s \in \sigma \wedge sa \in \mathsf{Plays}(A)\}\,,$$

$s \leq_{\hat{\sigma}} t$ iff $s \sqsubseteq t$, $sa \leq_{(\!|\hat{\sigma}|\!)} satb$ iff there is a chain of justifiers from b to a, and $\partial_{\hat{\sigma}}(sa) = a$.
Then $\hat{\sigma} = \langle |\hat{\sigma}|, \leq_{\hat{\sigma}}, \leq_{(\!|\hat{\sigma}|\!)}, \partial_{\hat{\sigma}} \rangle \in \mathsf{Aug}(A)$ is a causal strategy, and is total iff σ is total.

The proof appears as Lemmas 39 and 40 in Appendix A. As for configurations, so as to forget the concrete identity of events we consider augmentations up to *isomorphism*:

▶ **Definition 19.** *A* ***morphism*** *$\varphi : \mathscr{q} \to \mathscr{p}$ is a function $\varphi : |\mathscr{q}| \to |\mathscr{p}|$ satisfying:*

arena-preserving:	*$\partial_{\mathscr{p}} \circ \varphi = \partial_{\mathscr{q}}$,*
causality-preserving:	*for all $a_1, a_2 \in |\mathscr{q}|$, if $a_1 \rightarrowtriangle_{\mathscr{q}} a_2$ then $\varphi(a_1) \rightarrowtriangle_{\mathscr{p}} \varphi(a_2)$,*
configuration-preserving:	*for all $a_1, a_2 \in |\mathscr{q}|$, if $a_1 \rightarrowtriangle_{(\!|\mathscr{q}|\!)} a_2$ then $\varphi(a_1) \rightarrowtriangle_{(\!|\mathscr{p}|\!)} \varphi(a_2)$.*

An ***isomorphism*** *is an invertible morphism – we then write $\varphi : \mathscr{q} \cong \mathscr{p}$.*

Note that by *arena-preserving*, φ must send $\mathsf{init}(\mathscr{q})$ to $\mathsf{init}(\mathscr{p})$.

The reader may check that the construction of Proposition 18 applied to K_x and K_y yields, up to isomorphism, the (small) augmentations of Figures 12 and 13. The next fact shows that augmentations are indeed an alternative presentation of innocent strategies.

▶ **Lemma 20.** *For any finite innocent strategies σ, τ on arena A, then $\sigma = \tau$ iff $\hat{\sigma} \cong \hat{\tau}$.*

Proof. Clearly, $\sigma = \tau$ implies $\hat{\sigma} = \hat{\tau}$. Conversely, assume $\varphi : \hat{\sigma} \cong \hat{\tau}$. Take $s = s_1 \dots s_n \in \ulcorner\sigma\urcorner$, and write $s_{\leq i} = s_1 \dots s_i$. Then we have a chain $s_{\leq 1} \rightarrowtriangle_{\hat{\sigma}} s_{\leq 2} \rightarrowtriangle_{\hat{\sigma}} \dots \rightarrowtriangle_{\hat{\sigma}} s_{\leq n-1} \rightarrowtriangle_{\hat{\sigma}} s$, transported through φ to $t_{\leq 1} \rightarrowtriangle_{\hat{\tau}} \dots \rightarrowtriangle_{\hat{\tau}} t$. By *arena-preserving*, $t_i = s_i$ for all $1 \leq i \leq n$. Finally by *configuration-preserving*, s and t have the same pointers, hence $s = t$ and $s \in \tau$. Symmetrically, any P-view $t \in \ulcorner\tau\urcorner$ is in σ, hence $\ulcorner\sigma\urcorner = \ulcorner\tau\urcorner$ and $\sigma = \tau$ by innocence. ◀

3.3 Expansions of Causal Strategies

Besides including representations of innocent strategies, augmentations can also represent their *expansions*, *i.e.* arbitrary plays, with Opponent's scheduling factored out.

▶ **Definition 21.** *Consider A an arena, and $\mathscr{p} \in \mathsf{Aug}(A)$ a causal strategy. An* ***expansion*** *of $\mathscr{p}$, written $\mathscr{q} \in \exp(\mathscr{p})$, is $\mathscr{q} \in \mathsf{Aug}(A)$ such that:*

simulation: *there is a (necessarily unique) morphism $\varphi : \mathscr{q} \to \mathscr{p}$,*
+-obsessional: *for all $a^- \in |\mathscr{q}|$ and $\varphi(a^-) \rightarrowtriangle_{\mathscr{p}} b^+$, there is $a^- \rightarrowtriangle_{\mathscr{q}} a'$ s.t. $\varphi(a') = b^+$.*

The relationship between a causal strategy $\mathscr{p}$ and $\mathscr{q} \in \exp(\mathscr{p})$ is analogous to that between an arena A and a configuration $x \in \mathscr{C}(A)$: $\mathscr{q}$ explores a prefix of $\mathscr{p}$, possibly visiting the same branch many times. However, *determinism* ensures that only Opponent may cause duplications, and $+$-*obsessional* ensures that only Opponent may refuse to explore certain branches – if a Player move is available in $\mathscr{p}$, then it must appear in all corresponding branches of $\mathscr{q}$. Uniqueness of the morphism follows from $-$-linearity and determinism – see Lemma 42 in Appendix A. Figures 12 and 13 show expansions of (the causal strategies corresponding to) K_x and K_y.

Now, we set $⦇\mathscr{p}⦈ = \{⦇\mathscr{q}⦈ \mid \mathscr{q} \in \exp(\mathscr{p})\}$ the **positions** of a causal strategy $\mathscr{p}$, where $⦇\mathscr{q}⦈$ is the isomorphism class of $⦇\mathscr{q}⦈$. By Lemma 20, any innocent $\sigma : A$ yields a causal strategy $\hat{\sigma} : A$, so this leaves us with the task to prove that the two notions of position coincide.

▶ **Proposition 22.** *For any total finite innocent strategy $\sigma : A$, we have $⦇\sigma⦈ = ⦇\hat{\sigma}⦈$.*

Proof. Additional details appear in Appendix A. Any $\mathsf{x} \in ⦇\sigma⦈$ is the isomorphism class of $⦇s⦈$ for $s = s_1 \dots s_n \in \sigma$. We build an expansion $\mathscr{q}(s) \in \exp(\hat{\sigma})$ as follows. Its configuration is $⦇\mathscr{q}(s)⦈ = ⦇s⦈$ (see Definition 10) with events $|\mathscr{q}(s)| = \{1, \dots, n\}$. Its causal order is $i \leq_{\mathscr{q}(s)} j$ iff $j \geq i$ and s_i is reached in the computation of $\ulcorner s_{\leq j}\urcorner$. To show that $\mathscr{q}(s) \in \exp(\hat{\sigma})$ we must provide a morphism $\varphi : \mathscr{q}(s) \to \hat{\sigma}$, which is simply $\varphi(i) = \ulcorner s_{\leq i}\urcorner$. So, $\mathsf{x} = ⦇\mathscr{q}(s)⦈ \in ⦇\hat{\sigma}⦈$.

Reciprocally, take $\mathsf{x} \in ⦇\hat{\sigma}⦈$, obtained as the isomorphism class of some $⦇\mathscr{q}⦈$, for $\mathscr{q} \in \exp(\hat{\sigma})$. From the totality of σ, $\mathscr{q}$ has maximal events all positive – it has exactly as many Player as Opponent events, and admits a linear extension $s = s_1 \dots s_n$ which is *alternating*, *i.e.* $\lambda(s_i) \neq \lambda(s_{i+1})$ for all $1 \leq i \leq n-1$. Besides, for any $1 \leq i \leq n$, $\ulcorner s_{\leq i}\urcorner$ (treating s as a play on arena $⦇\mathscr{q}⦈$) coincides with $[s_i]_{\mathscr{q}} = \{s \in |\mathscr{q}| \mid s \leq_{\mathscr{q}} s_i\}$, totally ordered by $\leq_{\mathscr{q}}$. So, writing $\partial_{\mathscr{q}}(s) = \partial_{\mathscr{q}}(s_1) \dots \partial_{\mathscr{q}}(s_n) \in \mathsf{Plays}(A)$ with pointers inherited from $⦇\mathscr{q}⦈$, $\ulcorner\partial_{\mathscr{q}}(s)_{\leq i}\urcorner \in \ulcorner\sigma\urcorner$, hence $\partial_{\mathscr{q}}(s) \in \sigma$ by innocence and $⦇\partial_{\mathscr{q}}(s)⦈ \cong ⦇s⦈$. Therefore, $⦇\mathscr{q}⦈ = ⦇\partial_{\mathscr{q}}(s)⦈ \in ⦇\sigma⦈$. ◀

The idea is that plays in σ are exactly linearizations of expansions of $\hat{\sigma}$. From a play we get an expansion by factoring out Opponent's scheduling, mimicking the construction of P-views while keeping duplicated branches separate. Reciprocally, an expansion allows many (alternating) linearizations. For instance, the two plays of Section 2.6 are respectively linearizations of the expansions of Figures 12 and 13. This proposition fails if σ is not total, as expansions may then have trailing Opponent moves, preventing an alternating linearization.

Thanks to Proposition 22, we focus on positions reached by expansions of causal strategies.

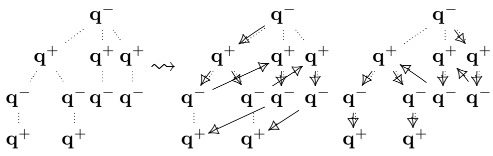


Figure 14 Non-unique causal explanation

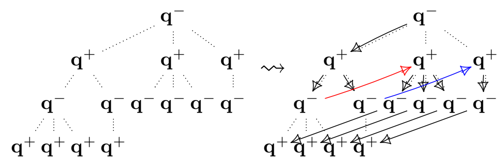


Figure 15 Unique causal explanation

4 Positional Injectivity

We now come to the main contribution of this paper, the proof of positional injectivity for total finite causal strategies. We start this section by introducing the proof idea.

4.1 Forks and Characteristic Expansions

Just from the static snapshot offered by positions, we must deduce the strategy.

Given $z \in \mathscr{C}(A)$, can we uniquely reconstruct its *causal explanation*, *i.e.* $\mathscr{q} \in \mathsf{Aug}(A)$ such that $z = \llparenthesis \mathscr{q} \rrparenthesis$? In general, there is no reason why $\mathscr{q}$ would be uniquely determined. Indeed, in Figure 14, we show on the left hand side the configuration z_1 underlying Figure 12 – up to iso it has exactly two causal explanations, shown on the right. The rightmost augmentation is not an expansion of K_x, so K_x is not the only strategy featuring (the isomorphism class of) z_1. However, we *can* find a position unique to K_x. Consider z_2 the configuration on the left hand side of Figure 15. The only possible augmentation (up to iso) yielding z_2 as a desequentialization appears on the right hand side (call it $\mathscr{q}$): every other attempt to guess causal wiring fails. In particular, the red and blue immediate causal links are forced by the cardinality of the subsequent duplications. But $\mathscr{q}$ is an expansion of the unique maximal branch of K_x – so it suffices to see z_2 in $\llparenthesis \sigma \rrparenthesis$ to know that $\sigma = K_x$.

This suggests a proof idea: given $\mathscr{p}_1, \mathscr{p}_2 : A$ causal strategies with $\llparenthesis \mathscr{p}_1 \rrparenthesis = \llparenthesis \mathscr{p}_2 \rrparenthesis$, we devise a *characteristic expansion* of $\mathscr{p}_1$ with duplications chosen to make the causal structure essentially unique; meaning it must be an expansion of $\mathscr{p}_2$ as well. We do this by using:

▶ **Definition 23.** *A **fork** in $\mathscr{q} \in \mathsf{Aug}(A)$ is a maximal non-empty set $X \subseteq |\mathscr{q}|$ s.t.:*

negative: *for all $a \in X$, $\lambda(a) = -$,*
sibling: *$X = \{\mathsf{init}(\mathscr{q})\}$ or there is $b \in |\mathscr{q}|$ such that for all $a \in X$, $b \rightarrowtriangle_{\mathscr{q}} a$,*
identical: *for all $a_1, a_2 \in X$, $\partial_{\mathscr{q}}(a_1) = \partial_{\mathscr{q}}(a_2)$.*

We write $\mathsf{Fork}(\mathscr{q})$ for the set of forks in augmentation $\mathscr{q}$.

