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On the ill-posedness of the triple deck model

Helge Dietert * † David Gérard-Varet * ‡

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Abstract

We analyze the stability properties of the so-called triple deck model, a classical refinement of the Prandtl equation to describe boundary layer separation. Combining the methodology introduced in [3], based on complex analysis tools, and stability estimates inspired from [4], we exhibit unstable linearizations of the triple deck equation. The growth rates of the corresponding unstable eigenmodes scale linearly with the tangential frequency. This shows that the recent result of Iyer and Vicol [14] of local well-posedness for analytic data is essentially optimal.

Keywords: Unsteady Boundary Layer, Triple-Deck model, analytic instabilities, spectral analysis
MSC Codes: 76D10 35P15

1 Introduction

Our concern in this paper is the triple deck model, introduced in the 1960's to describe the so-called boundary layer separation. The general concern behind this model is to understand the behaviour of Navier-Stokes solutions with velocity $\mathbf{u}_\nu = (u_\nu, v_\nu)$ and pressure p_ν near a rigid boundary, when the inverse Reynolds number ν goes to zero. Due to the no-slip condition at the boundary, it is well-known that this is a singular asymptotic problem: the Euler solution $\mathbf{u}_0 = (u_0, v_0)$ does not describe the dynamics near the wall. In a celebrated paper [22], Ludwig Prandtl tackled this problem through the use of matched asymptotic expansions. For planar flows in the half plane $\Omega = \mathbb{R} \times \mathbb{R}_+$, this means that two regions should be distinguished: one away from the wall, where

$$\mathbf{u}_\nu(T, X, Y) \approx \mathbf{u}_0(t, X, Y), \quad p_\nu(T, X, Y) \approx p_0(t, X, Y) \tag{1}$$

while close to the wall, in a boundary layer, one should have

$$\begin{aligned} u_\nu(T, X, Y) &\approx u_P\left(T, X, \frac{Y}{\sqrt{\nu}}\right), \quad v_\nu(T, X, Y) \approx \sqrt{\nu}\, v_P\left(T, X, \frac{Y}{\sqrt{\nu}}\right), \\ p_\nu(T, x, Y) &\approx p_P\left(T, X, \frac{Y}{\sqrt{\nu}}\right) \end{aligned} \tag{2}$$

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for boundary layer profiles $(u_P, v_P, p_P) = (u_P, v_P, p_P)(T, X, y)$. Moreover, by injecting the Prandtl boundary layer expansion in the Navier-Stokes equation and keeping the leading order terms, we end up with the Prandl system

$$\begin{aligned} \partial_T u_P + u_P \partial_X u_P + v_P \partial_y u_P - \partial_y^2 u_P + \partial_X p_P &= 0, \\ \partial_y p_P &= 0, \\ \partial_X u_P + \partial_y v_P &= 0, \\ u_P|_{y=0} = v_P|_{y=0} &= 0. \end{aligned} \tag{P}$$

This system is completed by the conditions at infinity

$$\lim_{y\to\infty} u_P(T, X, y) = u_0(T, X, 0), \qquad \lim_{y\to\infty} p_P(T, X, y) = p_0(T, X, 0)$$

which ensure the matching between the boundary layer and the upper inviscid region of the flow.

The Prandtl model has revealed very fruitful to understand steady Navier-Stokes flows in regions where boundary layers remain attached to the boundary. However, it is well-known that downstream of the flow, under an adverse pressure gradient, streamlines detach from the boundary and recirculation occur. Moreover, in the unsteady context, even upstream, Tollmien-Schlichting instabilities may destabilize the flow. All these hydrodynamic phenomena have consequences on the mathematical analysis of system (P), for which various negative results have been obtained: ill-posedness results [8, 19], blow-up results [6, 7, 17, 2], instability of Prandtl expansions at the level of the Navier-Stokes equations [10, 11, 12]. A common difficulty behind these works is the appearance of small tangential scales, that invalidate expansions of type (2), which are assumed to depend regularly on x. In order to capture the effect of these small scales, while still trying to obtain reduced models, several refinements of the Prandtl model were introduced in the 1960's and 1970's. The most famous ones are the triple deck model and the Interactive Boundary Layer model (IBL). The latter one, analyzed mathematically in the recent paper [3], consists in keeping additional $O(\sqrt{\nu})$ terms, resulting in a coupling between the inviscid equations for the upper region, and the (modified) Prandtl equation.

Here we focus on the triple deck model. We first extend the derivation given in [18, 25] to the unsteady setting. The basic idea is to study perturbations to the main Prandtl flow, in the vicinity of $T = T^*, X = X^*$ (typically the time and abscissa of separation), with small scale variations in T, X. Denoting ϵ the amplitude of the perturbation, and η, δ the small time and tangential scales, we write

$$\begin{aligned} u_\nu(T, X, Y) &\approx u_P\left(T, X, \frac{Y}{\sqrt{\nu}}\right) + \epsilon \tilde{u}\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}}\right) \\ &\approx u_P\left(T^*, X^*, \frac{Y}{\sqrt{\nu}}\right) + \epsilon \tilde{u}\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}}\right) + \mathcal{O}(\eta) + \mathcal{O}(\delta) \\ v_\nu(T, X, Y) &\approx \sqrt{\nu}\, v_P\left(T, X, \frac{Y}{\sqrt{\nu}}\right) + \sqrt{\nu}\,\frac{\epsilon}{\delta}\tilde{v}\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}}\right) \\ &\approx \sqrt{\nu}\,\frac{\epsilon}{\delta}\tilde{v}\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}}\right) + \mathcal{O}(\sqrt{\nu}) \\ p_\nu(T, X, Y) &\approx p_P(T, X) + \epsilon^2 \tilde{p}\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}}\right) \end{aligned}$$

(we anticipate that the amplitude of the pressure is ϵ^2, see below). Injecting the ansatz into the Navier-Stokes equations, we derive the relations satisfied by $(\tilde{u}, \tilde{v}, \tilde{p}) = (\tilde{u}, \tilde{v}, \tilde{p})(t, x, y)$. With

notation $U(y) := u_P(T^*, X^*, y)$, anticipating that $\eta \gg \delta$ and $\nu \ll \epsilon\delta^2$, we get

$$\partial_x \tilde{u} + \partial_y \tilde{v} = 0, \quad U\partial_x \tilde{u} + U'\tilde{v} = 0, \quad \partial_y \tilde{p} = 0.$$

The second identity reads $U^2 \partial_y \left(\frac{\tilde{v}}{U}\right) = 0$. Thanks to this relation and to the divergence-free condition, we can introduce a function $A = A(t,x)$ such that

$$\tilde{u}(t,x,y) = A(t,x)U'(y), \quad \tilde{v}(t,x,y) = -\partial_x A(t,x)U(y).$$

In particular, we see that $\tilde{u}(t,x,0) = A(t,x)U'(0)$ is non-zero. To restore the no-slip condition, one must add a sublayer. This sublayer is referred to as the *lower deck*, while the main one, corresponding originally to the Prandtl layer, is the *main deck*. Eventually, the upper region outside the $\mathcal{O}(\sqrt{\nu})$ boundary layer is called the *upper deck*. Let h be the typical length scale of the lower deck, and $z = y/h$. The velocity at the bottom of the main deck reads $U(y) + \epsilon\tilde{u}(t,x,y) \approx hU'(0)z + \epsilon\tilde{u}(t,x,0)$. For matching between the lower and main deck, it is therefore natural to take $h = \epsilon$, and to look for an asymptotics in the lower deck of the form:

$$\begin{aligned}
u_\nu(T,X,Y) &\approx \epsilon u\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}\epsilon}\right),\\
v_\nu(T,X,Y) &\approx \sqrt{\nu}\frac{\epsilon^2}{\delta} v\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}\epsilon}\right),\\
p_\nu(T,X,Y) &\approx \epsilon^2 p\left(\frac{T-T^*}{\eta}, \frac{X-X^*}{\delta}, \frac{Y}{\sqrt{\nu}\epsilon}\right)
\end{aligned}$$

with $(u,v) = (u,v)(t,x,z)$. Moreover, in order to match the effects of time variation, advection, and diffusion in the lower deck, $\partial_t \sim u\partial_X \sim \nu\partial_Y^2$, one has to take $\delta \sim \epsilon^3$, $\eta \sim \epsilon^2$. The amplitude $O(\epsilon^2)$ of the pressure term allows to retain it as well. This results in

$$\begin{aligned}
\partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u + \partial_x p &= 0,\\
\partial_z p &= 0,\\
\partial_x u + \partial_z v &= 0,\\
u|_{z=0} = v|_{z=0} &= 0.
\end{aligned}$$

These equations are the same as those in (P). But, the boundary conditions at infinity differ from the classical ones. Assume $U(\infty) = 1$, $U'(0) = 1$ for simplicity. On one hand, matching of the velocities of the lower and main desks yields

$$u(t,x,z) \sim U'(0)z + \tilde{u}(t,x,0) = z + A(t,x), \quad z \to +\infty.$$

On the other hand, as explained in [18] and apparent in the original Prandtl layer (P), the $\mathcal{O}(\epsilon^2)$ pressure should not change across the lower and main decks, and coincide with the trace of the pressure in the upper deck. In this upper deck, the dynamics is driven by the so-called blowing velocity, that is the normal component coming from the main deck: $\sqrt{\nu}\frac{\epsilon}{\delta}\,\tilde{v}(t,x,\infty) \sim \frac{\sqrt{\nu}}{\epsilon^2}$. Anticipating that the upper deck must have the same amplitude, we find $\epsilon^2 \sim \frac{\sqrt{\nu}}{\epsilon^2}$, that is $\epsilon = \nu^{1/8}$. Finally, in the upper deck, one looks for an asymptotics isotropic in X, Y of the form

$$\begin{aligned}
&u_\nu \approx 1 + \nu^{1/4}\overline{u}\left(\frac{T-T^*}{\nu^{1/4}}, \frac{X-X^*}{\nu^{3/8}}, \frac{Y}{\nu^{3/8}}\right), \quad v_\nu \approx \nu^{1/4}\overline{v}\left(\frac{T-T^*}{\nu^{1/4}}, \frac{X-X^*}{\nu^{3/8}}, \frac{Y}{\nu^{3/8}}\right),\\
&p_\nu \approx \nu^{1/4}\overline{p}\left(\frac{T-T^*}{\epsilon^2}, \frac{X-X^*}{\nu^{3/8}}, \frac{Y}{\nu^{3/8}}\right),
\end{aligned}$$

Plugging the asymptotic ansatz into the Navier-Stokes equations, yields the linearized Euler dynamics for $(\overline{u}, \overline{v}, \overline{p}) = (\overline{u}, \overline{v}, \overline{p})(t, x, \theta)$:

$$\partial_x \overline{u} + \partial_x \overline{p} = 0, \quad \partial_x \overline{v} + \partial_\theta \overline{p} = 0, \quad \partial_x \overline{u} + \partial_\theta \overline{v} = 0, \quad \overline{v}|_{\theta=0} = \tilde{v}(t, x, \infty) = -\partial_x A(t, x).$$

This system can be solved using Fourier transform in x as

$$\mathcal{F}\overline{p}|_{\theta=0}(\xi) = -i\mathrm{sign}(\xi)\mathcal{F}\partial_x A(t, \cdot)(\xi) = |\xi|\mathcal{F}A(t, \cdot)(\xi).$$

In physical variables this is

$$\overline{p}|_{\theta=0}(t, x) = |\partial_x|A(t, x) := \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{\partial_x A(t, \bar{x})}{x - \bar{x}}\mathrm{d}\bar{x},$$

where the right-hand side is the Hilbert transform of $\partial_x A$ in variable x and PV denotes the principal value.

