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# More on the Geometric Approach for Gravitation Coupled with Electromagnetism within a Riemann-Cartan Vacuum Spacetime

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**Abstract** The goal of this paper is to link the geometric variables of four-dimensional spacetime with electromagnetic and gravitational fields. For that purpose, assuming a Hilbert-Einstein action with Yang-Mills action, we derive both the gravitation and the electromagnetic equation within a Riemann-Cartan spacetime where curvature and torsion are present. We thus show that the gravitational and electromagnetic fields are respectively identified as geometric objects of such a spacetime, namely the curvature for gravitation which is a classical result, and the torsion as electromagnetic fields. Moreover, the torsion tensor of the spacetime is related to the so-called flux spin angular momentum of electromagnetic waves propagating within the spacetime.

**Keywords** Curvature, torsion · gravitation, electromagnetism · Riemann-Cartan manifold

## 1 Introduction

Deflection of light measured as earlier as 1919 was among the first experimental measurements to point out the interaction between gravitation and electromagnetic waves Dyson & al. [1920]. An important fact usually assumed as evident nowadays but that should not be forgotten is that both the gravitation and the electromagnetism either separately at their own or in interaction occur in a spacetime, and then illustrate the concept of field as action at distance phenomenon MacMullin [2002]. Unification of gravitation and electromagnetism draw back to at least one hundred years, as related in the review paper by Goenner e.g. Goenner [2004] where the most active period of unifying the two domains was between 1910 and 1933. Since then, it is however recognized that the unification of the gravitation and the electromagnetism within the same geometric framework remains

necessary to go further into the unification of field theories e.g. Hammond [1987]. Paraphrasing Tonnelat, there remains up to now a need of unified theory joining gravitation and electromagnetism whose basic equations represent the background required to design geometrical structure of the spacetime Tonnelat [1955], and it is prior to any other unification attempts. Since the pioneering work of Weyl one hundred years ago, there remains among the physics community a great interest to couple Maxwell electrodynamics to Einstein gravitation e.g. Weyl [1929]. The geometric approach for continuum physics includes at least two steps : development of the geometric base by means of manifold in the framework of differential geometry, and then definition of geometric objects and their identification of experimental quantities. As earlier as 1921, such approach was already adopted by Eddington to extend the Weyl's approach by using affine connection with non zero torsion to develop gravitation and electromagnetism theory within a unique framework e.g. Eddington [1921]. On the one hand, the development of the Einstein gravitation theory lies upon the Riemann geometry where the curvature of the spacetime is identified with the gravitation. More precisely, the geodesic equation shows that inertia and gravitation are unified in a unique framework of General Relativity. On the other hand, the Special Relativity unified the electricity and magnetism. Early, Weyl clearly thought that gravitation and electromagnetism could be reduced to the geometry of spacetime. Starting with Riemannian geometry, he slightly modified the metric compatibility equation to  $\nabla_{\alpha} g_{\beta\lambda} = \sigma_{\alpha} g_{\beta\lambda}$  by introducing a 1-form field  $\sigma_{\alpha} \neq 0$ , expecting that this 1-form would be an electromagnetic potential. However, recent studies highlight strong physics objection to this assumption, namely on the definition of the proper time and geodesic length in the spacetime resulting from the introduction of this 1-form e.g. Romero et al. [2019]. In general relativity, the metric is no longer a Minkowskian since  $\mathbf{g} = \mathbf{g}(\mathbf{x})$

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depends on the coordinates of the spacetime. In an attempt to formulate a unified theory of gravitation and electromagnetism, following the idea of Ferraris and Kijowski, Chrusciel proposed a Lagrangian function having depending only upon the skew-symmetric tensor  $\mathcal{F}_{\mu\nu} := \mathfrak{R}_{\mu\nu\lambda}^\lambda$  and on the symmetric part of the Ricci curvature tensor with nonzero torsion e.g. Chrusciel [1984]. The use of the curvature of the spacetime as the only one variable to sketch electromagnetism and gravitation might be questionable. Nevertheless, it could be checked that the two-covariant skew-symmetric tensor  $\mathcal{F}_{\mu\nu}$  satisfies the first set of Maxwell equations (Bianchi equations), and thus could be suggested as the combined electromagnetic field e.g. Hammond [1989], whereas the symmetric tensor obtained from Ricci curvature capture the gravitational fields. Implicitly the model proposed by Chrusciel suggests that the origin of the electromagnetism comes from the skew-symmetric part of the spacetime connection, and then of the torsion tensor. Recent works have shown that gravitation and electromagnetism could be reduced to geometric object of the spacetime, gravitation to curvature and electromagnetism to contortion tensor e.g. Fernando et al. [2012], or torsion tensor Hammond [2018], Rakotomanana [2018]. In Fernando et al. [2012], the authors assumed a particular Riemann-Cartan structure where the contortion tensor, is proportionnal to the product of the Faraday strength tensor with the electromagnetic potential. It seems however necessary to confirm or not this interesting assumption.

Historically, the interaction of relative gravitation and electromagnetism was often considered in a curved Riemannian spacetime e.g. Fernandez-Nunez & Bulashenko [2016]. It is usually assumed that the electromagnetic field is of the order of small perturbation of the spacetime metric. Only the influence of the metric on electromagnetic field is mostly accounted for, not the converse. Second, the influence of the Riemann-Cartan geometry on the electromagnetic field is not so easy. The hope to relate electromagnetism to affine connection of spacetime draw back to Einstein himself, a long debate among others between Einstein and Reichenbach relates the fact that Reichenbach showed that both the metric  $\mathbf{g}$  and the affine connection  $\nabla$  of the spacetime, which is independent on the metric  $\mathbf{g}$ , could be derived entirely from tetrads  $F_\alpha^i$  and their partial derivatives with respect to coordinates Reichenbach [1929] and they are respectively related to gravitation and electromagnetism. Implicitly, these results suggested that metric, torsion, and curvature of the spacetime constitute geometric objects of gravitation and electromagnetism. Although the relation between gravitation and spacetime curvature was clearly stated by Einstein's theory, the link between electromagnetism and other geometric variables remains not clear and evident.

A free electromagnetic field is suggested to not produce torsion e.g. Hehl et al. [1976], and there is in principle

no contribution from torsion in Maxwell equations. When a strong magnetic field coexists with matter distribution, there is however a possibility to induce spin polarization of individual particles composing the matter assumed as continuum e.g. Prasanna [1975a]. Some authors have even suggested that torsion play a keyrole in electromagnetism when considering electromagnetic field within twisted spacetime e.g. Hammond [1989]. They propose that the electromagnetic potential is represented by the torsion vector  $A_\alpha := \mathfrak{K}_\alpha = \mathfrak{K}_{\alpha\beta}^\beta$ . The influence of torsion tensor as cosmic dislocation (that is a singularity of the curvature tensor) was investigated in e.g. Dias & Moraes [2005], or some material defects such as screw dislocations Fumeron et al. [2015], or fluids with spin density e.g. Schutzhold et al. [2002]. A convenient method for deriving constitutive laws and conservation laws from a Lagrangian density lies on the concept of variation of an action, namely the Lagrangian variation and the Eulerian variation (Poincaré invariance) e.g. Utiyama [1956]. Covariance requires the use of metric, torsion, and curvature as arguments of Lagrangian function for a second gradient continuum e.g. Antonio & Rakotomanana [2011]. For the present paper, we are interested in the search of link between the electromagnetic fields and the gradient continuum geometric variables developed in e.g. Rakotomanana [2003] but extended to spacetime e.g. Rakotomanana [2019]. Electromagnetic waves, including light wave propagation, are described by Maxwell equations within Minkowskian, Riemannian or Riemann-Cartan spacetime. We consider in this work some elements of the theory of interaction between gravitation and electromagnetism respectively based on the classical Hilbert-Einstein action Hilbert [1915] and the Yang-Mills action Yang & Mills [1954]. For the sake of the clarity, we focus on the gravitation and electromagnetism within spacetime although results are easily extended to other continuum media at least from a conceptual point of view. Indeed, from a philosophical point of view, the analysis and then the physics of fields remains a most passionating domain of physics e.g. MacMullin [2002]. The apparent simplicity hides a deep concept surrounding action-at-distance. The paper is organized as follows: Section 2 is devoted to basic geometrical concepts of spacetimes. Section 3 deals with the classical electromagnetism within Minkowski spacetime. It also presents the combined gravitation theory with electromagnetism in a Riemann curved spacetime. In section 4, focus is on the geometric development of gravitation and electromagnetism deduced from the additional action including Hilbert-Einstein and Yang-Mills contributions. Section 5 considers the extension of worldline (autoparallel) deviation in presence of torsion and curvature. Last section gives some concluding remarks.

## 2 Basics on spacetime geometry

Space and time may be considered as the most fundamental concepts of physics and even most of natural sciences. Newton (1687) assumed that an absolute space and time (asymptotic and idealized) exists prior to any other concept. The theory of special relativity is based on two assumptions : (1) the laws of physics are the same for all inertial observers; and (2) the speed of light is constant and the same in a vacuum for all inertial observers independent of their motion relative to the source of the light. Theory of relative gravitation lies on two postulates: (1) the relativity principle saying that the local physics is governed by the theory of special relativity; and (2) the equivalence principle assuming that it is impossible to distinguish gravity and acceleration locally.

### 2.1 Affinely connected spacetime

Minkowski spacetime allows us to describe both particle mechanics and electromagnetism. It constitutes the fundamental geometry of special relativistic mechanics (without gravitation) and electrodynamics. Governing equations are derived in the local coordinates of Minkowski spacetime  $\mathcal{M}$  endowed with the metric denoted  $\hat{g}_{\mu\nu} := \{+1, -1, -1, -1\}$ . Coordinates within the spacetime manifold are denoted  $x^\mu := (x^0 = ct, x^1, x^2, x^3)$  where  $c$  is the light speed in a Minkowski spacetime, considered as vacuum. More generally, spacetime is described as a 4-dimensional manifold affinely connected where any event can be described by coordinates  $(x^0, x^1, x^2, x^3)$  of  $\mathbb{R}^4$ , where index 0 stands for time e.g. Rakotomanana [2019]. Namely, from Galilean view, space is a tridimensional Euclidean manifold and time is a one dimensional manifold, they are assumed to exist independently each other. In the Minkowskian point of view, spacetime is a four-dimensional manifold where time is coupled with space. Presence and motion of a structureless particle are assumed to not influence the spacetime geometry. In the presence of matter, an initially flat spacetime is bent. This results into a curved spacetime which can be understood as gravity. In a more extended point of view, affinely connected manifold constitutes the main geometry background for analyzing the spacetime structure with a more general purpose.

On an affinely connected manifold endowed with a metric  $\mathbf{g}$  and an independent connection  $\nabla$ , two tensors are associated to this connection (which is not a tensor), the torsion tensor  $\mathfrak{K}$  and the curvature tensor  $\mathfrak{R}$  defined by the following relations respectively e.g. Nakahara [1996]:

$$\begin{cases} \mathfrak{K}(\mathbf{u}, \mathbf{v}) := \nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}] \\ \mathfrak{R}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u}, \mathbf{v}]} \mathbf{w} \end{cases} \quad (1)$$

where  $[\cdot, \cdot]$  denotes the Lie-Jacobi bracket, and  $(\mathbf{u}, \mathbf{v}, \mathbf{w})$  an arbitrary triplet of vectors on the manifold e.g. Nakahara

[1996]. We consider the spacetime (or space) endowed with the metric  $\mathbf{g}$  which may depend or not of the coordinates  $\mathbf{x}$ . For gravitation theory, associated to this metric can be defined connection  $\bar{\nabla}$ , with zero torsion  $\mathfrak{K} \equiv 0$  but nonzero curvature  $\bar{\mathfrak{R}} \neq 0$ . However, to begin with, it is usual to start with the Minkowskian flat spacetime with the metric  $\hat{\mathbf{g}}$  with zero torsion, and zero curvature. Then we consider the actual spacetime metric  $\mathbf{g} := \hat{\mathbf{g}} + \mathbf{h}$ , where  $\mathbf{h}(\mathbf{x})$  is a perturbation. With the metric and the curvature, Einstein built the theory of general relativistic gravitation. The spacetime may also have non symmetric connection but compatible with the metric.

From the differential geometry point of view, it is then clear that three tensor variables constitute the underlying structure of the spacetime : the metric, the torsion and curvature. The metric compatibility condition is an essential property of a spacetime continuum to survive as a spacetime continuum after non holonomic perturbation  $\mathbf{h}$  e.g. Verçyn [1990]. By adopting the spacetime model with affinely connected manifold, we implicitly consider space of a aggregate of microcosms Gonseth [1926] where connection links each microcosm with its neighbour. Relative motions of microcosms may induce discontinuity of relative displacements and rotations e.g. Rakotomanana [1997], called translational and rotational dislocations. The non compatibility  $\nabla \mathbf{g} \neq 0$  between the metric and the connection leads to another more extended version of dislocations, called Somigliana dislocations, which is a continuum mechanics version of the Weyl theory Weyl [1929]. The geometric background associated to Somigliana field of dislocation is the the Weyl manifold where additional primal variables are the non-metricity tensor  $\mathbf{Q} := \nabla \mathbf{g}$ . A Somigliana dislocation is idealized in terms of a closed volume  $V$  of "added / subtracted matter" located within a continuum. After cutting the body and separating two opposite faces, the operation is represented by a small displacement field  $\mathbf{b}(\mathbf{x})$ , the empty space is then filled with "added matter" (or "subtracted matter" if the two faces penetrate each other after the cutting).