If $\mathscr{p}$ is a causal strategy, $\mathscr{q} \in \exp(\mathscr{p})$ and $X \in \mathsf{Fork}(\mathscr{q})$, the definition of expansions ensures that all Player moves caused by Opponent moves in X are copies. So if X has **cardinality** $\sharp X = n$, and if we find exactly one set of cardinality $\geq n$ of equivalent Player moves in $\llparenthesis \mathscr{q} \rrparenthesis$, we may deduce that there is a causal link. For instance, in Figure 15, the causal successors for the fork of cardinality 3 may be found so. In general though, several Opponent moves may cause indistinguishable Player moves, so that the cardinality of a set Y of duplicated Player moves is the sum of the cardinalities of the predecessor forks. To allow us to identify these predecessor sets uniquely, the trick is to construct the expansion so that all forks have cardinality a distinct power of 2, making it so that the predecessor forks can be inferred from the binary decomposition of $\sharp Y$. This brings us to the following definition.

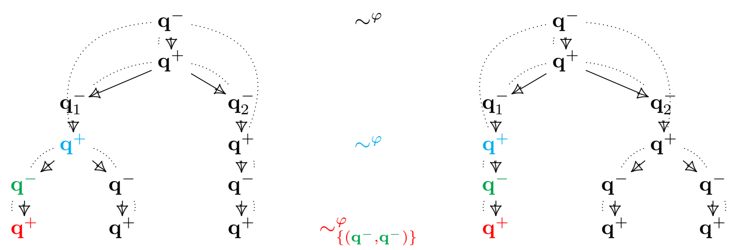


Figure 16 Distinct characteristic expansions reaching the same position

▶ **Definition 24.** *A **characteristic expansion** of* $\mathscr{p}$ *is* $\mathscr{q} \in \exp(\mathscr{p})$ *such that:*

$$\begin{array}{rl} \text{injective:} & \textit{for } X, Y \in \mathsf{Fork}(\mathscr{q}), \textit{ if } \sharp X = \sharp Y \textit{ then } X = Y, \\ \text{well-powered:} & \textit{for all } X \in \mathsf{Fork}(\mathscr{q}), \textit{ there is } n \in \mathbb{N} \textit{ such that } \sharp X = 2^n, \\ \text{−-obsessional:} & \textit{for all } a^+ \in |\mathscr{q}|, \textit{ if } \partial_{\mathscr{q}}(a^+) \rightarrow_A b^-, \textit{ there is } a^+ \rightarrow_{\mathscr{q}} a' \textit{ s.t. } \partial_{\mathscr{q}}(a') = b^-. \end{array}$$

This only constrains causal links in $\mathscr{q}$ from positives to negatives, but by *courteous* those are in $\mathscr{q}$ iff they are in $(\!|\mathscr{q}|\!)$. So for $\mathscr{q} \in \exp(\mathscr{p})$, that it is a characteristic expansion is in fact a property of $(\!|\mathscr{q}|\!)$. Furthermore it is stable under iso so that if $(\!|\mathscr{p}_1|\!) = (\!|\mathscr{p}_2|\!)$, for $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ characteristic there must be $\mathscr{q}_2 \in \exp(\mathscr{p}_2)$ characteristic too such that $(\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$ – so it makes sense to restrict our attention to positions reached by characteristic expansions.

How different can be characteristic $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ and $\mathscr{q}_2 \in \exp(\mathscr{p}_2)$ s.t. $(\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$? A first guess is *isomorphic*, but that is off the mark; $\mathscr{q}_1$ and $\mathscr{q}_2$ have some degree of liberty in swapping forks around (as in Figure 16): they have the “same branches, but with possibly different multiplicity”. A significant part of our endeavour has been to construct a relation between augmentations allowing such changes in multiplicity, while ensuring $\mathscr{p}_1 \cong \mathscr{p}_2$.

4.2 Bisimulations Across an Isomorphism

More than simply comparing augmentations, given $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$, $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$, we shall need a a predicate $a \sim b$ expressing that a and b have the same causal follow-up, up to the multiplicity of duplications. In particular, a and b must have “the same pointer”, but at first that makes no sense since a and b live in different ambient sets of events. So we also fix an isomorphism $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$ providing the translation, and aim to define $a \sim^\varphi b$ parametrized by φ. We give some examples in Figure 16, where φ is any of the two possible isomorphisms, assuming $\mathbf{q}_1^-$ and $\mathbf{q}_2^-$ correspond to different moves of the arena.

This is defined via a bisimulation game: for instance, establishing that the roots are in relation requires us to first match the blue nodes. But as the bisimulation unfolds, requiring all pointers to match up to φ is too strong: the pointers of red moves do *not* match – but seen from $\mathbf{q}^+$ this is fine as the justifiers for the red moves are encountered at the same step of the bisimulation game from $\mathbf{q}^+$. So our actual predicate has form $a \sim^\varphi_\Gamma b$ for Γ a *context*, stating a correspondence between negative moves established in the bisimulation game so far:

▶ **Definition 25.** *A **context** between* $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ *is* $\Gamma : \mathrm{dom}(\Gamma) \cong \mathsf{cod}(\Gamma)$ *a bijection s.t.* $\mathrm{dom}(\Gamma) \subseteq |\mathscr{q}|$, $\mathsf{cod}(\Gamma) \subseteq |\mathscr{p}|$, $\lambda_{\mathscr{q}}(\mathrm{dom}(\Gamma)) \subseteq \{-\}$, *and* $\forall a^- \in \mathrm{dom}(\Gamma)$, $\partial_{\mathscr{q}}(a) = \partial_{\mathscr{p}}(\Gamma(a))$.

We may now formulate a first notion of bisimulation across augmentations.

▶ **Definition 26.** *Consider* $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ *and an isomorphism* $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$.

For $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$ and Γ a context, we define a predicate $a \sim^{\varphi}_{\Gamma} b$ which holds if, firstly,

(a) $\partial_{\mathscr{q}}(a) = \partial_{\mathscr{p}}(b)$ *and* $\Gamma \vdash (a, b)$
(b) *if* $\mathsf{just}(a^+) \in \mathrm{dom}(\Gamma)$, *then* $\mathsf{just}(b) \in \mathsf{cod}(\Gamma)$ *and* $\Gamma(\mathsf{just}(a)) = \mathsf{just}(b)$,
(c) *if* $\mathsf{just}(a^+) \notin \mathrm{dom}(\Gamma)$, *then* $\mathsf{just}(b) \notin \mathsf{cod}(\Gamma)$ *and* $\varphi(\mathsf{just}(a)) = \mathsf{just}(b)$,

where $\Gamma \vdash (a, b)$ means that for all $a' \in \mathrm{dom}(\Gamma)$, $\neg(a' >_{\mathscr{q}} a)$ and for all $b' \in \mathsf{cod}(\Gamma)$, $\neg(b' >_{\mathscr{p}} b)$; and inductively, the following two bisimulation conditions hold:

(1) *if $a^+ \twoheadrightarrow_{\mathscr{q}} a'$, then there is $b^+ \twoheadrightarrow_{\mathscr{p}} b'$ with $a' \sim^{\varphi}_{\Gamma \cup \{(a', b')\}} b'$, and symmetrically,*
(2) *if $a^- \twoheadrightarrow_{\mathscr{q}} a'$, then there is $b^- \twoheadrightarrow_{\mathscr{p}} b'$ with $a' \sim^{\varphi}_{\Gamma} b'$, and symmetrically.*

As $\Gamma \vdash (a, b)$ implies $a' \notin \mathrm{dom}(\Gamma)$ and $b' \notin \mathsf{cod}(\Gamma)$, $\Gamma \cup \{(a', b')\}$ remains a bijection.

Of particular interest is the case $a \sim^{\varphi}_{\emptyset} b$ over an empty context, written simply $a \sim^{\varphi} b$. From this, we deduce a relation between augmentations: we write $\mathscr{q} \sim^{\varphi} \mathscr{p}$ iff $\mathsf{init}(\mathscr{q}) \sim^{\varphi} \mathsf{init}(\mathscr{p})$, for $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ and $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$. Resuming the discussion at the end of Section 4.1: bisimulations allow us to express that two characteristic expansions with isomorphic configurations are "the same". More precisely, in due course we will be able to prove:

▶ **Proposition 27.** *Consider $\mathscr{p}_1, \mathscr{p}_2 \in \mathsf{Aug}(A)$ causal strategies, $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ and $\mathscr{q}_2 \in \exp(\mathscr{p}_2)$ characteristic expansions with an isomorphism $\varphi : (\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$. Then, $\mathscr{q}_1 \sim^{\varphi} \mathscr{q}_2$.*

The proof is the core of our injectivity argument, which we will cover in Section 4.5. For now, we focus on how to conclude from $\mathscr{q}_1 \sim^{\varphi} \mathscr{q}_2$ that we have $\mathscr{p}_1 \cong \mathscr{p}_2$.

4.3 Compositional Properties of Bisimulations

To achieve that, we exploit compositional properties of bisimulations. More precisely, we show that $\mathscr{q}_i \in \exp(\mathscr{p}_i)$ induces a bisimulation $\mathscr{q}_i \sim \mathscr{p}_i$, and find a way to compose

$$\mathscr{p}_1 \sim \mathscr{q}_1 \quad \sim^{\varphi} \quad \mathscr{q}_2 \sim \mathscr{p}_2 \tag{1}$$

to deduce $\mathscr{p}_1 \sim \mathscr{p}_2$ in a sense yet to be defined, and $\mathscr{p}_1 \cong \mathscr{p}_2$ will follow. We start with:

▶ **Lemma 28.** *Consider augmentations $\mathscr{q}, \mathscr{p}, \mathscr{r} \in \mathsf{Aug}(A)$, isomorphisms $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$, $\psi : (\!|\mathscr{p}|\!) \cong (\!|\mathscr{r}|\!)$, events $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$, $c \in |\mathscr{r}|$, and contexts Γ, Δ. Then:*

reflexivity: $a \sim^{\mathrm{id}} a$,
transitivity: *if* $a \sim^{\varphi}_{\Gamma} b$ *and* $b \sim^{\psi}_{\Delta} c$ *with* $\mathsf{cod}(\Gamma) = \mathrm{dom}(\Delta)$, *then* $a \sim^{\psi\circ\varphi}_{\Delta\circ\Gamma} c$,
symmetry: *if* $a \sim^{\varphi}_{\Gamma} b$ *then* $b \sim^{\varphi^{-1}}_{\Gamma^{-1}} a$.

But in order to treat $\mathscr{q}_i \in \exp(\mathscr{p}_i)$ as a bisimulation between $\mathscr{q}_i$ and $\mathscr{p}_i$, Definition 26 does not do the trick: we cannot expect there to be an iso between $(\!|\mathscr{q}_i|\!)$ and $(\!|\mathscr{p}_i|\!)$ as $\mathscr{q}_i$ has by construction many more events. We therefore introduce a variant of Definition 26:

▶ **Definition 29.** *Consider $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$. For $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$, Γ, we have $a \sim_{\Gamma} b$ if*

(a) $\partial_{\mathscr{q}}(a) = \partial_{\mathscr{p}}(b)$ *and* $\Gamma \vdash (a, b)$,
(b) $\mathsf{just}(a^+) \in \mathrm{dom}(\Gamma)$ *and* $\Gamma(\mathsf{just}(a)) = \mathsf{just}(b)$,
(1) *if $a^+ \twoheadrightarrow_{\mathscr{q}} a'$, then $b^+ \twoheadrightarrow_{\mathscr{p}} b'$ with $a' \sim_{\Gamma \cup \{(a', b')\}} b'$, and symmetrically,*
(2) *if $a^- \twoheadrightarrow_{\mathscr{q}} a'$, then $b^- \twoheadrightarrow_{\mathscr{p}} b'$ with $a' \sim_{\Gamma} b'$, and symmetrically.*

This helps us relate $\mathscr{q}$ and $\mathscr{p}$ when $(\!|\mathscr{q}|\!)$ and $(\!|\mathscr{p}|\!)$ are not isomorphic: we set $\mathscr{q} \sim \mathscr{p}$ iff $\mathsf{init}(\mathscr{q}) \sim_{\{(\mathsf{init}(\mathscr{q}), \mathsf{init}(\mathscr{p}))\}} \mathsf{init}(\mathscr{p})$. A variation of Lemma 28 shows $\sim$ is an equivalence, and:

▶ **Proposition 30.** *Consider A an arena, $\mathcal{p} \in \mathsf{Aug}(A)$ a causal strategy, and $\mathcal{q} \in \mathsf{Aug}(A)$. Then, $\mathcal{q}$ is a $-$-obsessional expansion of $\mathcal{p}$ iff $\mathcal{q} \sim \mathcal{p}$.*

Proof. *If.* We simply construct $\varphi : \mathcal{q} \to \mathcal{p}$ for all $a \in |\mathcal{q}|$ by induction on $\leq_{\mathcal{q}}$. The image is provided by bisimulation, its uniqueness by *determinism* and $-$*-linearity*.