Thanks to this last condition, the triple deck model can be written

$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_z u - \partial_z^2 u + \partial_x|\partial_x|A &= 0, \\ \partial_x u + \partial_z v &= 0, \\ u|_{z=0} = v|_{z=0} &= 0. \\ \lim_{z\to\infty} u - z &= A. \end{aligned} \tag{TD}$$

The unknowns are $(u, v) = (u, v)(t, x, z)$ and $A = A(t, x)$. One must complete the system with an initial data $u|_{t=0} = u_0(x, z)$, consistent with the structure at infinity given by the last line of (TD). Note that if we let $z \to +\infty$ in the first equation, using $u = z + A + o(1)$, we obtain the redundant consistency equation:

$$\partial_t A + A\partial_x A + \partial_x|\partial_x|A = \partial_x \int_0^{+\infty} (u - A)\,\mathrm{d}z. \tag{3}$$

The different spatial scalings are shown in Fig. 1. Wrapping the derivation up, the overall idea is to consider a perturbation around a boundary layer (u_P, v_P) with small tangential scale δ around $X = X^*$ and assuming that away from the lower deck (which is where u_P is expected to loose monotonicity) the inviscid terms dominate.

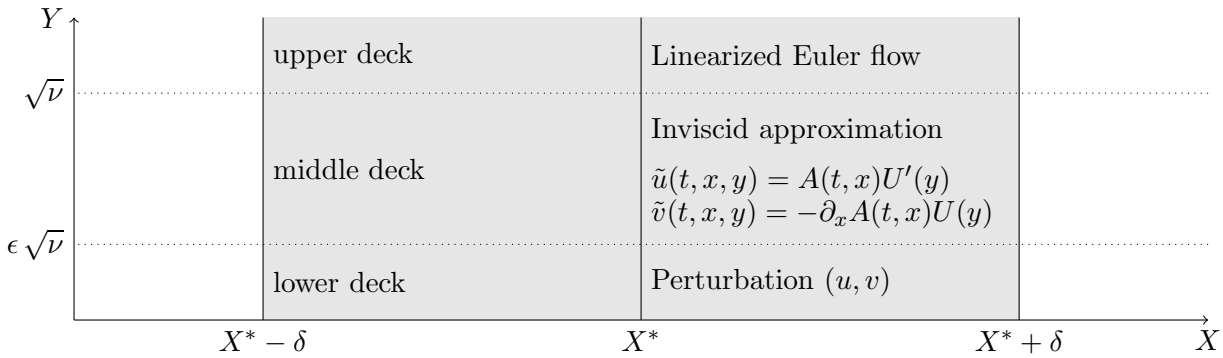

Figure 1: Spatial scales of the triple deck model where $\delta = \nu^{3/8}$ and $\epsilon = \nu^{1/8}$.

Although formulated in the 1960's, and extensively studied from a numerical viewpoint since then, the triple deck system (TD) has not been much investigated mathematically. In the steady case, one can mention the work [21] of L. Plantié, focused on a modification of the model: the

displacement velocity $A = A(x)$ is given while the pressure $p = p(x)$ is kept as an unknown. Well-posedness is established under an assumption of non-decreasing displacements, using Von Mises transform. In the unsteady case, the only work we are aware of is the recent paper [14] by S. Iyer and V. Vicol, which shows local in time well-posedness of (TD), for data u_0 that are analytic in x, Sobolev in y, with further Gaussian decay in y. Let us stress that although analytic well-posedness is well-known for the classical Prandtl equation, see [23, 16] extension of such result to (TD) is uneasy. Indeed, the evolution equation for u in (TD) contains the additional difficult term $\partial_x|\partial_x|A \approx \partial_x|\partial_x|u|_{z=\infty}$. This term is not skew-symmetric in $L^2(\mathbb{R}\times\mathbb{R}_+)$, with potential severe loss of two derivatives in x. To show a positive result in analytic regularity, the authors have to combine two main ingredients. On one hand, they control A thanks to equation (3), in which skew-symmetry of the Benjamin-Ono operator helps. On the other hand, they control $u - A$ thanks to its rapid decay in y and the use of clever time-dependent cut-offs. We refer to [14] for all necessary details.

Can we relax the assumption of analytic regularity for well-posedness? We remind that (P) is well-posed for any data with Gevrey 2 regularity [4], and for Sobolev data that are monotonic in y. The triple deck model being supposedly a refinement of the Prandtl one, it is natural to look for the same kind of stability results. Encouragement can be found from the analysis of the simplest linearization of (TD), namely around $u(z) = z$. The linearized system reads

$$\begin{aligned} \partial_t u + z\partial_x u + v - \partial_z^2 u + \partial_x|\partial_x|A &= 0,\\ \partial_x u + \partial_z v &= 0,\\ u|_{z=0} = v|_{z=0} &= 0.\\ \lim_{z\to\infty} u &= A. \end{aligned} \tag{4}$$

Explicit calculations, sketched in Appendix A, can be performed, and show that any family of eigenfunctions of the form

$$u_k(t,x,z) = e^{\lambda_k t} e^{ikx}\hat{u}_k(z), \quad k\in\mathbb{R} \tag{5}$$

satisfies $\mathcal{R}e\,\lambda_k = O(1)$ as $k\to\pm\infty$, which is consistent with Sobolev well-posedness.

Nevertheless, as we will show in this paper, there are monotonic shear flows $V_s(z) = z + U_s(z)$ such that the linearization of (TD) around u_s is ill-posed below analytic regularity. More precisely, we will prove that these linearized equations admit solutions of the form (5), with

$$\sigma_m := \liminf_{k\to+\infty} \mathcal{R}e\,\lambda_k/k > 0.$$

One could exhibit similarly solutions with $\liminf_{k\to-\infty} \mathcal{R}e\,\lambda_k/|k| > 0$. This prevents any general well-posedness statement for data u_0 that are not analytic in x. In short, for any $T > 0$, one needs to impose a bound of the form $\|\hat{u}_0(k,\cdot)\| \le Ce^{-(\sigma_m T)|k|}$, with some appropriate norm $\|\cdot\|$ in variable z, to ensure a bound on the solution over $(0,T)$. By the Paley-Wiener theorem, it is well-known that such exponential decay in $|k|$ corresponds to analytic regularity in x. In particular, one should not hope that the analytic result of [14] can be improved in general: we refer to [15, 9, 13] for examples on how strong linearized instabilities imply misbehaviour of the flow of the nonlinear equations. The next section is dedicated to the statement of our main results.

2 Results and strategy of proof

We investigate in this paper linearizations of system (TD), around shear flows of the form

$$u = z + U_s(z), v = 0.$$

Due to the diffusion term in (TD), non-affine shear flows are not solutions of the homogeneous triple deck equation. One way to circumvent this issue is of course to start from the inhomogeneous equation with source $-U_s''$. Another way to proceed is to consider time-dependent shear flows, and argue that the time variation of the flow is negligible at the time scale of the high frequency instabilities that we shall discuss here. We refer to [8] for a rigorous reasoning in this second direction.

We assume for simplicity that

$$U_s \in C^\infty(\mathbb{R}_+),\ U_s(0) = 0, \quad \lim_{z\to\infty} U_s = A_s \in \mathbb{R}, \quad \frac{\mathrm{d}^k}{\mathrm{d}z^k}(U_s - A_s)(z) = O(e^{-z}) \ \text{ for } k \leq 3. \tag{6}$$

Less stringent assumptions, notably on the decay of the profile at infinity, could be extracted from the proof. The linearized system reads

$$\begin{aligned} \partial_t u + (U_s + z)\partial_x u - (1 + U_s')\partial_x \int_0^z u + \partial_x|\partial_x|A - \partial_z^2 u = 0, \\ u|_{z=0} = 0, \\ \lim_{z\to\infty} u = A. \end{aligned} \tag{LTD}$$

We have expressed $v = -\int_0^z \partial_x u$ thanks to the divergence-free condition and the non-penetration condition $v|_{z=0} = 0$. We are interested in the spectral analysis of (LTD) in the high frequency regime, i.e. looking for eigenmodes of the form

$$u_k(t,x,y) = e^{-ik\mu_k t}e^{ikx}\hat{u}_k(y), \quad A_k(t,x) = \frac{1}{k}e^{-ik\mu_k t}e^{ikx}, \quad k \gg 1. \tag{7}$$

Note that by linearity, we are allowed to fix $\hat{A}_k = \frac{1}{k}$. We wish to exhibit a class of monotonic shear flows U_s such that (LTD) has non-trivial solutions of the form (7) for k large, satisfying $\liminf_{k\to+\infty} \mathcal{I}m\,\mu_k > 0$. To do so, we will follow the path introduced in [3] to analyze the stability properties of the linearized Interactive Boundary Layer model (IBL)

$$\begin{aligned} \partial_t u + (U_s + z)\partial_x u - (1 + U_s')\partial_x \int_0^z u - \partial_z^2 u = \partial_t u_e + u_e\partial_x u_e, \\ u_e - \sqrt{\nu}|\partial_x| \int_0^{+\infty} (u - u_e)dz = 0, \\ u|_{z=0} = 0, \\ \lim_{z\to\infty} u = u_e. \end{aligned} \tag{8}$$

This path goes through the following steps:

1. We plug the formula (7) in system (LTD), and reformulate our search for instability as a one-dimensional eigenvalue problem in variable z, with unknown eigenvalue μ_k.