*Cartan parallelogram* Figure (1) may be better interpreted with the concept of Cartan parallelogram. Say a scalar field  $\theta$  and a vector field  $\mathbf{w}$  within the spacetime. Consider two vectors  $\mathbf{f}_1$  and  $\mathbf{f}_2$  at any point  $M$  of  $\mathcal{M}$ , they define two paths of length  $\varepsilon_1$  and  $\varepsilon_2$  that may be considered as two adjacent sides of an infinitesimal parallelogram. Non zero torsion and curvature fields induce the following relationships Rakotomanana [1997]:

$$\begin{cases} \lim_{(\varepsilon_1, \varepsilon_2) \rightarrow 0} (\theta' - \theta'') / \varepsilon_1 \varepsilon_2 = \mathfrak{K}(\mathbf{f}_1, \mathbf{f}_2)[\theta] \\ \lim_{(\varepsilon_1, \varepsilon_2) \rightarrow 0} [\mathbf{f}'(\mathbf{w}') - \mathbf{f}''(\mathbf{w}'')] / \varepsilon_1 \varepsilon_2 = \mathfrak{R}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{w}, \mathbf{f}') - \mathbf{f}'(\nabla_{\mathfrak{K}(\mathbf{f}_1, \mathbf{f}_2)} \mathbf{w}) \end{cases} \quad (2)$$

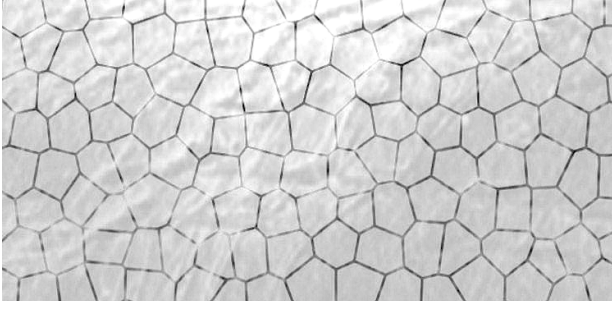


Fig. 1: Riemann-Cartan spacetime as a set of microcosms. The spacetime  $\mathcal{M}$  is viewed as connected chunks of micro-spacetimes having locally their own metric, torsion and curvature, and where relative displacements and rotations are allowed. The unusual feature not encountered before in Minkowski spacetime, namely a closure failure of parallelograms, implies the non vanishing of torsion and curvature.

where  $\theta(\mathbf{x})$ , and  $\mathbf{w}(\mathbf{x})$  are respectively a scalar field and a vector field on  $\mathcal{M}$ . We notice  $\theta' := \theta(M')$ , and  $\theta'' := \theta(M'')$ ; and  $\mathbf{w}' := \mathbf{w}(M')$ , and  $\mathbf{w}'' := \mathbf{w}(M'')$ . The system of equations (2) are the continuum extension of the discrete dislocation loop induced by the discontinuity of the displacement field within spacetime composed with microcosms. Fundamentally, the possibility of relative motion between two neighboured microcosms induces a non zero torsion and curvature.

## 2.2 Modelling continuum physics by Lagrangian

Action principles are widely used to express the laws of physics, namely those of general relativity and electromagnetism. Action is a functional basically defined on the spacetime. The Einstein-Cartan theory is the extension of the relativistic gravitation theory, allowing the spacetime to have nonzero torsion, it was suggested by Cartan in 1922. The application of the covariance theorem states that any Lagrangian function  $\mathcal{L}(\mathbf{g}, \nabla, \nabla)$  should be written as  $\mathcal{L}(\mathbf{g}, \mathfrak{T}, \mathfrak{R})$  to be diffeomorphism invariant Antonio & Rakotomanana [2011]. When the connection is that of Levi-Civita<sup>1</sup>, among the arguments of the Lagrangian function we should consider are second order derivatives of the metric as additional variable. Let a spacetime continuum modeled by a metric-affine manifold endowed with a connection compatible with the metric. (a)  $\mathcal{L}(\mathbf{g}, \mathfrak{T} = 0, \mathfrak{R} = 0)$  corresponds to an strain energy function in elastic spacetime; (b)  $\mathcal{L}(\mathbf{g}, \mathfrak{T})$  is associated to an elastic spacetime with dislocation singularity. (c)  $\mathcal{L}(\mathbf{g}, \mathfrak{T}, \mathfrak{R})$  is associated to an elastic spacetime with dislocation and disclination singularity. The elasticity refers to the metric as argument of the Lagrangian function. How-

<sup>1</sup>The Euclidean connection derived from the metric tensor of a spacetime body was mostly the connection used in continuum mechanics for over two centuries, e.g. Rakotomanana [2003].

ever the general form is far from tractable. It is worth to introduce the Ricci curvature tensor  $\mathfrak{R}_{\alpha\beta} := \mathfrak{R}^{\lambda}_{\lambda\alpha\beta}$ . For compatible connection, say  $\nabla \mathbf{g} \equiv 0$ , the Ricci curvature tensor is symmetric. Curvature of a three dimensional manifold is uniquely determined by the Ricci tensor. The scalar curvature is defined by the contraction  $\mathfrak{R} := g^{\alpha\beta} \mathfrak{R}_{\alpha\beta}$ . For the metric-affine gravity theory, the Lagrangian density takes the form of  $\mathcal{L}(\mathbf{g}, \mathfrak{R})$ , owing that metric and curvature are independent variables. In this paper we do not consider quantum physics which describes phenomena physics at the smallest scales of energy levels of atoms and subatomic particles. We rather consider classical physics in the sense where it describes phenomena at macroscopic and microscopic scale levels, namely gravitation and electromagnetism.

## 3 Gravitation and electromagnetism within curved spacetime

We remind in this section the covariant form of the laws of classical electromagnetism in spacetime which are compatible with gravitation. It should be stressed that originally, the Newton absolute space, and therefore Minkowski spacetime, enters implicitly into the general theory of gravitation and electromagnetism. We do not enter into the old debate whether vacuum space is full of substance or empty. We merely proceed step by step by considering a gradual complexity of the spacetime from Minkowski, Riemann and then Riemann-Cartan geometry.

### 3.1 Basic electromagnetism in a Minkowski spacetime

The Minkowski flat spacetime, say  $\mathbf{g} = \text{uniform}$ ,  $\mathfrak{T} \equiv 0$  and  $\mathfrak{R} \equiv 0$ , of special relativistic physics constitutes the right spacetime model for this purpose. Electromagnetism theory is built upon electric field  $\mathbf{E}$ , and magnetic field  $\mathbf{B}$ . Both of them depend on the space coordinate and the time in the general case. The general form of the Maxwell equations is intimately linked to the geometry of the spacetime characterized by the Minkowski metric, they constitute the fundamental basis of classical electrodynamics. Before going into the derivation of Maxwell equation, it is worth to remind the notion of proper time in the framework of special relativity. Consider a body  $I$  or a referential frame moving with a uniform velocity  $v$  with respect to  $\mathcal{M}$ . The proper time is given by  $d\tau := \sqrt{1 - (v^2/c^2)}dt$ , where the proper time  $\tau$  along a timelike world line in the spacetime  $\mathcal{M}$  is the laps of time measured by a clock following that line. As such, the Maxwell equations are the conservation laws, and then by accounting for constitutive laws, we obtain the partial differential equations where the electric and magnetic intensities are the unknowns. By using the form notation in the four-dimensional spacetime, the electromagnetic strength or also

Faraday tensor is a 2-form:

$$\mathcal{F} = E_1 dx^1 \wedge dx^0 + E_2 dx^2 \wedge dx^0 + E_3 dx^3 \wedge dx^0 \\ + B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2 \quad (3)$$

from which we can easily check the correspondence of the components  $\mathcal{F}_{\mu\nu}$  with the components of tri-dimensional space vector fields  $\mathbf{E}$  and  $\mathbf{B}$ . The covariant formulation of electromagnetism theory in the Minkowski spacetime  $\mathcal{M}$  is classically obtained by considering the Faraday tensor (3) from the four-dimensional electromagnetic potential  $\mathbf{A} = (A_\mu)$ :

$$\mathcal{F}_{\mu\nu} := \hat{\nabla}_\mu A_\nu - \hat{\nabla}_\nu A_\mu, \quad \mathcal{F}_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{bmatrix} \quad (4)$$

where  $\hat{\nabla}$  is the (flat) connection of the Minkowski spacetime, and where the combined electromagnetic field  $(\mathbf{E}, \mathbf{B})$  do not transform as 3-vectors but as the six components of the skew-symmetric tensor  $\mathcal{F}_{\mu\nu}$ . We conform here to the convention in e.g. Hehl [2008], Obukhov [2008]. In this way the skew symmetric tensor  $\mathcal{F}_{\mu\nu}$  is chosen as primal variables of the theory. Let us now define the dual variable  $\mathcal{H}^{\mu\nu}$  constructed from the electric displacement  $\mathbf{D}$  and the magnetic intensity  $\mathbf{H}$ . Independently on the constitutive laws, the classical electromagnetism theory considers the electromagnetic excitation as a two-form  $\mathcal{H}_{\mu\nu}$ :

$$\mathcal{H} := -H_1 dx^1 \wedge dx^0 - H_2 dx^2 \wedge dx^0 - H_3 dx^3 \wedge dx^0 \\ + D^3 dx^1 \wedge dx^2 + D^1 dx^2 \wedge dx^3 + D^2 dx^3 \wedge dx^1 \quad (5)$$

in the same way as the definition of the electromagnetic strength (3). The variable we are interested in is in fact the dual variable  $\mathcal{H}^{\mu\nu}$  in order to be able to link it with the primal variable  $\mathcal{F}_{\alpha\beta}$ . From (5), we easily obtain the two contravariant components as:

$$\mathcal{H}^{\mu\nu} = \begin{bmatrix} 0 & D^1 & D^2 & D^3 \\ -D^1 & 0 & H_3 & -H_2 \\ -D^2 & -H_3 & 0 & H_1 \\ -D^3 & H_2 & -H_1 & 0 \end{bmatrix} \quad (6)$$

*Remark 1* For Minkowski spacetime, the relations between fields are particular  $\mathbf{D} = \epsilon_0 \mathbf{E}$ , and  $\mathbf{B} = \mu_0 \mathbf{H}$ . We notice that that the electric permittivity and the magnetic permeability are related by  $\epsilon_0 \mu_0 = 1$  when we choose coordinate system with  $x^0 := ct$ . It is a particular linear constitutive law. Covariant formulation of constitutive laws may be derived by means of the electromagnetic Lagrangian, and the covariant Maxwell equations recast in 4-dimension accordingly:

$$\mathcal{L} := -\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}, \quad \hat{\nabla}_\mu \mathcal{F}^{\mu\nu} = 0 \quad (7)$$

where  $\mathcal{F}^{\mu\nu}$  denotes the electromagnetic tensor including the electric displacement field and the magnetic field.

By introducing the definitions (4) and (6) the three dimensional formulation of the Lagrangian density function reduces to:  $\mathcal{L} = \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{H})$ . From relations (4), (6), and (7), it is easy to derive the electromagnetic wave equation in the Minkowski spacetime e.g. Rakotomanana [2019]:

$$-g^{\nu\beta} \hat{\nabla}_\nu \hat{\nabla}_\beta A^\mu = 0 \quad (8)$$

by accounting for the Lorenz gauge condition  $\hat{\nabla}_\nu A^\nu \equiv 0$ . The equation (8) governs the electromagnetic wave travelling at the speed  $c$  within Minkowski spacetime.

*Remark 2* The covariance of the Maxwell 3D equations with respect to rotations, space reflection, time reversal, and charge conjugation (modification of positive charge to negative one) may be checked by means of a four-dimensional covariance analysis with respect to Lorentz group of transformations.

In presence of gravitation phenomena, the flat Minkowski spacetime becomes curved, metric components vary with location in the spacetime. Gravitation is related to the non null curvature,  $\mathfrak{R} \neq 0$ , within the spacetime. In the present paper, electromagnetic fields equations are formulated within curved spacetime of general relativity.

### 3.2 Gravitation within curved spacetime : Einstein fields equation

Although special relativity is mainly based on the Minkowski spacetime, which is a flat manifold, relative gravitation lies upon the Riemannian manifold (or more precisely pseudo-Riemannian). Let us consider a torsionless but curved spacetime  $(\mathcal{M}, \mathbf{g}, \nabla)$ . The spacetime structure is summarized by the Riemannian spacetime structure with the metric  $\mathbf{g}$ , the torsion  $\mathfrak{K}$ , and the curvature  $\mathfrak{R}$ :

$$\nabla \mathbf{g} = 0, \quad \mathfrak{K} \equiv 0, \quad \mathfrak{R} \neq 0 \quad (9)$$

The Lagrangian function  $\mathcal{L}$  usually depends on the metric, the torsion and the curvature in the general case. The relevant connection for Einstein relativistic gravitation is the Levi-Civita connection, with a non zero curvature describing the gravitational field, with the Lagrangian  $\mathcal{L}(\mathbf{g}, \mathfrak{R})$ . Let consider a Lagrangian corresponding to the Einstein-Hilbert action of classical relativistic gravitation Hilbert [1915], and with a retrospective analysis in Brading & Ryckman [2008]:<sup>2</sup>

$$\mathcal{S}_G := (1/2\chi) \int \mathcal{R} \omega_n \quad (10)$$

where  $\mathcal{R}$  is the scalar curvature and  $\omega_n$  a volume-form e.g. Nakahara [1996], for example  $\omega_n := \sqrt{\text{Det} \mathbf{g}} dx^0 \wedge \dots \wedge dx^3$

<sup>2</sup>By using a variational procedure, Hilbert preceded Einstein with the publication of the gravitational field equations of relative gravitation by five days, however both authors arrived at same field equations along very different paths.

when the spacetime is endowed with the metric  $\mathbf{g}$ . The constant factor  $\chi := 8\pi G/c^4$  is introduced to reproduce the classical mechanics of Newton when some matter is moving within this Minkowski spacetime reduced to an Euclidian space and a time interval e.g. Ryder [2009].

First we introduce the metric variation  $g_{\alpha\beta} \rightarrow g_{\alpha\beta} + \delta g_{\alpha\beta}$  (corresponding to the Eulerian variation of the metric at a fixed point of the spacetime). A straightforward calculus gives the variation of the connection and the Ricci curvature (we omit bar overline for connection and curvature for simplicity):

$$\delta \Gamma_{\alpha\beta}^\gamma = (1/2) g^{\gamma\lambda} (\nabla_\beta \delta g_{\alpha\lambda} + \nabla_\alpha \delta g_{\lambda\beta} - \nabla_\lambda \delta g_{\alpha\beta}) \quad (11)$$

$$\delta \mathfrak{R}_{\alpha\beta} = \nabla_\lambda (\delta \Gamma_{\beta\alpha}^\lambda) - \nabla_\beta (\delta \Gamma_{\lambda\alpha}^\lambda) \quad (12)$$

where the covariant derivative is related to the unperturbed (metric compatible) connection. The equation (12) is known as Palatini identity, showing that for torsionless spacetime the variation of the Ricci tensor may be transferred to the boundary condition terms via the divergence theorem. The principle of least action  $\delta \mathcal{S}_G \equiv 0$  for arbitrary variation of the metric gives the Einstein field equation of the general relativity e.g. Lovelock [1969]:

$$G_{\alpha\beta} := \mathfrak{R}_{\alpha\beta} - (1/2) \mathcal{R} g_{\alpha\beta} = 0 \quad (13)$$

which is the field equation within a Riemann spacetime, the Euler-Lagrange equations associated to the Einstein-Hilbert action. Metric components  $g_{\alpha\beta}(x^\mu)$  are the unknown variables in this field equation. To this end, given a curvature field  $\mathfrak{R}$  on a metric manifold  $\mathcal{M}$ , determination of the 10 metric components  $\mathbf{g}$  needs integration of system of 20 second order partial differential equations. For this to be possible, additional integrability must be satisfied for third-order derivatives of metric components assuming that they are  $\mathcal{C}^3$ . These are the Bianchi identities e.g. Rakotomanana [2003].

