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# An optimal instrumental variable approach for continuous-time multiple input-single output fractional model identification

Abir Mayoufi <sup>\*,\*\*</sup> Stéphane Victor <sup>\*</sup> Rachid Malti <sup>\*</sup>  
Manel Chetoui <sup>\*\*</sup> Mohamed Aoun <sup>\*\*</sup>

<sup>\*</sup> Université de Bordeaux, CNRS, IMS UMR 5218, Bordeaux  
INP/enseirb-matmeca – 351 cours de la Libération, 33405 Talence  
cedex, France

*firstname.lastname@ims-bordeaux.fr*

<sup>\*\*</sup> Université de Gabès, Avenue Omar Ibn El Khattab, Zrig  
Eddakhlania, 6029 Gabès, Tunisie

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**Abstract:** This paper proposes an instrumental variable approach for continuous-time system identification using fractional models with multiple input single output context. This work is an extension of the simplified refined instrumental variable approach (*srivcf*) developed for single input-single output fractional model identification (Malti et al. (2008a); Victor et al. (2013)) to the multiple input-single output case. Monte Carlo simulation analysis is used to demonstrate the performance of the proposed approach. A study is then provided to motivate differentiation order estimation, and more specifically, commensurate order estimation.

Keywords: system identification, continuous-time, instrumental variable, multiple inputs, state variable filter, fractional model.

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## 1. INTRODUCTION

System identification aims at providing mathematical models for dynamical systems based on observed inputs and outputs. Dynamical systems are normally formulated in continuous-time (CT), such as differential equations, unfortunately, most system identification algorithms have been developed in discrete-time (DT).

This paper focusses on Multiple Input Single Output (MISO) system identification, and especially for fractional models. Kortmann and Unbehauen (1987) have proposed system identification algorithms for a specific class of MISO nonlinear Hammerstein models. Westwick et al. (2006) proposed to use auto- and cross-correlation matrices with singular value decomposition to get robust estimation, unfortunately, the estimation provides high level of uncertainties in the estimation. Least squares algorithms can also be used for identifying MISO systems (see Zhang and Wang (2016)), leading to high variance on the estimated parameters when dealing with high level of noise. In Garnier et al. (2007), the **Refined Instrumental Variables for Continuous-time system** approach was extended to handle **MISO** models. Thanks to the instrumental variable mechanism, the estimates become consistent with low estimation variance. Two variants were developed: **rivc** (see Young and Jakeman (1980)) in presence of colored noise and the **Simplified rivc** in presence of white noise. An analysis of the consistency of the Simplified Refined Instrumental Variable method for Continuous-time systems has been discussed in Siqui et al. (2019).

Fractional system identification has become more and more important in different fields. First works, on fractional system identification, started in the late nineties. A state of the art on fractional system identification is carried out in Malti et al. (2008a). More recently, parameter estimation has been proposed by using least squares technique (see Chen et al. (2016); Zhao et al. (2017)) with the inconvenient of providing high estimation variance when measurements have high level of noise. Rahmani and Farrokhi (2019) have developed neuro-fractional-order Hammerstein model with a Lyapunov-based identification method by using state space representation, with a specific method. In order to eliminate bias and to reduce estimation variance, Cois et al. (2001) have proposed to add state variable filters to cut noise in high frequencies and also to use instrumental variable technique (**ivsvf**) for fractional model identification to remove bias in the estimation. The Simplified Refined Instrumental Variable (**sriv**) approach was extended to fractional models in Malti et al. (2008b); Victor et al. (2013) where estimates become consistent.

In this paper, a first contribution enables adjusting the cut-off frequency of state variable filters in order to reduce the estimation variance when using **ivsvf** method. Then, an extension of the **srivcf** approach is developed to handle MISO fractional model identification. Finally, a study is provided to motivate differentiation order estimation.

The paper is organized as follows. After introducing fractional calculus in section 1, the problem formulation is presented in section 2. The main contributions are presented in section 3. Section 4 validates the developments

on a simulation example. Finally a conclusion is drawn in section 5.

### Fractional calculus

A single-input-single-output (**SISO**) fractional mathematical model can be described by a differential equation containing operators of fractional order:

$$y(t) + a_1 p^{\alpha_1} y(t) + \dots + a_N p^{\alpha_N} y(t) = b_0 p^{\beta_0} u(t) + b_1 p^{\beta_1} u(t) + \dots + b_M p^{\beta_M} u(t) \quad (1)$$

where  $u(t), y(t)$  respectively are the input and the noise-free output,  $(a_i (i = 1, \dots, N), b_j (j = 0, \dots, M)) \in \mathbb{R}^2$  and the differentiation orders are positive real numbers such as:

$$\begin{cases} 0 < \alpha_1 < \alpha_2 < \dots < \alpha_N \\ 0 < \beta_0 < \beta_1 < \dots < \beta_M \end{cases}$$

$p$  refers to the differential operator ( $p = \frac{d}{dt}$ ).