2. We take the formal limit $k \to +\infty$ of the eigenvalue problem, and we derive a necessary and sufficient condition on U_s for the existence of an unstable eigenvalue μ_∞ to this limit eigenvalue problem. First, we show that eigenvalues μ_∞ are the zeroes of an holomorphic function Φ_∞. Namely, $\Phi_\infty(\mu) = \phi_{\mu,\infty}(0)$ for an explicit function $\phi_{\mu,\infty} = \phi_{\mu,\infty}(z)$. Let us remark that this inviscid condition $\Phi_\infty(\mu) = 0$ had already been derived in [26], and the existence of a zero in the unstable half-plane had been shown to hold numerically for a specific example of a steady flow beyond a bump (see also [1]). Here, we go much further, by using tools from complex analysis, inspired by the work of O. Penrose on

Vlasov-Poisson equilibria [20]. We show that the existence of a zero μ_∞ in the unstable half plane $\{\mathcal{I}m\,\mu > 0\}$ amounts to a condition on the number of crossings of the positive real axis by some explicit curve related to Φ_∞. Examples (both numerical and analytical) of shear flows satisfying this condition are given.

3. Eventually, we show that an instability at $k = +\infty$ persists at large but finite k. To do so, we express again the rescaled eigenvalue μ_k as the zero of a function Φ_k. Again, $\Phi_k(\mu) = \phi_{\mu,k}(0)$ for some function $\phi_{\mu,k} = \phi_{\mu,k}(z)$, but this function is no longer explicit. Roughly, it satisfies the resolvent equation of a Prandtl like operator. A keypoint of our analysis is to establish a stability estimate for this resolvent equation. Thanks to this estimate, we are then able to show that Φ_k is holomorphic in $\{\mathcal{I}m\,\mu > \delta\}$ for any $\delta > 0$ and k large enough. We are also able to show that for k large and μ in a compact set, the solution $\phi_{\mu,k}$ is close to $\phi_{\mu,\infty}$. We deduce from this that Φ_∞ and Φ_k are close in a neighborhood of μ_∞ for k large enough, and conclude by Rouché's theorem.

As a result of the analysis sketched above, we state our main theorem:

Theorem 1 (**Ill-posedness below analytic regularity**).
Assume that U_s satisfies (6) and that $V_s(z) := z + U_s(z)$ satisfies $V_s' > 0$ on $\mathbb{R}_+$. Let $g(u) := \frac{V_s''(y)}{(V_s'(y))^3}|_{y=V_s^{-1}(u)}$. Assume that g is strictly monotone in the neighborhood of each of its positive zeroes, and define

$$n_\pm := card\Big\{a > 0,\ g(a) = 0,\ -\frac{1}{V_s'(0)} + a\,\mathrm{PV}\int_0^\infty \frac{g(u)}{a-u}du > 0,\ \pm g \ \textit{strictly increasing near } a\Big\}.$$

Then, $n_\pm$ is finite, and if $n_+ - n_- \neq 0$, there exist solutions of (LTD) of type (7) with $\liminf_{k\to\infty} \mathcal{I}m\,\mu_k > 0$.
Moreover, there indeed exist shear flows U_s such that $n_+ - n_- \neq 0$.

The rest of the paper will be devoted to the proof of this result. Section 3 is dedicated to the first two steps alluded to above: rewriting of the problem as a 1-d eigenvalue problem, and sharp analysis of the case $k = +\infty$. Section 4 is devoted to the third step: we show how to go from an instability at infinite k to an instability at finite k. Eventually, Section 5 collects examples, either analytical or numerical, for which our instability criterion applies. Let us stress that despite the similarities between (LTD) and (8), a simple adaptation of the analysis carried in [3] is not enough to handle the triple deck model. The boundary conditions at infinity, and notably the fact that u is unbounded far away, create specific difficulties. In particular, we are unable to apply the kind of resolvent estimates used in [3] with such conditions at infinity. Instead, we adapt the stability estimates that we used in [4] to obtain Gevrey 2 bounds for solutions of the classical Prandtl equation.

3 The infinite frequency spectral problem

3.1 Reduction

We start by injecting solutions of type (7) in (LTD). From now on, we shall work in Fourier variables only, so we can use without confusion notation u_k instead of $\hat{u}_k$. We find

$$\Big(-\mu_k + V_s\Big)u_k - V_s'\int_0^z u_k - \frac{1}{ik}\partial_z^2 u_k = -1, \tag{9}$$

$$u_k|_{z=0} = 0, \tag{10}$$

$$\lim_{z\to\infty} u_k = \frac{1}{k}, \tag{11}$$

where we remind that $V_s(z) = z + U_s(z)$. Like in [3], we further write u_k in terms of a stream function ϕ_k as $u_k = (k^{-1} - \phi_k')$ yielding

$$\begin{aligned} (\mu_k - V_s)\phi_k' + V_s'\phi_k + \frac{1}{ik}\phi_k''' &= -1 + \frac{1}{k}\big(\mu_k - U_s + zU_s'\big), \\ \phi_k|_{z=0} = 0, \quad \phi_k'|_{z=0} &= \frac{1}{k}, \\ \lim_{z\to\infty} \phi_k'(z) &= 0. \end{aligned} \tag{12}$$

Note that from the consistency equation, we have

$$\phi_k(\infty) = -1 + k^{-1}\big(\mu_k - A_s\big) \tag{13}$$

which is again redundant to system (12). Let us note that the momentum equation is a third order ODE in z, and for general μ should require at most three boundary conditions for solvability. The fact that (12)-(13) contains four boundary conditions is reminiscent of the fact that it is an eigenvalue problem, with unknowns (μ_k, ϕ_k).

As explained in the previous section, in order to progress in the analysis of solutions (μ_k, ϕ_k) of (12), we shall consider the formal limit of this system as $k \to +\infty$. This raises a problem of boundary conditions: indeed, at $k = +\infty$, the viscous term $\frac{1}{ik}\phi_k'''$ disappears, and the operator in z becomes first order. Therefore, we drop the condition on $\phi_k'(0)$, and consider the following infinite frequency spectral problem

$$\begin{aligned} (\mu_\infty - V_s)\phi_\infty' + V_s'\phi_\infty &= -1, \\ \phi_\infty(0) = 0, \quad \lim_{z\to\infty} \phi_\infty'(z) &= 0. \end{aligned} \tag{14}$$

The formal limit of the consistency condition (13) is

$$\phi_\infty(\infty) = -1. \tag{15}$$

It is again redundant to (14): as ϕ_∞' still goes to zero at infinity, taking the limit $z \to +\infty$ in the momentum equation yields (15). This time, (14)-(15) contains three boundary conditions, for a first order system that would require *a priori* only one for solvability with an arbitrary given μ. We tackle a detailed analysis of this reduced eigenvalue problem in the next paragraph.

3.2 Spectral analysis of the reduced eigenvalue problem

We wish here to determine sharp conditions under which system (14) has a non-trivial solution $(\mu_\infty, \phi_\infty)$ with $\mathcal{I}m\,\mu_\infty > 0$. Given $\mu \in \mathbb{C} \setminus \mathbb{R}_+$, we denote by $\phi = \phi_{\mu,\infty}$ the solution of

$$\begin{aligned} (\mu - V_s)\phi' + V_s'\phi &= -1, \\ \lim_{z\to\infty} \phi'(z) &= 0. \end{aligned} \tag{16}$$

As mentioned before, for general μ, as the first equation is first order in z, one can only retain *a priori* one boundary condition for solvability. It is crucial that we retain here the condition on ϕ' at infinity, instead of the condition on ϕ at zero. This is a main difference with the treatment of the IBL model in [3]. Indeed, contrary to what happens in the IBL case, the solution of the equation $(\mu - V_s)\phi' - V_s'\phi = 1$ with $\phi(0) = 0$ is in general unbounded at infinity, due to the unboundedness of V_s. Hence, it could not help to solve the eigenvalue problem (14). More

generally no perturbative argument could be based on such solution, that is associated to the Dirichlet condition. On the contrary, system (16) has an explicit solution

$$\phi_{\mu,\infty}(z) = (\mu - V_s(z)) \int_z^\infty \frac{1}{(\mu - V_s(y))^2}\,\mathrm{d}y.$$

From this expression, one deduces that the consistency condition (15) is satisfied, as expected. Defining

$$\Phi_\infty(\mu) := \phi_{\mu,\infty}(0) = \mu \int_0^\infty \frac{1}{(\mu - V_s(y))^2}\,\mathrm{d}y, \tag{17}$$

we see that $(\mu_\infty, \phi_\infty)$ will be a solution of (14) if and only if

$$\Phi_\infty(\mu_\infty) = 0, \quad \phi_\infty := \phi_{\mu_\infty,\infty}.$$

The rest of the paragraph is devoted to the proof of the following proposition.

Proposition 2. *Let V_s, g and $n_\pm$ as in Theorem 1. Then, Φ_∞ has at least one zero in the unstable half-plane $\{\mathcal{I}m\,\mu > 0\}$ if and only if the condition $n_+ - n_- \neq 0$ is satisfied.*

We first state and prove two lemmas describing the behaviour of Φ_∞ in various regions of $\mathbb{C}\backslash\mathbb{R}_+$:

Lemma 3. *For any $\delta > 0$ there exists $R > 0$ such that*

$$|\Phi_\infty(\mu) + 1| \leq \delta$$

for all $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ with $|\mu| \geq R$.

Proof. We write

$$\Phi_\infty(\mu) = \mu \int_0^\infty \frac{1}{V_s'} \underbrace{\frac{V_s'}{(\mu - V_s)^2}}_{=\left(\frac{1}{\mu - V_s}\right)'} \mathrm{d}y = -\frac{1}{V_s'(0)} - \mu \int_0^\infty \frac{V_s''}{V_s'^2} \frac{1}{V_s - \mu}\,\mathrm{d}y \tag{18}$$

By elementary calculation, $\frac{-1}{V_s'(0)} + \int_0^\infty \frac{V_s''}{V_s'^2}\,\mathrm{d}y = \frac{-1}{V_s'(0)} - \left[\frac{1}{V_s'}\right]_0^\infty = -1$. Hence we find

$$\Phi_\infty(\mu) + 1 = \int_0^\infty \frac{V_s''}{V_s'^2} \frac{V_s}{\mu - V_s}\,\mathrm{d}y = \int_0^\infty \frac{g(u)u}{\mu - u}\,\mathrm{d}u,$$

where g is the function in Theorem 1. Note that g decays fast at infinity because $V_s'' = U_s''$ does, and because $V_s' \to 1$ at infinity. We decompose:

$$\int_0^\infty \frac{g(u)u}{\mu - u} du = \int_0^{+\infty} 1_{\{|u - \mathcal{R}e\,\mu| \geq 1\}} \frac{g(u)u}{\mu - u} \mathrm{d}u + \int_0^{+\infty} 1_{\{|u - \mathcal{R}e\,\mu| \leq 1\}} \frac{g(u)u}{\mu - u} \mathrm{d}u.$$