*Remark 3* First, the derivation of the field equation of general relativity, is obtained accounting that the variation of the connection  $\delta \Gamma_{\alpha\beta}^\gamma$  is shifted to the boundary by means of the divergence operator. Second, the Einstein's gravitational equations are invariant under the group of (passive) diffeomorphisms (covariance), and not only for the group of Lorentz transformations. Under an arbitrary change of coordinate  $\tilde{x}^\alpha = \tilde{x}^\alpha(x^\mu)$ , the same shape of equations is obtained to give exactly  $\tilde{G}_{\alpha\beta} = 0$ .

### 3.3 Linearized gravitation equations : Gravitational waves

In this subsection, we derive the basic equations due to linear perturbation of the Minkowskian metric as for linear gravity phenomenon. Gravity is the consequence of how massive object deforms the spacetime. Near any massive body, the spacetime becomes curved following the change of the

spacetime metric. The deformation does not stay only near the massive body. The field equations of Einstein suggested that the deformation can propagate throughout the entire spacetime. The main difference compared to seismic waves is that gravitational waves can travel in empty space at the light speed. This is typical example where the gauge invariance is useful for deriving the wave equations of relativistic gravitation. The method is based on linear perturbation of the metric, the Ricci curvature tensor, and the Einstein tensor.

Consider a weak field gravitation where metric is close to Minkowski metric  $g_{\alpha\beta} \simeq \hat{g}_{\alpha\beta} + 2\varepsilon_{\alpha\beta}$ , with  $\|\varepsilon_{\alpha\beta}\| \ll 1^3$ . It is also usual to assume that at large distance from sources, the spacetime becomes Minkowskian e.g. Dixon [1975]. Theory of special gravitation allows us to obtain the conservation laws associated to the linearized part of Hilbert-Einstein Lagrangian  $\mathcal{L}(g_{\alpha\beta}, \partial_\gamma \partial_\lambda g_{\alpha\beta}) := (1/2\chi) \mathcal{R}$  with:

$$\overline{\mathfrak{R}}_{\alpha\beta\mu}^\lambda = \hat{g}^{\lambda\sigma} (\partial_\mu \partial_\alpha \varepsilon_{\sigma\beta} - \partial_\mu \partial_\beta \varepsilon_{\sigma\alpha} + \partial_\sigma \partial_\beta \varepsilon_{\mu\alpha} - \partial_\sigma \partial_\alpha \varepsilon_{\mu\beta}) \quad (14)$$

The equation (13) governs the dynamics of Riemann spacetime in relativistic gravitation. the unknowns are the metric components. In the presence of moving bodies, the problem in relativistic gravitation is to solve, at the same time, the gravitation fields induced by the bodies and the motion of the bodies e.g. Papapetrou [1951]. From the previous equation, we deduce the linear scalar curvature e.g. Rakotomanana [2019]:

$$\mathcal{R} = 2 [\partial^\alpha \partial^\nu \varepsilon_{\alpha\nu} - \hat{g}^{\alpha\nu} \partial_\alpha \partial_\nu (\text{Tr} \varepsilon)] \quad (15)$$

The linearized Einstein's equation of gravity  $G_{\beta\mu} = 0$  is obtained accordingly after some rearrangements to highlight the D'Alembertian operator:

$$G_{\beta\mu} = - \underbrace{\hat{g}^{\alpha\nu} \partial_\alpha \partial_\nu [\varepsilon_{\beta\mu} - (\text{Tr} \varepsilon) \hat{g}_{\beta\mu}]}_{\text{D'Alembertian}} + \partial^\nu \partial_\mu \varepsilon_{\beta\nu} + \partial_\beta \partial^\nu \varepsilon_{\nu\mu} - \partial_\beta \partial_\mu (\text{Tr} \varepsilon) - (\partial^\alpha \partial^\nu \varepsilon_{\alpha\nu}) \hat{g}_{\beta\mu} \quad (16)$$

Despite the fact that we have ten equations for ten unknowns, it is not yet possible, at this step, to solve them because we have first to define a coordinate system. Indeed, the decomposition of the metric into two terms as the flat Minkowskian metric  $\eta_{\beta\mu}$ , and a perturbation  $2\varepsilon_{\beta\mu}$  is not unique. Depending on the choice of a coordinate system ( $x^\lambda$ ) the shape of the perturbation may be different. For this purpose, let introduce the Lorentz gauge (also called Einstein gauge, Hilbert

<sup>3</sup>Application of the Lagrangian formalism in general relativity may induce some difficulties, because physical quantities in classical or special relativity framework require fixed geometric background (Newtonian or Minkowskian spacetime). Indeed, for general relativity the spacetime geometry is itself a dynamical object. Separation of the metric into two parts that may be respectively assigned to inertia and gravity is an affair of taste e.g. Shen & Moritz [1996].

gauge, de Donder gauge or Fock gauge). Starting from the coefficients of connection  $\Gamma_{\mu\nu}^\lambda$ , the Lorentz gauge imposes that the skew-symmetry part of the connection is equal to zero  $g^{\mu\nu} \Gamma_{\mu\nu}^\lambda \equiv 0$ , together with its linearized version:

$$\begin{aligned} \hat{g}^{\mu\nu} \hat{g}^{\rho\lambda} (\partial_\mu \varepsilon_{\rho\nu} + \partial_\nu \varepsilon_{\mu\rho} - \partial_\rho \varepsilon_{\mu\nu}) \\ = \hat{g}^{\lambda\rho} \partial^\mu \varepsilon_{\rho\mu} - (1/2) \partial^\lambda \text{Tr}(\varepsilon) \equiv 0 \end{aligned} \quad (17)$$

The requirement of gauge invariance condition based on the *Lorentz gauge*, the linearized gravitation takes the form of after some straightforward calculus :

$$\hat{g}^{\alpha\nu} \partial_\alpha \partial_\nu \bar{\varepsilon}_{\beta\mu} = 0 \implies \hat{g}^{\alpha\nu} \hat{\nabla}_\alpha \hat{\nabla}_\nu \bar{\varepsilon}_{\beta\mu} = 0 \quad (18)$$

This is a wave equation in the Minkowskian spacetime where the perturbed metric is the unknown. Obviously, it takes the form of classical equation of a wave (18) travelling at the speed  $c$  of the light as electromagnetic wave.

*Remark 4* The source-free field equations (13) and accordingly the gravitational wave equation (18) were obtained along independent paths by Hilbert and Einstein. To account for the sources (“forces”) terms other than gravitation, the associated Lagrangian  $\mathcal{L}$  may be split into the sum of two contributions, a gravitational part given by the Riemann curvature scalar and a source part which they left unspecified except for the postulation that it depend only on the components of the metric and the components of the electromagnetic vector potential and its first derivatives e.g. Sauer [2005]. Both Hilbert and Einstein had left the source term undetermined to some extent. Einstein had not specified his source term at all. Hilbert proposed that the source term depend only on the electromagnetic variables and implicitly assumed that all matter is of electromagnetic origin. In the following, electromagnetism coupled with gravitation is investigated.

### 3.4 Covariant Maxwell equations in curved spacetime

Various phenomenae may have effects on electromagnetic waves such as the presence of gravitational field. The constitutive equations should be worthily changed to account for the modification of the spacetime environment in presence of gravity for instance (curvature). Here, we thus consider the electromagnetic constitutive laws  $\mathbf{D} = \varepsilon \mathbf{E}$  and  $\mathbf{B} = \mu \mathbf{H}$  where  $\varepsilon$  and  $\mu$  are the electric and magnetic parameters of the spacetime replacing  $\varepsilon_0$  and  $\mu_0$ . For the sake of the simplicity, let us consider the simplest example of action for free electromagnetic field without sources and occurring within a Riemann curved spacetime where we assume an action including both Hilbert-Einstein Hilbert [1915] and a Yang-Mills Yang & Mills [1954] contributions, say e.g. Hammond [1987].:

$$\mathcal{S} := \int_{\mathcal{M}} \mathcal{L} \omega_n \quad \text{with} \quad \mathcal{L} := -\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2\chi} \bar{\mathcal{R}} \quad (19)$$

where no matter action is considered, and where the Faraday tensor and the scalar curvature are defined by the relationships:

$$\mathcal{F}_{\mu\nu} := \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu, \quad \bar{\mathcal{R}} := g^{\mu\nu} \bar{\mathcal{R}}_{\mu\nu} \quad (20)$$

where the Faraday tensor  $\mathcal{F}^{\mu\nu}$  is calculated with the connection with zero torsion. First, the Lagrangian variation of the action (19) allows us to obtain the expression:

$$\begin{aligned} \Delta \mathcal{S} = \int_{\mathcal{M}} \left\{ -\frac{1}{2} \mathcal{F}^{\mu\nu} \Delta \mathcal{F}_{\mu\nu} + \frac{1}{4} \mathcal{F}_{\mu\nu} \left( g^{\mu\lambda} \mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho} g^{\lambda\nu} \right) \Delta g_{\lambda\rho} \right. \\ \left. + \frac{1}{2\chi} \left( \bar{\mathcal{R}}^{\lambda\rho} - \frac{\bar{\mathcal{R}}}{2} g^{\lambda\rho} \right) \Delta g_{\lambda\rho} + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} \Delta g_{\lambda\rho} \right. \\ \left. + \frac{1}{2\chi} g^{\mu\nu} \left[ \bar{\nabla}_\lambda \left( \Delta \bar{\Gamma}_{\mu\nu}^\lambda \right) - \bar{\nabla}_\mu \left( \Delta \bar{\Gamma}_{\lambda\nu}^\lambda \right) \right] \right\} \omega_n \end{aligned}$$

For the variation of the Lagrangian (19), it is worth to remind the independent variations of the metric and the four-potential vector. The Lagrangian variation of the Faraday tensor takes the form of :

$$\Delta \mathcal{F}_{\mu\nu} = \bar{\nabla}_\mu (\Delta A_\nu) - \bar{\nabla}_\nu (\Delta A_\mu) \quad (21)$$

This relation is obtained by directly writing:

$$\begin{aligned} \Delta \mathcal{F}_{\mu\nu} &= \Delta \left( \partial_\mu A_\nu - \bar{\Gamma}_{\mu\nu}^\rho A_\rho \right) - \Delta \left( \partial_\nu A_\mu - \bar{\Gamma}_{\nu\mu}^\rho A_\rho \right) \\ &= \left( \partial_\mu \Delta A_\nu - \bar{\Gamma}_{\mu\nu}^\rho \Delta A_\rho - \Delta \bar{\Gamma}_{\mu\nu}^\rho A_\rho \right) \\ &\quad - \left( \partial_\nu \Delta A_\mu - \bar{\Gamma}_{\nu\mu}^\rho \Delta A_\rho - \Delta \bar{\Gamma}_{\nu\mu}^\rho A_\rho \right) \end{aligned}$$

accounting for that the connection variation  $\Delta \bar{\Gamma}_{\mu\nu}^\rho$  induces a variation of the field  $\Delta \mathcal{F}_{\mu\nu}$ . Second, the two systems of conservation laws associated to the unknown primal variables (say the 4-vector potential  $A_\mu$ , and the Riemannian metric  $g_{\mu\nu}$ ) are derived by varying the Lagrangian along the Lie-derivative variations  $\mathcal{L}_\xi A_\mu$ , and  $\mathcal{L}_\xi g_{\mu\nu}$ . Shifting the divergence terms at the boundary of the spacetime and assuming a zero divergence at this boundary allow us to obtain the conservation laws. We can rearrange the Lagrangian variation of the action to give:

$$\begin{aligned} \Delta \mathcal{S} = \int_{\mathcal{M}} \bar{\nabla}_\nu \mathcal{F}^{\mu\nu} \Delta A_\mu \omega_n + \int_{\mathcal{M}} \left[ \frac{1}{2\chi} \left( \bar{\mathcal{R}}^{\lambda\rho} - \frac{\bar{\mathcal{R}}}{2} g^{\lambda\rho} \right) \right. \\ \left. + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} + \frac{\mathcal{F}_{\mu\nu}}{4} \left( g^{\mu\lambda} \mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho} g^{\lambda\nu} \right) \right] \Delta g_{\lambda\rho} \omega_n \end{aligned}$$

owing that the Faraday tensor is *in fine* expressed in terms of the potential  $A_\mu$  by means of equation (20). Due to the arbitrariness of the metric and potential variations, we obtain the classical (and covariant) Einstein-Maxwell equations:

$$\begin{cases} \bar{\nabla}_\nu \mathcal{F}^{\mu\nu} = 0 \\ \frac{1}{2\chi} \left( \bar{\mathcal{R}}^{\lambda\rho} - \frac{\bar{\mathcal{R}}}{2} g^{\lambda\rho} \right) + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} \\ - \frac{1}{4} \left( g^{\lambda\mu} \mathcal{F}_{\mu\nu} \mathcal{F}^{\rho\nu} + g^{\lambda\nu} \mathcal{F}_{\nu\mu} \mathcal{F}^{\mu\rho} \right) = 0 \end{cases} \quad (23)$$



where the first equation is the covariant Maxwell equations in a Riemann spacetime, such as second gradient continuum. The second equation governs the interaction of electromagnetism with gravitation. The unknowns in first term of the second equation are the spacetime metric. The electromagnetic source (including both the second and the third terms) in the second equation constitutes the energy-momentum tensor. They influence the gravitation field and *vice versa* the metric field has also some influence on the electromagnetic field via the Levi-Civita covariant derivative  $\bar{\nabla}$ . Maxwell equations (23) (first row) are used to analyze the electromagnetic wave within a Riemann spacetime. The first equation may be re-written:

$$\begin{aligned}\bar{\nabla}_\nu \mathcal{F}^{\mu\nu} &= \bar{\nabla}_\nu (g^{\mu\alpha} g^{\alpha\nu} \mathcal{F}_{\alpha\beta}) = \bar{\nabla}_\nu [g^{\mu\alpha} \bar{\nabla}_\alpha A^\nu - g^{\nu\beta} \bar{\nabla}_\beta A^\mu] \\ &= g^{\mu\alpha} [\bar{\nabla}_\alpha \bar{\nabla}_\nu A^\nu + \mathfrak{R}_{\nu\alpha\gamma}^\nu A^\gamma] - g^{\nu\beta} \bar{\nabla}_\nu \bar{\nabla}_\beta A^\mu = 0\end{aligned}$$

where we have used the Schouten relations *e.g.* Rakotomanana [2003] with a zero torsion. Maxwell equations include a classical wave part, a divergence term, and the contribution of the Ricci curvature of the spacetime:

$$-g^{\nu\beta} \bar{\nabla}_\nu \bar{\nabla}_\beta A^\mu + g^{\mu\alpha} \bar{\nabla}_\alpha \bar{\nabla}_\nu A^\nu + g^{\mu\alpha} \bar{\mathfrak{R}}_{\alpha\gamma} A^\gamma = 0 \quad (24)$$

The first term expresses a D'Alembertian operator. The second term may be dropped if we assume a Lorenz gauge  $\bar{\nabla}_\nu A^\nu = 0$ . We then obtain the electromagnetic wave propagation equation within curved spacetime:

$$-g^{\nu\beta} \bar{\nabla}_\nu \bar{\nabla}_\beta A^\mu + g^{\mu\alpha} \bar{\mathfrak{R}}_{\alpha\gamma} A^\gamma = 0 \quad (25)$$

in which we notice the direct influence of the gravitation (represented by Ricci curvature) on the electromagnetic wave propagation. In the following we will consider an extension of the equation (25) in the framework of Riemann-Cartan spacetime.