The differentiation to an arbitrary order  $\nu$  of a function  $f(t)$  in the sense of Grünwald-Letnikov is defined by:

$$p^\nu f(t) = \frac{1}{T^\nu} \sum_{h=0}^H (-1)^h \binom{\nu}{h} f(t - Th) \quad (2)$$

with  $H = \lfloor \frac{t}{T} \rfloor$  ( $\lfloor \cdot \rfloor$  is the floor operator),  $T$  is the sampling time and  $\binom{\nu}{k}$  is the Newton's binomial generalized to fractional orders:

$$\binom{\nu}{k} = \frac{\Gamma(\nu+1)}{\Gamma(k+1)\Gamma(\nu-k+1)} = \frac{\nu(\nu-1)\dots(\nu-k+1)}{k!}.$$

The Laplace transform of a  $\nu$  order derivative, of a causal function  $f$  (null  $\forall t \leq 0$ ), is given by :

$$\mathcal{L}\{p^\nu x(t)\} = s^\nu X(s). \quad (3)$$

This leads to the following non-integer transfer function:

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^M b_j s^{\beta_j}}{1 + \sum_{i=1}^N a_i s^{\alpha_i}}. \quad (4)$$

**Definition 1.1.** (Commensurate system). For a commensurate system, transfer function (4) can be rewritten in the following form:

$$G(s) = \frac{b_0 (s^\nu)^{\frac{\beta_0}{\nu}} + b_1 (s^\nu)^{\frac{\beta_1}{\nu}} + \dots + b_M (s^\nu)^{\frac{\beta_M}{\nu}}}{1 + a_1 (s^\nu)^{\frac{\alpha_1}{\nu}} + \dots + a_N (s^\nu)^{\frac{\alpha_N}{\nu}}}, \quad (5)$$

where all powers of  $s^\nu$  are integers.

In this case, equation (4) takes the following form:

$$G(s) = \frac{\sum_{j=0}^m \tilde{b}_j s^{j\nu}}{1 + \sum_{i=1}^n \tilde{a}_i s^{i\nu}}, \quad \nu \in \mathbb{R}^+, \quad (6)$$

with  $j = 1, \dots, m$ ,  $m = \frac{\beta_M}{\nu} \in \mathbb{N}$ ,  $i = 1, \dots, n$ ,  $n = \frac{\alpha_N}{\nu} \in \mathbb{N}$ , and

$$\begin{cases} \tilde{b}_j = b_j & \text{if } \exists j \in \{0, 1, \dots, M\} \text{ such that } j\nu = \beta_j \\ \tilde{b}_j = 0 & \text{otherwise} \\ \tilde{a}_i = a_i & \text{if } \exists i \in \{1, \dots, N\} \text{ such that } i\nu = \alpha_i \\ \tilde{a}_i = 0 & \text{otherwise.} \end{cases}$$

For commensurate order systems, the most well-known stability criterion was introduced by Matignon (1998) who

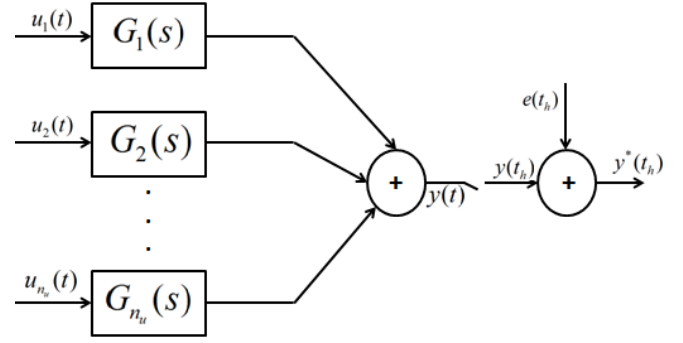


Fig. 1. MISO model

demonstrated the stability of a commensurate system with an order between 0 and 1. This theorem has been extended by Moze and Sabatier (2005) for a commensurate order between 1 and 2.