Only if. For $\varphi : \mathcal{q} \to \mathcal{p}$ and $a \in |\mathcal{q}|$, write $[a]^-_{\mathcal{q}} = \{a' \in |\mathcal{q}| \mid a' \leq_{\mathcal{q}} a \ \& \ \lambda(a') = -\}$; it is totally ordered by $\leq_{\mathcal{q}}$ as $\mathcal{q}$ is *forestial*. From the conditions on φ, it is direct that it induces an order-iso $[a]^-_{\mathcal{q}} \cong [\varphi(a)]^-_{\mathcal{p}}$, *i.e.* a context $\Gamma\langle a\rangle : [a]^-_{\mathcal{q}} \cong [\varphi(a)]^-_{\mathcal{p}}$. Then, by Lemma 45, we obtain $a \sim_{\Gamma\langle a\rangle} \varphi(a)$ for all $a \in |\mathcal{q}|$. We then apply this to $\mathsf{init}(\mathcal{q})$. ◀

This vindicates Definition 29. But for (1), we must compose two kinds of bisimulations, following Definitions 26 and 29. Fortunately, whenever both definitions apply, they coincide:

▶ **Lemma 31.** *Consider $\mathcal{q}, \mathcal{p} \in \mathsf{Aug}(A)$, and $\varphi : (\!|\mathcal{q}|\!) \cong (\!|\mathcal{p}|\!)$. Then, $\mathcal{q} \sim^{\varphi} \mathcal{p}$ iff $\mathcal{q} \sim \mathcal{p}$.*

Proof. *If.* Straightforward from Definitions 26 and 29: case *(c)* is never used.

Only if. We actually prove that for all $a \in |\mathcal{q}|$, $b \in |\mathcal{p}|$, for all context Γ which is *complete* in the sense that $[a]^-_{\mathcal{q}} \subseteq \mathrm{dom}(\Gamma)$ and $[b]^-_{\mathcal{p}} \subseteq \mathsf{cod}(\Gamma)$, if $a \sim^{\varphi}_{\Gamma} b$ then $a \sim_{\Gamma} b$. The proof is immediate by induction: the clause *(c)* is never used from the hypothesis that Γ is complete. Finally, we apply this to the roots of $\mathcal{q}, \mathcal{p}$ with context $\{(\mathsf{init}(\mathcal{q}), \mathsf{init}(\mathcal{p}))\}$. ◀

Altogether, we have:

▶ **Proposition 32.** *Consider $\mathcal{p}_1, \mathcal{p}_2 \in \mathsf{Aug}(A)$ causal strategies, $\mathcal{q}_1 \in \exp(\mathcal{p}_1), \mathcal{q}_2 \in \exp(\mathcal{p}_2)$ characteristic expansions with an iso $\varphi : (\!|\mathcal{q}_1|\!) \cong (\!|\mathcal{q}_2|\!)$. If $\mathcal{q}_1 \sim^{\varphi} \mathcal{q}_2$, then $\mathcal{p}_1 \cong \mathcal{p}_2$.*

Proof. By Lemma 31, $\mathcal{q}_1 \sim \mathcal{q}_2$. As characteristic expansions, $\mathcal{q}_1$ and $\mathcal{q}_2$ are $-$-obsessional, so by Proposition 30, $\mathcal{q}_1 \sim \mathcal{p}_1$ and $\mathcal{q}_2 \sim \mathcal{p}_2$. So $\mathcal{p}_1 \sim \mathcal{q}_1 \sim \mathcal{q}_2 \sim \mathcal{p}_2$ but $\sim$ is an equivalence, so $\mathcal{p}_1 \sim \mathcal{p}_2$. By Proposition 30, we have $\varphi : \mathcal{p}_1 \to \mathcal{p}_2$ and $\psi : \mathcal{p}_2 \to \mathcal{p}_1$ composing to $\psi \circ \varphi : \mathcal{p}_1 \to \mathcal{p}_1$. But by Lemma 42, there is only one morphism from $\mathcal{p}_1$ to itself, the identity, so $\psi \circ \varphi = \mathrm{id}$. Likewise $\varphi \circ \psi = \mathrm{id}$, hence $\varphi : \mathcal{p}_1 \cong \mathcal{p}_2$ as required. ◀

4.4 Clones

In Section 4.1, we introduced *characteristic expansions* which, via duplications with well-chosen cardinalities, constrain the causal structure. More precisely, if $\mathcal{q} \in \exp(\mathcal{p})$ is characteristic, looking at a set of duplicated Player moves in $(\!|\mathcal{q}|\!)$ of cardinality n as in Figure 17, decomposing $n = \sum_{i \in I} 2^i$, we can deduce that the causal predecessors of the $\mathbf{q}^+_j$'s are among the forks with cardinality 2^i for $i \in I$. But that is not enough: this does not tell us how to distribute the $\mathbf{q}^+_j$'s to the forks, and not all the choices will work: while the $\mathbf{q}^+_j$'s are copies, their respective causal follow-ups might differ. So the idea is simple: imagine that the causal follow-ups for the $\mathbf{q}^+_j$'s are already reconstructed. Then we may compare them using bisimulation, and replicate the same reasoning as above on bisimulation equivalence classes.

So we are left with the task of leveraging bisimulation to define an adequate equivalence relation on $|\mathcal{q}|$. This leads to the notion of *clones*, our last technical tool.

▶ **Definition 33.** *Consider $\mathcal{q}, \mathcal{p} \in \mathsf{Aug}(A)$, $\varphi : (\!|\mathcal{q}|\!) \cong (\!|\mathcal{p}|\!)$, and $a \in |\mathcal{q}|$, $b \in |\mathcal{p}|$.*

We say that a and b are ***clones*** *through φ, written $a \approx^{\varphi} b$, if there is a context Γ preserving pointers (*i.e. *for all $a' \in \mathrm{dom}(\Gamma)$, $\varphi(\mathsf{just}(a')) = \mathsf{just}(\Gamma(a'))$) such that $a \sim^{\varphi}_{\Gamma} b$.*

This allows a and b (and their follow-ups) to change their pointers through some unspecified Γ. Indeed, the picture painted by Figure 17 is limited: a fork might trigger Player moves with different pointers, as in Figure 18. As $a \approx^{\varphi} b$ quantifies existentially over contexts,

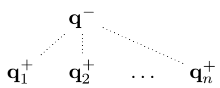


Figure 17 A set of copied Player moves

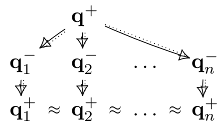


Figure 18 A set of clones switching pointers

compositional properties of clones are more challenging. Nevertheless, via a canonical form for contexts and leveraging Lemma 28, we show that $a \approx^{\mathrm{id}} a$, that $a \approx^{\varphi} b$ and $b \approx^{\psi} c$ imply $a \approx^{\psi\circ\varphi} c$, and that $a \approx^{\varphi} b$ implies $b \approx^{\varphi^{-1}} a$ whenever these typecheck – see Appendix B.2. Instantiating Definition 33 with $\mathscr{q} = \mathscr{p}$ and $\varphi = \mathrm{id}$, we get an equivalence relation $\approx$ on $|\mathscr{q}|$.

Moreover, we have the crucial property that forks generate clones (see Appendix B.2):

▶ **Lemma 34.** *Consider $\mathscr{q}$ a $-$-obsessional expansion of causal strategy $\mathscr{p}$ on arena A. Then, for all $a_1^-, a_2^- \in X \in \mathsf{Fork}(\mathscr{q})$, for all $a_1^- \rightarrowtriangle_{\mathscr{q}} b_1^+$ and $a_2^- \rightarrowtriangle_{\mathscr{q}} b_2^+$, $b_1 \approx b_2$.*

By Lemma 34, if a clone class includes a positive move, it also has all its cousins triggered by the same fork – so clone classes may be partitioned following forks:

▶ **Lemma 35.** *Let $\mathscr{q}$ be a characteristic expansion of causal strategy $\mathscr{p}$, and Y a clone class of positive events in $|\mathscr{q}|$, with $\sharp Y = \sum_{i\in I} 2^i$ for $I \subseteq \mathbb{N}$ finite. Then, for all $i \in \mathbb{N}$, $i \in I$ iff there is $X_i \in \mathsf{Fork}(\mathscr{q})$ with $\sharp X_i = 2^i$ and $a^- \in X_i$, $b^+ \in Y$ such that $a^- \rightarrowtriangle_{\mathscr{q}} b^+$.*

Proof. For any $i \in \mathbb{N}$, we write X_i the fork of $\mathscr{q}$ of cardinality 2^i, if it exists.

Consider the set $J := \{j \in \mathbb{N} \mid X_j \text{ exists}, \exists a \in X_j,\ \exists b \in Y,\ a \rightarrowtriangle_{\mathscr{q}} b\}$. Any $b \in Y$ is positive and so the unique (by determinism) successor of some negative event a. Moreover a appears in a fork X and by Lemma 34, all events of X are predecessors of events of Y. Hence, we have $Y = \bigcup_{j\in J} \mathsf{succ}(X_j)$, where the union is disjoint since $\mathscr{q}$ is forest-shaped. Therefore,

$$\sharp Y \;=\; \sum_{j\in J} \sharp\mathsf{succ}(X_j) \;=\; \sum_{j\in J} \sharp X_j \;=\; \sum_{j\in J} 2^j,$$

where the second equality is obtained by determinism. By uniqueness of the binary decomposition, $J = I$, which proves the lemma by definition of J. ◀

4.5 Positional Injectivity

We are finally in a position to prove the core of the injectivity argument.

▶ **Lemma 36** (Key lemma)**.** *Consider $\mathscr{p}_1, \mathscr{p}_2 \in \mathsf{Aug}(A)$ causal strategies, $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ and $\mathscr{q}_2 \in \exp(\mathscr{p}_2)$ characteristic expansions, and $\varphi : (\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$. Then, $\forall a^+ \in |\mathscr{q}_1|, a \approx^{\varphi} \varphi(a)$.*

Proof. The **co-depth** of $a \in |\mathscr{q}_i|$ is the maximal length k of $a = a_1 \rightarrowtriangle_{\mathscr{q}_i} \dots \rightarrowtriangle_{\mathscr{q}_i} a_k$ a causal chain in $\mathscr{q}_i$. We show by induction on k the two symmetric properties:

(a) for all $a^+ \in |\mathscr{q}_1|$ of co-depth $\leq k$, we have $a \approx^{\varphi} \varphi(a)$,
(b) for all $a^+ \in |\mathscr{q}_2|$ of co-depth $\leq k$, we have $a \approx^{\varphi^{-1}} \varphi^{-1}(a)$.

Take $a^+ \in |\mathscr{q}_1|$ of co-depth k. If a is maximal in $\mathscr{q}_1$, so is $\varphi(a)$ in $\mathscr{q}_2$ and $a \approx \varphi(a)$. Else, the successors of a partition as $G_1, \dots, G_n \subseteq \mathsf{Fork}(\mathscr{q}_1)$, where $G_i = \{b_{i,1}^-, \dots, b_{i,2^{p_i}}^-\}$; likewise the successors of $\varphi(a)$ in $\mathscr{q}_2$ are the forks $\varphi(G_i)$. For all $1 \leq i \leq n$ and $1 \leq j \leq 2^{p_i}$, we claim:

$$\text{for all } b_{i,j} \rightarrowtriangle_{\mathscr{q}_1} c_{i,j}, \text{ there is } \varphi(b_{i,j}) \rightarrowtriangle_{\mathscr{q}_2} d_{i,j} \text{ satisfying } c_{i,j} \approx^{\varphi} d_{i,j}. \tag{2}$$

Write $X = [c_{i,j}]_\approx$ the clone class of $c_{i,j}$ in $\mathscr{q}_1$. It is easy to prove that the clone relation preserves co-depth, so it follows from the induction hypothesis and Lemma 50 that $\varphi(X)$ is a clone class in $\mathscr{q}_2$. By Lemma 35, $\sharp X$ has 2^{p_i} in its binary decomposition – and as φ is a bijection, so does $\sharp(\varphi(X))$. So by Lemma 35, there is $\varphi(b_{i,j}) \in \varphi(G_i)$ and $d_{i,j} \in \varphi(X)$ such that $\varphi(b_{i,j}) \rightarrowtriangle_{\mathscr{q}_2} d_{i,j}$. Since $\varphi(c_{i,j}), d_{i,j} \in \varphi(X)$ they are clones, so using $c_{i,j} \approx^\varphi \varphi(c_{i,j})$ by induction hypothesis, $c_{i,j} \approx^\varphi d_{i,j}$. Likewise, the mirror property of (2) also holds.