### 3.5 Some remarks on electromagnetic fields

The second row of system (23) is the field equation which extends the Einstein equation for Riemann-Cartan spacetime, where the term represents the energy momentum analogous of the Maxwell energy-momentum for the space part, and with nonsymmetric property when considering the time-like part:

$$T^{\lambda\rho} := -\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} - \frac{\mathcal{F}^{\mu\nu}}{2} (g^{\mu\lambda} \mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho} g^{\lambda\nu}) \quad (26)$$

It is the Minkowski energy-momentum tensor due to electromagnetic field. It modifies the gravitational field as source whereas the spacetime modifies the electromagnetic field according to (25). The temporal component of the energy-momentum (26) holds:  $T^{00} = \frac{1}{2} (D^i E_i + B^i H_i)$ , which is exactly the electromagnetic energy density

$$T^{00} = (1/2) (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) := \mathcal{E}$$

in a three-dimensional formulation. By introducing the electromagnetic tensors (4) and (6) into the expression of the energy-momentum tensor, we have the Minkowski tensor  $T^{\mu\nu}$  is not symmetric when considering the time index 0. It is worth to express the energy momentum as:

$$T_M^{\mu\nu} = \begin{bmatrix} \mathcal{E} & \mathbf{E} \times \mathbf{H} \\ \mathbf{D} \times \mathbf{B} & \mathbf{T}^M \end{bmatrix} \quad (27)$$

where  $\mathcal{E}$  is the energy, and  $\mathbf{T}^M$  is the Maxwell tensor with contravariant components  $T^{ij}$ . The (nonsymmetric) energy-momentum such defined is called Minkowski energy momentum.

*Remark 5* The Poynting vector  $\mathbf{S} := \mathbf{E} \times \mathbf{H}$  (originally discovered by JH Poynting in 1884) represents the rate of energy in the  $i$ -direction. However, it is well-known that the Maxwell equations admit only the fundamental fields  $\mathbf{E}$  and  $\mathbf{B}$ , without considering any model of material media where the electromagnetic fields occur. In such a case, the Poynting vector is rigorously defined as  $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ , where  $\mu_0$  is the vacuum permeability (Minkowski spacetime). This is considered as the general expression of the Poynting vector. The moment of the Poynting vector  $\mathbf{J} := \int \mathbf{x} \times \mathbf{S} dv$  is defined as the angular momentum including the so-called orbital angular momentum and the spin angular momentum *e.g.* Barnett [2002].

We relate the electromagnetic fields  $T^{\lambda\rho}$  as source of the bending of the spacetime, we can multiply this equation by the covariant components of the metric  $g_{\lambda\rho}$  to obtain without difficulty the Ricci curvature and then the curvature of the spacetime:

$$-\bar{\mathfrak{R}} = \mathcal{S} := g_{\lambda\rho} T_M^{\lambda\rho} = -2\chi \mathcal{F}_{\lambda\rho} \mathcal{F}^{\lambda\rho} = \chi (\mathbf{D} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{H}) \quad (28)$$

which is exactly  $\chi$  times twice of the electromagnetic part of the Lagrangian.

*Remark 6* In this section the goal is not reached because the electromagnetic variables should be considered as additional variables. They are not directly related to the spacetime geometry. One should add the electromagnetic potential  $\mathbf{A}$  (or the electromagnetic Faraday strength  $\mathcal{F}$ ) as arguments of the Yang-Mills action.

## 4 Gravitation and electromagnetism within twisted and curved spacetime

Analysis of electromagnetic fields in presence of extremely massive gravitation remains a relevant topic in relativistic astrophysics. Propagation of electromagnetic waves governed by Maxwell equations within a curved spacetime constitutes

a fundamental basis for studying signals received from neutron stars and black holes to name but a few in astrophysics. Other methods consist in measuring the signal due to gravitational waves. It is now admitted that the influence of the non-Minkowskian metric of the curved spacetime is much stronger on the electromagnetic field  $\mathcal{F}_{\mu\nu}(x^\alpha)$  than the influence of this field on the bending of the spacetime  $\mathcal{M}$ . Including the torsion of continuum recently gains interest in the propagation of light in twisted medium e.g. Zhang & al. [2019]. In this section we consider the gravitation electromagnetism interaction within a Riemann-Cartan spacetime endowed with metric  $g_{\alpha\beta}(x^\mu)$  and connection  $\Gamma_{\alpha\beta}^\gamma(x^\mu)$ . We consider curved spacetime  $(\mathcal{M}, \mathbf{g}, \nabla)$  with non zero torsion. As previously, the spacetime structure is characterized by the Riemann-Cartan spacetime with the metric  $\mathbf{g}$ , the torsion  $\mathfrak{T}$ , and the curvature  $\mathfrak{R}$ :

$$\nabla \mathbf{g} = 0, \quad \mathfrak{T} \neq 0, \quad \mathfrak{R} \neq 0 \quad (29)$$

In the following, we derive the Maxwell equations in a curved and twisted manifold  $\mathcal{M}$ . By using a formalism based on exterior calculus, Maxwell equations were established for various continua (Minkowski, Riemann, and almost post-Riemann) Puntigam et al. [1997] where they considered as basic axioms the conservation of electric charge and the conservation of magnetic flux. This allows them to put aside the connection structure of the spacetime. Third, either for metric-based energy, or metric-torsion based energy, it is worth to define a Lagrangian  $\mathcal{L}(\mathcal{F}_{\mu\nu}, \dots)$  associated to the electromagnetic fields when we face the question of variational formulation. To relate electromagnetism with relativistic gravitation, it is interesting to remind that application of the gauge invariance principle for the group of translation (corresponding to torsion) of the spacetime  $\mathcal{M}$  with Yang-Mills type Lagrangian, quadratic in the field strengths  $\mathcal{F}_{\mu\nu}$  (as for electromagnetism), allows us to deduce the Einstein's theory of gravitation, based on the Einstein-Hilbert action e.g. Cho [1976a].

#### 4.1 Faraday tensor in twisted spacetime

Formulation of Maxwell equations by means of differential forms may be not equivalent to formulation by means connection in Riemann-Cartan spacetime e.g. Vandyck [1996]. In a Riemann-Cartan spacetime, the Faraday tensor is calculated as follows e.g. Prasanna [1975a], Smalley [1986]:  $\mathcal{F}_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathfrak{T}_{\mu\nu}^\rho A_\rho$ . It is rather different if calculated by means of an exterior derivative of the 1-form  $\mathbf{A} = (A_\mu)$  e.g. Prasanna [1975a]:

$$\mathcal{F} := d\mathbf{A} \quad \implies \quad \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (30)$$

where, in such a case, we have exactly the same form of Faraday tensor in either Minkowski spacetime or Riemann

spacetime. In this framework, two of the Maxwell equations  $d\mathcal{F} = 0$  would be expected since the Faraday tensor 2-form  $\mathcal{F}$  is exact, say  $\mathcal{F} := d\mathbf{A}$ , and hence closed,  $d\mathcal{F} = d(d\mathbf{A}) = 0$ . In order to investigate electromagnetic waves within twisted and curved spacetime matter (which may be considered as a Riemann-Cartan manifold), it is assumed that the electromagnetic field is described by an electromagnetic 2-form  $\mathcal{F}_{\mu\nu}$ . It constitutes an extended model of electromagnetism within curved spacetime as earlier as in e.g. Plebanski [1960], and in the framework of differential forms e.g. Prasanna [1975a]. Prasanna [1975a] has derived the Maxwell equations in a Riemann-Cartan spacetime. The Faraday tensor (minimally coupled to the gravitation in a Riemann-Cartan spacetime via the torsion) and the scalar curvature are defined by the relationships:

$$\begin{aligned} \mathcal{F}_{\mu\nu} &:= \nabla_\mu A_\nu - \nabla_\nu A_\mu = \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu - \mathfrak{T}_{\mu\nu}^\rho A_\rho, \\ \mathcal{R} &:= g^{\mu\nu} \mathfrak{R}_{\mu\nu} \end{aligned} \quad (31)$$

where the contravariant components of  $\mathcal{F}^{\mu\nu}$  are calculated by means the connection with torsion e.g. Smalley & Krisch [1992]. Definition of Faraday tensor (30) in Riemann spacetime holds for both Euclidean and (pseudo)-Riemannian and also even proposed in some post Riemannian spacetimes e.g. Puntigam et al. [1997]. As extension the definition (31) is valid for both Euclidean, Riemannian and Riemann-Cartan spacetime. This again illustrates the fact that the extension of physical variables as  $\mathcal{F}_{\mu\nu}$  can be done in many ways (as a 2-form in e.g. Puntigam et al. [1997] or as a twice the skew-symmetric part of the gradient in e.g. Smalley & Krisch [1992]).

#### 4.2 Field equations, wave equations

For the sake of the simplicity, consider the action for free gravito-electromagnetic field without sources and occurring within a Riemann-Cartan spacetime (curved and twisted), the Yang-Mills and Hilbert-Einstein action e.g. Charap & Duff [1977],:

$$\mathcal{S} := \int_{\mathcal{M}} \mathcal{L} \omega_n \quad \text{with} \quad \mathcal{L} := -\frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2\chi} \mathcal{R} \quad (32)$$

where no action due to matter is considered. For the variation of (32), it is worth to remind that the metric and the torsion are independents primal variables as well as the electromagnetic four-potential. The Lagrangian variation of this 2-form and curvature include both the variation of the potential  $A_\mu$ , the variation of the Riemann metric  $g_{\alpha\beta}$ , and also the variation of the connection  $\Gamma_{\alpha\beta}^\gamma$ . First, the Lagrange vari-

ation of the action (32) allows us to obtain :

$$\Delta \mathcal{S} = \int_{\mathcal{M}} \left\{ -\frac{1}{2} \mathcal{F}^{\mu\nu} \Delta \mathcal{F}_{\mu\nu} + \frac{1}{4} \mathcal{F}_{\mu\nu} \left( g^{\mu\lambda} \mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho} g^{\lambda\nu} \right) \Delta g_{\lambda\rho} \right. \\ \left. + \frac{1}{2\chi} \left( \mathfrak{R}^{\lambda\rho} - \frac{\mathcal{R}}{2} g^{\lambda\rho} \right) \Delta g_{\lambda\rho} + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} \Delta g_{\lambda\rho} \right. \\ \left. + \frac{g^{\mu\nu}}{2\chi} \left[ \nabla_{\lambda} \left( \Delta \Gamma_{\mu\nu}^{\lambda} \right) - \nabla_{\mu} \left( \Delta \Gamma_{\lambda\nu}^{\lambda} \right) - \mathfrak{K}_{\lambda\mu}^{\rho} \Delta \Gamma_{\rho\nu}^{\lambda} \right] \right\} \omega_n = 0 \quad \begin{cases} \nabla_{\nu} \mathcal{F}^{\mu\nu} = 0 \\ \frac{1}{2\chi} \left( \mathfrak{R}^{\lambda\rho} - \frac{\mathcal{R}}{2} g^{\lambda\rho} \right) + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} \\ - \frac{1}{4} \left( g^{\lambda\mu} \mathcal{F}_{\mu\nu} \mathcal{F}^{\nu\rho} + g^{\lambda\nu} \mathcal{F}_{\nu\mu} \mathcal{F}^{\mu\rho} \right) = 0 \\ (\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu}) A_{\lambda} + \frac{1}{\chi} \mathfrak{K}_{\lambda\rho}^{\mu} g^{\rho\nu} = 0 \end{cases} \quad (36)$$

where the last line is deduced from extended version of Palatini relation. Indeed, we can formulate the variation of primal independent variables as  $\delta g_{\alpha\beta}$  and the connection  $\delta \Gamma_{\alpha\beta}^{\gamma}$ . For this purpose, we remind the relations e.g. Rakotomanana [1997]:

$$\Delta \mathfrak{K}_{\alpha\beta}^{\gamma} = \Delta \Gamma_{\alpha\beta}^{\gamma} - \Delta \Gamma_{\beta\alpha}^{\gamma}, \\ \Delta \mathfrak{K}_{\alpha\beta\mu}^{\lambda} = \nabla_{\alpha} \left( \Delta \Gamma_{\beta\mu}^{\lambda} \right) - \nabla_{\beta} \left( \Delta \Gamma_{\alpha\mu}^{\lambda} \right) - \mathfrak{K}_{\alpha\beta}^{\nu} \Delta \Gamma_{\nu\mu}^{\lambda} \quad (33)$$

where the covariant derivatives use the connection with non zero torsion. The second equation (33) extends the Palatini identity when continuum has torsion. The Lagrangian variation of the Faraday tensor takes the form of :

$$\Delta \mathcal{F}_{\mu\nu} = \nabla_{\mu} (\Delta A_{\nu}) - \nabla_{\nu} (\Delta A_{\mu}) + \Delta \mathfrak{K}_{\mu\nu}^{\rho} A_{\rho} \quad (34)$$