**Theorem 1.1. Stability theorem:**

Let  $G$  be a commensurate transfer function (6) and  $\nu$  its commensurate order.  $G(s) = \frac{Q_\nu(s)}{P_\nu(s)}$  is **BIBO** (bounded input bounded output) stable if and only if:

$$0 < \nu < 2 \quad (7)$$

and

$$\forall s_k \in \mathbb{C}, P_\nu(s_k) = 0 \quad \text{such as} \quad |\arg(s_k)| > \nu \frac{\pi}{2}. \quad (8)$$

## 2. PROBLEM FORMULATION

Consider a MISO fractional system (see Fig. 1) described by:

$$\begin{cases} y_{u_k}(t) = G_k(p) u_k(t) \\ y(t) = \sum_{k=1}^{n_u} y_{u_k}(t) \\ y^*(t_h) = y(t_h) + e(t_h) \end{cases} \quad (9)$$

where  $n_u$  is the number of inputs and  $u(t) = [u_1(t) \dots u_{n_u}(t)]$ . The input-output dataset  $(u(t), y(t))$  are collected at regular samples. The input signals  $u_1(t), u_2(t), \dots, u_{n_u}(t)$  must be uncorrelated. The output measurement  $y(t)$  is tainted by a white noise  $e(t_h)$  and  $G_k(p)$  is the fractional transfer function that relates the input signal  $u_k(t)$  to its noise-free output  $y_{u_k}(t)$ , defined as follows:

$$G_k(p) = \frac{B_k(p, \theta_k)}{A_k(p, \theta_k)} = \frac{\sum_{j=0}^{M_k} b_{j,k} p^{\beta_{j,k}}}{1 + \sum_{i=1}^{N_k} a_{i,k} p^{\alpha_{i,k}}}. \quad (10)$$

The objective is to estimate the parameters of the model described by equation (9) using  $H$  samples of inputs/output data  $\{u_1(t_h), \dots, u_{n_u}(t_h), y^*(t_h)\}_{h=1}^H$ . The fractional commensurate order is supposed known a priori and only the linear coefficients are estimated.

The parameter vector is defined as

$$\begin{cases} \theta_k = [b_{0,k}, b_{1,k}, \dots, b_{M_k,k}, a_{1,k}, \dots, a_{N_k,k}]^T, \quad k = 1, \dots, n_u \\ \theta = [\theta_1, \dots, \theta_{n_u}]. \end{cases} \quad (11)$$

It is well known that the methods based on least squares give biased parameters in presence of noise affecting the output measurements. To solve this problem, the use of methods based on instrumental variable is proposed in this work.

### 3. REFINED INSTRUMENTAL VARIABLE FOR CONTINUOUS-TIME FRACTIONAL MODELS

#### 3.1 *srivcf* approach for SISO models

The *srivcf* approach is developed in Victor et al. (2013) for **SISO** system identification by fractional models. It is based on the use of an auxiliary model.

The error function  $\varepsilon$  is given by the output error:

$$\varepsilon(t, \theta) = y^*(t) - \frac{B(p, \theta)}{A(p, \theta)} u(t), \quad (12)$$

which can be rewritten as,

$$\varepsilon(t, \theta) = A(p, \theta) \left( \frac{1}{A(p, \theta)} y^*(t) \right) - B(p, \theta) \left( \frac{1}{A(p, \theta)} u(t) \right). \quad (13)$$

Thus a low-pass filter is applied to both input and output signals.

Setting  $y_f^*(t) = \frac{1}{A(p, \theta)} y^*(t)$  and  $u_f(t) = \frac{1}{A(p, \theta)} u(t)$ , equation (13) takes the following form:

$$\varepsilon(t, \theta) = A(p, \theta) y_f^*(t) - B(p, \theta) u_f(t). \quad (14)$$

The original output error is put under an equation error function, which yields to optimal Instrumental Variable (**IV**) estimates, (as defined in Young (2011)). The filter transfer function is defined as

$$F_{opt}(p) = \frac{1}{A(p, \theta)}. \quad (15)$$

However, in practice  $\theta$  is unknown and so is  $A(p, \theta)$ . Therefore this problem is solved in an iterative optimization procedure, which is intended to adjust the estimates iteratively until convergence. Hence, the following filter is initialized and computed iteratively:

$$F^{iter}(p) = \frac{1}{\hat{A}(p, \hat{\theta}^{iter})} = \frac{1}{1 + \sum_{i=1}^N \hat{a}_i p^{\alpha_i}}, \quad (16)$$

where *iter* is the iteration number, *iter* = 1, 2, ... until convergence, and  $\hat{a}_i$  is the estimate at each iteration.