Deducing $a \approx^\varphi \varphi(a)$ requires some care: cloning is defined via a context, and the $c_{i,j} \approx^\varphi \varphi(c_{i,j})$ might not share the same. However, the contexts can be put into canonical forms that are shown to agree – Lemma 54 allows us to prove $a \approx^\varphi \varphi(a)$ from (2) and its mirror property. Finally, *(b)* is proved symmetrically. ◀

Now, consider $\mathscr{p}_1, \mathscr{p}_2, \mathscr{q}_1, \mathscr{q}_2, \varphi$ as in Proposition 27. If the $\mathscr{q}_i$'s are empty or singleton trees, there is nothing to prove. Otherwise $\mathscr{q}_i$ starts with $a_i^- \rightarrowtriangle_{\mathscr{q}_i} b_i^+$ with a_i^- initial. But then $[b_i^+]_\approx$ is the only singleton clone class in $\mathscr{q}_i$. As φ preserves clone classes, $\varphi(b_1^+) = b_2^+$. By Lemma 36, $b_1 \approx^\varphi b_2$. Thus $b_1 \sim^\varphi b_2$, so $a_1 \sim^\varphi a_2$ and $\mathscr{q}_1 \sim^\varphi \mathscr{q}_2$. This concludes the proof of Proposition 27. Putting everything together, we obtain:

▶ **Theorem 37.** *For $\mathscr{p}_1, \mathscr{p}_2 \in \mathsf{Aug}(A)$ causal strategies s.t. $(\!|\mathscr{p}_1|\!) = (\!|\mathscr{p}_2|\!)$, then $\mathscr{p}_1 \cong \mathscr{p}_2$.*

Proof. Consider $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ a characteristic expansion. By hypothesis, there must be $\mathscr{q}_2 \in \exp(\mathscr{p}_2)$ and $\varphi : (\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$; necessarily $\mathscr{q}_2$ is also a characteristic expansion of $\mathscr{p}_2$. By Proposition 27, we have $\mathscr{q}_1 \sim^\varphi \mathscr{q}_2$. By Proposition 32, we have $\mathscr{p}_1 \cong \mathscr{p}_2$. ◀

Finally, Theorem 15 follows from Theorem 37, Proposition 22 and Lemma 20.

Theorem 15 only concerns *total* finite innocent strategies. In contrast, Theorem 37 requires no totality assumption: totality comes in not in the injectivity argument, but in Proposition 22 linking standard and causal strategies. Without totality, expansions of $\hat{\sigma}$ might not have as many Opponent as Player moves, and so may not be linearizable via alternating plays. Intuitively, in alternating plays Opponent may only explore converging parts of the strategy, whereas in the causal setting Opponent is free to explore simultaneously many branches, including divergences. Positional injectivity for *partial* finite innocent strategies may be studied causally by restricting to $+$*-covered* expansions, *i.e.* with only Player maximal events. But then we must also abandon $-$*-obsessionality* as Opponent moves leading to divergence will not be played, breaking our proof (Lemma 34 fails) in a way for which we see no fix.

5 Beyond Total Finite Strategies

Finally, we show some subtleties and partial results on generalizations of Theorem 15.

First, positional injectivity fails in general. Consider the infinitary terms $f : o \to o \to o \vdash T_1, T_2, L, R : o$ recursively defined as $T_1 = f\, T_2\, R, T_2 = f\, L\, T_1, L = f\, L\, \bot$ and $R = f\, \bot\, R$ in an infinitary simply-typed λ-calculus with divergence $\bot$. The corresponding strategies differ: their causal representations appear in Figures 19 and 20, infinite trees represented via loops.

We consider positions reached by plays – or equivalently, by $+$-covered expansions of Figures 19 and 20. In fact, both strategies admit all *balanced* positions on $[\![(o \to o \to o) \to o]\!]$, *i.e.* with as many Opponent as Player moves. Ignoring the initial $\mathbf{q}^-$, a position is a multiset of **bricks** as in Figure 21, with $i \in \mathbb{N}$ occurrences of $\mathbf{q}_1^-$ and $j \in \mathbb{N}$ of $\mathbf{q}_2^-$. A brick with $i = j = 0$ is a **leaf**. The position is balanced if it has as many Opponent as Player moves.

Now, any position can be realized in $[\![\lambda f^{o\to o\to o}. T_i]\!]$ by first placing bricks with occurrences of both $\mathbf{q}_1^-$ and $\mathbf{q}_2^-$ greedily alongside the *spine*, shown in red in Figures 19 and 20. At each step, we continue from only one of the copies opened, leaving others dangling. If this

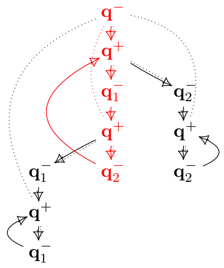


Figure 19 $[\![\lambda f^{o\to o\to o}.\,T_1]\!]$

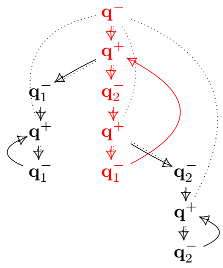


Figure 20 $[\![\lambda f^{o\to o\to o}.\,T_2]\!]$

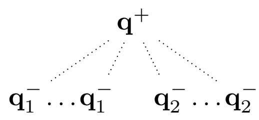


Figure 21 Bricks

gets stuck, apart from leaves we are left with only $\mathbf{q}_1^-$'s, or, only $\mathbf{q}_2^-$'s, but there is always a matching non-spine infinite branch available. Finally, leaves can always be placed as their number matches that of trailing negative moves by the balanced hypothesis.

We have $(\!|[\![\lambda f^{o\to o\to o}.\,T_1]\!]|\!) = (\!|[\![\lambda f^{o\to o\to o}.\,T_2]\!]|\!)$ as both strategies can realize *all* balanced positions on the arena $[\![o \to o \to o]\!]$, and *exactly* those: positional injectivity fails.

Positionality for *finite* innocent strategies remains open. We could only prove:

▶ **Theorem 38.** *Let $\sigma_1, \sigma_2 : A$ be finite innocent strategies with $(\!|\sigma_1|\!) = (\!|\sigma_2|\!)$. Then, σ_1 and σ_2 have the same P-views of maximal length.*

For the proof (see Appendix C), we assume σ_1 has a P-view s of maximal length n. We perform an expansion of s where each Opponent branching at co-depth $2d+1$ has arity $d+1$. By a combinatorial argument on trees, the only way to reassemble its nodes exhaustively in a tree with depth bounded by d is to rebuild exactly the same tree. Hence the tree is also in $\exp(\hat{\sigma_2})$, and $s \in \sigma_2$. This steers us into conjecturing that positional injectivity holds for partial finite innocent strategies, but our proof attempts have remained inconclusive.

6 Conclusion

Though innocent strategies in the Hyland-Ong sense are not positional, total finite innocent strategies satisfy *positional injectivity* – however, the property fails in general.

Beyond its foundational value, we believe this result may be helpful in the game semantics toolbox. Game semantics can be fiddly; in particular, proofs that two terms yield the same strategy are challenging to write in a concise yet rigorous manner. This owes a lot to the complexity of *composition*: proving that a play s is in $[\![M\,N]\!]$ involves constructing an "interaction witness" obtained from plays in $[\![M]\!]$ and $[\![N]\!]$ plus an adequate "zipping" of the two. Manipulations of plays with pointers are tricky and error-prone, and the link between plays and terms is obfuscated by the multi-layered interpretation.

In contrast, Theorem 15 lets us prove innocent strategies equal by comparing their positions. Now, constructing a position of $[\![MN]\!]$ simply involves exhibiting matching positions for $[\![M]\!]$ and $[\![N]\!]$. Side-stepping the interpretation, this can be presented as typing terms with positions or configurations – combining Section 2.5 and the link between relational semantics and non-idempotent intersection type systems [11]. For instance, in this way, finite definability, a basic result seldom presented in full formal details, boils down to typing the defined term with the same positions as the original strategy.

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A Augmentations and Strategies: Proofs from Section 3

In the sequel, A is a fixed arena. For any augmentation $\mathscr{q} \in \mathsf{Aug}(A)$ and event $a \in |\mathscr{q}|$, we define $\mathsf{succ}(a) := \{b \mid a \rightarrowtriangle_{\mathscr{q}} b\}$ the set of immediate successors of a in $\leq_{\mathscr{q}}$. We also define $\uparrow a$ the set of descendants of a, i.e. $\uparrow a := \{a' \mid a \leq_{\mathscr{q}} a'\}$.

In this first section, we provide more detailed proofs for augmentations as defined in Section 3, studying their expansions and their links with strategies. We start by proving Proposition 18 of Section 3.2, which allows us to see traditional innocent strategies as causal strategies, with the following two lemmas. For any non-empty pointing string s, we use the notation s_ω for the last element of s.

▶ **Lemma 39.** *For $\sigma : A$ finite innocent on A well-opened, $\hat{\sigma}$ is a causal strategy,* i.e. *a receptive $-$-linear augmentation.*

Proof. First we must check that $(\!|\hat{\sigma}|\!) = \langle |\hat{\sigma}|, \leq_{(\!|\hat{\sigma}|\!)}, \partial_{\hat{\sigma}}\rangle$ is a configuration. It is clear from the definition and the fact that A is well-opened that $\langle |\hat{\sigma}|, \leq_{(\!|\hat{\sigma}|\!)}\rangle$ is a finite tree. The well-openedness of A also ensures *minimality-respecting.*

Causality-preserving. Consider $s, t \in |\hat{\sigma}|$ such that $s \rightarrowtriangle_{(\!|\hat{\sigma}|\!)} t$. Then t_ω points to s_ω by definition, so by rigidity $\partial_{\hat{\sigma}}(s) \rightarrowtriangle_A \partial_{\hat{\sigma}}(t)$.

Hence, $(\!|\hat{\sigma}|\!)$ is a configuration. We now check that $\hat{\sigma}$ is an augmentation.

Rule-abiding. Let $t \in |\hat{\sigma}|$ and $s \leq_{(\!|\hat{\sigma}|\!)} t$. By definition, there exists a chain of pointers from t_ω to s_ω. This implies that $s \sqsubseteq t$, hence $s \leq_{\hat{\sigma}} t$.

Courteous. Let $s \rightarrowtriangle_{\hat{\sigma}} t$ such that $\lambda_A(s) = +$ or $\lambda_A(t) = -$. By definition of $\leq_{\hat{\sigma}}$, $s = s'b^+$ and $t = s'ba^-$ since A is alternating. By definition of a P-view, a points to b and $s \rightarrowtriangle_{(\!|\hat{\sigma}|\!)} t$.

Deterministic. Let $s^- \rightarrowtriangle_{\hat{\sigma}} t_1^+$ and $s^- \rightarrowtriangle_{\hat{\sigma}} t_2^+$. By definition of $\rightarrowtriangle_{\hat{\sigma}}$, we have $s = s'a^-$, $t_1 = s'ab_1^+$, $t_2 = s'ab_2^+$, and $b_1 = b_2$ (with the same pointer) by determinism of σ.

So $\hat{\sigma}$ is an augmentation. Finally, we check the two additional conditions.

Receptive. Let $s \in |\hat{\sigma}|$ such that $\partial_{\hat{\sigma}}(s) \rightarrowtriangle_A a^-$. Since A is alternating, we know $s = s'b^+$, and $s \in \sigma$ by definition of $\hat{\sigma}$. Furthermore, $\partial_{\hat{\sigma}}(s) \rightarrowtriangle_A a^-$ implies that $sa \in \mathsf{Plays}(A)$, where a points to s_ω. So $sa \in |\hat{\sigma}|$, with $s \rightarrowtriangle_{\hat{\sigma}} sa$ and $\partial_{\hat{\sigma}}(sa) = a$.

$-$-linear. Let $s \rightarrowtriangle_{\hat{\sigma}} t_1^-$ and $s \rightarrowtriangle_{\hat{\sigma}} t_2^-$ such that $\partial_{\hat{\sigma}}(t_1) = \partial_{\hat{\sigma}}(t_2)$. By definition of $\leq_{\hat{\sigma}}$, $t_1 = sa_1^-$ and $t_2 = sa_2^-$, where both a_i point to s_ω by courtesy. Moreover, by definition of $\partial_{\hat{\sigma}}$, $a_1 = a_2$. Hence $t_1 = t_2$. ◀

▶ **Lemma 40.** *For $\sigma : A$ finite innocent on A well-opened, $\hat{\sigma}$ is total iff σ is total.*

Proof. From the previous lemma, $\hat{\sigma}$ is always receptive.