This relation is obtained by directly writing:

$$\Delta \mathcal{F}_{\mu\nu} = \Delta \left( \partial_{\mu} A_{\nu} - \Gamma_{\mu\nu}^{\rho} A_{\rho} \right) - \Delta \left( \partial_{\nu} A_{\mu} - \Gamma_{\nu\mu}^{\rho} A_{\rho} \right) \\ = \left( \partial_{\mu} \Delta A_{\nu} - \Gamma_{\mu\nu}^{\rho} \Delta A_{\rho} - \Delta \Gamma_{\mu\nu}^{\rho} A_{\rho} \right) \\ - \left( \partial_{\nu} \Delta A_{\mu} - \Gamma_{\nu\mu}^{\rho} \Delta A_{\rho} - \Delta \Gamma_{\nu\mu}^{\rho} A_{\rho} \right)$$

accounting for that the variation of the geometric structure, say  $\Delta \mathfrak{K}_{\mu\nu}^{\rho}$ , induces a variation of the field  $\Delta \mathcal{F}_{\mu\nu}$ . At a second step, the three systems of conservation laws associated to the unknown primal variables (say the 4-vector potential  $A_{\mu}$ , the Riemannian metric  $g_{\mu\nu}$ , and the torsion  $\mathfrak{K}_{\mu\nu}^{\rho}$ ) are derived by varying the Lagrangian along the Lie-derivative variations  $\mathcal{L}_{\xi} A_{\mu}$ ,  $\mathcal{L}_{\xi} g_{\mu\nu}$ , and  $\mathcal{L}_{\xi} \mathfrak{K}_{\mu\nu}^{\rho}$ . Now we factorize the variation with respect to the Lagrangian variations of the electromagnetic potential  $\Delta A_{\mu}$ , the metric  $\Delta g_{\lambda\rho}$ , and the connection  $\Delta \Gamma_{\mu\nu}^{\lambda}$  respectively. The presence of the term  $\Delta \Gamma_{\mu\nu}^{\lambda}$  means that the torsion and curvature may evolve since they are independent primal variables of the theory. By shifting divergence terms at the boundary of the spacetime  $\mathcal{M}$  we can rearrange the Lagrangian variation of the action to give:

$$\Delta \mathcal{S} = \int_{\mathcal{M}} \nabla_{\nu} \mathcal{F}^{\mu\nu} \Delta A_{\mu} \omega_n + \int_{\mathcal{M}} \left[ \frac{1}{2\chi} \left( \mathfrak{R}^{\lambda\rho} - \frac{\mathcal{R}}{2} g^{\lambda\rho} \right) \right. \\ \left. + \frac{1}{8} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} + \frac{\mathcal{F}_{\mu\nu}}{4} \left( g^{\mu\lambda} \mathcal{F}^{\rho\nu} + \mathcal{F}^{\mu\rho} g^{\lambda\nu} \right) \right] \Delta g_{\lambda\rho} \omega_n \\ - \int_{\mathcal{M}} \left( (\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu}) A_{\lambda} + \frac{1}{\chi} g^{\rho\nu} \mathfrak{K}_{\lambda\rho}^{\mu} \right) \Delta \Gamma_{\mu\nu}^{\lambda} \omega_n \quad (35)$$

owing that the Faraday tensor is *in fine* expressed in terms of the potential  $A_{\mu}$  by means of (31). Due to the arbitrariness

of the variation of primal variables, we deduce the system of partial differential equations:

where we notice a slightly extension of the fields equations in Charap & Duff [1977] for Riemann-Cartan spacetime. The first row of the system (36) expresses the Maxwell equations in Riemann-Cartan spacetime, and it should be stressed that for this Lagrangian (model), the potential  $A_{\mu}$  may be apparently calculated independently on the gravitation (except eventual coupling at the boundary  $\partial \mathcal{M}$ ). The connection approach for Faraday tensor is equivalent to the differential form approach when the spacetime is Riemannian without torsion (Vandyck [1996]), or when the non metricity of the connection is traceless. The Maxwell equations of (36) show that the connection approach with torsion is "naturally" deduced from a variation principle, and the same form as the form approach is obtained.

*Remark 7* It is stressed that for non-symmetric tensors, care should be taken about the placement of the indices.

*Non symmetry of the Ricci tensor* First, the difference between the spacetime connection connection and the Levi-Civita connection is the contortion tensor:

$$\mathcal{T}_{\alpha\beta}^{\gamma} := \Gamma_{\alpha\beta}^{\gamma} - \bar{\Gamma}_{\alpha\beta}^{\gamma} \quad (37)$$

where  $\bar{\Gamma}_{\alpha\beta}^{\gamma}$  are the Christoffel symbols associated to the metric  $g_{\alpha\beta}$ . The contortion tensor describes the deviation of the matter geometry from the Riemannian geometry one, whose connection reduces to the Christoffel symbols. By choosing a metric connection on the manifold, we have:

$$\mathfrak{K}_{\alpha\beta\lambda}^{\gamma} = \partial_{\alpha} (\bar{\Gamma}_{\beta\lambda}^{\gamma} + \mathcal{T}_{\beta\lambda}^{\gamma}) - \partial_{\beta} (\bar{\Gamma}_{\alpha\lambda}^{\gamma} + \mathcal{T}_{\alpha\lambda}^{\gamma}) \\ - (\bar{\Gamma}_{\alpha\lambda}^{\mu} + \mathcal{T}_{\alpha\lambda}^{\mu}) (\bar{\Gamma}_{\beta\mu}^{\gamma} + \mathcal{T}_{\beta\mu}^{\gamma}) \\ + (\bar{\Gamma}_{\beta\lambda}^{\mu} + \mathcal{T}_{\beta\lambda}^{\mu}) (\bar{\Gamma}_{\alpha\mu}^{\gamma} + \mathcal{T}_{\alpha\mu}^{\gamma})$$

We arrive to the expression of the curvature strain which is defined as the difference between the matter and the curvature calculated with Levi-Civita connection:

$$\mathfrak{K}_{\alpha\beta\lambda}^{\gamma} = \bar{\mathfrak{K}}_{\alpha\beta\lambda}^{\gamma} + \bar{\nabla}_{\alpha} \mathcal{T}_{\beta\lambda}^{\gamma} - \bar{\nabla}_{\beta} \mathcal{T}_{\alpha\lambda}^{\gamma} - (\mathcal{T}_{\beta\mu}^{\gamma} \mathcal{T}_{\alpha\lambda}^{\mu} - \mathcal{T}_{\alpha\mu}^{\gamma} \mathcal{T}_{\beta\lambda}^{\mu}) \quad (38)$$

where another interest appears when we calculate the Ricci curvature which is obviously non symmetric :

$$\mathfrak{R}_{\beta\lambda} := \mathfrak{K}_{\alpha\beta\lambda}^{\alpha} = \bar{\mathfrak{R}}_{\beta\lambda} + \bar{\nabla}_{\alpha} \mathcal{T}_{\beta\lambda}^{\alpha} - \bar{\nabla}_{\beta} \mathcal{T}_{\alpha\lambda}^{\alpha} \\ - (\mathcal{T}_{\beta\mu}^{\alpha} \mathcal{T}_{\alpha\lambda}^{\mu} - \mathcal{T}_{\alpha\mu}^{\alpha} \mathcal{T}_{\beta\lambda}^{\mu}) \quad (39)$$

where we notice that the Levi-Civita covariant derivative is used for calculating the contortion. In view of the relation (39), we notice the possibility of the Ricci tensor to be not symmetric in presence of torsion. The gravitational part in the equation (36–2) is then not symmetric too, and this is coherent to the fact that the Minkowski energy-momentum may be not symmetric too.

The curvature (39) allows us to calculate de scalar curvature  $\mathcal{R} := g^{\beta\lambda} \mathfrak{R}_{\beta\lambda}$ . Since the both two connections are initially assumed metric compatible, and by shifting the divergence terms to the spacetime boundary  $\partial\mathcal{M}$ , we notice that the gravitation part of the Hilbert-Einstein action seem to be not equivalent when using the two connections : Levi-Civita and the connection with non zero torsion. However, the symmetric  $\mathfrak{R}_{\beta\lambda}^S$  and anti-symmetric  $\mathfrak{R}_{\beta\lambda}^A$  parts of the Ricci curvature are obtained from (39):

$$\begin{cases} \mathfrak{R}_{\beta\lambda}^S = \bar{\mathfrak{R}}_{\beta\lambda} \\ \mathfrak{R}_{\beta\lambda}^A = \bar{\nabla}_\alpha \mathcal{T}_{\beta\lambda}^\alpha - \bar{\nabla}_\beta \mathcal{T}_{\alpha\lambda}^\alpha - (\mathcal{T}_{\beta\mu}^\alpha \mathcal{T}_{\alpha\lambda}^\mu - \mathcal{T}_{\alpha\mu}^\alpha \mathcal{T}_{\beta\lambda}^\mu) \end{cases}$$

where we notice that the symmetric part reduces to the Ricci curvature associated to the Levi-Civita connection whereas the skew-symmetric part includes all the remaining terms of the equation (39). We easily deduce that the two scalar curvatures are equal  $\mathcal{R} := g^{\beta\lambda} \mathfrak{R}_{\beta\lambda} = g^{\beta\lambda} \bar{\mathfrak{R}}_{\beta\lambda} = \bar{\mathcal{R}}$  and the same Lagrangian density of the gravitation by using the two connections. We nevertheless draw attention that the choice of the volume-form  $\omega_n$  for integrating 4-forms should be undertaken carefully to ensure compatibility e.g. Mosna & Saa [2005].

*Electromagnetic wave propagation* The Maxwell equations (36) (first row) may be used to analyze the electromagnetic wave propagation within a twisted and curved spacetime. Let consider a spacetime  $\mathcal{M}$  endowed with a metric  $g_{\alpha\beta}$  and a connected with  $\Gamma_{\alpha\beta}^\gamma$ , this later is compatible with the metric. Maxwell equations may be re-written as follows:

$$\begin{aligned} \nabla_\nu \mathcal{F}^{\mu\nu} &= \nabla_\nu (g^{\mu\alpha} g^{\alpha\nu} \mathcal{F}_{\alpha\beta}) = \nabla_\nu [g^{\mu\alpha} \nabla_\alpha A^\nu - g^{\nu\beta} \nabla_\beta A^\mu] \\ &= g^{\mu\alpha} [\nabla_\alpha \nabla_\nu A^\nu - \mathfrak{K}_{\nu\alpha}^\gamma \nabla_\gamma A^\nu + \mathfrak{K}_{\nu\alpha}^\gamma \nabla_\gamma A^\nu] \\ &\quad - g^{\nu\beta} \nabla_\nu \nabla_\beta A^\mu = 0 \end{aligned}$$

where we used the Schouten' relations e.g. Rakotomanana [2003]. By arranging the previous relationships, we notice that the Maxwell equations include, as for elastic wave propagation, a classical wave part, a divergence term, and the contribution of the twisting and the Ricci curvature of the spacetime:

$$-g^{\nu\beta} \nabla_\nu \nabla_\beta A^\mu - g^{\mu\alpha} \mathfrak{K}_{\nu\alpha}^\gamma \nabla_\gamma A^\nu + g^{\mu\alpha} \mathfrak{R}_{\alpha\gamma}^\gamma A^\gamma = 0 \quad (40)$$

where the condition  $\nabla_\nu A^\nu \equiv 0$  was used to extend the Lorenz condition in the framework of Riemann-Cartan geometry, more specifically in the way of Gauss units system. The first term expresses a D'Alembertian operator. The second term

may be dropped if we assume a null divergence as a gauge condition. For a non twisted and non curved spacetime, the electromagnetic wave propagation equation reduces to  $\square A^\mu = 0$ . The third term introduces a first covariant derivative which leads to a diffusion of the wave (spacetime attenuation), and the last term points out a breathing mode whenever the boundary conditions allow it e.g. Futhazar et al. [2014]. What should be observed too is that the torsion and Ricci curvature influence the wave propagation linearly. It should be stressed that the spacetime geometry and *in fine* the gravitation is in fact tightly linked to the electromagnetism phenomenon. This may not be perceived at a first sight. In sum, the second row of the system (36) gives the coupling equation of the electromagnetic field and the gravitational field one. The electromagnetic terms act as a source-term for the gravitation. They act as a kind of electromagnetic energy generating evolution of the spacetime metric. We recognize the Einstein field equation in the absence of the electromagnetic field. Despite its apparent relative simplicity, the system of partial differential equations (36) remains complex since the connection, and by the way the Ricci and total curvatures, includes both the (gravitational) metric  $g_{\mu\nu}$  and the contortion tensor  $\mathcal{T}_{\mu\nu}^\gamma$ .

*Remark 8* Most importantly, the torsion  $\mathfrak{K}$  of the spacetime is related to the flux spin angular momentum of electromagnetic waves e.g. Allen & al. [1992], Barnett [2002], Milonni & Boyd [2010] as shown in the equation (36-3). This equation is deduced directly from a variational principle by assuming an additive action including both the Hilbert-Einstein and Yang-Mill actions. In a recent work, by using another meyhod based on the fields equations, it has also been shown that electromagnetic spin creates torsion within spacetime Hammond [2018], and accordingly the presence of the torsion as argument of the Lagrangian  $\mathcal{L}$  is required to ensure gauge invariance of the electromagnetism theory. Electromagnetic waves and particularly, light beams have angular momentum, corresponding to the Poynting vector. There are two particularly manners in which a electromagnetic waves can rotate: if every polarization vector rotates, the wave has spin; if the phase structure rotates, the wave has orbital angular momentum. What would be interesting too is that spin angular moment as well as orbital angular momentum have mechanical effects, this may open to some research in the domain of relativistic optomechanics at least from a basic theoretical point of view.

#### 4.3 Remarks on the extended electromagnetism wave equation

The third row of (36) gives the equation to calculate the torsion field. It is striking its analogy with the relation assumed by Fernando et al. [2012] by considering a partic-

ular Riemann-Cartan spacetime and working with contortion tensor. It is a link between electromagnetic fields and the twisting of the spacetime. What is interesting is that the electromagnetic field allows us to calculate with an algebraic explicit formula the torsion field by means of the third row. Once the torsion is obtained, we can apply covariant derivative within Riemann-Cartan geometry. By multiplying with  $g_{\nu\sigma}$ , the explicit formula for calculating the torsion is obtained accordingly by means of an algebraic relation:

$$\begin{aligned}\mathfrak{K}_{\lambda\sigma}^{\mu} &= -\chi g_{\sigma\nu} (\mathcal{F}^{\mu\nu} - \mathcal{F}^{\nu\mu}) A_{\lambda} \\ &= -2\chi g_{\sigma\nu} (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) A_{\lambda} \nabla_{\alpha} A_{\beta}\end{aligned}\quad (41)$$

owing the expression of the electromagnetic strength in terms of potential. It may be noticed that the contribution of the electromagnetic potential to the torsion field is of second order " $A_{\lambda} \nabla_{\alpha} A_{\beta}$ ".