Hence, expression (14) can be rewritten as:

$$\varepsilon(t, \theta) = y_f^*(t) - \varphi_f(t) \theta, \quad (17)$$

where  $\varphi_f(t)$  is the regression vector defined as:

$$\varphi_f(t) = \begin{bmatrix} p^{\beta_0} u_f(t), \dots, p^{\beta_M} u_f(t) \\ -p^{\alpha_1} y_f^*(t), \dots, -p^{\alpha_N} y_f^*(t) \end{bmatrix}^T. \quad (18)$$

Solving this problem with least squares minimization will lead to biased estimates as the output is corrupted by additive noise.

To avoid this problem, an instrumental vector  $\varphi_f^{iv}(t, \theta)$  is built at each iteration:

$$\varphi_f^{iv}(t, \hat{\theta}^{iter}) = \begin{bmatrix} p^{\beta_0} u_f(t) \dots p^{\beta_M} u_f(t) \\ -p^{\alpha_1} y_{u,f}^{iter}(t) \dots -p^{\alpha_N} y_{u,f}^{iter}(t) \end{bmatrix}^T. \quad (19)$$

The noise-free output estimates  $y_{u,f}^{iter}(t)$  are obtained from an auxiliary model calculated at each iteration:

$$y_{u,f}^{iter}(t, \hat{\theta}^{iter}) = \frac{y_u^{iter}(t)}{\hat{A}(p, \hat{\theta}^{iter})} = \frac{\hat{B}(p, \hat{\theta}^{iter})}{\hat{A}(p, \hat{\theta}^{iter})^2} u(t). \quad (20)$$

The optimal IV estimates are computed at each iteration:

$$\hat{\theta}^{iter+1} = [\Phi_f^{iv} \Phi_f^T]^{-1} \Phi_f^{iv} \mathbf{Y}_f^* \quad (21)$$

where:

$$\Phi_f^{iv} = [\varphi_f^{iv}(t_1, \theta^{iter}), \dots, \varphi_f^{iv}(t_H, \theta^{iter})] \quad (22)$$

$$\Phi_f = [\varphi_f(t_1), \dots, \varphi_f(t_H)] \quad (23)$$

$$\mathbf{Y}_f^* = [y_f^*(t_1), \dots, y_f^*(t_H)]. \quad (24)$$

The *srivcf* algorithm is iterated until convergence:

$$\sum_{j=1}^{\dim(\hat{\theta})} \left( \frac{\hat{\theta}_j^{iter+1} - \hat{\theta}_j^{iter}}{\hat{\theta}_j^{iter+1}} \right)^2 < \ell \quad (25)$$

where  $\ell$  is a given precision factor.

An estimation of the covariance matrix on the estimation  $\hat{\theta}$  can be computed:

$$\hat{P}_\theta = \hat{\sigma}^2 [\Phi_f^{iv} (\Phi_f^{iv})^T]^{-1} \quad (26)$$

where  $\hat{\sigma}^2$  is the empirical estimation of the noise variance and  $\Phi_f^{iv}$  is the instrumental vector computed at the last iteration.

Convergence properties of the refined IV methods can be found in Liu et al. (2011).

**Suboptimal instrumental variable with state variable filters** The least squares-based state-variable filter (**lssvf**) or the suboptimal instrumental variable state-variable filter (**ivsvf**) developed by Cois et al. (2001) and Malti et al. (2008a) can be used by replacing the filter (16) by a state variable filter defined as a Poisson filter:

$$F_{svf}(p) = \frac{1}{\left(1 + \frac{p}{\omega_c}\right)^{N_c}}, \quad (27)$$

$\omega_c$  and  $N_c$  being two parameters to be established by the user.  $N_c$  is determined as the integer above the maximum of  $\alpha_N$  and  $\beta_M$ . As the system is strictly proper, this means  $\alpha_N \geq \beta_M$ , hence  $N_c = \lfloor \alpha_N \rfloor + 1$ . Concerning the cut-off frequency  $\omega_c$ , the main idea of using the state-variable filter is to let all the signal harmonics that are lower than  $\omega_c$  by attenuating all frequencies above it.

#### 3.2 *srivcf* approach for MISO models

For a MISO model, an algorithm is proposed to optimize the parameters. The main idea is to decompose the MISO model, described in (9), into  $n_u$  SISO models. In this case, the error function takes the following form:

$$\varepsilon_k(t, \theta) = x_{k,f}(t, \theta) - y_{u_k,f}(t, \theta_k), \quad k = 1, \dots, n_u \quad (28)$$

where,  $y_{u_k}$  is the noise-free part of  $x_k$ :

$$x_k(t, \theta) = y^*(t) - \sum_{\substack{n=1 \\ n \neq k}}^{n_u} y_{u_n}(t, \theta_n). \quad (29)$$

At each iteration, a “for” loop is launched to estimate each parameter vector  $\theta_k$  ( $k = 1, \dots, n_u$ ), assuming that  $\theta_n$  (with  $n \neq k$ ) are known, by applying the SISO version of the *srivcf* algorithm. The same steps described in Section 3.1 are used.

