



HAL
open science

The extended binary quadratic residue code of length 42 holds a 3–design

Alexis Bonnecaze, Patrick Solé

► To cite this version:

Alexis Bonnecaze, Patrick Solé. The extended binary quadratic residue code of length 42 holds a 3–design. *Journal of Combinatorial Designs*, 2021, 29 (8), pp.528-532. 10.1002/jcd.21782 . hal-03217485

HAL Id: hal-03217485

<https://hal.science/hal-03217485>

Submitted on 4 May 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The extended binary quadratic residue code of length 42 holds a 3–design

A. Bonnecaze*, P. Solé[†], [‡]

Abstract

The codewords of weight 10 of the $[42, 21, 10]$ extended binary quadratic residue code are shown to hold a design of parameters $3 - (42, 10, 18)$. Its automorphism group is isomorphic to $PSL(2, 41)$. Its existence can be explained neither by a transitivity argument, nor by the Assmus-Mattson theorem.

Keywords: designs, quadratic residue codes, Assmus-Mattson theorem

AMS Math Sc. Cl. (2010): 94 B15, 62K10

1 Introduction

The quadratic residue codes have been known to hold designs of large strength since the seminal paper [1], which constructs in particular the Witt designs by coding theoretic techniques. The argument in that famous paper (see also [2] for background) is the so-called Assmus-Mattson theorem which derives the existence of designs from certain facts on the weight distribution of the dual code [13, Chap. 6, Th. 29].

In the present note we produce, by electronic calculation, a $3 - (42, 10, 18)$ design which is not in the table [7, pp. 82–83], cannot be derived from the Assmus-Mattson Theorem, and does not follow by the standard transitivity

*alexis.bonnecaze@univ-amu.fr

[†]corresponding author, sole@enst.fr

[‡]AB and PS are with Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France.

argument. Its blocks are the supports of the codewords of weight 10 in the [42, 21, 10] extended binary quadratic residue code. Its automorphism group coincides with that of that code. Thus it is isomorphic to $PSL(2, 41)$ which is not 3-homogeneous. Note that $PSL(2, q)$ is 3-homogeneous in general for $q \equiv 3 \pmod{4}$ [8]. Thus our design cannot be obtained by the methods of [6]. Neither can it be obtained from the orbits glueing methods of [3, Th. 5], or [10, Th. 5.1], since $41 \equiv 9 \pmod{16}$. Even though the code we are using is a formally self-dual even code, the results of [9] do not apply, because the minimum distance is not high enough.

2 Background material

2.1 Permutation groups

A permutation group G acting on a set X is *transitive* on X if it has a single orbit on X . Given an integer s it is *s-fold transitive* if it is transitive on ordered s -uples of distinct elements of X . It is *s-fold homogeneous* if it is transitive on unordered s -uples of distinct elements of X . Let p denote an odd prime. The group $PSL(2, p)$ is defined in its action on the projective line $\mathbb{F}_p \cup \infty$ as the set of all transforms

$$y \mapsto \frac{a + by}{c + dy}$$

with $ad - bc = 1$, and the conventions $\frac{0}{0} = 0$ and $\frac{\neq 0}{0} = \infty$. See [13, Chap. 16, §5], or [14, Chap 17] for background.

2.2 Designs

A block design of parameters $t - (b, v, k, r, \lambda)$ (shortly a $t - (v, k, \lambda)$ design) is an incidence structure $(\mathcal{P}, \mathcal{B}, I)$ satisfying the following axioms.

1. The set \mathcal{P} has v points.
2. The set \mathcal{B} has b blocks.
3. Each block is incident to k points.
4. Each point is incident to r blocks.
5. Each t -tuple of points is incident in common with λ blocks.

2.3 Codes

Let $\mathbb{F}_2 = \{0, 1\}$ denote the finite field of order 2. A binary code of length n is a \mathbb{F}_2 subspace of \mathbb{F}_2^n . The *weight* of a vector of \mathbb{F}_2^n is the number of its nonzero coordinates. The *weight distribution* of a code C is the sequence A_w of number of codewords of C of weight w . It is written in Magma [12] notation as the list with generic element $\langle w, A_w \rangle$ where w ranges over the weights of C . A binary code is *cyclic* if it is invariant under the cyclic shift. Cyclic codes are in one to one correspondence with ideals of the residue class ring $\mathbb{F}_2[x]/(x^n - 1)$. The generator polynomial of a cyclic code is then the generator of the corresponding ideal. The *Quadratic residue codes* are the cyclic codes of length p , with p an odd prime, defined for $p \equiv \pm 1 \pmod{8}$ by the generator polynomial of degree $\frac{p-1}{2}$