If. Assume σ is total. Consider $s \in |\hat{\sigma}|$ maximal for $\leq_{\hat{\sigma}}$, and assume $\lambda(s) = -$. Then by definition s is a P-view and $s = s'a^-$, where $s' \in \sigma$ by innocence and $s'a \in \mathsf{Plays}(A)$. By totality of σ, there exists b^+ such that $s'ab \in \sigma$, and since s is already a P-view we have $\ulcorner s'ab \urcorner = s'ab$. Hence $s \rightarrowtriangle_{\hat{\sigma}} s'ab$, which contradicts the maximality of s.

Only if. Assume $\hat{\sigma}$ is total. Consider $s \in \sigma$ such that $sa \in \mathsf{Plays}(A)$. Let $t = \ulcorner s \urcorner$, then $t \in \sigma$ by innocence of σ. Moreover, $ta = \ulcorner sa \urcorner \in |\hat{\sigma}|$. By totality of $\hat{\sigma}$, ta cannot be maximal, so there exists b^+ such that $ta \rightarrowtriangle_{\hat{\sigma}} tab$. Since $tab \in |\hat{\sigma}|$ and $tab = \ulcorner tab \urcorner$, we also have $tab \in \sigma$. By innocence of σ, $sab \in \sigma$, and σ is total. ◀

Those two lemmas prove Proposition 18.

In the development, we focus on expansions of causal strategies (Definition 21). We prove minimality-preservation and uniqueness of morphisms.

▶ **Lemma 41.** *Consider $\mathscr{q} \in \mathsf{Aug}(A)$ and $a \in |\mathscr{q}|$.*
Then $\partial_{\mathscr{q}}(a) \in \min(A)$ iff $a = \mathsf{init}(\mathscr{q})$.

Proof. *If.* Assume that $a = \mathsf{init}(\mathcal{q})$. Assuming that $\partial_{\mathcal{q}}(a)$ is not minimal, then by condition *minimality-preserving* of Definition 9, a is not minimal in $(\!|\mathcal{q}|\!)$. But then by condition *rule-abiding* of Definition 16, a cannot be minimal in $\mathcal{q}$ either, contradiction.

Only if. Assume $\partial_{\mathcal{q}}(a) \in \min(A)$. In particular, a has negative polarity since A is negative. Assuming that a is not minimal in $\mathcal{q}$, then it has an antecedent $a' \rightarrowtriangle_{\mathcal{q}} a$. By courtesy, we have $a' \rightarrowtriangle_{(\!|\mathcal{q}|\!)} a$ as well. Hence, $\partial_{\mathcal{q}}(a') \rightarrowtriangle_A \partial_{\mathcal{q}}(a)$, contradiction. ◀

In particular, this proves that morphisms preserve initial events (by arena-preservation).

▶ **Lemma 42.** *Consider $\mathcal{p} \in \mathsf{Aug}(A)$ a causal strategy and $\mathcal{q} \in \exp(\mathcal{p})$. Then there exists a unique morphism $\varphi : \mathcal{q} \to \mathcal{p}$.*

Proof. The existence is given by the definition of $\mathcal{q} \in \exp(\mathcal{p})$.

Consider $\varphi, \psi : \mathcal{q} \to \mathcal{p}$ two morphisms from $\mathcal{q}$ to $\mathcal{p}$. Consider $a \in |\mathcal{q}|$ minimal such that $\varphi(a) \neq \psi(a)$. If a is minimal for $\leq_{\mathcal{q}}$, this contradicts Lemma 41 (by arena-preservation). Therefore, there is a (necessarily unique) antecedent $a' \rightarrowtriangle_{\mathcal{q}} a$ for which $\varphi(a') = \psi(a')$. By causality-preservation of morphisms, we have $\varphi(a') \rightarrowtriangle_{\mathcal{p}} \varphi(a)$ and $\varphi(a') \rightarrowtriangle_{\mathcal{p}} \psi(a)$. If a is positive, then $\varphi(a) = \psi(a)$ by determinism of $\mathcal{p}$, contradiction. If a is negative, then likewise we remark that $\partial_{\mathcal{p}}(\varphi(a)) = \partial_{\mathcal{q}}(a)$ and $\partial_{\mathcal{p}}(\psi(a)) = \partial_{\mathcal{q}}(a)$, contradicting $-$-linearity of $\mathcal{p}$. ◀

This lemma ensures the uniqueness condition of Definition 21.

▶ **Lemma 43.** *Let $\sigma : A$ a total finite innocent strategy and $\mathsf{x} \in (\!|\sigma|\!)$. Then $\mathsf{x} \in (\!|\hat{\sigma}|\!)$.*

Proof. Consider $\mathsf{x} \in (\!|\sigma|\!)$, it is the isomorphism class of $(\!|s|\!)$ for some $s = s_1 \dots s_n \in \sigma$. We build an augmentation $\mathcal{q}(s)$ as follows: its underlying configuration is $(\!|\mathcal{q}(s)|\!) = (\!|s|\!)$, with $|\mathcal{q}(s)| = \{1, \dots, n\}$. Its causal order is defined inductively with for any $i, j \in |\mathcal{q}(s)|$,

$$\begin{array}{lcllll} i \leq_{\mathcal{q}(s)} 2j & \Leftrightarrow & i = 2j & \text{or} & i \leq_{\mathcal{q}(s)} 2j-1\ ; \\ i \leq_{\mathcal{q}(s)} 2j+1 & \Leftrightarrow & i = 2j+1 & \text{or} & i \leq_{\mathcal{q}(s)} k \text{ where } s_{2j+1} \text{ points to } s_k \text{ in } s. \end{array}$$

Remark that since s is alternating and negative, for any $i \in |\mathcal{q}(s)|$, s_i is negative iff i is odd. So $i \leq_{\mathcal{q}(s)} j$ means that s_i is reached in the computation of $\ulcorner s_{\leq j} \urcorner$. It is clear that $\langle |\mathcal{q}(s)|, \leq_{\mathcal{q}(s)} \rangle$ is a tree; we must check that $\mathcal{q}(s)$ is an augmentation.

Rule-abiding. For any $i \leq_{(\!|\mathcal{q}(s)|\!)} j$, there exists a chain of justifiers from s_i to s_j. Since s is P-visible, s_i must appear in $\ulcorner s_1 \dots s_j \urcorner$. But $\leq_{\mathcal{q}(s)}$ is inductively defined to follow the construction of a P-view, so $i \leq_{\mathcal{q}(s)} j$.

Courteous. For all $i \rightarrowtriangle_{\mathcal{q}(s)} j$, if $\mathrm{pol}(s_i) = +$ (resp. $\mathrm{pol}(s_j) = -$), then i is even (resp. j is odd), and by definition of $\leq_{\mathcal{q}(s)}$, we know that s_j points to s_i. Hence $i \rightarrowtriangle_{(\!|\mathcal{q}(s)|\!)} j$.

Deterministic. For any negative i, then i is odd and $i \rightarrowtriangle_{\mathcal{q}(s)} j$ implies $j = i + 1$.

We must now check that $\mathcal{q}(s) \in \exp(\hat{\sigma})$. Consider $\varphi : |\mathcal{q}(s)| \to |\hat{\sigma}|$ such that $\varphi(i) = \ulcorner s_{\leq i} \urcorner$. Then φ is a morphism:

Arena-preserving. It is clear that $\partial_{\hat{\sigma}} \circ \varphi = \partial_{\mathcal{q}(s)}$, since the P-view preserves the last move.

Configuration-preserving. For all $i \rightarrowtriangle_{(\!|\mathcal{q}(s)|\!)} j$, we know that s_j points to s_i since $(\!|q(s)|\!) = (\!|s|\!)$. By innocence of σ, s is P-visible and s_j keeps its pointer in $\ulcorner s_{\leq j} \urcorner$ (and this pointer is still s_i). Hence, $\varphi(i) \rightarrowtriangle_{(\!|\hat{\sigma}|\!)} \varphi(j)$.

Causality-preserving. By definition, for any $j \in |\mathcal{q}(s)|$ such that $j \geq 1$, we have $2j - 1 \rightarrowtriangle_{\mathcal{q}(s)} 2j$. Moreover, s_{2j} is positive, so $\varphi(2j) = \ulcorner s_{\leq 2j} \urcorner = \ulcorner s_{\leq 2j-1} \urcorner s_{2j}$, where $s_2 j$ keeps its pointer. So $\varphi(2j-1) \rightarrowtriangle_{\hat{\sigma}} \varphi(2j)$. Likewise, for any $i, 2j+1 \in |\mathcal{q}(s)|$, we have $i \rightarrowtriangle_{\mathcal{q}} 2j+1$ iff s_{2j+1} points to s_i in s. In that case, we have $\varphi(2j+1) = \ulcorner s_{\leq 2j+1} \urcorner = \ulcorner s_{\leq i} \urcorner s^{-}_{2j+1}$, where s_{2j+1} is negative since $2j+1$ is odd. So $\varphi(i) \rightarrowtriangle_{\hat{\sigma}} \varphi(2j+1)$.

$+$-*obsessional.* For all $i^- \in |\mathscr{q}(s)|$, if $\varphi(i) \twoheadrightarrow_{\hat{\sigma}} t^+$, then it means $t = \ulcorner s_{\leq i} \urcorner b^+ \in \sigma$. Since $s \in \sigma$, s is of even length, so $i+1 \in |\mathscr{q}|$ and by determinism and innocence, $s_{i+1} = b$. Therefore, $\varphi(i+1) = t$, and $i \twoheadrightarrow_{\mathscr{q}(s)} i+1$.

Finally, $\mathscr{q}(s) \in \exp(\hat{\sigma})$, which means $\mathsf{x} = (\!|\mathscr{q}(s)|\!) \in (\!|\hat{\sigma}|\!)$. ◀

▶ **Lemma 44.** *Consider $\sigma : A$ a total finite innocent strategy, and $\mathsf{x} \in (\!|\hat{\sigma}|\!)$. Then $\mathsf{x} \in (\!|\sigma|\!)$.*

Proof. Consider $\mathsf{x} \in (\!|\hat{\sigma}|\!)$, then there exists $\mathscr{q} \in \exp(\hat{\sigma})$ such that $(\!|\mathscr{q}|\!) = \mathsf{x}$. From the totality of $\hat{\sigma}$ (Lemma 40), $\mathscr{q}$ has maximal events all positive - hence by determinism it has exactly as many Player as Opponent moves. Moreover, by courtesy, any $a \twoheadrightarrow_{\mathscr{q}} a'$ implies that $\lambda(a) \neq \lambda(a')$, since A is alternating. Therefore, there exists an alternating sequentialization of $\mathscr{q}$, which we construct inductively. We start by the minimal event of $\mathscr{q}$, negative by Lemma 41, and its only successor (exists by totality, unique by determinism). If a positive event b^+ has one or more successors, we inductively construct sequentializations for each subtree of root $a^- \in \mathsf{succ}(b)$. All sequences are alternating, start with a negative event, and end with a positive event. Hence, we can concatenate them in an arbitrary order. We obtain an alternating sequence $s_1^-, s_2^+, \ldots, s_n^+$ such that:

$$|\mathscr{q}| = \{s_i \mid 1 \leq i \leq n\}.$$

We can see $s := s_1 \ldots s_n$, with pointers imported from $(\!|\mathscr{q}|\!)$ (*i.e.* $(\!|s|\!) = (\!|\mathscr{q}|\!)$), as a play on arena $(\!|\mathscr{q}|\!)$. Then for any $1 \leq i \leq n$, it holds that $\ulcorner s_{\leq i} \urcorner$ coincides with $[s_i]_{\mathscr{q}}$, the causal dependency of s_i in $\mathscr{q}$ (clear by induction on i). Writing $\partial_{\mathscr{q}}(s) = \partial_{\mathscr{q}}(s_1) \ldots \partial_{\mathscr{q}}(s_n)$ with pointers imported from $(\!|\mathscr{q}|\!)$, it follows that $\partial_{\mathscr{q}}(s) \in \mathsf{Plays}(A)$ and $\ulcorner(\partial_{\mathscr{q}}(s))_{\leq i}\urcorner \in \ulcorner\!\ulcorner \sigma \urcorner\!\urcorner$ (by definition of $\hat{\sigma}$). Hence $\partial_{\mathscr{q}}(s) \in \sigma$ by innocence, with $(\!|\partial_{\mathscr{q}}(s)|\!) \cong (\!|s|\!)$. Therefore, $(\!|\partial_{\mathscr{q}}(s)|\!) \cong (\!|\mathscr{q}|\!)$, *i.e.* $(\!|\mathscr{q}|\!) = (\!|\partial_{\mathscr{q}}(s)|\!) \in (\!|\sigma|\!)$. ◀

▶ **Proposition 22.** *For any total finite innocent strategy $\sigma : A$, we have $(\!|\sigma|\!) = (\!|\hat{\sigma}|\!)$.*

Proof. Immediate by Lemmas 43 and 44. ◀

B Positional Injectivity: Proofs from Section 4

B.1 Compositional Properties of Bisimulations (Section 4.3)

In this section, we prove some technical lemmas about bisimulations.