In this case, (17) can be expressed in the following form:

$$\varepsilon_k(t, \theta) = x_k(t, \theta_k) - \varphi_{k,f}(t) \theta_k, \quad k = 1, \dots, n_u \quad (30)$$

where,

$$\varphi_{k,f}(t, \theta_k) = \begin{bmatrix} p^{\beta_0} u_{k,f}(t) \dots p^{\beta_M} u_{k,f}(t) \\ -p^{\alpha_1} x_{k,f}(t, \theta_k) \dots -p^{\alpha_N} x_{k,f}(t, \theta_k) \end{bmatrix}^T \quad (31)$$

noting that:

$$x_{k,f}(t, \theta_k) = F_k(p) x_k(t, \theta_k) \quad (32)$$

and

$$F_k(p) = \frac{1}{1 + \sum_{i=1}^{N_k} a_{i,k} p^{\alpha_{i,k}}}, \quad (33)$$

where  $F_k(p)$  is the filter that yields the optimal **IV** estimates of  $\theta_k$ . In practice, the estimates are unknown, therefore  $F_k$  is updated at each iteration with  $F_k^{iter}(p)$  as defined in (16).

The MISO version of **srivcf** approach is summarized by the following steps.

**Step1:**

Initialize the parameters vector  $\theta_0$  to calculate the auxiliary model outputs  $y_{u_k}$ :

$$y_{u_k}(t, \theta_k^0) = \frac{B_k(p, \theta_k^0)}{A_k(p, \theta_k^0)} u_k(t), \quad k = 1, \dots, n_u \quad (34)$$

**Step2:**

$iter = 1, \dots, N_{iter}$

**A)** for  $k = 1, \dots, n_u$

a) Compute the response  $x_k^{iter}(t, \theta_k^{iter})$  to  $u_k(t)$ :

$$x_k^{iter}(t, \hat{\theta}_k^{iter}) = y^*(t) - \sum_{\substack{n=1 \\ n \neq k}}^{n_u} y_{u_n}(t, \hat{\theta}_n^{iter}). \quad (35)$$

b) Filter the noisy response of  $x_k^{iter}(t)$ , the auxiliary model outputs  $y_{u_k}$ , and the input signals  $u_k$ :

$$x_{k,f}^{iter}(t) = F_k^{iter}(p) x_k^{iter}(t) \quad (36)$$

$$y_{u_k,f}^{iter}(t) = F_k^{iter}(p) y_{u_k}^{iter}(t) \quad (37)$$

$$u_{k,f}(t) = F_k^{iter}(p) u_k(t) \quad (38)$$

c) Form the instrumental variable  $\varphi_{k,f}^{iv}$  and the regression  $\varphi_{k,f}$  vectors:

$$\varphi_{k,f}^{iv}(t) = \begin{bmatrix} p^{\beta_0} u_{k,f}(t) \dots p^{\beta_{M_k}} u_{k,f}(t) \\ -p^{\alpha_1} y_{u_k,f}^{iter}(t) \dots -p^{\alpha_{N_k}} y_{u_k,f}^{iter}(t) \end{bmatrix}^T \quad (39)$$

$$\varphi_{k,f}(t) = \begin{bmatrix} p^{\beta_0} u_{k,f}(t) \dots p^{\beta_{M_k}} u_{k,f}(t) \\ -p^{\alpha_1} x_{k,f}^{iter}(t) \dots -p^{\alpha_{N_k}} x_{k,f}^{iter}(t) \end{bmatrix}^T \quad (40)$$

d) Compute the **IV**-based solution  $\hat{\theta}_k^{iter+1}$

$$\hat{\theta}_k^{iter+1} = [\Phi_{k,f}^{iv} \Phi_{k,f}^T]^{-1} \Phi_{k,f}^{iv} X_{k,f}^{iter}, \quad (41)$$

with

$$\Phi_{k,f}^{iv} = [\varphi_{k,f}^{iv}(t_1), \dots, \varphi_{k,f}^{iv}(t_H)] \quad (42)$$

$$\Phi_{k,f} = [\varphi_{k,f}(t_1), \dots, \varphi_{k,f}(t_H)] \quad (43)$$

$$X_{k,f}^{iter} = [x_{k,f}^{iter}(t_1), \dots, x_{k,f}^{iter}(t_H)] \quad (44)$$

e) Generate the auxiliary model output  $y_{u_k}(t)$  using  $\hat{\theta}_k^{iter+1}$ :

$$y_{u_k}(t, \hat{\theta}_k^{iter+1}) = \frac{B_k(p, \hat{\theta}_k^{iter+1})}{A_k(p, \hat{\theta}_k^{iter+1})} u_k(t), \quad (45)$$