By dominated convergence, the first term goes to zero when $|\mu| \to +\infty$ in the region $\mathbb{C} \setminus \mathbb{R}_+$, and the second one goes to zero when $|\mu| \to +\infty$ in the region $\{\mathcal{R}e\,\mu \leq -1\} \cup \{\mathcal{I}m\,\mu > 1\}$. Eventually, for $0 < |\mathcal{I}m\,\mu| \leq 1$, and $\mathcal{R}e\,\mu \geq -1$, we write the second term as follows ($\mu = a + ib$):

$$\int_0^{+\infty} 1_{\{|u - \mathcal{R}e\,\mu| \leq 1\}} \frac{g(u)u}{\mu - u} \mathrm{d}u = \int_{-1}^1 \frac{g(v+a)(v+a) - g(a)a}{ib - v} \mathrm{d}v \; + \; g(a)a \int_{-1}^1 \frac{1}{ib - v} \mathrm{d}v$$

The first term vanishes when $|\mu| \to +\infty$, that is $a \to +\infty$, invoking again dominated convergence, and the second one goes to zero as well taking into account that $g(a)a \to 0$ and that $\lim_{b \to 0^+} \int_{-1}^1 \frac{1}{ib - v} dv$ exists by Plemelj formula. This concludes the proof of the lemma. □

Lemma 4. *For all $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ with $\mathcal{R}e\,\mu \leq 0$,*

$$\mathcal{R}e\,\Phi_\infty(\mu) < 0.$$

Moreover,

$$\lim_{\substack{\mu \to 0 \\ \mu \in \mathbb{C}\setminus\mathbb{R}_+}} \mathcal{R}e\,\Phi_\infty(\mu) = -\frac{1}{V_s'(0)} < 0.$$

Proof. From the definition (17), denoting $\mu = a + ib$, we infer:

$$\mathcal{R}e\,\Phi_\infty(\mu) = \int_0^\infty \frac{a((a-V_s)^2 - b^2) + 2b^2(a - V_s)}{|\mu - V_s(y)|^4}\,\mathrm{d}y.$$

For the numerator we find

$$a((a-V_s)^2 - b^2) + 2b^2(a - V_s) = a(a-V_s)^2 + b^2(a - 2V_s) \leq 0$$

as $a \leq 0$. Unless $a = b = 0$ there exists also always z for which it is strictly negative, which concludes the proof of the first inequality. The other one is a simple consequence of the expression

$$\Phi_\infty(\mu) = -\frac{1}{V_s'(0)} + \mu \int_0^\infty \frac{g(u)}{\mu - u}\,\mathrm{d}y, \tag{19}$$

see (18) and the definition of g in Theorem 1. The integral $\int_0^\infty \frac{g(u)}{\mu-u}\,\mathrm{d}u$ only diverges logarithmicaly as $\mu \to 0$, hence the result. □

We have now all the ingredients to conclude our analysis of the zeroes of Φ_∞ in $\{\mathcal{I}m\,\mu > 0\}$. By Lemma 3, there exists $R > 0$ such that

$$|\Phi_\infty(\mu) + 1| \leq \frac{1}{4}, \quad \mathcal{I}m\,\mu > 0, \quad |\mu| \geq R.$$

Let

$$\Omega_\epsilon := \{\mu \in \mathbb{C},\ \mathcal{I}m\,\mu > \epsilon,\ |\mu| \leq R\}$$

With our choice of R, Φ_∞ has a zero in $\{\mathcal{I}m\,\mu > 0\}$ if and only if it has one in Ω_ϵ for some $\epsilon > 0$ small enough. As Φ_∞ is a holomorphic function, its zeroes in $\{\mathcal{I}m\,\mu > 0\}$ are isolated, so that we can restrict to ϵ along a sequence going to zero and such that Φ_∞ does not vanish at $\partial\Omega_\epsilon$. Then, the number n_ϵ of its zeroes in Ω_ϵ, counted with multiplicity, is given by

$$n_\epsilon = \frac{1}{2i\pi} \oint_{\partial\Omega_\epsilon} \frac{\Phi_\infty'(\zeta)}{\Phi_\infty(\zeta)}\,\mathrm{d}\zeta.$$

Let γ_ϵ be a direct parametrisation of the curve $\partial\Omega_\epsilon$. We have

$$\frac{1}{2i\pi} \oint_{\partial\Omega_\epsilon} \frac{\Phi_\infty'(\zeta)}{\Phi_\infty(\zeta)}\,\mathrm{d}\zeta = \frac{1}{2i\pi} \int \frac{\Phi_\infty'(\gamma_\epsilon(t))}{\Phi_\infty(\gamma_\epsilon(t))}\gamma_\epsilon'(t)\,\mathrm{d}t = \frac{1}{2i\pi} \int \frac{(\Phi_\infty \circ \gamma_\epsilon)'(t)}{(\Phi_\infty \circ \gamma_\epsilon)(t)}\,\mathrm{d}t = \frac{1}{2i\pi} \int_{\Phi_\infty(\partial\Omega_\epsilon)} \frac{1}{\xi}\,\mathrm{d}\xi$$

so that the number of roots equals the winding number of the curve $\Phi_\infty(\partial\Omega_\epsilon)$ around 0. To compute this winding number, one can choose a complex logarithm with a branch cut along the positive real axis. The winding number is given by the sum of the jumps of this logarithm, which corresponds to the number of crossings of the curve $\Phi_\infty(\partial\Omega_\epsilon)$ with the positive real axis. More precisely,

$$\frac{1}{2i\pi} \oint_{\partial\Omega_\epsilon} \frac{\Phi_\infty'(\zeta)}{\Phi_\infty(\zeta)}\,\mathrm{d}\zeta = \text{ number of crossings from below} - \text{number of crossings from above}. \tag{20}$$

We remind that the intersection of $\Phi_\infty(\partial\Omega_\epsilon \cap \{|\mu| = R\})$ with the positive real axis is empty. Hence it remains to understand the crossings of $\Phi_\infty([-R, R] + i\epsilon)$, for $\epsilon > 0$ going to zero. By Lemma 4, there exists $\rho > 0$, such that $\mathcal{R}e\,\Phi_\infty < 0$ for $-R < a < \rho$, so that we can restrict to $a \in [\rho, R]$. Over this interval, formula (19) yields that

$$\begin{aligned}\Phi_\infty(a+ib) &= -\frac{1}{V_s'(0)} + a\int_0^{+\infty} \frac{g(u)}{a+ib-u}\,\mathrm{d}u + ib\int_0^{+\infty} \frac{g(u)}{a+ib-u}\,\mathrm{d}u \\ &\xrightarrow[b\to 0^+]{} -\frac{1}{V_s'(0)} + a\,\mathrm{PV}\int_0^{+\infty} \frac{g(u)}{a-u}\,\mathrm{d}u \;-\; i\pi a g(a)\end{aligned} \tag{21}$$

using Plemelj formula, where the convergence is uniform in $a \in [\rho, R]$. We extend the definition of Φ_∞ over $\mathbb{R}_+^*$ by

$$\Phi_\infty(a) := -\frac{1}{V_s'(0)} + a\,\mathrm{PV}\int_0^{+\infty} \frac{g(u)}{a-u}\,\mathrm{d}u \;-\; i\pi a g(a), \quad a > 0.$$

We notice that the quantity n_+, resp. n_-, defined in Theorem 1, corresponds to the number of crossings from above, resp. from below, of the curve $\Phi_\infty(\mathbb{R}_+^*)$ with the positive real axis. Lemma 3 being uniform in b, we still have $\Phi_\infty(a) \to -1$ as $a \to +\infty$. Also, as in Lemma 4, $\lim_{a\to 0^+} \mathcal{R}e\,\Phi_\infty(a) = -\frac{1}{V_s'(0)} < 0$. It follows that $n_\pm$ is finite and coincides with the crossings of $\Phi_\infty([\rho, R])$ up to taking ρ smaller and R larger. Finally, from the uniform convergence in (21), we deduce that for $\epsilon > 0$ small enough, (20) is equal to $n_+ - n_-$. This concludes the proof of Proposition 2.

4 Persistence of the instability at finite k

This section is devoted to the proof of Theorem 1, except for the last statement, which will be considered in the next section. We assume that $n_+ - n_- \neq 0$, see the statement of the theorem. Our goal is to show that (LTD) has solutions of type (7) with $\liminf_{k\to+\infty} \mathcal{I}m\,\mu_k > 0$. In other words, we need to prove that for all k large enough, there exists μ_k, ϕ_k solving (12) with $\mathcal{I}m\,\mu_k \geq \delta > 0$ for some δ independent of k.

4.1 Resolvent Estimate

The first step is to consider the following resolvent problem

$$\begin{cases}(\mu - V_s)\psi' + (V_s')\psi + \dfrac{1}{ik}\psi''' = F, \\ \psi'|_{z=0} = 0, \lim\limits_{z\to\infty} \psi'(z) = 0, \lim\limits_{z\to\infty} \psi = 0.\end{cases} \tag{22}$$

Note that we do not prescribe the value of ψ at zero. Let $\rho(z) = 1 + z^m$, m large. We denote by $\langle\,,\rangle_\rho$ the $L^2(\rho)$ scalar product, and set $\|\cdot\| = \langle\cdot,\cdot\rangle_\rho^{1/2}$.

Proposition 5. *Let $\mu_m > 0$. There exists k_m, depending on μ_m such that for any μ with $\mathcal{I}m\,\mu \geq \mu_m$, for any $k \geq k_m$ and any $F \in L^2(\rho)$, the system (22) has a unique solution $\psi = \psi_{\mu,k}$ of the form*

$$\psi = \Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B, \quad B'|_{z=0} = 0, \quad \lim_{z\to\infty} B = 0.$$

where $B \in H^3(\mathbb{R}_+)$ satisfies the estimates

$$\mathcal{I}m\,\mu\,\|B'\|^2 + \frac{1}{k}\|B''\|^2 \leq \frac{4}{(\mathcal{I}m\,\mu)^3}\|F\|^2 \tag{23}$$

and

$$\mathcal{I}m\,\mu\,\|B''\|_{L^2}^2 + \frac{1}{k}\|B'''\|_{L^2}^2 \leq C\Big(\frac{\sqrt{k}}{(\mathcal{I}m\,\mu)^{9/2}} + \frac{1}{(\mathcal{I}m\,\mu)^2}\Big)\|F\|^2 \tag{24}$$

for a constant C depending on μ_m. Moreover, the map $\mu \to \psi_{\mu,k}(0)$ is analytic in $\{\mathcal{I}m\,\mu > \mu_m\}$.