Recall that for $\mathscr{q} \in \mathsf{Aug}(A)$, $a \in |\mathscr{q}|$ there is an order-iso $\Gamma\langle a\rangle : [a]^-_{\mathscr{q}} \cong [\varphi(a)]^-_{\mathscr{p}}$, with $[a]^-_{\mathscr{q}}$ the totally ordered set of negative dependencies of a in $\mathscr{q}$. Recall also that the *co-depth* of $a \in |\mathscr{q}|$ is the maximal length of a causal chain $a = a_1 \twoheadrightarrow_{\mathscr{q}} \ldots \twoheadrightarrow_{\mathscr{q}} a_k$ in $\mathscr{q}$.

▶ **Lemma 45.** *Consider $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ where $\mathscr{p}$ is a causal strategy and $\mathscr{q} \in \exp(\mathscr{p})$ is a $-$-obsessional expansion with the morphism $\varphi : \mathscr{q} \to \mathscr{p}$. Then*

$$a \sim_{\Gamma\langle a\rangle} \varphi(a)\,.$$

Proof. By induction on the co-depth of $a \in |\mathscr{q}|$. We must check that $a \sim_{\Gamma\langle a\rangle} \varphi(a)$.

First, *(a)* is immediate by the definition of $\Gamma\langle a\rangle$.

(b). We know that if a is positive, then $\mathsf{just}(a^+) \in [a]^-_{\mathscr{q}} = \mathrm{dom}(\Gamma\langle a\rangle)$ and $\mathsf{just}(\varphi(a)) \in [\varphi(a)]^-_{\mathscr{p}} = \mathsf{cod}(\Gamma\langle a\rangle)$. Moreover $\mathsf{just}(\varphi(a)) = \varphi(\mathsf{just}(a))$ since φ is *configuration-preserving.*

(1). Assume $a^+ \twoheadrightarrow_{\mathscr{q}} b^-$. Then $\varphi(a) \twoheadrightarrow_{\mathscr{p}} \varphi(b)$. By induction hypothesis, $b \sim_{\Gamma(b)} \varphi(b)$. But $[b^-]^-_{\mathscr{q}} = [a]^-_{\mathscr{q}} \cup \{b\}$ and $[\varphi(b)^-]^-_{\mathscr{p}} = [\varphi(a)]^-_{\mathscr{p}} \cup \{\varphi(b)\}$, so finally

$$b \sim_{\Gamma\langle a\rangle \cup \{(b,\varphi(b))\}} \varphi(b)\,.$$

The same reasoning applies for the symmetric condition. Assume $\varphi(a)^+ \twoheadrightarrow_{\mathscr{p}} b$, then $\varphi^{-1}(b)$ exists by receptivity of $\mathscr{q}$ and $-$-linearity of $\mathscr{p}$.

(2). Same as for *(1)*, except $[b^+]^-_{\mathscr{q}} = [a]^-_{\mathscr{q}}$ and $[\varphi(b)^+]^-_{\mathscr{p}} = [\varphi(a)]^-_{\mathscr{p}}$. The same reasoning applies for the symmetric condition. Assume $\varphi(a)^- \twoheadrightarrow_{\mathscr{p}} b$, then $\varphi^{-1}(b)$ exists by $+$-obsessionality of $\mathscr{q}$. ◀

In the following proofs, we will need a few additional properties on bisimulations and contexts, to define a *minimal context*.

▶ **Lemma 46.** *Consider $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ with $\varphi : \mathscr{q} \cong \mathscr{p}$. Consider $a \sim^{\varphi}_{\Gamma} b$ for some Γ. Then for any $a' \in \uparrow a$, there exists $b' \in \uparrow b$ such that $a' \sim^{\varphi}_{\Gamma \cup \Delta} b'$, where*

$$a = a_0 \twoheadrightarrow_{\mathscr{q}} a_1 \twoheadrightarrow_{\mathscr{q}} \ldots \twoheadrightarrow_{\mathscr{q}} a' = a_n\,, \qquad\qquad b = b_0 \twoheadrightarrow_{\mathscr{p}} b_1 \twoheadrightarrow_{\mathscr{p}} \ldots \twoheadrightarrow_{\mathscr{p}} b' = b_n\,,$$

and Δ is the context defined as $\Delta = \{(a_i, b_i) \mid 0 \leq i \leq n$ and $\mathrm{pol}(a_i) = -\}$.

Moreover, if $a \sim^{\varphi}_{\Gamma'} b$ for a context Γ', we also have $a' \sim^{\varphi}_{\Gamma' \cup \Delta} b'$.

Proof. Immediate by induction of the co-depth of a' and by definition of bisimulation. ◀

▶ **Definition 47.** *Consider $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$, $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$, $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$ with $a \sim^{\varphi}_{\Gamma} b$ for some context Γ. We define $\Gamma_{a,b}$ the **minimal context** for $a \sim^{\varphi}_{\Gamma} b$ as the restriction of Γ s.t.*

$$c \in \mathrm{dom}(\Gamma_{a,b}) \qquad \Leftrightarrow \qquad \begin{cases} \exists a' \in \uparrow a, \mathsf{just}(a') = c & \text{(a)} \\ \Gamma(c) \neq \varphi(c) & \text{(b)} \end{cases}$$

for all $c \in |\mathscr{q}|$,and symmetrically the mirror condition applies to any $d \in |\mathscr{p}|$.

▶ **Lemma 48.** *Consider $\mathscr{q}, \mathscr{p} \in \mathsf{Aug}(A)$ with $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$. Consider $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$ and Γ, Γ' two contexts such that $a \sim^{\varphi}_{\Gamma} b$ and $a \sim^{\varphi}_{\Gamma'} b$.*

Then $\Gamma_{a,b} = \Gamma'_{a,b}$. Moreover, $\Gamma_{a,b}$ is the minimal (for inclusion) context s.t. $a \sim^{\varphi}_{\Gamma_{a,b}} b$.

Proof. The equality comes from Lemma 46 and the definition of $\Gamma_{a,b}$ and $\Gamma'_{a,b}$. By induction, $a \sim^{\varphi}_{\Gamma_{a,b}} b$, since we can safely remove from $\mathrm{dom}(\Gamma)$ all c that are never "used", *i.e.* such that there exists no $a' \in \uparrow a$ having c as pointer; and all c such that $\Gamma(c) = \varphi(c)$, because then we can use condition (c) of Definition 26 instead of condition (b). Finally, for any context Γ'' such that $a \sim^{\varphi}_{\Gamma''} b$, we have $\Gamma_{a,b} = \Gamma''_{a,b} \subseteq \Gamma''$, so $\Gamma_{a,b}$ is minimal for inclusion. ◀

This lemma allows us to write *the minimal context for a, b* without mentioning Γ.

B.2 Clones (Section 4.4)

A key notion in the proof of positional injectivity is the notion of clones, a variation of bisimulation. Although the added constraint on contexts makes transitivity more challenging, we can still prove a variation of Lemma 28. We use the same notation that for the usual bisimulation: for any a, b events of an augmentation $\mathscr{q}$, $a \approx b$ means $a \approx^{\mathrm{id}} b$.

▶ **Lemma 49.** *Consider $\mathscr{q}, \mathscr{p}, \mathscr{r} \in \mathsf{Aug}(A)$, $\varphi : (\!|\mathscr{q}|\!) \cong (\!|\mathscr{p}|\!)$ and $\psi : (\!|\mathscr{p}|\!) \cong (\!|\mathscr{r}|\!)$, and:*

$$a \approx^{\varphi} b \qquad\qquad b \approx^{\psi} c$$

for some $a \in |\mathscr{q}|$, $b \in |\mathscr{p}|$, and $c \in |\mathscr{r}|$. Then, we also have $a \approx^{\psi \circ \varphi} c$.

Proof. Consider Γ_1 and Γ_2 the minimal contexts such that $a \sim^{\varphi}_{\Gamma_1} b$ and $b \sim^{\psi}_{\Gamma_2} c$. If $\mathsf{cod}(\Gamma_1) = \mathrm{dom}(\Gamma_2)$, the result is immediate by Lemma 28: we get $a \sim^{\psi\circ\varphi}_{\Gamma_2\circ\Gamma_1} c$ with, for any $e \in \mathrm{dom}(\Gamma_1) = \mathrm{dom}(\Gamma_2 \circ \Gamma_1)$,

$$\psi(\varphi(\mathsf{just}(e))) = \psi(\mathsf{just}(\Gamma_1(e))) = \mathsf{just}(\Gamma_2(\Gamma_1(e)))$$

so $\Gamma_2 \circ \Gamma_1$ preserves pointers and $a \approx^{\psi\circ\varphi} c$.

Now, assume there exists $e \in \mathsf{cod}(\Gamma_1)$ such that $e \notin \mathrm{dom}(\Gamma_2)$. Since Γ_1 is minimal, there exists $b' \in \uparrow b$ such that $\mathsf{just}(b') = e$. By $b \sim^{\psi}_{\Gamma_2} c$ and Lemma 46, there exists a matching $c' \in \uparrow c$ such that $b' \sim^{\psi}_{\Gamma_2\cup\Delta}$, with Δ the negative moves between b and b', paired with the negative moves between c and c'. Since $e \in \mathsf{cod}(\Gamma_1)$, $\neg(e \geq_{\sigma} b)$, so $e \notin \mathrm{dom}(\Delta)$. Hence, $\mathsf{just}(c') = \psi(e)$ and $\psi(e) \notin \mathsf{cod}(\Gamma_2)$. So we can write

$$b \sim^{\psi}_{\Gamma_2\cup\{(e,\psi(e))\}} c$$

where $\Gamma_2 \cup \{e, \psi(e)\}$ preserves pointers. Likewise, for any $e' \in \mathrm{dom}(\Gamma_2)$, $e' \notin \mathsf{cod}(\Gamma_1)$, we have $\Gamma_1 \cup \{\varphi^{-1}(e'), e'\}$ well-defined and pointer-preserving, such that

$$a \sim^{\varphi}_{\Gamma_1\cup\{(\varphi^{-1}(e'),e')\}} b\,.$$

This allows us to define the following pointer-preserving contexts:

$$\begin{array}{ll}\Gamma_1' & := \Gamma_1 \cup \{(\varphi^{-1}(e'), e') \mid e' \in \mathrm{dom}(\Gamma_2), e' \notin \mathsf{cod}(\Gamma_1)\} \\ \Gamma_2' & := \Gamma_2 \cup \{(e, \psi(e)) \mid e \in \mathsf{cod}(\Gamma_1), e \notin \mathrm{dom}(\Gamma_2)\}\end{array}$$

Then $\Gamma_2' \circ \Gamma_1'$ preserves pointers, and by Lemma 28 we have $a \sim^{\psi\circ\varphi}_{\Gamma_2'\circ\Gamma_1'} c$, so $a \approx^{\psi\circ\varphi} c$. ◀

This allows us to prove equivalence properties for the clone relation.

▶ **Lemma 50.** *Consider $q, p, r \in \mathsf{Aug}(A)$ augmentations, with $\varphi : (\!|q|\!) \cong (\!|p|\!)$ and $\psi : (\!|p|\!) \cong (\!|r|\!)$ two isomorphisms, and events $a \in |q|$, $b \in |p|$, $c \in |r|$:*

reflexivity:	$a \approx^{\mathrm{id}} a$,
transitivity:	*if* $a \approx^{\varphi} b$ *and* $b \approx^{\psi} c$, *then* $a \approx^{\psi\circ\varphi} c$,
symmetry:	*if* $a \approx^{\varphi} b$ *then* $b \approx^{\varphi^{-1}} a$.

Proof. *Reflexivity.* By Lemma 28, $a \sim^{\mathrm{id}} a$, which implies $a \approx^{\mathrm{id}} a$.
Transitivity. See Lemma 49.
Symmetry. Immediate by Lemma 28: if Γ preserves pointers, so does Γ^{-1}. ◀

Clones through id in characteristic expansions will be especially interesting, because then we can partition equivalence classes of $\approx^{\mathrm{id}}$ into successors of forks.

▶ **Lemma 34.** *Consider q a $-$-obsessional expansion of causal strategy p on arena A. Then, for all $a_1^-, a_2^- \in X \in \mathsf{Fork}(q)$, for all $a_1^- \rightarrow_q b_1^+$ and $a_2^- \rightarrow_q b_2^+$, $b_1 \approx b_2$.*

Proof. If $X = \{\mathsf{init}(q)\}$, $a_1 = a_2$ and the result is immediate by determinism and reflexivity. Otherwise, we note b the predecessor of both a_1 and a_2 for $\leq_q$.