**B)** Iterate step **(A)** until convergence:

$$test_k = \sum_{j=1}^{\dim(\hat{\theta}_k)} \left( \frac{\hat{\theta}_{k,j}^{iter+1} - \hat{\theta}_{k,j}^{iter}}{\hat{\theta}_{k,j}^{iter+1}} \right)^2 < \ell \quad (46)$$

where  $\ell$  is a sufficiently small fixed edge.

**Step3:**

Compute the parametric covariance matrix  $P_k$ , after convergence  $\hat{\theta}^{iter}$  is found:

for  $k = 1, \dots, n_u$

$$\hat{P}_k = \sigma_{opt}^2 \mathcal{H}_k^{-1} \quad (47)$$

with  $\hat{\sigma}^2$  is the empirical estimation of the noise variance, given by:

$$\hat{\sigma}^2 = y^*(t) - \sum_{k=1}^{n_u} y_{u_k}^{iter}(t) \quad (48)$$

where  $y_{u_k}^{iter}(t)$  is calculated at the last iteration.

$\mathcal{H}_k$  is the approximate Hessian given by:

$$\mathcal{H}_k = \frac{\partial \varepsilon}{\partial \theta_k}^T \frac{\partial \varepsilon}{\partial \theta_k}, \quad (49)$$

where  $\frac{\partial \varepsilon}{\partial \theta_k}$  is the error sensitivity function given by:

$$\frac{\partial \varepsilon}{\partial \theta_k} = -\frac{\partial \hat{y}_{u_k}}{\partial \theta_k} = -\left[ \frac{\partial \hat{y}_{u_k}}{\partial b_{0,k}}, \dots, \frac{\partial \hat{y}_{u_k}}{\partial b_{M_k,k}}, \frac{\partial \hat{y}_{u_k}}{\partial a_{1,k}}, \dots, \frac{\partial \hat{y}_{u_k}}{\partial a_{N_k,k}} \right]^T. \quad (50)$$

Here  $\frac{\partial \hat{y}_{u_k}}{\partial b_{j,k}}$  and  $\frac{\partial \hat{y}_{u_k}}{\partial a_{i,k}}$  are the output sensitivity functions, given by:

$$\frac{\partial \hat{y}_{u_k}}{\partial b_{j,k}} = \frac{\hat{p}^{\hat{\beta}_j}}{1 + \sum_{i=1}^{N_k} \hat{a}_{i,k} \hat{p}^{\hat{\alpha}_i}} u_k(t), \quad \forall j = 0, \dots, M_k, \quad (51)$$

$$\frac{\partial \hat{y}_{u_k}}{\partial a_{i,k}} = \frac{\sum_{j=0}^{M_k} \hat{b}_{j,k} \hat{p}^{\hat{\beta}_j + \hat{\alpha}_i}}{\left( 1 + \sum_{i=1}^{N_k} \hat{a}_{i,k} \hat{p}^{\hat{\alpha}_i} \right)^2} u_k(t), \quad \forall i = 0, \dots, N_k. \quad (52)$$

**Initializing the srivcf MISO algorithm** The **srivcf** algorithm is an iterative algorithm that can be initialized by a suboptimal method: the **(lssvf)** or the **(ivsvf)** by using state-variable filters such as described in section 3.2.

The **svf** as defined in relation (27) takes the form of a Poisson filter. Such as implemented in the CRONE toolbox, the cut-off frequency  $\omega_c$  can be set either by the user or automatically as  $\omega_s/10$ , where  $\omega_s$  is the Shannon's frequency.

A procedure is now proposed to enhance the setting of the filter cut-off frequency. The **svf** is set to each different frequency of  $\Omega = [\frac{\omega_s}{2}, \frac{\omega_s}{5}, \frac{\omega_s}{10}, \frac{\omega_s}{15}]$ , around Shannon's frequency  $\omega_s$ , then a quadratic error criterion  $J_i$  is computed

$$J_i = \|\varepsilon_i(t)\|_2, \quad i = 1, \dots, \text{length}(\Omega), \quad (53)$$

with  $\varepsilon(t) = y^*(t) - y_{est}(t)$ . A table containing  $\Omega$  and the associated criterion value  $J_i$  is defined and the minimum value is chosen as it provides the optimal cut-off frequency.