Proof of the well-posedness statement in Proposition 5. We only detail the *a priori* estimates leading to the well-posedness. For the detailed construction of solutions in a similar context, see [3]. Inspired by our work [4], we introduce the solution B of the system

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B = \psi, \quad B'|_{z=0} = 0, \quad \lim_{z\to\infty} B = 0. \tag{25}$$

Through differentiation in z, we get

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B' - V_s'B = \psi'.$$

Inserting the two previous identities for ψ and ψ' in Equation (22), we find

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)^2 B' - \Big[\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big), V_s'\Big]B = F$$

that is

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)^2 B' - \Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, U_s'\Big]B = F \tag{26}$$

We introduce the solution φ of

$$\Big(\overline{\mu} - V_s - \frac{1}{ik}\Big(\frac{d}{dz} + \rho'\rho^{-1}\Big)^2\Big)\varphi = B', \quad \varphi|_{z=0} = 0, \quad \lim_{z\to\infty}\varphi = 0. \tag{27}$$

Taking the scalar product of (26) with φ, we find

$$\Big\langle\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B', B'\Big\rangle_\rho - \frac{1}{ik}V_s'(0)B(0)\overline{\varphi'(0)}\rho(0) - \Big\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, U_s'\Big]B, \varphi\Big\rangle_\rho = \langle F, \varphi\rangle_\rho.$$

The boundary term at $z = 0$ comes from the integration by parts of the diffusion term, taking into account that

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B'(0) = \psi'(0) + V_s'(0)B(0) = V_s'(0)B(0).$$

We perform one more integration by parts and take the imaginary part to find

$$\begin{aligned}\mathcal{I}m\,\mu\|B'\|^2 + \frac{1}{k}\|B''\|^2 &= \mathcal{I}m\,\frac{1}{ik}\langle B'', \rho'\rho^{-1}B'\rangle_\rho - \mathcal{I}m\,\frac{1}{ik}V_s'(0)B(0)\overline{\varphi'(0)}\rho(0)\\ &\quad + \mathcal{I}m\,\Big\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, U_s'\Big]B, \varphi\Big\rangle_\rho + \mathcal{I}m\,\langle F, \varphi\rangle_\rho.\end{aligned} \tag{28}$$

It remains to estimate the four terms at the right-hand side. Clearly,

$$\mathcal{I}m\,\frac{1}{ik}\langle B'', \rho'\rho^{-1}B'\rangle_\rho \leq \frac{1}{k}\|\rho'\rho^{-1}\|_\infty\|B''\|\,\|B'\| \leq \frac{1}{2k}\|B''\|^2 + \frac{1}{2k}\|\rho'\rho^{-1}\|_\infty^2\|B'\|^2. \tag{29}$$

To bound the last terms, we first need to relate norms of φ to norms of B. We claim:

Lemma 6. *For k large enough (depending on μ_m and ρ), the solution φ of (27) satisfies*

$$\|\varphi\| \le \frac{\sqrt{2}}{\mathcal{I}m\,\mu}\|B'\|, \quad \|\varphi\|_{L^2} \le \frac{\sqrt{2}}{\mathcal{I}m\,\mu}\|B'\|_{L^2},$$
$$\|\varphi'\| \le \frac{\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'\|, \quad \|\varphi'\|_{L^2} \le \frac{\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'\|_{L^2}, \quad \|\varphi''\|_{L^2} \le Ck\|B'\|_{L^2},$$

where C depends also on μ_m.

We take the $L^2(\rho)$ scalar product of (27) with φ, and retain the imaginary part:

$$\begin{aligned}\mathcal{I}m\,\mu\|\varphi\|^2 + \frac{1}{k}\|\varphi'\|^2 &= \mathcal{I}m\,\frac{1}{ik}\big\langle \rho'\rho^{-1}\varphi, \varphi'\big\rangle_\rho - \mathcal{I}m\,\big\langle B', \varphi\big\rangle_\rho \\ &\le \frac{1}{2k}\|\rho'\rho^{-1}\|_\infty^2\|\varphi\|^2 + \frac{1}{2k}\|\varphi'\|^2 + \frac{1}{2\,\mathcal{I}m\,\mu}\|B'\|^2 + \frac{\mathcal{I}m\,\mu}{2}\|\varphi\|^2.\end{aligned}$$

For $\frac{1}{2k}\|\rho'\rho^{-1}\|_\infty^2 \le \frac{\mu_m}{4}$, we get in particular

$$\frac{\mathcal{I}m\,\mu}{4}\|\varphi\|^2 \le \frac{1}{2\,\mathcal{I}m\,\mu}\|B'\|^2$$

which implies the first estimate. We also find

$$\frac{1}{2k}\|\varphi'\|^2 \le \frac{1}{2\,\mathcal{I}m\,\mu}\|B'\|^2$$

which implies the second estimate. Similar (and even simpler) calculations yield

$$\|\varphi\|_{L^2} \le \frac{\sqrt{2}}{\mathcal{I}m\,\mu}\|B'\|_{L^2}, \quad \|\varphi'\|_{L^2} \le \frac{\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'\|_{L^2}.$$

To obtain the last inequality of the lemma, we multiply by φ'' and integrate. Denoting $\langle\,,\rangle_{L^2}$ the classical L^2 scalar product, we get

$$\begin{aligned}\frac{1}{k}\|\varphi''\|_{L^2}^2 + \mathcal{I}m\,\mu\|\varphi'\|_{L^2}^2 &= \mathcal{I}m\,\frac{1}{ik}\langle\big(2\big(\rho'\rho^{-1}\big)'\varphi' + \big(\rho'\rho^{-1}\big)''\varphi, \varphi''\rangle_{L^2} + \mathcal{I}m\,\langle V_s'\varphi, \varphi'\rangle_{L^2} + \mathcal{I}m\,\langle B', \varphi''\rangle_{L^2} \\ &\le \frac{1}{4k}\|\varphi''\|_{L^2}^2 + \frac{1}{k}\|2\big(\rho'\rho^{-1}\big)'\|_\infty^2\|\varphi'\|_{L^2}^2 + \frac{1}{4k}\|\varphi''\|_{L^2}^2 + \frac{1}{k}\|\big(\rho'\rho^{-1}\big)''\|_\infty^2\|\varphi\|_{L^2}^2 \\ &\quad + \frac{\mathcal{I}m\,\mu}{4}\|\varphi'\|_{L^2}^2 + \frac{1}{\mathcal{I}m\,\mu}\|V_s'\|_\infty^2\|\varphi\|_{L^2}^2 \\ &\quad + \frac{1}{4k}\|\varphi''\|_{L^2}^2 + k\|B'\|_{L^2}^2.\end{aligned}$$

Using the previous estimate for $\|\varphi\|_{L^2}$, we deduce that for k large enough

$$\frac{1}{4k}\|\varphi''\|_{L^2}^2 + \frac{\mathcal{I}m\,\mu}{4}\|\varphi'\|_{L^2}^2 \le \big(C + k\big)\|B'\|_{L^2}^2$$

for a constant C depending on μ_m. The last estimate of the lemma follows.

We now go back to the identity (28), where we have to bound the last three terms at the right-hand side. The easiest is

$$\mathcal{I}m\,\big\langle F, \varphi\big\rangle_\rho \le \frac{1}{(\mathcal{I}m\,\mu)^3}\|F\|^2 + \frac{(\mathcal{I}m\,\mu)^3}{4}\|\varphi\|^2 \le \frac{1}{(\mathcal{I}m\,\mu)^3}\|F\|^2 + \frac{\mathcal{I}m\,\mu}{2}\|B'\|^2. \tag{30}$$

The commutator term splits into

$$\begin{aligned}\mathcal{I}m\,\Big\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2},U_s'\Big]B,\varphi\Big\rangle_\rho &= \mathcal{I}m\,\Big\langle\frac{2}{ik}U_s''B',\varphi\Big\rangle_\rho + \mathcal{I}m\,\Big\langle\frac{1}{ik}U_s'''B,\varphi\Big\rangle_\rho\\ &\le \frac{2}{k}\|U_s''\|_\infty\|B'\|\,\|\varphi\| + \frac{C_H}{k}\|U_s'''(1+z)\|_\infty\|B'\|\,\|\varphi\|,\end{aligned}$$

where we used the Hardy inequality $\|\frac{B}{1+y}\|\le C_H\|B'\|$. Using the first estimate in the lemma, we end up with

$$\mathcal{I}m\,\Big\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2},U_s'\Big]B,\varphi\Big\rangle_\rho \le \frac{C}{(\mathcal{I}m\,\mu)k}\|B'\|^2. \tag{31}$$

Eventually, for the trace term, we have through Sobolev imbedding:

$$\begin{aligned}\mathcal{I}m\,\frac{1}{ik}(U_s'(0)+1)B(0)\overline{\varphi'(0)}\rho(0) &\le \frac{C}{k}\|B\|_\infty\|\varphi'\|_{L^2}^{1/2}\|\varphi''\|_{L^2}^{1/2}\\ &\le \frac{C}{k}\|\rho^{-1/2}\|_{L^2}\|B'\|\,\|\varphi'\|_{L^2}^{1/2}\,\|\varphi''\|_{L^2}^{1/2}\\ &\le \frac{C'}{k^{1/4}(\mathcal{I}m\,\mu)^{1/4}}\|B'\|\,\|B'\|_{L^2}\end{aligned} \tag{32}$$

for constants C, C' depending on μ_m.

Collecting (29)-(30)-(31), we find that for k large enough, depending on μ_m,

$$\mathcal{I}m\,\mu\|B'\|^2 + \frac{1}{k}\|B''\|^2 \le \frac{4}{(\mathcal{I}m\,\mu)^3}\|F\|^2$$

that is (23).