First, we prove that b_1 and b_2 are bisimilar. Since q is a $-$-obsessional expansion of p, there exists a (unique, by Lemma 42) morphism $\varphi : q \to p$. By Lemma 45,

$$b_1 \sim_{\Gamma(b_1)} \varphi(b_1) \qquad \text{and} \qquad b_2 \sim_{\Gamma(b_2)} \varphi(b_2)\,.$$

By $-$-linearity of $\mathcal{p}$, we know that $\varphi(a_1) = \varphi(a_2)$, which implies $\varphi(b_1) = \varphi(b_2)$ by determinism. So $\mathsf{cod}(\Gamma(b_1)) = \mathsf{cod}(\Gamma(b_2))$, and by Lemma 28,

$$b_1 \sim_{\Gamma(b_2)^{-1}\circ\Gamma(b_1)} b_2 \ .$$

Writing $\Gamma_1 := \Gamma(b_1)$, $\Gamma_2 := \Gamma(b_2)$ and $\Gamma := \Gamma_2^{-1} \circ \Gamma_1$, it remains to check that Γ preserves pointers. Consider $c \in \mathrm{dom}(\Gamma) = [b_1]^{-}_{\mathcal{q}}$, either $c = a_1$ or $c \leq_{\mathcal{q}} b$. If $c = a_1$, then $\Gamma_1(a_1) = \varphi(a_1) = \Gamma_2(a_2)$, so $\Gamma(a_1) = a_2$ (and both have the same pointer b by courtesy). Otherwise, $c \leq_{\mathcal{q}} b$, so $c \in \mathrm{dom}(\Gamma_2)$ and $\Gamma_1(c) = \varphi(c) = \Gamma_2(c)$, hence $\Gamma(c) = c$. In both cases, Γ preserves pointers, so finally $b_1 \approx b_2$. ◀

B.3 Positional Injectivity (Section 4.5)

In this section, we prove additional lemmas needed in the proof of Lemma 36.

▶ **Lemma 51.** *Consider $\mathcal{q}, \mathcal{p} \in \mathsf{Aug}(A)$ two augmentations such that there exists an isomorphism $\varphi : (\!|\mathcal{q}|\!) \cong (\!|\mathcal{p}|\!)$. Consider $a \in |\mathcal{q}|$, $b \in |\mathcal{p}|$ and Γ a pointing context s.t. $a \sim^{\varphi}_{\Gamma} b$.*
Then a and b have the same co-depth.

Proof. Straightforward by induction. ◀

▶ **Lemma 52.** *Consider $\mathcal{q}, \mathcal{p} \in \mathsf{Aug}(A)$ two augmentations such that there exists an isomorphism $\varphi : (\!|\mathcal{q}|\!) \cong (\!|\mathcal{p}|\!)$. Consider $a \in |\mathcal{q}|$, $b \in |\mathcal{p}|$ and Γ a pointing context such that $a \sim^{\varphi}_{\Gamma} b$. Then for any $c \in |\mathcal{q}|$ such that $c \notin \mathrm{dom}(\Gamma)$, $\varphi(c) \notin \mathsf{cod}(\Gamma)$ and $\neg(c \in {\uparrow} a)$, $\neg(\varphi(c) \in {\uparrow} b)$, we have $a \sim^{\varphi}_{\Gamma\cup\{(c,\varphi(c))\}} b$. Moreover, for any $c \in |\mathcal{q}|$ and $d \in |\mathcal{p}|$ such that*

$$\begin{array}{lll} c \notin \mathrm{dom}(\Gamma)\ , & \neg(c \in {\uparrow} a)\ , & \forall a' \in {\uparrow} a, \mathsf{just}(a') \neq c\ , \\ d \notin \mathsf{cod}(\Gamma)\ , & \neg(d \in {\uparrow} b)\ , & \forall b' \in {\uparrow} b, \mathsf{just}(b') \neq d\ , \end{array}$$

then we also have $a \sim^{\varphi}_{\Gamma\cup\{(c,d)\}} b$.

Proof. Straightforward by induction. Either c is never used in the bisimulation (no one in ${\uparrow} a$ points to c), and we can pair it with any d which is not used either and add (c, d) to Γ (as long as we still have $(\Gamma \cup \{(c, d)\}) \vdash (a, b)$); or it is used with condition *(c)* of Definition 26 and we can add $(c, \varphi(c))$ to Γ and use condition *(b)* instead. ◀

▶ **Lemma 53.** *Consider $\mathcal{q} \in \mathsf{Aug}(A)$ and $a, b \in |\mathcal{q}|$ such that $a \approx b$.*
Then the minimal context for a and b is either empty or $\Gamma : \{c\} \cong \{b\}$.

Proof. Assume, seeking a contradiction, that the minimal context Γ has at least two distinct elements $c_1, c_2 \in \mathrm{dom}(\Gamma)$. First, we can remark that since $a \approx b$, there exists Γ' a pointers-preserving context such that $a \sim_{\Gamma'} b$, and since Γ is a restriction of Γ', Γ also preserves pointers.

By condition *(a)* of Definition 47, $c_1 \leq_{\mathcal{q}} a$ and $c_2 \leq_{\mathcal{q}} a$. Therefore, $c_1 \leq_{\mathcal{q}} c_2$ or $c_2 \leq_{\mathcal{q}} c_1$ – assume *w.l.o.g.* that it is the former. By courtesy, $\mathsf{just}(c_1) \leq_{\mathcal{q}} \mathsf{just}(c_2)$ as well. For the same reason, $\Gamma(c_1) \leq_{\mathcal{q}} \Gamma(c_2)$ or $\Gamma(c_2) \leq_{\mathcal{q}} \Gamma(c_1)$.

If it is the latter, this entails that $\mathsf{just}(\Gamma(c_2)) \leq_{\mathcal{q}} \mathsf{just}(\Gamma(c_1))$ by courtesy; *i.e.*, since Γ preserves pointers, $\mathsf{just}(c_2) \leq_{\mathcal{q}} \mathsf{just}(c_1)$. So $\mathsf{just}(c_1) = \mathsf{just}(c_2)$, and because $c_1, c_2 \leq_{\mathcal{q}} a$, we have $c_1 = c_2$, contradiction.

So, $\Gamma(c_1) \leq_{\mathcal{q}} \Gamma(c_2)$, and $\Gamma(c_1) \neq \Gamma(c_2)$ by hypothesis. By courtesy, this entails that $\Gamma(c_1) \leq_{\mathcal{q}} \mathsf{just}(\Gamma(c_2))$. Likewise, $c_1 \leq_{\mathcal{q}} c_2$ entails $c_1 \leq_{\mathcal{q}} \mathsf{just}(c_2)$. Moreover, Γ preserves pointers, so $\mathsf{just}(c_2) = \mathsf{just}(\Gamma(c_2))$. Hence, we have both

$$\Gamma(c_1) \leq_{\mathcal{q}} \mathsf{just}(c_2) \qquad\qquad c_1 \leq_{\mathcal{q}} \mathsf{just}(c_2)\ ,$$

so c_1 and $\Gamma(c_1)$ are comparable for $\leq_{\mathcal{q}}$ since $\mathcal{q}$ is a forest. But they are negative, so they have the same antecedent by courtesy. This implies $c_1 = \Gamma(c_1)$, which contradicts condition *(b)* of Definition 47. ◀

▶ **Lemma 54.** *Consider $\mathcal{q}, \mathcal{p} \in \mathsf{Aug}(A)$, $\varphi : (\!|\mathcal{q}|\!) \cong (\!|\mathcal{p}|\!)$. Consider also $a^+ \in |\mathcal{q}|$ s.t. $\mathsf{succ}(a) = \bigcup_{i\in I} G_i$, where $I \subseteq \mathbb{N}$ and for $i \in I$, $G_i = \{b_{i,1}, \ldots, b_{i,2^i}\} \in \mathsf{Fork}(\mathcal{q})$ with $\sharp G_i = 2^i$. Then we have $a \approx^\varphi \varphi(a)$, provided the two conditions hold:*

$$\textit{if } b_{i,j} \rightarrowtriangle_{\mathcal{q}} c_{i,j}, \textit{ then } \varphi(b_{i,j}) \rightarrowtriangle_{\mathcal{p}} d_{i,j} \textit{ and } c_{i,j} \approx^\varphi d_{i,j}\ , \tag{3}$$

$$\textit{if } \varphi(b_{i,j}) \rightarrowtriangle_{\mathcal{p}} d_{i,j}, \textit{ then } b_{i,j} \rightarrowtriangle_{\mathcal{q}} c_{i,j} \textit{ and } c_{i,j} \approx^\varphi d_{i,j}\ . \tag{4}$$

Proof. First, remark that $\partial_{\mathcal{q}}(a) = \partial_{\mathcal{p}}(\varphi(a))$ and $\varphi(\mathsf{just}(a)) = \mathsf{just}(\varphi(a))$.

For any $i \in I$, $1 \leq j \leq 2^i$, let $\Gamma_{i,j}$ be the minimal context for $b_{i,j}$ and $\varphi(b_{i,j})$. Such a context exists since either $b_{i,j}$ has no successors, and by (4) neither does $\varphi(b_{i,j})$, either $b_{i,j}$ has only one (by determinism) and $c_{i,j} \approx^\varphi d_{i,j}$ by (3). In both cases, $b_{i,j} \approx^\varphi \varphi(b_{i,j})$.

We wish to take the union of all $\Gamma_{i,j}$ as the context for a and $\varphi(a)$, but this is only possible if they are *compatible*. More precisely, we must ensure that for all $e \in \mathcal{q}, i, k \in I, 1 \leq j \leq 2^i$ and $1 \leq l \leq 2^k$, if there are $c'_{i,j} \in \uparrow b_{i,j}$ and $c'_{k,l} \in \uparrow b_{k,l}$ having both e as justifier, then their matching $d'_{i,j} \in \uparrow\varphi(b_{i,j})$ and $d'_{k,l} \in \uparrow\varphi(b_{k,l})$ also have the same justifier. This can only be a problem if e appears in $\mathrm{dom}(\Gamma_{i,j})$ or in $\mathrm{dom}(\Gamma_{k,l})$ as otherwise both justifiers are $\varphi(e)$.

For all i, j, $\Gamma_{i,j}$ has either one or zero element by Lemma 53. If all $\Gamma_{i,j}$ are empty, we can directly lift the clone relation to a. Otherwise, consider i, j s.t. $\Gamma_{i,j} : \{e_{i,j}\} \cong \{f_{i,j}\}$. From Definition 47, $e_{i,j} \in [b_{i,j}]^-_{\mathcal{q}}$ and $f_{i,j} \in [\varphi(b_{i,j})]^-_{\mathcal{p}}$. Actually we have $f_{i,j} \in [\varphi(a)]^-_{\mathcal{p}}$: indeed $f_{i,j} \neq \varphi(b_{i,j})$, since $e_{i,j}$ and $f_{i,j}$ have the same justifier through φ and the only $e \in [b_{i,j}]^-_{\mathcal{q}}$ s.t. $\varphi(\mathsf{just}(e)) = \mathsf{just}(\varphi(b_{i,j}))$ is $b_{i,j}$, which contradicts Definition 47.

Now, assume that for some k, l, there exists $c'_{k,l} \in \uparrow b_{k,l}$ s.t. $\mathsf{just}(c'_{k,l}) = e_{i,j}$. Since $b_{k,l} \approx^\varphi \varphi(b_{k,l})$, there is a matching $d'_{k,l} \in \uparrow\varphi(b_{k,l})$ such that

$$\varphi(\mathsf{just}(e_{i,j})) = \mathsf{just}(\mathsf{just}(d'_{k,l}))\ .$$

For $b_{i,j} \sim^\varphi_{\Gamma_{i,j}} \varphi(b_{i,j})$ and $b_{k,l} \sim^\varphi_{\Gamma_{k,l}} \varphi(b_{k,l})$ to be compatible, we need $\mathsf{just}(d'_{k,l}) = f_{i,j}$. But since $\Gamma_{i,j}$ preserves pointers,

$$\varphi(\mathsf{just}(e_{i,j})) = \mathsf{just}(f_{i,j})\ .$$

Putting both equalities together, we obtain

$$\mathsf{just}(\mathsf{just}(d'_{k,l})) = \mathsf{just}(f_{i,j})\ ,$$

where $\mathsf{just}(d'_{k,l}) \in [d'_{k,l}]^-_{\mathcal{p}}$ and $f_{i,j} \in [\varphi(a)]^-_{\mathcal{p}}$. But $[\varphi(a)]^-_{\mathcal{p}} \subseteq [d'_{k,l}]^-_{\mathcal{p}}$, which is a fully ordered set for $\leq_{\mathcal{p}}$, so $\mathsf{just}(d'_{k,l})$ and $f_{i,j}$ are comparable. Moreover, they are negative, so by courtesy

$$\mathsf{just}(\mathsf{just}(d'_{k,l})) = \mathsf{just}(f_{i,j}) \text{ iff } \mathsf{pred}(\mathsf{just}(d'_{k,l})) = \mathsf{pred}(f_{i,j})$$

where pred is the predecessor for $\leq_{\mathcal{p}}$. Hence, $\mathsf{just}(d'_{k,l}) = f_{i,j}$ (see Figure 22, where $\rightarrowtriangle$ represents $\rightarrowtriangle_{\mathcal{q}}$, $\cdots$ represents $\rightarrowtriangle_{(\!|\mathcal{q}|\!)}$, and $\rightarrow$ represents $\leq_{\mathcal{q}}$ (and the same applies for $\mathcal{p}$)).