$$\prod_{r=\square} (x - \alpha^r),$$

with α a primitive root of order p . Since 2 is a quadratic residue modulo an odd prime p this polynomial is indeed in $\mathbb{F}_2[x]$. As a simple example consider the case of $p = 7$ when the polynomial

$$(x - \alpha)(x - \alpha^2)(x - \alpha^4) = x^3 + x + 1$$

generates the Hamming code [7, 4, 3]. See [13, Chap.16] for background.

The *automorphism* group a binary code of length n is the subgroup of the symmetric group on the n coordinate places that leaves the code wholly invariant.

The Assmus-Mattson Theorem is as follows for binary codes [13, Chap. 6, Th. 29] .

Theorem 1 (*Assmus-Mattson*) *If C is a binary $[n, k, d]$ code such that the weight distribution of its dual code contains at most $d - t$ nonzero weights $\leq n - t$ then the codewords of weight d of C form a t -design.*

A folklore theorem is that if the automorphism group of a binary code is t -homogeneous then the codewords of any given weight $\geq t$ hold t -designs [14, p. 308]. .

3 Construction

Let Q be the binary quadratic residue code of prime length 41, and denote by C its extension by an overall parity-check. The code C is not self-dual,

but it is equivalent to its dual [13, Chap. 16, Fig. 16.3]. Designs in even formally self-dual codes have been explored in [9], but the codes there are extremal, which would require a [42, 21, 12].

The weight distribution of C is also its dual weight distribution

$$\begin{aligned} & \langle 0, 1 \rangle, \langle 10, 1722 \rangle, \langle 12, 10619 \rangle, \langle 14, 49815 \rangle, \langle 16, 157563 \rangle, \langle 18, 341530 \rangle, \\ & \langle 20, 487326 \rangle, \langle 22, 487326 \rangle, \langle 24, 341530 \rangle, \langle 26, 157563 \rangle, \langle 28, 49815 \rangle, \langle 30, 10619 \rangle, \\ & \langle 32, 1722 \rangle, \langle 42, 1 \rangle. \end{aligned}$$

Thus the Assmus-Mattson theorem, which requires at most $10 - 3 = 7$ non zero dual weights $\leq 42 - 3 = 39$, cannot apply here.

Theorem 2 *The 1722 codewords of weight 10 of C hold a 3-(1722, 42, 10, 410, 18) block design \mathcal{D} .*

Proof. Magma computation [12]. ■

Its automorphism group is a permutation group of order 34440 with generators

$$\begin{aligned} & (3, 30, 29, 31)(4, 9, 18, 7)(5, 24, 25, 17)(6, 22, 38, 42)(8, 34, 11, 28) \\ & (10, 36, 16, 33)(12, 32, 23, 21)(13, 15, 14, 41)(19, 20, 39, 27)(26, 35, 40, 37), \\ & (3, 8, 6, 33, 15, 29, 11, 38, 36, 41)(4, 35, 27, 21, 5, 18, 37, 20, 32, 25) \\ & (7, 26, 39, 23, 17, 9, 40, 19, 12, 24)(10, 14, 31, 28, 42, 16, 13, 30, 34, 22), \\ & (1, 32)(2, 21)(3, 36)(5, 10)(6, 26)(7, 38)(8, 20)(9, 22)(11, 25)(12, 23)(13, 33)(14, 28) \\ & (15, 39)(16, 27)(17, 30)(18, 37)(19, 31)(24, 41)(29, 34)(40, 42), \\ & (2, 32, 23, 12, 36, 29, 37, 14, 24, 10, 8, 15, 40, 6, 4, 31, 41, 18, 26, 19) \\ & (3, 42, 11, 34, 9, 28, 20, 17, 30, 33, 39, 7, 16, 22, 25, 38, 35, 27, 5, 21), \end{aligned}$$

This group can be characterized as an abstract group as follows.