*Remark 9* The investigation of the interaction of electromagnetic masses with Riemann-Cartan continuum was done by numerous authors for charged and spinning "static" dust (here static means here no displacement of the center of mass), for perfect fluids with spin density e.g. Smalley & Krisch [1992]. By analyzing the solutions of Maxwell equations, the torsion field together with the spin of Einstein-Cartan gravitation theory may be suggested as produced by the electromagnetic field e.g. Tiwari & Ray [1997]. Paraphrasing these authors, it was concluded that in the absence of electromagnetic fields, the body has a vanishing spin density which itself is associated to the continuum torsion e.g. Hehl & von der Heyde [1973]. The third algebraic equation of (36) conforms this conclusion concerning the torsion field.

*Remark 10* From the system of equations (36), we notice that the electromagnetic energy-momentum in a vacuum has the same shape as for as for electromagnetic within a continuum matter Obukhov & Hehl [2003]:

$$T^{\lambda\rho} = \frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} g^{\lambda\rho} - \frac{1}{2} (g^{\lambda\mu} \mathcal{F}_{\mu\nu} \mathcal{F}^{\nu\rho} + g^{\lambda\nu} \mathcal{F}_{\nu\mu} \mathcal{F}^{\mu\rho}) \quad (42)$$

This is a (non) symmetric Minkowski (canonical) energy-momentum tensor e.g. Obukhov [2008] for the free electromagnetic field occurring within spacetime. There is a controversy between the version of Minkowski and that of Abraham, not deduced from a Lagrangian. We do not enter into this long last debate, which was done in the past. We have just to remind that the Minkowski version is defined in the framework of Lagrange-Noether conforming to the invariance approach we adopt in this work. Obukhov & Hehl suggested the adoption of the Minkowski version (42) which is motivated by the Lagrangian axiomatic approach, and by the experimental evidence conducted in the past by Walker & Walker (which is based on experimental measurements of

dielectric disk placed in a crossed oscillating radial electric and longitudinal magnetic fields), and James (which is based on a similar experimental jig but with radial electric field and azimuthal magnetic field) Obukhov & Hehl [2003].

*Remark 11* In the previous equations, the torsion does not propagate. In order to account for the torsion propagation, i.e., a well-known method would be to add a scalar bilinear term of the covariant derivatives of the torsion e.g. Hammond [1987] where the trace of the torsion  $\mathfrak{K}_{\nu}^{\mu} := \mathfrak{K}_{\nu\mu}^{\mu}$  can be considered as the electromagnetic four-potential, and the skew-symmetric part of the Ricci curvature tensor as proportional to the electromagnetic Faraday tensor. For that purpose, he has considered the Lagrangian:  $\mathcal{S} := \int_{\mathcal{M}} (\frac{\mathcal{R}}{\chi} + a \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu})$  with  $\mathcal{G}_{\mu\nu} := \partial_{\mu} \mathfrak{K}_{\nu} - \partial_{\nu} \mathfrak{K}_{\mu}$  where arbitrariness of the metric and the torsion variations hold. In his approach the electromagnetic variables are also deduced from spacetime geometry.

To go further let us consider the first and third equations of the system of equations relating gravitation and electromagnetism. As for the Maxwell equations within Minkowski spacetime, the above equation may be formulated by means of four-potential vector  $A^{\mu}$  by introducing properties of curvature tensor, the metricity of the connection and the Lorenz gauge ( $\nabla_{\nu} A^{\nu} \equiv 0$ ). The third equation may be re-arranged to isolate the torsion. The first and third equations thus give, by assuming a null divergence for the potential  $\nabla_{\nu} A^{\nu} = 0$ :

$$\begin{cases} -g^{\nu\beta} \nabla_{\nu} \nabla_{\beta} A^{\mu} - g^{\mu\alpha} \mathfrak{K}_{\nu\alpha}^{\gamma} \nabla_{\gamma} A^{\nu} + g^{\mu\alpha} \mathfrak{K}_{\alpha\gamma}^{\gamma} A^{\gamma} = 0 \\ 2\epsilon\chi (\nabla^{\gamma} A_{\alpha} - \nabla_{\alpha} A^{\gamma}) A_{\beta} = \mathfrak{K}_{\alpha\beta}^{\gamma}\end{cases}\quad (43)$$

where, in the Maxwell equations, the first term represents a wave equation, the second term a diffusion contribution due to the torsion field, and the last term with the Ricci curvature introduces a "breathing" mode due to the non vanishing of curvature tensor. It should be pointed out that the torsion field is of second-order with respect to the potential  $A^{\alpha}$ .

*Remark 12* In the present model, we define the Faraday tensor as  $\mathcal{F}_{\alpha\beta} := \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha}$  where the connection has torsion. The  $U(1)$  gauge invariance of Maxwell equations may be violated without cautions with this choice. Indeed by modifying the potential as  $A_{\beta} \rightarrow A_{\beta} + g_{\beta\gamma} \nabla^{\gamma} \Lambda$  where  $\Lambda(x^{\mu})$  is an arbitrary function, we get:  $\mathcal{F}_{\alpha\beta} = \nabla_{\alpha} A_{\beta} - \nabla_{\beta} A_{\alpha} - \mathfrak{K}_{\alpha\beta}^{\gamma} \nabla_{\gamma} \Lambda$  where the last term vanishes if and only if the torsion is zero or the function  $\Lambda$  is covariantly uniform. Some previous authors propose to define  $\mathcal{F}_{\alpha\beta} := \bar{\nabla}_{\alpha} A_{\beta} - \bar{\nabla}_{\beta} A_{\alpha}$  as Faraday tensor even in Riemann-Cartan spacetime e.g. de Andrade & Pereira [1999], Smalley [1986]. Further analyses are required in the future. Results in the framework of Riemann-Cartan Gravitation (e.g. Sotiriou & Liberati [2007]) may highlight some problems of gauge invariance since the

Minkowski		Special Relativity
Spacetime metric	$g_{\alpha\beta} := \{+, -, -, -\}$	$g := \sqrt{ \text{Det}g_{\alpha\beta} }$
Faraday tensor	$\Gamma_{\alpha\beta}^\gamma \equiv 0$	$\mathcal{F}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$
Constitutive laws	$\mathcal{L} := -\frac{1}{4}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}g$	$\mathcal{F}^{\alpha\beta} = \epsilon_0 g^{\alpha\mu}g^{\beta\nu}\mathcal{F}_{\mu\nu}$
Conservation laws	$\nabla_\beta \mathcal{F}^{\alpha\beta} = 0$	$g^{\nu\beta}\nabla_\nu \nabla_\beta A^\mu = 0$
Riemann		Einstein Gravitation
Spacetime metric	$g_{\alpha\beta} := g_{\alpha\beta}(x^\mu)$	$g := \sqrt{ \text{Det}g_{\alpha\beta} }$
Faraday tensor	$\bar{\Gamma}_{\alpha\beta}^\gamma$	$\mathcal{F}_{\alpha\beta} = \bar{\nabla}_\alpha A_\beta - \bar{\nabla}_\beta A_\alpha$
Constitutive laws	$\mathcal{L} := -\frac{1}{4}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}g$	$\mathcal{F}^{\alpha\beta} = \epsilon_0 g^{\alpha\mu}g^{\beta\nu}\mathcal{F}_{\mu\nu}$
Conservation laws	$\bar{\nabla}_\beta \mathcal{F}^{\alpha\beta} = 0$	$g^{\nu\beta}\bar{\nabla}_\nu \bar{\nabla}_\beta A^\mu - g^{\mu\alpha}\mathfrak{R}_{\alpha\gamma}A^\gamma = 0$
Riemann-Cartan		Einstein-Cartan Gravitation
Spacetime metric	$g_{\alpha\beta} := g_{\alpha\beta}(x^\mu)$	$g := \sqrt{ \text{Det}g_{\alpha\beta} }$
Faraday tensor	$\Gamma_{\alpha\beta}^\gamma$	$\mathcal{F}_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$
Constitutive laws	$\mathcal{L} := -\frac{1}{4}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}g$	$\mathcal{F}^{\alpha\beta} = \epsilon_0 g^{\alpha\mu}g^{\beta\nu}\mathcal{F}_{\mu\nu}$
Conservation laws	$\nabla_\beta \mathcal{F}^{\alpha\beta} = 0$	$g^{\nu\beta}\nabla_\nu \nabla_\beta A^\mu + g^{\mu\alpha}\mathfrak{K}_{\nu\alpha}^\gamma \nabla_\gamma A^\nu - g^{\mu\alpha}\mathfrak{R}_{\alpha\gamma}A^\gamma = 0$

Table 1: Theories of electromagnetism interacting with gravitation in twisted and curved spacetime : Minkowski(flat), Riemann (curved) and Riemann-Cartan (twisted and curved)

electromagnetic tensor does not satisfy the Lorenz gauge invariance (say  $U(1)$  gauge invariance) e.g. Puntigam et al. [1997]. Choosing the Faraday tensor as  $\mathcal{F} := dA$  or equivalently  $\mathcal{F}_{\mu\nu} := \bar{\nabla}_\mu A_\nu - \bar{\nabla}_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu - \mathfrak{K}_{\mu\nu}^\rho A_\rho$  allows to obtain a  $U(1)$  invariant model but induces the following field equations:  $\nabla_\nu \mathcal{F}^{\mu\nu} - 2\mathfrak{K}_{\rho\nu}^\mu \mathcal{F}^{\rho\nu} = 0$  and  $\mathfrak{K}_{\mu\nu}^\rho = 0$  replacing the first Maxwell equations and the third equation coupling the torsion within spacetime and the electro-

magnetic fields. It means that the torsion is identically zero within the spacetime. Further investigations should be done about the definition of the Faraday tensor, there is yet a lot to be done in this domain. This may hurt at first sight, however, more investigations should be conducted since the concept of magnetic monopole enters into the discussion because the Gauss law on magnetic flux should be re-analyzed in such a case e.g. Fernando et al. [2012].

## 5 Geodesic and autoparallel deviation for gravitational and electromagnetic waves

Detection of gravitational waves illustrates a well-known application of the geodesic deviation equation e.g. Nieto et al. [2007]. The measuring of the separation of two neighbored geodesic curves in a Riemann spacetime may be evaluated by means of the separation acceleration. Some previous studies have extended this deviation equation to include the relativistic top moving in a gravitational field e.g. Nieto et al. [2007], or to reformulate the geodesic deviation in terms of teleparallel gravity Darabi et al. [2015]. Geodesic deviation equation describes the relative motion of two structureless particles determined by the spacetime geometry with non zero curvature. When particles are charged electrically, the deviation equation should account for the charge  $q$  e.g. Balakin et al. [2000] where geodesic curves are called autoparallel curves or worldlines e.g. Balakin et al. [2000]. In the present subsection, we are interested in developing the extension of the deviation equation for geodesics to autoparallel curves where the spacetime is curved with torsion. In view of the previous results where the torsion tensor of the spacetime is directly related, identified, with spin angular momentum of the electromagnetic field by means of the equation (41), it is necessary to check what about the influence of the torsion on the deviation of autoparallel curves.

### 5.1 Geodesic deviation equation in Riemannian manifold

First, let consider two geodesic curves in the pseudo-Riemann spacetime  $(\mathcal{M}, g)$  denoted by  $\gamma_0$  and  $\gamma_1$  respectively. At the same proptertime  $\tau$ , we define the separation four-vector  $\xi$  of the spacetime  $\mathcal{M}$  which connects a point (event)  $x(\tau)$  of the geodesic  $\gamma_0$  to a point (event)  $x(\tau) + \xi(\tau)$  of a nearby geodesic  $\gamma_1$ . The separation  $\xi$  is small in such a way that any expansion of tensor function of  $\xi$  with respect to  $\xi$  can be truncated to only the first-order terms. The relativistic acceleration  $\mathbf{a}$  of two material points is defined as the second derivative of the separation vector  $\xi$  as the two material points move along their respective geodesics. Let remind the separation velocity and deduce the separation acceleration along the geodesic curve as follows:

$$\mathbf{v} := \nabla_{\mathbf{u}} \xi, \quad \mathbf{a} := \nabla_{\mathbf{u}} \mathbf{v} \quad (44)$$

where the components  $\mathbf{u} := d\mathbf{x}/d\tau$  define the four-vector velocity (timelike vector). Then the separation acceleration takes the general form :

$$\frac{D^2\xi}{D\tau^2} := \nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\xi \quad (45)$$

which remains valid for either Riemann or Riemann-Cartan spacetime. For this purpose, let us worthly remind some basic assumptions and relations:

1. A geodesic on the spacetime  $\mathcal{M}$  endowed with an affine connection  $\nabla$  is defined as a curve  $\gamma(t)$  such that parallel transport along the curve preserves the tangent vector to the curve. Say  $\mathbf{u}$  a four-vector along a geodesic curve, then we have :

$$\nabla_{\mathbf{u}}\mathbf{u} = 0 \quad (46)$$

2. The orthogonality condition implies the nullity of the Lie derivative of the separation vector  $\xi$  along the vector tangent to the geodesic curve:  $\mathcal{L}_{\mathbf{u}}\xi := \nabla_{\mathbf{u}}\xi - \nabla_{\xi}\mathbf{u} - \mathfrak{K}(\xi, \mathbf{u}) = 0$ . This allows us to deduce two equivalent relations :

$$\begin{cases} \nabla_{\xi}\mathbf{u} = \nabla_{\mathbf{u}}\xi - \mathfrak{K}(\xi, \mathbf{u}) \\ \nabla_{\mathbf{u}}\xi = \nabla_{\xi}\mathbf{u} + \mathfrak{K}(\xi, \mathbf{u}) \end{cases} \quad (47)$$

3. We also remind the basic relation for the curvature operator for any three vectors :

$$\nabla_{\mathbf{u}}\nabla_{\mathbf{v}}\mathbf{w} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}}\mathbf{w} - \nabla_{[\mathbf{u}, \mathbf{v}]}\mathbf{w} = \mathfrak{R}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad (48)$$

From the equation (48), we have:

$$\nabla_{\mathbf{u}}\nabla_{\xi}\mathbf{u} - \nabla_{\xi}\nabla_{\mathbf{u}}\mathbf{u} - \nabla_{[\mathbf{u}, \xi]}\mathbf{u} = \mathfrak{R}(\mathbf{u}, \xi, \mathbf{u}) \quad (49)$$

where the second and the third terms vanish because vector  $\mathbf{u}$  is tangent to a geodesic, and the Lie derivative of  $\xi$  along the geodesic curve is zero respectively. We deduce the intrinsic Jacobi equation so-called geodesic deviation equation, expressed by means of the Levi-Civita connection e.g. Levi-Civita [1927] (1927), Synge [1934],:

$$\frac{D^2\xi}{D\tau^2} = \overline{\mathfrak{R}}(\mathbf{u}, \xi, \mathbf{u}), \quad \frac{D^2\xi^\alpha}{D\tau^2} = \overline{\mathfrak{R}}_{\delta\gamma\beta}^\alpha u^\beta u^\delta \xi^\gamma \quad (50)$$

which is valid within a Riemannian spacetime without torsion.  $\overline{\mathfrak{R}}(\mathbf{u}, \cdot, \mathbf{u})$  is called Jacobi operator along the curve  $\gamma(\tau)$  with the tangent  $\mathbf{u}$ . The geodesic deviation equation (50) shows that the curvature produces acceleration of the separation between two neighboring geodesics  $\gamma_0$  and  $\gamma_1$ . This provides a geometrical interpretation of the curvature tensor. The geodesic deviation equation constitutes a fundamental equation for relativistic gravitation since it relates the relativistic acceleration of two nearby particles in presence of gravitation field. In a flat spacetime, the separation will be linear. Equation (50) allows us to analyze numerous motions

of particles in gravitational field, such as the chaotic behavior of particles orbits but they are not well-suited to study spinning particles, either for microscopic with intrinsic spin or macroscopic bodies with intrinsic spin e.g. Leclerc [2005].