#### 4. SIMULATION EXAMPLE

A simulation example is considered in this section.

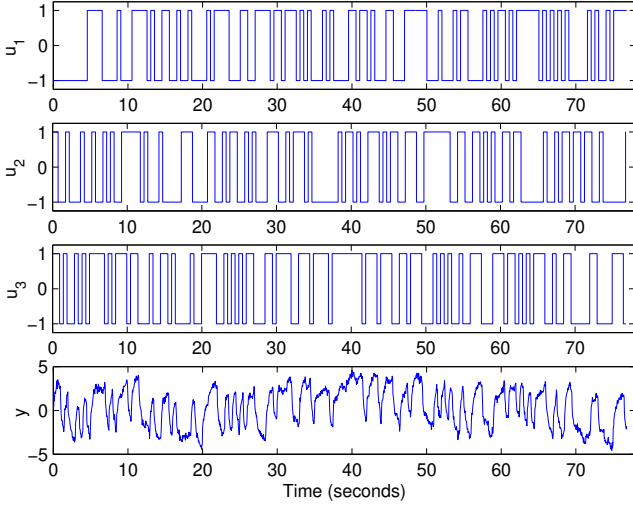


Fig. 2. The input/output data used for system identification of system (54)

Data are generated using the following MISO system:

$$S : \begin{cases} y_u(t) = \frac{1}{3p^{0.5}+1}u_1(t) + \frac{2}{2p^{0.5}+1}u_2(t) + \frac{5}{p^{0.5}+1}u_3(t), \\ y(t_h) = y_u(t_h) + e(t_h). \end{cases} \quad (54)$$

Three uncorrelated pseudo random binary sequences (prbs), with amplitudes fixed between  $-1$  and  $1$ , are applied to system (54) as input signals  $u_i$ ,  $i = 1, 2, 3$ . The length of each prbs is set to  $3 \times \tau_{max}$ , where  $\tau_{max} = \max(\tau_1, \tau_2, \tau_3)$  is the maximum time constant of each subsystem (54). The noise-free output  $y_u(t)$ , being the sum of the three output responses, is corrupted by a zero mean Gaussian white noise, with a noise to signal ratio NSR set to  $-20$ dB, to give the output measurement  $y^*(t)$  (see Figure 1 with  $n_u = 3$ ). In this simulation, the sampling period is set to  $T = 0.05s$  (see Fig.2).

#### 4.1 Coefficient estimation with known differentiation orders

Assume that all the differentiation orders are known, set to  $\nu_1 = \nu_2 = \nu_3 = 0.5$  such as the true system (54), and estimate all the coefficients. A Monte Carlo analysis, for  $N_{exp} = 75$  experiments, is carried out to illustrate the efficiency of the MISO **srivcf** method.

Table 1 illustrates the numerical results of the Monte Carlo simulation and the performances of **srivcf** and **ivsvf** methods. Thanks to the **IV** mechanism, both methods, **ivsvf** and **srivcf**, provide unbiased estimates with lesser bias with the **srivcf** method. Comparing the estimate variances, the **srivcf** method provides better results as the estimates have lower variance (up to 2.6 times less). Therefore, the **srivcf** provides more consistent estimates.

#### 4.2 Coefficient estimations with unknown differentiation orders

##### Unique commensurate order influence

Varying the commensurate order, between  $\nu = 0.25$  and  $\nu = 1$ , and applying the **srivcf** method on the MISO system (54), the cost function is computed as:

Table 1. Monte Carlo simulation with 75 runs ( $\bar{\theta}$  being the mean,  $\hat{\sigma}(\theta)$  being the standard deviation of the parameter estimates).

True		ivsvf		srivcf	
		$\bar{\theta}$	$\hat{\sigma}(\theta)$	$\bar{\theta}$	$\hat{\sigma}(\theta)$
$b_{0,1}$	1	1.0159	0.0233	1.0078	0.0089
$a_{1,1}$	3	3.1159	0.3045	3.0451	0.1397
$b_{0,2}$	2	1.9762	0.0071	2.0100	0.0043
$a_{1,2}$	2	1.9801	0.0134	2.0087	0.0081
$b_{0,3}$	5	5.0184	0.0012	4.9969	0.0009
$a_{1,3}$	1	1.0136	0.0001	0.9995	0.0001

