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The extended binary quadratic residue code of length 42 holds a 3–design

A. Bonnecaze*, P. Solé[†], [‡]

Abstract

The codewords of weight 10 of the $[42, 21, 10]$ extended binary quadratic residue code are shown to hold a design of parameters $3 - (42, 10, 18)$. Its automorphism group is isomorphic to $PSL(2, 41)$. Its existence can be explained neither by a transitivity argument, nor by the Assmus-Mattson theorem.

Keywords: designs, quadratic residue codes, Assmus-Mattson theorem

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1 Introduction

The quadratic residue codes have been known to hold designs of large strength since the seminal paper [1], which constructs in particular the Witt designs by coding theoretic techniques. The argument in that famous paper (see also [2] for background) is the so-called Assmus-Mattson theorem which derives the existence of designs from certain facts on the weight distribution of the dual code [13, Chap. 6, Th. 29].

In the present note we produce, by electronic calculation, a $3 - (42, 10, 18)$ design which is not in the table [7, pp. 82–83], cannot be derived from the Assmus-Mattson Theorem, and does not follow by the standard transitivity

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argument. Its blocks are the supports of the codewords of weight 10 in the [42, 21, 10] extended binary quadratic residue code. Its automorphism group coincides with that of that code. Thus it is isomorphic to $PSL(2, 41)$ which is not 3-homogeneous. Note that $PSL(2, q)$ is 3-homogeneous in general for $q \equiv 3 \pmod{4}$ [8]. Thus our design cannot be obtained by the methods of [6]. Neither can it be obtained from the orbits glueing methods of [3, Th. 5], or [10, Th. 5.1], since $41 \equiv 9 \pmod{16}$. Even though the code we are using is a formally self-dual even code, the results of [9] do not apply, because the minimum distance is not high enough.

2 Background material

2.1 Permutation groups

A permutation group G acting on a set X is *transitive* on X if it has a single orbit on X . Given an integer s it is *s-fold transitive* if it is transitive on ordered s -uples of distinct elements of X . It is *s-fold homogeneous* if it is transitive on unordered s -uples of distinct elements of X . Let p denote an odd prime. The group $PSL(2, p)$ is defined in its action on the projective line $\mathbb{F}_p \cup \infty$ as the set of all transforms

$$y \mapsto \frac{a + by}{c + dy}$$

with $ad - bc = 1$, and the conventions $\frac{0}{0} = 0$ and $\frac{\neq 0}{0} = \infty$. See [13, Chap. 16, §5], or [14, Chap 17] for background.

2.2 Designs

A block design of parameters $t - (b, v, k, r, \lambda)$ (shortly a $t - (v, k, \lambda)$ design) is an incidence structure $(\mathcal{P}, \mathcal{B}, I)$ satisfying the following axioms.

1. The set \mathcal{P} has v points.
2. The set \mathcal{B} has b blocks.
3. Each block is incident to k points.
4. Each point is incident to r blocks.
5. Each t -tuple of points is incident in common with λ blocks.

2.3 Codes

Let $\mathbb{F}_2 = \{0, 1\}$ denote the finite field of order 2. A binary code of length n is a \mathbb{F}_2 subspace of \mathbb{F}_2^n . The *weight* of a vector of \mathbb{F}_2^n is the number of its nonzero coordinates. The *weight distribution* of a code C is the sequence A_w of number of codewords of C of weight w . It is written in Magma [12] notation as the list with generic element $\langle w, A_w \rangle$ where w ranges over the weights of C . A binary code is *cyclic* if it is invariant under the cyclic shift. Cyclic codes are in one to one correspondence with ideals of the residue class ring $\mathbb{F}_2[x]/(x^n - 1)$. The generator polynomial of a cyclic code is then the generator of the corresponding ideal. The *Quadratic residue codes* are the cyclic codes of length p , with p an odd prime, defined for $p \equiv \pm 1 \pmod{8}$ by the generator polynomial of degree $\frac{p-1}{2}$

$$\prod_{r=\square} (x - \alpha^r),$$

with α a primitive root of order p . Since 2 is a quadratic residue modulo an odd prime p this polynomial is indeed in $\mathbb{F}_2[x]$. As a simple example consider the case of $p = 7$ when the polynomial

$$(x - \alpha)(x - \alpha^2)(x - \alpha^4) = x^3 + x + 1$$

generates the Hamming code [7, 4, 3]. See [13, Chap.16] for background.

The *automorphism* group a binary code of length n is the subgroup of the symmetric group on the n coordinate places that leaves the code wholly invariant.

The Assmus-Mattson Theorem is as follows for binary codes [13, Chap. 6, Th. 29] .

Theorem 1 (*Assmus-Mattson*) *If C is a binary $[n, k, d]$ code such that the weight distribution of its dual code contains at most $d - t$ nonzero weights $\leq n - t$ then the codewords of weight d of C form a t -design.*

A folklore theorem is that if the automorphism group of a binary code is t -homogeneous then the codewords of any given weight $\geq t$ hold t -designs [14, p. 308]. .