*Remark 13* For Newton mechanics, a geodesic is a curve along which a particle moves as free falling particle. The concept of geodesic deviation is based on the comparison of two geodesic curves in the spacetime  $(t, x^a)$  for Newton spacetime. Say two particles within a potential  $\Phi(x^\mu)$ , their motion equations hold respectively:

$$\ddot{x}^a(t) = -\partial^a\Phi(P), \quad \ddot{z}^a(t) = -\partial^a\Phi(Q) = -\partial^a\Phi(P) + \ddot{\xi}(t) \quad (51)$$

Expanding the potential about the point  $P$  gives:  $-\Phi(Q) = -\Phi(P) - \partial_b\Phi(P)\xi^b - \mathcal{O}(\xi)$ . This implies the expression of the acceleration of the geodesic deviation:

$$\frac{d^2\xi^a}{dt^2} = -\partial^a\partial_b\Phi\xi^b \quad (52)$$

which represents the distance acceleration between two particles falling in a nonuniform gravitation field  $\Phi(t, x^a)$ . Extension of the special non-relativistic spacetime to include gravitation is first due to Cartan [1986] and later in e.g. Havas [1964], and for continuum mechanics in e.g. Duval & Kunzle [1978]. In weak field condition (for earth gravitation this means that  $GM/(c^2R) \ll 1$ ) ( $G$  is the constant of gravitation,  $M$  earth mass,  $R$  earth radius, and  $c$  light speed) and the low speed motion i.e.  $v/c \ll 1$  the difference between Newton gravitation and general relativistic gravitation may be neglected e.g. Shen & Moritz [1996]. For relating the Newtonian description with the relativistic approach, let us remind the time variable as  $d\tau = cdt$  where  $c$  is the light speed, and to rewrite the geodesic equation of Newtonian limit case as:

$$\frac{d^2\xi^a}{d\tau^2} = -\mathfrak{R}_{b00}^a \xi^b \quad (53)$$

where we introduce  $\mathfrak{R}_{b00}^a = c^2 \partial^a\partial_b\Phi$ . Such equation was used to analyse the oblateness of the earth e.g. Greenberg [1974].

## 5.2 Autoparallel deviation in Riemann-Cartan spacetime

The idea is now to detect the relativistic acceleration of two nearby particles when the spacetime is curved with torsion. First of all, it is worth to remind that in a Riemann-Cartan manifold the deviation from an autoparallel curve is obtained from the definition of the deviation (component and intrinsic forms):

$$\frac{D^2\xi}{D\tau^2} := \nabla_{\mathbf{u}}(\nabla_{\mathbf{u}}\xi) \quad (54)$$

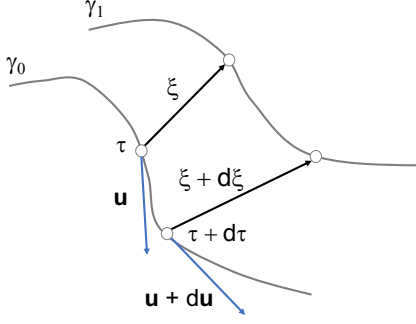


Fig. 2: Autoparallel curves. The gap vector  $\xi$  separating autoparallel timelike curves respectively  $\gamma_0$  and  $\gamma_1$ , defined by  $\nabla_{\mathbf{u}}\mathbf{u} = 0$ .

where the connection have torsion and curvature. In the following, this intrinsic definition can be also used to obtain with a straightforward calculus the result (50) on a Riemann manifold. Let now extend to Riemann-Cartan spacetime.

**Theorem 1** *Let  $(\mathcal{M}, \mathbf{g}, \Gamma)$  a Riemann-Cartan spacetime with  $\mathbf{u} := d\mathbf{x}/d\tau$ , four-vector velocity (timelike vector), and  $\xi$  the separation between two autoparallel curves  $\gamma_0$ , and  $\gamma_1$ . We assume the Lie derivative of  $\xi$  along  $\mathbf{u}$  vanishes (as for classical assumption in general relativity)  $\mathcal{L}_{\mathbf{u}}\xi \equiv 0$ . Then the acceleration of the separation between two auto-parallel curves  $\gamma_0$ , and  $\gamma_1$  takes the form of:*

$$\frac{D^2\xi}{D\tau^2} = \mathfrak{R}(\mathbf{u}, \xi, \mathbf{u}) + \mathfrak{K}\left(\frac{D\xi}{D\tau}, \mathbf{u}\right) + \nabla_{\mathbf{u}}\mathfrak{K}(\xi, \mathbf{u}) \quad (55)$$

which extends the equation (50) from Riemann to Riemann-Cartan spacetime.

*Proof:* The relation (48) induces  $\nabla_{\mathbf{u}}\nabla_{\xi}\mathbf{u} = \mathfrak{R}(\mathbf{u}, \xi, \mathbf{u})$ . The nullity of the Lie derivative also holds:  $\nabla_{\mathbf{u}}\xi = \nabla_{\xi}\mathbf{u} + \mathfrak{K}(\xi, \mathbf{u})$ . The separation acceleration writes in such a case:

$$\begin{aligned} \frac{D^2\xi}{D\tau^2} &= \nabla_{\mathbf{u}}(\nabla_{\mathbf{u}}\xi) = \nabla_{\mathbf{u}}(\nabla_{\xi}\mathbf{u} + \mathfrak{K}(\xi, \mathbf{u})) \\ &= \mathfrak{R}(\mathbf{u}, \xi, \mathbf{u}) + \nabla_{\mathbf{u}}[\mathfrak{K}(\xi, \mathbf{u})] \end{aligned}$$

For the sake of clarity, we calculate separately the last term :

$$\nabla_{\mathbf{u}}[\mathfrak{K}(\xi, \mathbf{u})] = (\nabla_{\mathbf{u}}\mathfrak{K})(\xi, \mathbf{u}) + \mathfrak{K}(\nabla_{\mathbf{u}}\xi, \mathbf{u}) + \mathfrak{K}(\xi, \nabla_{\mathbf{u}}\mathbf{u})$$

where the third last term vanishes since  $\mathbf{u}$  is tangent to the geodesic. Owing that  $\nabla_{\mathbf{u}}\xi := D\xi/D\tau$ , we get the result  $\square$

*Remark 14* Each term in the equation (55) has its physical interpretation when identified with the results of Balakin et al. [2000] (page 5011, equation (8)), where the torsion  $\mathfrak{K}$  which is proportional to the Faraday tensor  $\mathcal{F}$  appears as well as its covariant derivative  $\nabla\mathfrak{K}$  related to  $\nabla\mathcal{F}$ . Assuming a Riemann-Cartan manifold allows us to entirely include

Theory	Potential	Geodesic deviation
Newton	$\Phi$	$\frac{d^2\xi^a}{d\tau^2} = \mathfrak{R}_{00b}^a \xi^b$
Einstein	$\mathbf{g}, \bar{\nabla}$	$\frac{D^2\xi}{D\tau^2} = \bar{\mathfrak{R}}(\mathbf{u}, \xi, \mathbf{u})$
Einstein-Cartan	$\mathbf{g}, \nabla$	$\frac{D^2\xi}{D\tau^2} = \mathfrak{R}(\mathbf{u}, \xi, \mathbf{u}) + \mathfrak{K}\left(\frac{D\xi}{D\tau}, \mathbf{u}\right) + (\nabla_{\mathbf{u}}\mathfrak{K})(\xi, \mathbf{u})$

Table 2: Expression of the geodesic and autoparallel deviation equation for Newton (N), Einstein (E), and Einstein-Cartan (EC) theories. So, the acceleration of separation is completely determined by the curvature and the torsion of the spacetime, associated to gravitation and electromagnetic fields respectively.

in the geometric backgrounds of the spacetime two kinds of action-at-distance physics fields : gravitation and electromagnetism. In view of the system of equations (43), it is obvious that the torsion of the spacetime is due to the spin of angular momentum, and the solving of such equation remains a great challenge. Notice that gravitational waves are described by linearized gravitation field equation deduced from curvature (18) whereas electromagnetic waves may also described by further linearized version of (43) involving torsion. For the sake of the completeness, elastic waves are associated to linearized equations of Euler continuum mechanics equations. Indeed, by considering the relation between torsion tensor and electromagnetism (41), it is hopefully guessed that the last two terms of the (55) represent forces due to electromagnetism. Symmetries may reduce the number of components. For example, within Einstein-Cartan spacetime with a spherical symmetry where the metric is of Schwarzschild type, the simplest case where the spins of individual particles, or fluid elements composing the continuum are all aligned in the radial direction, only the component  $\mathfrak{K}_{23}^0 = -\mathfrak{K}_{32}^0 := \mathfrak{K}_0$  is not equal to zero e.g. Prasanna [1975b].

### 5.3 Summary

Geodesic deviation is present for any gravitational theory. We can sketch the analogy between Newton, Einstein, and Einstein-Cartan gravitation in the table below. In the following table we resume the different expressions of the geodesic deviation where  $\tau$  is the proper time of relativistic theory. These three formulae express how the spacetime curvature and torsion influence two nearby geodesic or autoparallel curves, making them converge to or diverge from each other. The right-hand side terms may be considered as tidal forces. The analogies between the tidal forces resulting from the



previous three theories is obvious<sup>4</sup>. For Newtonian gravitation, the tidal forces do not explicitly depend on the velocity  $\mathbf{u}$  conversely to Einstein and to Einstein-Cartan gravitation.

*Remark 15* In the deviation equations (53), (50), and (55), the vector  $\xi$  may be physically interpreted as the vector separation of two moving objects (ideally two mass points) near each other, and vector  $\mathbf{u}$  represents their initial motions. The second term is linear with respect to the separation vector constitutes the influence of the spacetime geometry on this separation acceleration. For Einstein-Cartan spacetime, we again observe and stress that a non curved spacetime with torsion may induce a separation acceleration between the two moving objects due to gravitation and electromagnetism, but described entirely and solely with spacetime geometry.

## 6 Discussion

The unified theory in physics gave rise to long debate in science history. The need of unified theory joining gravitation and electromagnetism prior to any other unification attempts remains up-to-date in order to design the geometrical structure of the spacetime. The Einstein-Cartan theory is probably the most natural extension of the relative gravitation theory in order to include the non symmetry of the affine connection, considered as independent variable with respect to metric field. Cartan introduced additionally the torsion tensor as arguments of the theory in 1922 and it was proposed before the discovery of the electron spin. On the one hand, numerous works devoted to the extension of relative gravitation did not treat the electromagnetic part and might face a problem of lack of gauge invariance. On the other hand, there are many studies on the coupling of gravitation with electromagnetism in a Riemann spacetime. With this approach, it is necessary to have the presence of matter (by the way the electron spin) in order to point out the torsion tensor. In the present paper, we choose to avoid introduction of matter and then provide a spacetime geometry as initial background without reference to matter either continuous or discrete. The guideline is to electromagnetism interacting with gravity such as source of change of Riemann spacetime of gravitation theory to Riemann-Cartan spacetime. We have considered in the present study the Hilbert-Einstein action (which is the simplest case among numerous gravitation theories) combined to Yang-Mills action (again the simplest case of electromagnetism theories) to relate electromagnetism and gravitation within the extended spacetime with torsion. In the present paper, we mainly consider the intimate link between the electrodynamics and the geometry of the spacetime where the electromagnetic waves are

propagating. Considering a very simple shape of the Lagrangian (the same form for all the models), we extend the geometry structure from the curved spacetime and then to curved spacetime with torsion. In order to analyze the interaction of electromagnetism and gravitation, the development of the Maxwell equations within curved spacetime shows the electromagnetism and gravitation mutual influence by means of the geometry characterized by metric, Levi-Civita connection, and associated Ricci curvature. When dealing with spacetime analogous to continuum with continuous distribution of singularity e.g. Rakotomanana [1997], where abrupt gradients of physical properties may occur, the extension of the Maxwell equations, namely the resulting wave propagation, is necessary to account for the non zero torsion  $\mathfrak{K} \neq 0$  and non zero curvature  $\mathfrak{R} \neq 0$ . Among numerous approaches, the use of Riemann-Cartan manifold as underlying geometrical structure seems worth. First, "geometrization" of gravity developed by Einstein by considers the dependence of the Lagrangian on the curvature tensor as the starting point for deriving the field equations (Einstein equations). Second, introducing the tensor Faraday including electric and magnetic fields - within the Lagrangian, it is recognized that the light, a particular case of electromagnetic waves, bends if viewed from a uniformly accelerating frame and then accordingly that the gravity would therefore bend the light. The interaction of gravitation and electromagnetic waves are described Einstein-Maxwell equations. The "geometrization" of the electromagnetic fields constitutes the third step when these fields are present in the spacetime. For that purpose, we have considered an extended spacetime where curvature and torsion are present, a Riemann-Cartan spacetime. By observing the fields equation, we find that the gravitational and electromagnetic fields are respectively identified as geometric objects of such a spacetime, namely the curvature  $\mathfrak{R}^\gamma_{\alpha\beta\lambda}$  and the torsion  $\mathfrak{K}^\gamma_{\alpha\beta}$ . Further studies are required for the invariance aspects. The equation (36) we obtained, is analogous to the particular contortion tensor found in a paper by Fernando et al. [2012]. Indeed, they have deduced that a particular connection defined by  $\Gamma^\gamma_{\alpha\beta} := \bar{\Gamma}^\gamma_{\alpha\beta} + \mathcal{T}^\gamma_{\alpha\beta}$  in Riemann-Cartan spacetime allowed them to derive Maxwell equations. It is shown that the torsion is nothing more than the spin angular momentum of an electromagnetic wave. The first goal of this work is reached in the sense that the electromagnetic variables are defined entirely from the spacetime geometry. More precisely the geometry of the spacetime is deduced from the electromagnetic variables. The work should not be considered as a "geometrization" of gravitation and electromagnetism, but should be considered as a unification of the geometric structure of the spacetime underlying continuum physics. Gravitation and electromagnetic fields are merged into a set of geometric variables of spacetime, as curvature and torsion. As a second goal, wave propagation equations were derived by accounting for the

<sup>4</sup>The geodesic deviation equation is also called the Jacobi equation in the framework of differential geometry.

geometric structure of the spacetime, they conform to previous theories in the literature and has been extended for curved spacetime with torsion. For instance, following another path Poplawski suggested to define the four-potential as a part of the trace of the torsion itself e.g. Poplawski [2010]. All these aspects will certainly constitute future research topics. We observe that both the torsion and the curvature influence the electromagnetic wave propagation in a Riemann-Cartan spacetime. Despite the crucial point on the Lorenz gauge invariance, this model seems to extend and thus include all previous models.