So all contexts $\Gamma_{i,j}$ are compatible. Writing $\Gamma = \cup_{i,j}\Gamma_{i,j}$ it follows that $b_{i,j} \sim^\varphi_\Gamma \varphi(b_{i,j})$ via Lemma 52; which entails that $a \sim^\varphi_\Gamma \varphi(a)$ by two steps of the bisimulation game. This implies $a \approx^\varphi \varphi(a)$ since all $\Gamma_{i,j}$ preserve pointers. ◀

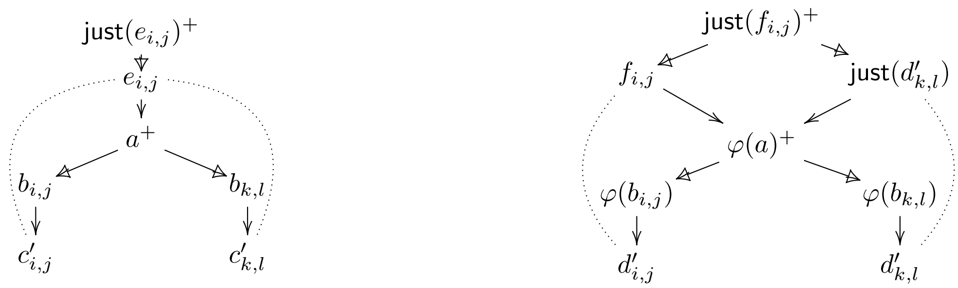


Figure 22 Justifiers in $\mathscr{q}$ and $\mathscr{p}$

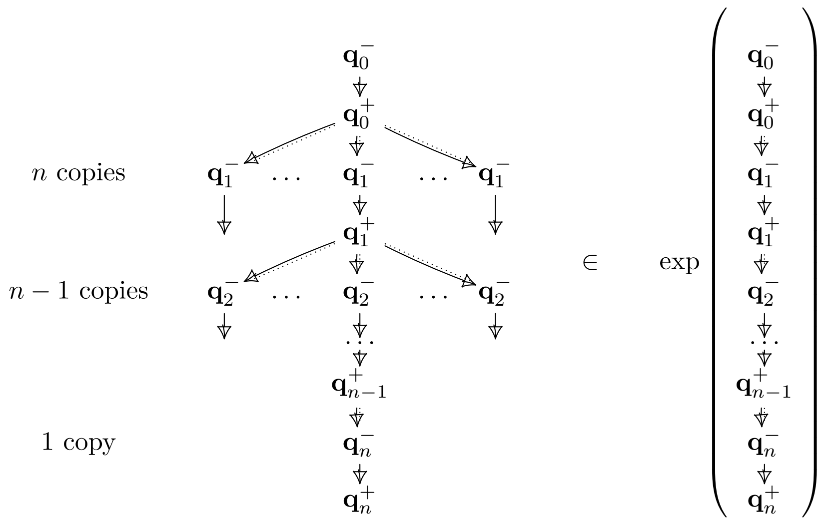


Figure 23 Wide expansion of a P-view

C Beyond Total Finite Strategies: Proofs from Section 5

We now give the proof of Theorem 38.

Consider $\sigma_1, \sigma_2 : A$ finite (but not necessarily total) innocent strategies. If they are empty, there is nothing to prove. Otherwise, let $2n+2$ be the length of s the longest P-view among them. *W.l.o.g.*, assume that $s \in \ulcorner \sigma_1 \urcorner$.

Consider $\mathscr{p}_1$ the sub-augmentation of $\hat{\sigma_1}$ restricted to prefixes of s – it is a linear augmentation of length $2n+2$, as shown on the right hand side of Figure 23. We build the **wide expansion** $\mathscr{q}_1 \in \exp(\mathscr{p}_1)$ as in the left hand side of Figure 23. It is the unique $-$-obsessional and $+$-obsessional expansion of $\mathscr{p}_1$ s.t. each fork of co-depth $2k$ has cardinality k (except for the initial move). So for any $1 \leq k \leq n$, they are $\frac{n!}{(n-k)!}$ copies of $\mathbf{q}_k^+$.

Now, since $(\!|\sigma_1|\!) = (\!|\sigma_2|\!)$, Proposition 22 entails $(\!|\hat{\sigma_1}|\!) = (\!|\hat{\sigma_2}|\!)$. Therefore, there is $\mathscr{q}_2 \in \exp(\hat{\sigma_2})$ along with some isomorphism $\varphi : (\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$. By abuse of notation, we keep referring to events of $|\mathscr{q}_2|$ with the same naming convention as in Figure 23, this is justified by the isomorphism φ.

Now, we study the shape of $\mathscr{q}_2$. It is a tree starting with the unique initial move $\mathbf{q}_0^-$, and by courtesy it cannot break the causal links from positives to negatives; so we may regard it as a tree whose nodes are the $\mathbf{q}_k^+$'s. For each $0 \leq k \leq n$, it has exactly $n!/k!$ nodes of arity k

(by *arity*, we mean the number of children in the tree structure) and by hypothesis its depth is bounded by $n + 1$. The essence of the situation is captured by the following simplified setting:

Fix $n \in \mathbb{N}$. **Simple trees** are finite trees made of nodes k of arity k for $0 \leq k \leq n$. We set $T_0 = 0$, and for $k > 0$, T_k is the tree with root k and k copies of T_{k-1} as children. If t is a simple tree, its **size** $\sharp t$ is its number of nodes, and its **depth** is the maximal number of nodes reached in a path. For instance, the depth of T_k is $k + 1$ and

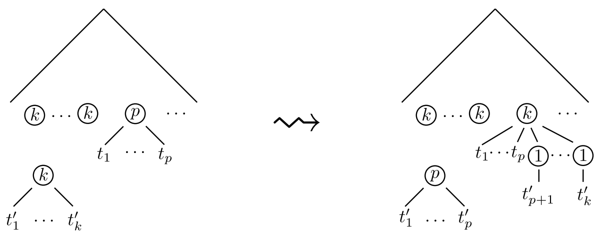
$$\sharp T_k = k! \sum_{i=0}^{k} \frac{1}{i!} \,.$$

Now, let us consider the set $\mathsf{Trees}(n)$ of simple trees of depth $\leq n + 1$, and having, for $2 \leq k \leq n$, $\frac{n!}{k!}$ nodes k, and arbitrarily many nodes 1 and 0. We prove:

▶ **Lemma 55.** *Let $t \in \mathsf{Trees}(n)$ of maximal size. Then, $t = T_n$.*

Proof. Seeking a contradiction, assume t is distinct from T_n. Consider a minimal node where they differ, *i.e.* closest to the root – say t has some p at the row corresponding to k's in T_n. If $k = 0$ then $p > 0$ and this contradicts that the depth of t is less than n. So, $k \geq 1$. If $p > k$, then $p \geq 2$. But by minimality, t is the same as T_n for all rows closer to the root, so all p for $p > k$ are exhausted. Hence, $p < k$. If $k = 1$ and $p = 0$, then we may replace p with T_1, yielding $t' \in \mathsf{Trees}(n)$ of size strictly greater than $\sharp t$, contradicting maximality. Otherwise, $k \geq 2$. Then the number of nodes k is fixed, there are fewer of those on this row as for T_n, and they cannot occur on rows closer to the root. Therefore, there is an occurrence of k strictly deeper in t.

We then perform the transformation as in the diagram:



This yields $t' \in \mathsf{Trees}(n)$. But $\sharp t' > \sharp t$, contradicting the maximality of t. ◀

Now, from $\mathscr{q}_2$ we extract a simple tree $t(\mathscr{q}_2) \in \mathsf{Trees}(n)$ as follows. For each $0 \leq k \leq n$, to each $\mathbf{q}^+_{n-k}$ we associate a node k, with edges as in $\mathscr{q}_2$. Because all P-views in σ_2 have length lesser or equal to $2n + 2$ and $\mathscr{q}_2 \in \exp(\hat{\sigma_2})$, $t(\mathscr{q}_2)$ has depth $\leq n + 1$. The constraints on the number of each node are ensured by the isomorphism $\varphi : (\!|\mathscr{q}_1|\!) \cong (\!|\mathscr{q}_2|\!)$. Therefore $t(\mathscr{q}_2) \in \mathsf{Trees}(n)$, and by Lemma 55, $t(\mathscr{q}_2) = T_n$.

This induces directly an isomorphism ψ between $(\mathscr{q}_1, \leq_{\mathscr{q}_1})$ and $(\mathscr{q}_2, \leq_{\mathscr{q}_2})$. Note that there is a priori no reason why ψ and φ would coincide, so to conclude, we must still check that ψ preserves $\rightarrowtriangle_{(\!|\mathscr{q}_1|\!)}$, *i.e.* justification pointers. Assume $\mathbf{q}^-_j \rightarrowtriangle_{(\!|\mathscr{q}_1|\!)} \mathbf{q}^+_i$. Then, $\mathbf{q}^+_i$ has arity $n - i$, and $\mathsf{just}(\mathsf{just}(\mathbf{q}^+_i)) = \mathbf{q}^+_j$ of arity $n - j$. But then, by construction, it follows that for any move $a^+ \in |\mathscr{q}_1|$ of arity $n - i$, $\mathsf{just}(\mathsf{just}(a))$ has arity $n - j$. This is transported by the isomorphism φ, so this property also holds for $\mathscr{q}_2$. Now, consider $\psi(\mathbf{q}^+_i) \in |\mathscr{q}_2|$. Its justifier is some $b^- \in |\mathscr{q}_2|$ such that $\mathsf{just}(b^-)$ has arity $n - j$. But as arity is preserved by ψ, there is only one move with this property in the causal history of $\psi(\mathbf{q}^+_i)$, namely $\psi(\mathbf{q}^-_j)$. So, ψ preserves pointers.

Finally, ψ also preserves the image in the arena: by construction of $\mathscr{q}_1$, all positive moves with the same arity have the same image, and all negative moves whose justifiers have the same arity also have the same image. Hence, the image only depends on the arity, which is a property of $\llparenthesis \mathscr{q}_1 \rrparenthesis$; and since $\llparenthesis \mathscr{q}_1 \rrparenthesis$ and $\llparenthesis \mathscr{q}_2 \rrparenthesis$ are isomorphic, the same holds for $\mathscr{q}_2$. Since ψ preserves arity and justifiers, it also preserves the image in the arena.

We have constructed an isomorphism $\psi : \mathscr{q}_1 \cong \mathscr{q}_2$. Consider a maximal branch

$$\mathbf{q}_0^- \rightarrowtriangle_{\mathscr{q}_2} \mathbf{q}_0^+ \rightarrowtriangle_{\mathscr{q}_2} \dots \rightarrowtriangle_{\mathscr{q}_2} \mathbf{q}_n^+$$

of $\mathscr{q}_2$. Since $\mathscr{q}_2 \in \exp(\hat{\sigma_2})$, we have a morphism $\nu : \mathscr{q}_2 \to \hat{\sigma_2}$. Its image by ν is

$$s_1 \rightarrowtriangle_{\hat{\sigma_2}} s_1 s_2 \rightarrowtriangle_{\hat{\sigma_2}} \dots \rightarrowtriangle_{\hat{\sigma_2}} s_1 \dots s_{2n+2}$$

a sequence of prefixes of a P-view $s_1 \dots s_{2n+2}$, which, using that $\mathscr{q}_1$ is an expansion of $\mathscr{p}_1$ and $\psi : \mathscr{q}_1 \cong \mathscr{q}_2$, is immediately seen to be exactly s. Hence, $s \in \sigma_2$ as claimed.