3 Construction

Let Q be the binary quadratic residue code of prime length 41, and denote by C its extension by an overall parity-check. The code C is not self-dual,

but it is equivalent to its dual [13, Chap. 16, Fig. 16.3]. Designs in even formally self-dual codes have been explored in [9], but the codes there are extremal, which would require a [42, 21, 12].

The weight distribution of C is also its dual weight distribution

$$\begin{aligned} & \langle 0, 1 \rangle, \langle 10, 1722 \rangle, \langle 12, 10619 \rangle, \langle 14, 49815 \rangle, \langle 16, 157563 \rangle, \langle 18, 341530 \rangle, \\ & \langle 20, 487326 \rangle, \langle 22, 487326 \rangle, \langle 24, 341530 \rangle, \langle 26, 157563 \rangle, \langle 28, 49815 \rangle, \langle 30, 10619 \rangle, \\ & \langle 32, 1722 \rangle, \langle 42, 1 \rangle. \end{aligned}$$

Thus the Assmus-Mattson theorem, which requires at most $10 - 3 = 7$ non zero dual weights $\leq 42 - 3 = 39$, cannot apply here.

Theorem 2 *The 1722 codewords of weight 10 of C hold a 3-(1722, 42, 10, 410, 18) block design \mathcal{D} .*

Proof. Magma computation [12]. ■

Its automorphism group is a permutation group of order 34440 with generators

$$\begin{aligned} & (3, 30, 29, 31)(4, 9, 18, 7)(5, 24, 25, 17)(6, 22, 38, 42)(8, 34, 11, 28) \\ & (10, 36, 16, 33)(12, 32, 23, 21)(13, 15, 14, 41)(19, 20, 39, 27)(26, 35, 40, 37), \\ & (3, 8, 6, 33, 15, 29, 11, 38, 36, 41)(4, 35, 27, 21, 5, 18, 37, 20, 32, 25) \\ & (7, 26, 39, 23, 17, 9, 40, 19, 12, 24)(10, 14, 31, 28, 42, 16, 13, 30, 34, 22), \\ & (1, 32)(2, 21)(3, 36)(5, 10)(6, 26)(7, 38)(8, 20)(9, 22)(11, 25)(12, 23)(13, 33)(14, 28) \\ & (15, 39)(16, 27)(17, 30)(18, 37)(19, 31)(24, 41)(29, 34)(40, 42), \\ & (2, 32, 23, 12, 36, 29, 37, 14, 24, 10, 8, 15, 40, 6, 4, 31, 41, 18, 26, 19) \\ & (3, 42, 11, 34, 9, 28, 20, 17, 30, 33, 39, 7, 16, 22, 25, 38, 35, 27, 5, 21), \end{aligned}$$

This group can be characterized as an abstract group as follows.

Proposition 1 *The automorphism group G of \mathcal{D} is isomorphic to $PSL(2, 41)$.*

Proof. The automorphism group of C is known to be isomorphic to $PSL(2, 41)$ by [14, Chap. 17]. Since the blocks of \mathcal{D} consist of all the codewords of weight 10 of C , they are wholly invariant under the automorphism

group of C . A Magma computation [12] shows that the order of the automorphism group \mathcal{D} is $34440 = \frac{1}{2}41 \times (41^2 - 1)$, the known order of $PSL(2, 41)$ [13, Chap. 16, Th. 9 (b)]. The result follows. ■

The group G is *not* 3-homogeneous.

Proposition 2 *The group G partitions the set of triples into two orbits of equal size.*

Proof. A Magma computation shows that the orbits of $\{1, 2, 3\}$, and $\{1, 3, 8\}$, are disjoint and both of size $5740 = \binom{42}{3}/2$. ■

Remarks:

1. The codewords of weights 9 and 10 in the quadratic residue code of length 41 hold 2-designs of parameters $2 - (41, 9, 18)$ and $2 - (41, 10, 72)$. They cannot be explained by the Assmus-Mattson theorem or group action. The $2 - (41, 9, 18)$ is a derived design of \mathcal{D} .
2. It can be shown that the linear span of \mathcal{D} is C . This gives an alternative proof of Proposition 1.
3. The codewords of minimum weight in the extended quadratic residue code of length 74 do not hold 3-designs.
4. The codewords of weight 10 and 12 in the duadic codes of length 74 of [11] do not hold 3-designs.

4 Conclusion

In this work, we have constructed a new 3-design in the minimum weight codewords of a quadratic residue codes. Its existence cannot be explained by the Assmus-Mattson Theorem, nor by group action.

The arguments are based on machine calculation. It would be more satisfying to have a conceptual derivation. A method by multivariate weight enumerators and invariant theory in the spirit of [5], would be more conceptual, but certainly not computer-free. The fact that the low weight codewords in the extended quadratic residue code of length 74 or in the duadic codes

of that length do not hold 3-designs suggests that the design in this note is exceptional enough that a general conjecture is impossible to formulate.

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