To establish estimate (24), we introduce the solution φ_1 of

$$\Big(\overline{\mu} - V_s - \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)\varphi_1 = B''', \quad \varphi_1|_{y=0}=0, \quad \lim_{y\to\infty}\varphi_1 = 0. \tag{33}$$

Taking the usual L^2 scalar product of (26) with φ_1, we obtain

$$\langle\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B', B'''\rangle_{L^2} + \frac{1}{ik}V_s'(0)B(0)\overline{\varphi_1'(0)} - \langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2},U_s'\Big]B,\varphi_1\rangle_{L^2} = \langle F,\varphi_1\rangle_{L^2}.$$

Hence,

$$\begin{aligned}\mathcal{I}m\,\mu\|B''\|_{L^2}^2 + \frac{1}{k}\|B'''\|_{L^2}^2 &= \mathcal{I}m\,\langle V_s'B', B''\rangle_{L^2}\\ &\quad + \mathcal{I}m\,\frac{1}{ik}V_s'(0)B(0)\overline{\varphi_1'(0)}\rho(0)\\ &\quad + \mathcal{I}m\,\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2},U_s'\Big]B,\varphi_1\rangle_{L^2} - \mathcal{I}m\,\langle F,\varphi_1\rangle_{L^2}.\end{aligned}$$

The last three terms can be treated like before, replacing φ by φ_1. First, the estimates in Lemma 6 are replaced by

$$\|\varphi_1\|_{L^2} \le \frac{\sqrt{2}}{\mathcal{I}m\,\mu}\|B'''\|_{L^2}, \quad \|\varphi_1'\|_{L^2} \le \frac{\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'''\|_{L^2}, \quad \|\varphi_1''\|_{L^2} \le Ck\|B'''\|_{L^2},$$

with the same proof. Then, estimate (31) becomes

$$\mathcal{I}m\,\langle\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2},U_s'\Big]B,\varphi_1\rangle_{L^2}\leq\frac{C}{(\mathcal{I}m\,\mu)k}\|B'\|\,\|B'''\|_{L^2}\leq\frac{1}{4k}\|B'''\|_{L^2}^2+\frac{C^2}{(\mathcal{I}m\,\mu)^2k}\|B'\|^2.$$

Proceeding as in (32), we find

$$\mathcal{I}m\,\frac{1}{ik}V_s'(0)B(0)\overline{\varphi_1'(0)}\leq\frac{C}{k^{1/4}(\mathcal{I}m\,\mu)^{1/4}}\|B'\|\,\|B'''\|_{L^2}\leq\frac{1}{4k}\|B'''\|_{L^2}^2+\frac{C^2\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'\|^2.$$

Also,

$$-\mathcal{I}m\,\langle F,\varphi_1\rangle_{L^2}\leq\|F\|_{L^2}\,\|\varphi_1\|_{L^2}\leq\|F\|_{L^2}\frac{\sqrt{2}}{\mathcal{I}m\,\mu}\|B'''\|_{L^2}\leq\frac{1}{4k}\|B'''\|_{L^2}^2+\frac{2}{(\mathcal{I}m\,\mu)^2}\|F\|_{L^2}^2.$$

The remaining term is controlled by

$$\mathcal{I}m\,\langle V_s'B',B''\rangle\leq C\|B'\|_{L^2}\,\|B''\|_{L^2}\leq\frac{C^2}{2(\mathcal{I}m\,\mu)}\|B'\|^2\,+\frac{\mathcal{I}m\,\mu}{2}\|B''\|_{L^2}.$$

Collecting these bounds, we end up with

$$\frac{\mathcal{I}m\,\mu}{2}\|B''\|_{L^2}^2+\frac{1}{4k}\|B'''\|_{L^2}^2\leq C\Big(\frac{\sqrt{k}}{\sqrt{\mathcal{I}m\,\mu}}\|B'\|^2+\frac{1}{\mathcal{I}m\,\mu}\|B'\|^2+\frac{2}{(\mathcal{I}m\,\mu)^2}\|F\|_{L^2}^2\Big).$$

From this bound and (23), we deduce (24)

$$\mathcal{I}m\,\mu\|B''\|_{L^2}^2+\frac{1}{k}\|B'''\|_{L^2}^2\leq C\Big(\frac{\sqrt{k}}{(\mathcal{I}m\,\mu)^{9/2}}+\frac{1}{(\mathcal{I}m\,\mu)^2}\Big)\|F\|^2$$

for some constant C depending on μ_m.

Proof of the analyticity statement in Proposition 5. The last thing to be shown is the analyticity of the map $\mu\to\psi_{\mu,k}(0)$. As before, we write $\psi=\psi_{\mu,k}$. We proceed as follows: let $\chi_n(z)=\chi\big(\frac{z}{n}\big)$, for some smooth non-negative χ which is one near the origin and zero in the large. We consider the approximate problem

$$\begin{cases}(\mu-(U_s+z\chi_n))\psi'+(U_s'+(z\chi_n)')\psi+\dfrac{1}{ik}\psi'''=F,\\ \psi'|_{z=0}=0,\ \lim\limits_{z\to\infty}\psi'(z)=0,\ \lim\limits_{z\to\infty}\psi=0.\end{cases}\tag{34}$$

Above calculations apply to to this approximate system as well, and yield a solution

$$\psi_n=\Big(\mu-(U_s+z\chi_n)+\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B_n,\quad B_n'|_{z=0}=0,\quad\lim_{z\to\infty}B_n=0$$

where B_n satisfies the same estimates (23)-(24) uniformly in n. In particular, $B_n'\in L^2(\rho)$. The difference with the original system is that this implies $\psi_n'\in L^2(\rho)$: it can be deduced from the formula

$$\psi_n'=\Big(\mu-(U_s+z\chi_n)+\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B_n'-(U_s'+(z\chi_n)')B_n$$

as the base flow $U_s+z\chi_n$ is not diverging at infinity. Using the bounds for B_n, one obtains an estimate of the type $\|\psi_n'\|\leq C_n\|F\|$, with a bound that may depend on n through χ_n, but is

uniform in μ inside $\{\mathcal{I}m\,\mu > \mu_m\}$. Using further the equation in (34), we find that $\psi_n' \in H^2(\rho)$, and a bound of the form $\|\psi_n'\|_{H^2(\rho)} \leq C_n\|F\|$. Introducing the operator

$$\mathcal{L}_n : H^2(\rho) \cap H_0^1(\rho) \to L^2(\rho), \quad u \mapsto -(U_s + z\chi_n)u - (U_s' + (z\chi_n)')\int_z^\infty u + \frac{1}{ik}u''$$

we then know that its resolvent $(\mu + \mathcal{L}_n)^{-1}$ is well-defined for $\mathcal{I}m\,\mu > \mu_m$, and as any resolvent operator is analytic in μ. Hence, $\psi_n' = (\mu + \mathcal{L}_n)^{-1}F$ is analytic in μ with values in $H^2(\rho)$, and by imbedding $\mu \to \psi_n(0)$ is analytic as well.

To conclude that $\mu \to \psi(0)$ is analytic, it remains to show that $\psi_n(0) \xrightarrow[n\to+\infty]{} \psi(0)$ uniformly on the compact sets of $\{\mathcal{I}m\,\mu > \mu_m\}$. Note that, by (25), we have

$$\begin{aligned}|\psi_n(0) - \psi(0)| &= |\mu(B_n(0) - B(0)) + \frac{1}{ik}\partial_z^2(B_n - B)(0)| \\ &\leq C\Big(\|B_n' - B'\|_{L^2(\tilde\rho)} + \|B_n'' - B''\|_{L^2(\tilde\rho)} + \|B_n''' - B'''\|_{L^2}\Big)\end{aligned}$$

for all μ in a compact set K, for any weight function $\tilde\rho \geq 1$ such that $\frac{1}{\tilde\rho} \in L^1(\mathbb{R}_+)$, where the constant C depends on K and k. The last step is to establish that the right-hand side goes to zero (uniformly in $\mu \in K$), which can be done through an estimate of the difference. Namely, combining (26) and its analogue

$$\Big(\mu - (U_s + z\chi_n) + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)^2 B_n' - \Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, U_s' + (z\chi_n)'\Big]B_n = F$$

we see that

$$\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)^2 (B_n - B)' - \Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, U_s'\Big](B_n - B) = R_n$$

where, denoting $\psi_n = z(\chi_n - 1)$:

$$\begin{aligned}R_n = &\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, \psi_n'\Big]B_n' \\ &- \Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)(\psi_n B_n) - \psi_n\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B_n \\ &- \psi_n^2 B_n.\end{aligned}$$

Applying (23) and (24) with weight $\tilde\rho = (1+z)^{-8}\rho$ instead of ρ, we find that

$$\|B_n' - B'\|_{L^2(\tilde\rho)} + \|B_n'' - B''\|_{L^2(\tilde\rho)} + \|B_n''' - B'''\|_{L^2} \leq C\|R_n\|_{L^2(\tilde\rho)}$$

for a constant C that again may depend on K or k. Eventually, using that B_n' is bounded uniformly in n in $H^1(\rho)$, one can check that $\|R_n\|_{L^2(\tilde\rho)}$ goes to zero as $n \to \infty$. For instance,

$$\begin{aligned}\Big\|\Big[\frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}, \psi_n'\Big]B_n'\Big\|_{L^2(\tilde\rho)} &\lesssim \|\psi_n'''(1+z)^{-2}\|_\infty\|B_n'\|_{L^2(\rho)} + \|\psi_n''(1+z)^{-2}\|_\infty\|B_n''\|_{L^2(\rho)} \\ &\lesssim \|\psi_n'''(1+z)^{-4}\|_\infty + \|\psi_n''(1+z)^{-4}\|_\infty \lesssim \frac{1}{n}.\end{aligned}$$

All other terms defining R_n can be treated with similar ideas.

4.2 Conclusion

We remind that our goal is to find solutions (μ_k, ϕ_k) of (12) with $\liminf_{k\to+\infty} \mu_k > 0$. From the analysis of Section 3, notably Proposition 2, we already know that the system (14) has a solution $(\mu_\infty, \phi_\infty)$ with $\mathcal{I}m\, \mu_\infty > 0$. We shall find μ_k, ϕ_k with μ_k close to μ_∞. We fix $\mu_m = \frac{1}{2}\mathcal{I}m\, \mu_\infty$.

We shall first prove that for any μ with $\mathcal{I}m\, \mu \geq \mu_m$ and $k \geq k_m$, with k_m given by Proposition 5, one can construct a solution $\phi = \phi_{\mu,k}$ of

$$\begin{aligned} (\mu - V_s)\phi' + V_s'\phi + \frac{1}{ik}\phi''' &= -1 + \frac{1}{k}(\mu - U_s + zU_s'), \\ \phi'|_{z=0} = \frac{1}{k}, \quad \lim_{z\to\infty} \phi'(z) &= 0. \end{aligned} \tag{35}$$

The first equation implies the compatibility condition $\phi(\infty) = -1 + \frac{1}{k}(\mu - A_s)$. We look for an approximation of ϕ under the form

$$\phi_{app} = \phi_{\mu,\infty} + \phi_{bl}$$

where $\phi_{\mu,\infty}$ solves (16) and where $\phi_{bl} = \phi_{bl}(z)$ is a boundary layer term that allows to recover the right Neumann boundary condition at $z = 0$ and the right Dirichlet condition at infinity. Namely, ϕ_{bl} is the solution of

$$\mu\phi_{bl}' + \frac{1}{ik}\phi_{bl}''' = 0, \quad \phi_{bl}'(0) = -\phi_{\mu,\infty}'(0) + \frac{1}{k}, \quad \phi_{bl}'(\infty) = 0, \quad \phi_{bl}(\infty) = \frac{1}{k}(\mu - A_s).$$

The solution is explicitly given by

$$\phi_{bl}(z) = \frac{1}{k}(\mu - A_s) + \int_\infty^z e^{-\sqrt{-ik\mu}\, y} \mathrm{d}y \Big(-\phi_{\mu,\infty}'(0) + \frac{1}{k}\Big). \tag{36}$$

Thanks to this choice, it is straightforward to check that the difference $\psi = \phi - \phi_{app}$ satisfies a system of type (22) with

$$F := V_s\phi_{bl}' - V_s'\phi_{bl} - \frac{1}{ik}\phi_{\mu,\infty}''' + \frac{1}{k}(\mu - U_s + zU_s'). \tag{37}$$