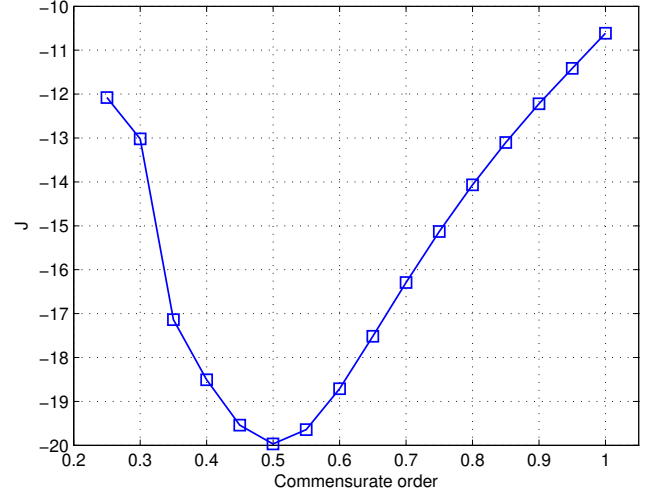


Fig. 3. Cost function versus commensurate order

$$J_{dB} = 10 \log_{10} \left( \frac{\sum_{h=1}^H (y_{est}(ht_s) - y^*(ht_s))^2}{\sum_{h=1}^H y_{est}(ht_s)^2} \right) \quad (55)$$

where  $y_{est}$  is the estimated output.

To study the influence of the commensurate order, the cost function  $J_{dB}$  is plotted versus the commensurate order. As shown in Fig.3, the minimum of the cost function is found at  $\nu = 0.5$  which corresponds to the true commensurate order. In fact, this minimal value corresponds to the applied NSR of  $-20$ dB, there is no modeling error for  $\nu = 0.5$ .

Moreover, as illustrated in Fig.3, for  $\nu = 0.8$  the cost function is around  $-14$ dB, consequently the modeling error can be evaluated to  $6$ dB.

##### Different commensurate order influence

Consider now the following MISO system where each model has a different commensurate order:

$$S : \begin{cases} y_u(t) = \frac{1}{3p^{0.25}+1}u_1(t) + \frac{2}{2p^{0.5}+1}u_2(t) + \frac{5}{p^{0.75}+1}u_3(t), \\ y(t_h) = y_u(t_h) + e(t_h). \end{cases} \quad (56)$$

The input/output data plotted in Figure 2, are again used for the system identification. As a reminder, each model has a different differentiation order in (56).



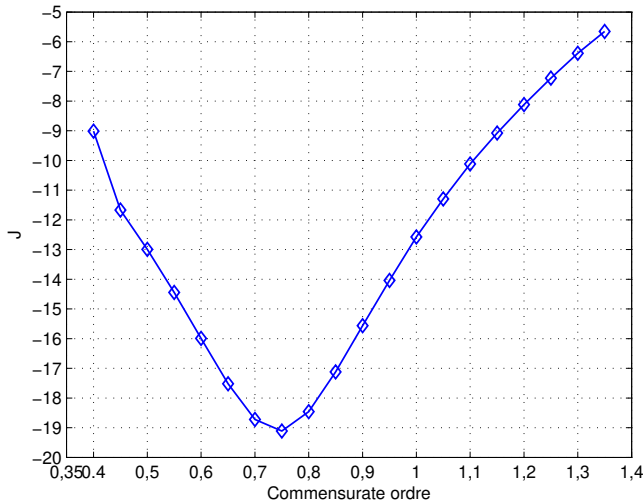


Fig. 4. Cost function versus a unique commensurate order

The influence of estimating a unique commensurate order is evaluated.

Varying the commensurate order, between  $\nu = 0.4$  and  $\nu = 1.35$ , and applying the *srivcf* method on MISO system (56), the cost function, defined by equation (55), is computed in order to study the influence of a unique commensurate order. The cost function  $J_{dB}$  is plotted versus the commensurate order on Fig.4. The minimum of the cost function is found at  $\nu = 0.75$  where the cost function is -19dB. Consequently the modeling error is of 1dB. For  $\nu = 0.6$  the cost function is around -16dB, and the modeling error is around 4dB.

## 5. CONCLUSION

In this paper, the simplified refined instrumental variable for continuous-time fractional systems (*srivcf*) has been extended to multiple input models. A process to better initialize the *srivcf* algorithm is proposed by varying the cut-off frequency of the state variable filters. In order to validate the MISO *srivcf* method, a numerical example with Monte Carlo simulation is provided. Finally a study of the influence of the commensurate orders on the parameter estimation have been realized. For future works, estimating the commensurate order could be investigated.

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