Last but not least, we investigate the influence of torsion and curvature on the deviation equation. It is found that the derivation of the autoparallel deviation equation within a Riemann-Cartan spacetime allows us to obtain a tidal force including terms representing gravitation and electromagnetic forces which are similar to results obtained by other approach. The main interest of our results would be that these results are obtained without assuming the presence of electromagnetic field, but solely by accounting the geometric structure, metric, torsion and curvature, of the spacetime. Minkowski spacetime is the emptiest state, Riemann is in the middle whereas Riemann-Cartan constitutes the fullest state of the spacetime where gravitation and electromagnetic phenomena are present.

## 7 Conclusion

Both gravitation and electromagnetism are shown to be manifestation of the curvature and torsion of the Riemann-Cartan spacetime. This geometric approach might have interest in the continuum approach for physics of the vacuum spacetime.

## References

- Acedo L. Autoparallel vs. Geodesic trajectories in a model of torsion gravity, *Universe* 1, 2015, pp 422-445.
- Allen L, Beijbergen MW, Spreeuw RJC, Woerdman JP. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes *Physical Review A* 45(11), 1992, pp 8185-8189.
- de Andrade VC, Pereira JG. Torsion and the electromagnetic field, *International Journal of Modern Physics D* 8(2), 1999, pp 141-151.
- Antonio Tamarasselvame N, Rakotomanana L. On the form-invariance of Lagrangean function for higher gradient continuum, in *Mechanics of Generalized Continua*, Ed. by H Altenbach, G Maugin, V Erofeev, Springer-Verlag, 2011, pp 291-322.
- Balakin A, van Holten JW, Kerner R. Motions and world-line deviations in Einstein-Maxwell theory, *Classical and Quantum Gravitation* 17, 2000, pp 5009-5023.
- Barnett SM. Optical angular-momentum flux, *Journal of Optics B : Quantum and Semiclassical Optics* 4, 2002, pp 7-16.
- Brading KA, Ryckman TA. Hilbert's "Foundations of Physics" : Gravitation and electromagnetism within the axiomatic method, *Studies in History and Philosophy of Modern Physics* 39, 2008, pp 102-153.
- Capozziello S, Cardone VF, Piedipalumbo E, Sereno M, Troisi A. Matching torsion Lambda-term with observations, *International Journal of Modern Physics D* 12, 2003, pp 381-394.
- Cartan E. *On manifolds with an affine connection and the theory of general relativity*, English translation of the French original by A. Magnon and A. Ashtekar, Bibliopolis, Napoli, 1986.
- Charap JM, Duff MJ. Gravitational effects on Yang-Mills topology, *Physics Letters* 69B(4), 1977, pp 445-447.
- Cho YM. Einstein Lagrangian as the translational Yang-Mills Lagrangian, *Physical Review D* 14(10), 1976, pp 2521-2525.
- Chrusciel PT. On the unified affine electromagnetism and gravitation theories, *Acta Physica Polonica* B15, 1984, pp 35-51.
- Darabi F, Mousavi M, Atazadeh K. Geodesic deviation equation in  $f(T)$  gravity, *Physical Review D* 91, 2015, pp 084023/ 1-11.
- Dias L, Moraes F. Effects of torsion on electromagnetic fields, *Brazilian Journal of Physics* 35 (3A), 2005, pp 636-640.
- Dixon WG. On the uniqueness of the Newtonian theory as a geometric theory of gravitation, *Communications in Mathematical Physics*, 45, 1975, pp 167-182.
- Duval Ch, Kunzle HP. Dynamics of continua and particles from general covariance of Newtonian gravitation theory, *Reports on Mathematical Physics*, 13(3), 1978, pp 351-368.
- Dyson FW, Eddington AS, Davidson C. A Determination of the Deflection of the Light by the Sun's Gravitational Field from Observations made at the Total Eclipse of May 29, 1919, *Philosophical Transactions A*, vol 20, 579, 1920, pp 291-330.
- Eddington AS. A Generalisation of Weyl's Theory of the Electromagnetic and Gravitational Fields, *Proceedings of the Royal Society A*, 99, 697, 1921, pp 104-122.
- Fernandez-Nunez I, Bulashenko O. Anisotropic metamaterial as an analogue of a black hole, *Physics Letters A*, 380, 2016, pp1-8.
- Fernando J, Giglio T and Rodrigues Jr WA. Gravitation and electromagnetism as Geometrical objects of Riemann-Cartan Spacetime structure, *Advanced Applied Clifford*

- Algebras*, 22, 2012, pp 640-664.
- Fumeron S, Pereira E, Moraes F. Generation of optical vorticity from topological defects, *Physica B*, 476, 2015, pp 19-23.
- Futhazar G, Le Marrec L, Rakotomanana LR. Covariant gradient continua applied to wave propagation within deformed material, *Archive for Applied Mechanics*, 2014.
- Goenner HFM. On the History of Unified Field Theories, Part. I *Living Review in Relativity*, 7, 2004, <http://www.livingreviews.org/lrr-2004-2>
- Gonseth F. *Les fondements des mathématiques : De la géométrie d'Euclide à la relativité générale et à l'intuitionisme* Ed. Albert Blanchard, Paris, 1926.
- Greenberg PJ. The equation of geodesic deviation in Newtonian theory and the oblateness of the earth, *Il Nuovo Cimento*, 24(2), 1974, pp 272-286.
- Hammond RT. Gravitation, Torsion, and Electromagnetism, *General Relativity and Gravitation* 20(8), 1987, pp 813-827.
- Hammond RT. Einstein-Maxwell theory from torsion, *Classical Quantum and Gravitation* 6, 1989, pp195-198.
- Hammond RT. Electromagnetic spin creates torsion, *International Journal of Modern Physics D* vol 17, 14, 2018, 1847005.
- Havas P. Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity *Reviews of Modern Physics*, October 1964, pp 938-965.
- Hehl FW, von der Heyde P. Spin and the structure of spacetime, *Annales de l'Institut Henri Poincaré*, section A, 19 (2), 1973, pp 179-196.
- Hehl FW, von der Heyde P, Kerlick GD, Nester JM. General relativity with spin and torsion : Foundations and prospects, *Reviews of Modern Physics* 48(3), 1976, pp 393-416.
- Hehl FW. Maxwell's equations in Minkowski's world: their premetric generalization and the electromagnetic energy-momentum tensor, *Annalen der Physik* 17(0-10), 2008, pp 691-704.
- Hilbert D. Die Grundlagen der Physik, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen – Mathematisch-Physikalische Klasse*, 3, 1915, pp 395–407
- Kleinert H. *Multivalued Fields: in Condensed matter, Electromagnetism, and Gravitation*, World Scientific, Singapore, 2008.
- Kovetz A. *Electromagnetic theory* Oxford Science Publications, New-York, 2000.
- Leclerc M. Mathisson-Papapetrou equations in metric and gauge theories of gravity in a Lagrangian formulation, *Classical and Quantum Gravitation* 22, 2005, pp 3203-3221.
- Levi-Civita T. Sur l'écart géodésique, *Mathematische Annalen* 97, 1927, pp 291-320.
- Lovelock D. The uniqueness of the Einstein field equations in a four-dimensional space, *Archive of Rational Analysis and Mechanics* 33, 1969, pp 54-70.
- Mathisson M. New mechanics of material system, *General Relativity and Gravitation*, 42, 2010, pp 1011-1048./ Translated by A Ehlers from the Original Paper : Neue mechanik materieller Systeme, *Acta Physica Polonica* 6, 1937, pp 163-200
- McMullin E. The Origins of the Field Concept in Physics, *Physics Perspectives*, 4, 2002, pp 13-39.
- Milonni PW, Boyd RW. Momentum of light on a dielectric medium, *Advances in Optics and Photonics* 2, 2010, pp 519-553.
- Mosna RA, Saa R. Volume elements and torsion, *Journal of Mathematical Physics* 46, 2005, pp 112502-1/10. <https://doi.org/10.1063/1.2121207>
- Nakahara M. Geometry, Topology and Physics, *Graduate Student Series in Physics*, edited by D.F. Brewer, Institute of Physics Publishing, Bristol, 1996.
- Nieto JA, Saucedo J, Villanueva VM, Geodesic deviation equation for relativistic tops and the detection of gravitational waves, *Revista Mexicana de Física S* 53(2), 2007, pp 141-145.
- Obukhov YN. Electromagnetic energy and momentum in moving media, *Annalen der Physik. Berlin* 17(9-10), 2008, pp 830-851.
- Obukhov YN, Hehl FW. Electromagnetic energy-momentum and forces in matter, *Physics Letters A* 311, 2003, pp 277-284.
- Papapetrou A. Spinning test-particles in general relativity I, *Proceedings of the Royal Society of London A* 209, 1951, pp248-258.
- Philipp D, Perlick V, Lammerzahl C, Deshpande K. On geodesic deviation in Schwarzschild spacetime, *Metrology for Aerospace IEEE* 2015, pp 198-203.
- Plebanski J. Electromagnetic waves in gravitational fields, *Physical Review* 118/5, 1960, pp 1396-1408.
- Poplawski NJ. Torsion as electromagnetism and spin, *International Journal of Theoretical Physics* 49(7), 2010, pp1481-1488.
- Prasanna AR. Static fluid spheres in Einstein-Cartan theory, *Physical Review D* 11(8), 1975, pp 2076-2082.
- Prasanna AR. Maxwell's equations in Riemann-Cartan space  $U_4$ , *Physics Letters A* 54(1), 1975, pp 17-18.
- Puntigam RA, Lämmerzahl C, Hehl FW. Maxwell's theory on a post-Riemannian spacetime and the equivalence principle, *Classical and Quantum Gravitation* 14, 1997, pp 1347-1356.
- Rakotomanana RL. Contribution à la modélisation géométrique et thermodynamique d'une classe de milieux faiblement continus, *Archives for Rational Mechanics and Analysis* 141, 1997, pp 199-236.

- Rakotomanana RL. *A geometric approach to thermomechanics of dissipating continua*, in *Progress in Mathematical Physics Series*, Birkhäuser, Boston, 2003.
- Rakotomanana RL. Second Gradient Continuum : Role of Electromagnetism Interacting with the Gravitation on the Presence of Torsion and Curvature, In: Altenbach H., Pouget J., Rousseau M., Collet B., Michelitsch T. (eds) *Generalized Models and Non-classical Approaches in Complex Materials 1. Advanced Structured Materials*, vol 89. 2018, Springer, Cham.
- Rakotomanana RL. *Covariance and Gauge Invariance in Continuum Physics : Application to Mechanics, Gravitation, and Electromagnetism*, in *Progress in Mathematical Physics*, Birkhäuser, Basel, 2019.
- Reichenbach H. Zur Einordnung des neuen Einsteinschen Ansatzes über Gravitation und Elektrizität, *Zeitschrift für Physik* 53(9), 1929, pp 683-689. <https://doi.org/10.1007/BF01330826>
- Romero C, Lima RG, Sanomiya TAT. One hundred years of Weyl's (unfinished) unified field theory, *Studies in history and Philosophy of Modern Physics* 66, 2019, pp 180-185.
- Ryder L. *Introduction to general relativity* Cambridge University Press, New-York, 2009.
- Sauer T. Einstein Equations and Hilbert Action : What is missing on page 8 of the proofs of Hilbert's First communication on the Foundations of Physics, *Archives of History of Exact Sciences* 59, 20015, pp 577-590.
- Shen W, Moritz H. On the separation of gravitation and inertia and the determination of the relativistic gravity field in the case of free motion, *Journal of Geodesy* 70, 1996, pp 633-644.
- Schutzhold R, Plunien G, Soff G. Dielectric black hole analogs, *Physical Review Letters* 88(6), 2002, pp 061101/1- 061101/4.
- Smalley LL. On the extension of geometric optics from Riemannian to Riemann-Cartan spacetime, *Physics Letters A* 117(6), 1986, pp 267-269.
- Smalley LL, Krisch JP. Minimal coupling of electromagnetic fields in Riemann-Cartan space-times for perfect fluids with spin density, *Journal of Mathematical Physics* 33(3), 1992, pp 1073-1081.
- Sotiriou TP, Liberati S. Metric-affine  $f(R)$  theories of gravity, *Annals of Physics* 322, 2007, pp 935-966.
- Synge JL. On the deviation of geodesics and null-geodesics, particularly in relation to the properties of spaces of constant curvatures and indefinite line-element, *Annals of Mathematics* 35(4), 1934, pp 705-713.
- Tiwari RN, Ray S. Static spherical charged dust electromagnetic mass models in Einstein-Cartan theory, *General Relativity and Gravitation*, 29(6) 1997, pp 683-690.
- Tonnellat MA. *La théorie du champ unifié d'Einstein et quelques-uns de ses développements*, Gauthier-Villars, Paris, 1955.
- Utiyama R. Invariant theoretical interpretation of interaction. *Physical Review* 101, 1956, pp 1597-1607.
- Vandyck MA. Maxwell's equations in spaces with non-metricity and torsion. *J Physics A : Math Gen* 29, 1996, pp 2245-2255.
- Verçyn A. Metric-torsion gauge theory of continuum line defects, *International Journal of Theoretical Physics* 29, 1, 1990, pp 7-21.
- Weyl H. Gravitation and the electron, *Proceedings of the National Academy of Sciences* 15, 1929, pp 323-334.
- Yang CN, Mills RL. Conservation of isotopic spin and isotopic gauge invariance *Physical Review* 96, 1, 1954, p191-195.
- Zhang Y, Shi L, Zhang RY, Duan J, Ng J, Chan CT, Fung KH. Metric-Torsion Duality of Optically Chiral Structures, *Physical Review Letters* 122, 2019, 200201-1-6.