By Proposition 5, there exists a solution ψ to this system, so that

$$\phi_{\mu,k} = \phi_{app} + \psi = \phi_{\mu,\infty} + \phi_{bl} + \psi$$

defines a solution of (35). Defining

$$\Phi_k(\mu) := \phi_{k,\mu}(0).$$

We shall prove that for k large enough, there exists μ_k in the disk $D(\mu_\infty, \frac{1}{2}\mathcal{I}m\, \mu_\infty) := \{z \in \mathbb{C} : |\mu_\infty - z| < \frac{1}{2}\mathcal{I}m\, \mu_\infty\}$ such that $\Phi_k(\mu_k) = 0$. Hence, $(\mu_k, \phi_k := \phi_{\mu_k,k})$ will be the desired solution to (12). More precisely, we state

Proposition 7. *There exist constants $C, K > 0$ such that Φ_k is holomorphic in $D(\mu_\infty, \frac{1}{2}\mathcal{I}m\, \mu_\infty)$ and*

$$|\Phi_k(\mu) - \Phi_\infty(\mu)| = |\phi_{\mu,k}(0) - \phi_{\mu,\infty}(0)| \leq Ck^{-1/4}, \qquad \forall \mu \in D(\mu_\infty, \frac{1}{2}\mathcal{I}m\, \mu_\infty), k \geq K.$$

Before proving this proposition, let us show how it implies the existence of a zero μ_k of Φ_k. We already know that Φ_∞ is holomorphic, and that μ_∞ is one of its zeroes, therefore isolated. Hence, for $\delta > 0$ small enough, $\epsilon := \inf_{|\mu-\mu_\infty|=\delta} |\Phi_\infty(\mu)| > 0$. For all k large enough so that $|Ck^{-1/4}| \leq \frac{\epsilon}{2}$, we conclude by Proposition 7 and Rouché's theorem that Φ_k has a zero in $D(\mu_\infty, \delta)$.

Proof of Proposition 7. We have

$$\Phi_k(\mu) - \Phi_\infty(\mu) = \phi_{\mu,k}(0) - \phi_{\mu,\infty}(0) = \phi_{bl}(0) + \psi(0).$$

First, from (36),

$$\phi_{bl}(0) = \frac{1}{k}(\mu - A_s) - \frac{1}{\sqrt{-ik\mu}}\Big(- \phi_{\mu,\infty}(0) + \frac{1}{k}\Big) \tag{38}$$

so that for all $\mu \in D(\mu_\infty, \delta)$ it holds that $|\phi_{bl}(0)| \leq Ck^{-1/2}$. Then, with the notations of Proposition 5,

$$\begin{aligned} |\psi(0)| &= \Big|\Big(\mu - V_s + \frac{1}{ik}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big)B(0)\Big| = \Big|\mu B(0) + \frac{1}{ik}B''(0)\Big| \\ &\leq C\left(\|B'\| + \|B''\|_{L^2}^{1/2}\|B'''\|_{L^2}^{1/2}\right) \\ &\leq Ck^{1/2}\|F\| \end{aligned}$$

where the last inequality is a consequence of the estimates in Proposition 5, with F defined in (37). Note that

$$\begin{aligned} |F(z)| \leq \Big\|\frac{V_s}{z}\Big\|_{L^\infty}|z\phi_{bl}'(z)| + \|V_s'\|_{L^\infty}\big|\phi_{bl}(z) - \frac{1}{k}(\mu - A_s)\big| + |V_s'(z) - 1|\ \frac{1}{k}(\mu - A_s) \\ + \frac{1}{k}|U_s(z) - A_s| + \frac{1}{k}|zU_s'(z)|. \end{aligned}$$

Using this inequality, one gets $\|F\| \leq Ck^{-3/4}$, and eventually $|\psi(0)| \leq Ck^{-1/4}$. The estimate of Proposition 7 follows. As regards the analyticity of $\mu \mapsto \Phi_k$, it follows from the analyticity of $\mu \mapsto \phi_{bl}(0)$ and of $\mu \to \psi(0)$. The former is deduced directly from formula (38), having in mind the analyticity of $\phi_{\mu,\infty}(0) = \Phi_\infty(0)$. The latter is deduced from the analyticity statement of Proposition 5. More precisely, the statement is given there for a source term F that is independent of μ, but it is still true, with the same proof, for an F analytic in μ, which is the case here, *cf.* (37). This concludes the proof of the proposition. □

5 Examples of instabilities

In this final part of the paper, we exhibit examples, both numerical and analytical, for which the quantity $n_+ - n_-$ mentioned in Theorem 1 is indeed non-zero.

For the theoretical existence of unstable modes, we note that (17) can be written as

$$\Phi_\infty(\mu) = \int_0^\infty q_\mu(V_s(y))\,\mathrm{d}y$$

with the function $q_\mu : u \mapsto \mu(\mu - u)^{-2}$. For a fixed μ, we now look at $q_\mu([0,\infty))$ and by the idea of Section 6 of [3], we can construct smooth profiles V_s with $\Phi_\infty(\mu) = 0$ if and only if the origin is in the interior of the convex hull of $q_\mu([0,\infty))$. Indeed for $\mathcal{R}e\,\mu > 0$ and $\mathcal{I}m\,\mu > 0$, the origin is in the convex hull of $q_\mu([0,\infty))$ so that unstable modes exist, see Figure 2 as an example.

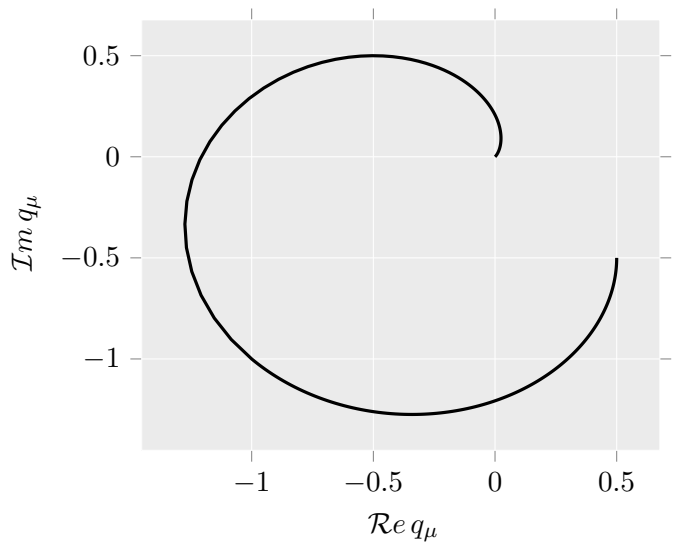

Figure 2: Image of $q_\mu([0,\infty))$ for $\mu = 1 + i$.

The advantage of the stability criterion is that it is explicitly computable and has an immediate visual interpretation as the image of $\Phi_\infty(\mathbb{R} + i\epsilon)$ for $\epsilon > 0$.

As an example, we consider

$$V_s(y) = x + 4x\, e^{-2x}. \tag{39}$$

In this case we find the resulting curve is shown in Figure 3. Here we see that it is stable as it is not crossing the positive real axis.

As another example, we consider

$$V_s(y) = \sin(2x)\, e^{-x} + x\,(1 - e^{-x}). \tag{40}$$

In this case we find the resulting curve is shown in Figure 4. Here we see that unstable modes exists as the positive real axis is crossed once.

A Stability for $U_s = 0$

The assumptions of Theorem 1 are not satisfied in the case where U_s vanishes identically. On the contrary, one can show in this setting that any family of solutions of (LTD) of type

$$u_k(t,x,z) = e^{\lambda_k t} e^{ikx} \hat{u}_k(z), \quad A_k(t,x) = e^{\lambda_k t} e^{ikx}, \quad k \in \mathbb{R},$$

has growth rate $(\mathcal{R}e\, \lambda_k)^+ = O(1)$ as $|k| \to +\infty$. We now sketch the proof of this claim. It builds upon classical works on the linear stability of Couette or Blasius flows within Navier-Stokes, *cf.* [5, 24, 27].

The trick is to differentiate the linearised equation once by z and express the result in terms of $\omega(z) := \hat{u}_k'(z)$. Then the linearised evolution (4) yields the eigenmode equation

$$(\lambda_k + ikz)\omega - \omega'' = 0$$

with the boundary conditions

$$\omega'(0) = ik|k|, \lim_{z\to\infty} \omega = 0$$

and the consistency equation

$$1 = \int_0^\infty \omega(z)\, \mathrm{d}z \tag{41}$$

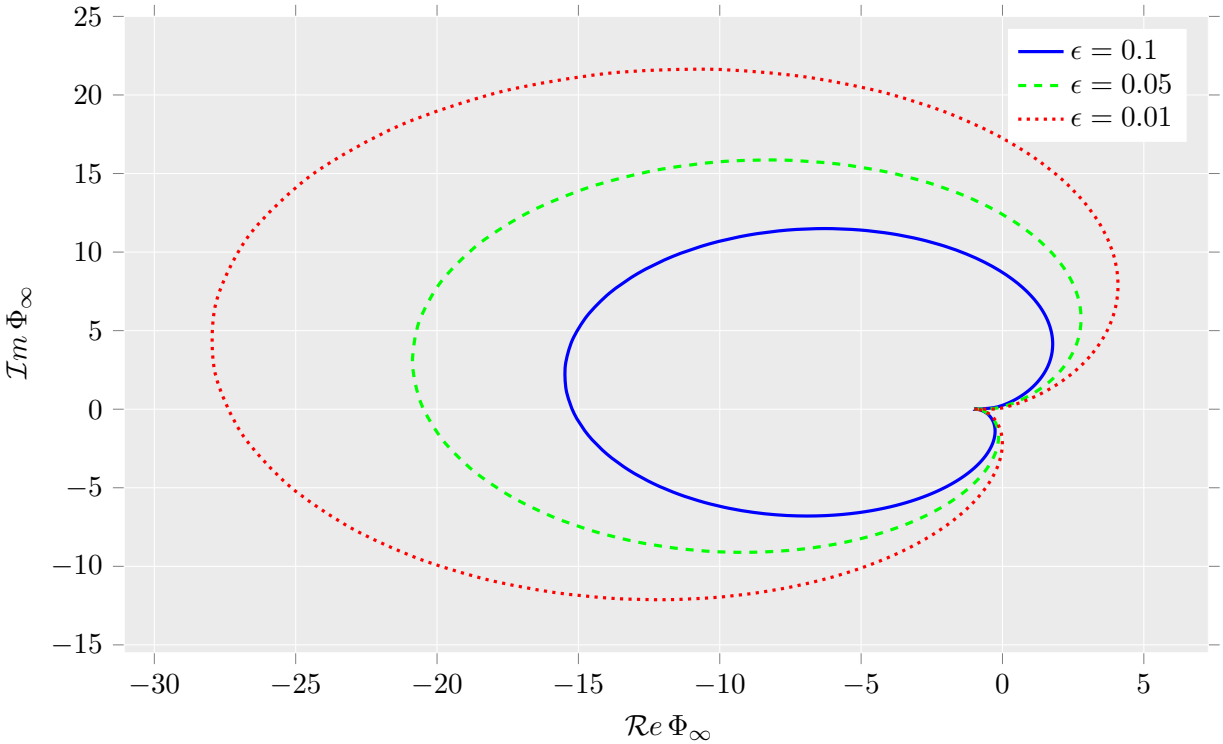

Figure 3: Image of $\Phi_\infty(\mathbb{R} + i\epsilon)$ for V_s from (39) with $\epsilon = 0.1$, $\epsilon = 0.5$ and $\epsilon = 0.01$.