Proposition 1 *The automorphism group G of \mathcal{D} is isomorphic to $PSL(2, 41)$.*

Proof. The automorphism group of C is known to be isomorphic to $PSL(2, 41)$ by [14, Chap. 17]. Since the blocks of \mathcal{D} consist of all the codewords of weight 10 of C , they are wholly invariant under the automorphism

group of C . A Magma computation [12] shows that the order of the automorphism group \mathcal{D} is $34440 = \frac{1}{2}41 \times (41^2 - 1)$, the known order of $PSL(2, 41)$ [13, Chap. 16, Th. 9 (b)]. The result follows. ■

The group G is *not* 3-homogeneous.

Proposition 2 *The group G partitions the set of triples into two orbits of equal size.*

Proof. A Magma computation shows that the orbits of $\{1, 2, 3\}$, and $\{1, 3, 8\}$, are disjoint and both of size $5740 = \binom{42}{3}/2$. ■

Remarks:

1. The codewords of weights 9 and 10 in the quadratic residue code of length 41 hold 2-designs of parameters $2 - (41, 9, 18)$ and $2 - (41, 10, 72)$. They cannot be explained by the Assmus-Mattson theorem or group action. The $2 - (41, 9, 18)$ is a derived design of \mathcal{D} .
2. It can be shown that the linear span of \mathcal{D} is C . This gives an alternative proof of Proposition 1.
3. The codewords of minimum weight in the extended quadratic residue code of length 74 do not hold 3-designs.
4. The codewords of weight 10 and 12 in the duadic codes of length 74 of [11] do not hold 3-designs.

4 Conclusion

In this work, we have constructed a new 3-design in the minimum weight codewords of a quadratic residue codes. Its existence cannot be explained by the Assmus-Mattson Theorem, nor by group action.

The arguments are based on machine calculation. It would be more satisfying to have a conceptual derivation. A method by multivariate weight enumerators and invariant theory in the spirit of [5], would be more conceptual, but certainly not computer-free. The fact that the low weight codewords in the extended quadratic residue code of length 74 or in the duadic codes

of that length do not hold 3-designs suggests that the design in this note is exceptional enough that a general conjecture is impossible to formulate.

Acknowledgement: The authors thank Professor Jenny Key for helpful discussions.

References

- [1] E.F. Assmus jr, H.F. Mattson jr, New 5-designs, *J. of Combinatorial Theory A*, **6**, (1969), 122–151.
- [2] E.F. Assmus jr, H.F. Mattson jr, R. Turyn, Research to develop the algebraic theory of codes, Sylvania report AFCRL-67-0365, (1967), available from <https://apps.dtic.mil/dtic/tr/fulltext/u2/656783.pdf>.
- [3] N. Balachandran, D. Ray-Chaudhuri, Simple 3-designs and $PSL(2, q)$ with $q \equiv 1 \pmod{4}$. *Des. Codes Cryptogr.* **44** (2007), 263–274.
- [4] T. Beth, D. Jungnickel, H. Lenz, *Design Theory*, 2nd edition, vol. I, (1999), Cambridge University Press.
- [5] A. Bonnetcaze, B. Mourrain, P. Solé, Jacobi Polynomials, Type II Codes, and Designs. *Des. Codes Cryptogr.* **16**, (1999), 215–234.
- [6] P.J. Cameron, H.R. Maimanid, G.R. Omidic, B. Tayfeh-Rezaie, 3-Designs from $PSL(2, q)$, *Discrete Mathematics* 306 (2006) 3063–3073.
- [7] C.J. Colbourn, J.H. Dinitz (2007), *Handbook of Combinatorial Designs* (2nd ed.), Boca Raton: Chapman & Hall/ CRC, ISBN 1-58488-506-8.
- [8] C. Ding, *Designs from linear codes*, World Scientific (2018) Singapore.
- [9] G.T. Kennedy and V. Pless, On designs and formally self-dual codes, *Designs Codes Cryptography* **4**, (1994) 43–55.
- [10] W. Liu, J-X. Tang, Y-X. Wu, Some new 3-designs from $PSL(2, q)$ with $q \equiv 1 \pmod{4}$, *Science in China*, **55**,(2012), 1901–1911.
- [11] J.S. Leon, J. M. Masley, V. Pless, On duadic codes, *IEEE Trans. on Information Theory*, (1984), 709–714.
- [12] <http://magma.maths.usyd.edu.au/calc/>

- [13] F.J. MacWilliams, N.J. A. Sloane, *The theory of Error Correcting Codes*, North-Holland, Amsterdam (1981).
- [14] V.S. Pless, W.C. Huffman, *Handbook of Coding theory*, North Holland (1998), Amsterdam.