by using that $\omega = \hat{u}_k'(z)$ and $\lim_{z\to\infty} \hat{u}_k(z) = 1$. The idea is then to make a change of variable in the complex plane to get back to the Airy equation

$$\xi\varphi(\xi) - \partial_\xi^2\varphi(\xi) = 0.$$

We set

$$\eta_k := (ik)^{-2/3}\lambda_k, \quad \xi := (ik)^{1/3}y, \quad W(\xi) := \omega(y)$$

where the roots are chosen with positive real part so that

$$(\xi + \eta_k)W - W'' = 0.$$

Using the boundary condition on $\omega'(0)$ and the decay condition of ω at infinity, we get

$$W(\xi) = ik|k|(ik)^{-1/3}\frac{\mathrm{Ai}(\xi + \eta_k)}{\mathrm{Ai}(\eta_k, -1)}$$

where Ai is the so-called Airy function of the first kind, and $\mathrm{Ai}(\cdot, -1)$ denotes its derivative, *cf.* [5]. Then the consistency equation (41) yields the dispersion relation

$$1 = ik|k|\frac{\mathrm{Ai}(\eta_k, 1)}{\mathrm{Ai}(\eta_k, -1)}, \tag{42}$$

where $\mathrm{Ai}(\cdot, 1)$ is the antiderivative of Ai that vanishes at $+\infty$, and we recall $\eta_k = (ik)^{-2/3}\lambda_k$.

We will now use these relations to determine the asymptotic behaviour of unstable eigenvalues λ_k (meaning $\mathcal{R}e\,\lambda_k > 0$) when $|k| \to +\infty$. We distinguish between three regimes: η_k goes to zero, goes to infinity, or is $O(1)$ as $|k| \to +\infty$.

- If $\eta_k \to 0$

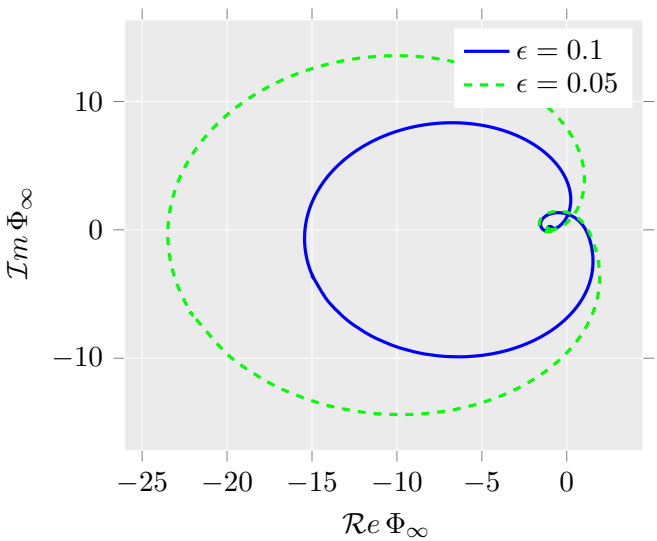

Figure 4: Image of $\Phi_\infty(\mathbb{R} + i\epsilon)$ for V_s from (40) with $\epsilon = 0.1$ and $\epsilon = 0.5$.

We find

$$1 \sim ik|k|\frac{\mathrm{Ai}(0,1)}{\mathrm{Ai}(0,-1)}$$

where $\frac{\mathrm{Ai}(0,1)}{\mathrm{Ai}(0,-1)} = 3^{-2/3}\Gamma(1/3)$, see [5, equation (A11)]. This yields a contradiction.

- If $|\eta_k| \to +\infty$

We use the asymptotic expansion given in [5], see equations (A12)-(A13)-(A14). In the case $k > 0$, we find

$$1 \sim \frac{ik|k|}{\eta_k}\frac{(1 - \frac{3a(1)}{2}(\eta_k)^{-3/2})}{(1 - \frac{3a(-1)}{2}(\eta_k)^{-3/2})}$$

where $a(p) = \frac{1}{72}(12p^2 + 24p + 5)$. Using the $\eta_k = (ik)^{-2/3}\lambda_k$ we find $\mathcal{R}e\,\lambda_k \leq 0$.

- If $\eta_k \sim O(1)$

Then

$$\mathrm{Ai}(\eta_k, 1) = \mathrm{Ai}(\eta_k, -1)ik|k|.$$

Hence we see that a subsequence of η_k should converge to some η^0 satisfying $\mathrm{Ai}(\eta^0, 1) = 0$. As $\mathcal{R}e\,\lambda_k > 0$, one finds that $-5\pi/6 < \arg(\eta_k) < \pi/6$ and thus $-5\pi/6 \leq \arg(\eta^0) \leq \pi/6$. Similarly, in the case $k < 0$, one should have $-\pi/6 \leq \arg(\eta^0) \leq 5\pi/6$.

These two scenarios are excluded by the following proposition, which can be found in [27]: *the function* $\mathrm{Ai}(\cdot, 1)$ *has no zero in the closed sector* $-5\pi/6 \leq \arg(\eta) \leq 5\pi/6$. This concludes the proof of spectral stability.

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References

[1] R. J. Bodonyi and F. T. Smith, *On short-scale inviscid instabilities in the flow past surface-mounted obstacles*, in Stability of Time Dependent and Spatially Varying Flows, D. L. Dwoyer and M. Y. Hussaini, eds., New York, NY, 1987, Springer New York, pp. 148–163.

[2] C. Collot, T.-E. Ghoul, and N. Masmoudi, *Singularities and unsteady separation for the inviscid two-dimensional prandtl system*, Archive for Rational Mechanics and Analysis, (2021).

[3] A.-L. Dalibard, H. Dietert, D. Gérard-Varet, and F. Marbach, *High frequency analysis of the unsteady interactive boundary layer model*, SIAM J. Math. Anal., 50 (2018), pp. 4203–4245.

[4] H. Dietert and D. Gérard-Varet, *Well-posedness of the Prandtl equations without any structural assumption*, Ann. PDE, 5 (2019), pp. Paper No. 8, 51.

[5] P. Drazin and W. Reid, *Hydrodynamic stability*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, second ed., 2004. With a foreword by John Miles.

[6] W. E and B. Engquist, *Blowup of solutions of the unsteady Prandtl's equation*, Comm. Pure Appl. Math., 50 (1997), pp. 1287–1293.

[7] F. Gargano, M. Sammartino, and V. Sciacca, *Singularity formation for Prandtl's equations*, Phys. D, 238 (2009), pp. 1975–1991.

[8] D. Gérard-Varet and E. Dormy, *On the ill-posedness of the Prandtl equation*, J. Amer. Math. Soc., 23 (2010), pp. 591–609.

[9] D. Gérard-Varet and T. Nguyen, *Remarks on the ill-posedness of the Prandtl equation*, Asymptot. Anal., 77 (2012), pp. 71–88.

[10] E. Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Comm. Pure Appl. Math., 53 (2000), pp. 1067–1091.

[11] E. Grenier, Y. Guo, and T. Nguyen, *Spectral instability of characteristic boundary layer flows*, Duke Math. J., 165 (2016), pp. 3085–3146.

[12] E. Grenier and T. T. Nguyen, *L^∞ instability of Prandtl layers*, Ann. PDE, 5 (2019), pp. Paper No. 18, 36.

[13] Y. Guo and T. Nguyen, *A note on Prandtl boundary layers*, Comm. Pure Appl. Math., 64 (2011), pp. 1416–1438.

[14] S. Iyer and V. Vicol, *Real analytic local well-posedness for the triple deck*, Communications on Pure and Applied Mathematics, (2020).

[15] V. Kamotski and G. Lebeau, *On 2D Rayleigh-Taylor instabilities*, Asymptot. Anal., 42 (2005), pp. 1–27.

[16] I. Kukavica and V. Vicol, *On the local existence of analytic solutions to the Prandtl boundary layer equations*, Commun. Math. Sci., 11 (2013), pp. 269–292.

[17] I. Kukavica, V. Vicol, and F. Wang, *The van Dommelen and Shen singularity in the Prandtl equations*, Adv. Math., 307 (2017), pp. 288–311.

[18] P.-Y. Lagrée, *Interacting boundary layers*, in Asymptotic methods in fluid mechanics: survey and recent advances, vol. 523 of CISM Courses and Lect., SpringerWienNewYork, Vienna, 2010, pp. 247–286.

[19] C.-J. Liu and T. Yang, *Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay*, J. Math. Pures Appl. (9), 108 (2017), pp. 150–162.

[20] O. Penrose, *Electrostatic instabilities of a uniform non-maxwellian plasma*, Physics of Fluids, 3 (1960), pp. 258–265.

[21] L. Plantié, *Existence result for the boundary layer of triple deck type with known displacement*, Math. Models Methods Appl. Sci., 16 (2006), pp. 1319–1346.

[22] L. Prandtl, *Über Flüssigkeitsbewegung bei sehr kleiner Reibung*, in Proceedings of the International Mathematical Congress *Verh. III Int. Math.-Kongr.*, Leipzig: Teubner(1905), Heidelberg, 1904, pp. 484–491. Reprinted in *Gesammelte Abhandlungen* II, 575–584 (1961).

[23] M. Sammartino and R. E. Caflisch, *Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations*, Comm. Math. Phys., 192 (1998), pp. 433–461.

[24] F. Smith, *On the non-parallel flow stability of the Blasius boundary layer*, Proc. Roy. Soc. London Ser. A, 366 (1979), pp. 91–109.

[25] F. T. Smith, *On the high Reynolds number theory of laminar flows*, IMA J. Appl. Math., 28 (1982), pp. 207–281.

[26] O. Tutty and S. Cowley, *On the stability and the numerical solution of the unsteady interactive boundary-layer equation*, Journal of Fluid Mechanics, 168 (1986), pp. 431–456.

[27] W. Wasow, *On small disturbances of plane couette flow*, Journal of Research of the National Bureau of Standards, 51 (1953), pp. 195–202.