



**HAL**  
open science

# Variance Optimization and Control Regularity for Mean-Field Dynamics

Benoît Bonnet, Francesco Rossi

► **To cite this version:**

Benoît Bonnet, Francesco Rossi. Variance Optimization and Control Regularity for Mean-Field Dynamics. 2021. hal-03216326v1

**HAL Id: hal-03216326**

**<https://hal.science/hal-03216326v1>**

Preprint submitted on 3 May 2021 (v1), last revised 29 Jul 2021 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Variance Optimization and Control Regularity for Mean-Field Dynamics <sup>\*</sup>

Benoît Bonnet <sup>\*</sup> Francesco Rossi <sup>\*\*</sup>

<sup>\*</sup> *Inria Paris, Laboratoire Jacques-Louis Lions, Sorbonne Université,  
Université Paris-Diderot SPC, CNRS, Inria, 75005 Paris, France  
(e-mail benoit.a.bonnet@inria.fr)*

<sup>\*\*</sup> *Dipartimento di Matematica “Tullio Levi-Civita”, Università degli  
Studi di Padova, 63 via Trieste, Padova, Italy  
(e-mail: francesco.rossi@math.unipd.it)*

---

**Abstract.** We study a family of optimal control problems in which one aims at minimizing a cost that mixes a quadratic control penalization and the variance of the system, both for finite sets of agents and for the limit problem as their number goes to infinity. While the solution for finitely many agents always exists in a unique and explicit form, the behavior of the corresponding limit problem is very sensitive to the magnitude of the time horizon and penalization parameter. For the variance minimization, there exist Lipschitz-in-space optimal controls for the infinite dimensional problem, which can be obtained as a suitable limit of the optimal controls for the finite-dimensional problems. The same holds true for the variance maximization with a sufficiently small final time. Instead, for large final times (or equivalently large penalizations of the variance), Lipschitz-regular optimal controls do not exist for the macroscopic problem.

*Keywords:* Optimal control, Conservation Laws, Distributed Parameter Systems, Mean-Field Control, Large Scale Complex Systems

---

## 1. INTRODUCTION

The mathematical analysis of collective behavior in large-scale systems of interacting agents has received an increasing attention during the past decades. Multi-agent systems are ubiquitous in applications, ranging from networked control to animal flocks and swarms, see e.g. Bullo et al. (2009); Cucker and Smale (2007). In this context, a multi-agent system is usually described by a family of ordinary differential equations (ODEs for short), of the form

$$\dot{x}_i(t) = \mathbf{v}_N(t, \mathbf{x}(t), x_i(t)), \quad (1)$$

where  $\mathbf{x} = (x_1, \dots, x_N)$  denotes the state of all the agents and  $\mathbf{v}_N : [0, T] \times (\mathbb{R}^d)^N \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a *non-local velocity field* depending both on the running agent and on the whole state of the system. However general and useful, the intrinsic dependence of such models on the number  $N \geq 1$  of agents makes most of the classical computational approaches practically intractable for realistic scenarios.

One of the most natural ideas to circumvent this limitation is to approximate the large system in (1) by a single infinite-dimensional dynamics. This process, called the *mean-field limit* procedure, describes the dynamics when the number  $N$  of agents tend to infinity, in a specific way (see e.g. the survey Golse (2016)). In this setting, agents are supposed to be identical or indistinguishable, and the assembly of particles is described by means of its *spatial density*  $\mu(\cdot)$ , which is represented by a measure. The evolution in time of this global quantity is prescribed by a

*non-local continuity equation*, that is a partial differential equation (PDE for short) of the form

$$\partial_t \mu(t) + \nabla \cdot (v(t, \mu(t), \cdot) \mu(t)) = 0. \quad (2)$$

This approach has been successfully used e.g. to model pedestrian dynamics and biological systems (Camazine et al. (2001); Cristiani et al. (2011)), as well as to transpose the study of classical patterns such as consensus or flocking to macroscopic approximations of discrete multi-agent systems (Carrillo et al. (2010); Ha and Liu (2009)).

In addition to the modeling and analysis of this class of dynamics, the problem of controlling multi-agent systems is relevant in a growing number of applications, see e.g. Caponigro et al. (2015); Burger et al. (2014); Leonard (2013). Motivated by implementability and efficiency considerations, many contributions have aimed at generalising relevant notions of control theory to PDEs of the form (2), serving as mean-field approximations of the discrete systems (1) (see e.g. Berman et al. (2009); Elamvazhuthi et al. (2018); Elamvazhuthi and Berman (2019)). A few articles have been dealing with controllability results (see Duprez et al. (2019, 2020)) or explicit syntheses of control laws (e.g. Caponigro et al. (2015); Piccoli et al. (2015)). On the other hand, the major part of the literature has been focusing on *mean-field optimal control problems*, with contributions ranging from existence results (Bonnet and Frankowska (2021b); Fornasier et al. (2019, 2014); Fornasier and Solombrino (2014)) to first-order optimality conditions (Bongini et al. (2017); Bonnet (2019); Bonnet and Frankowska (2021a); Bonnet and Rossi (2019); Cagnari et al. (2020); Pogodaev (2016)).

---

<sup>\*</sup> This research was partially supported by the Padua University grant SID 2018 “Controllability, stabilizability and infimum gaps for control systems”, prot. BIRD 187147.

In this article, we consider optimal control problems formulated both in the ODE and PDE settings, and discuss the possibility of applying the mean-field approach (i.e. to let  $N \rightarrow +\infty$ ) in the context of optimal control. We shall restrict our attention to a very simple class of problems, which anyway shows the most important issues arising in this setting. Indeed, we aim to prove the following idea: if optimal controls at the discrete level can be written as Lipschitz functions of the state, with a Lipschitz constant that is uniform with respect to the number  $N \geq 1$  of agents, then such controls pass to the limit, and the resulting mean-field optimal controls are Lipschitz as well. Instead, when the Lipschitz constant of the optimal control explodes as  $N \rightarrow +\infty$ , then there does not exist a Lipschitz regular optimal control for the mean-field problem.

To this end, we write one of the simplest optimal control problem possible for (1) and (2). We assume that the agents evolve on the real line  $\mathbb{R}$ , and that the controls act linearly on each of them. We assume that there is no interaction between the agents at the dynamical level (i.e.  $v \equiv 0$ ), and that a final cost promotes either the minimization or the maximization of the variance (both in the finite and infinite-dimensional setting) at the final time  $T$ . Moreover, a running cost encodes an  $L^2$ -penalization of the controls. The relative weight between these two costs is represented by a scalar quantity  $\lambda \neq 0$ , which relative value with respect to  $T$  plays a fundamental role in regularity of optimal controls.

In the sequel, we will therefore consider the following discrete multi-agent optimal control problem.

$(\mathcal{P}_N)$

Minimize the cost functional

$$\mathcal{C}_N(\mathbf{x}^0, \mathbf{u}) := \frac{1}{2N} \sum_{i=1}^N \int_0^T u_i^2(t) dt - \frac{1}{2\lambda} \text{Var}(\mathbf{x}(T))$$

where

- ◇ the controls  $\mathbf{u} : [0, T] \rightarrow [-1, 1]^N$  are Lebesgue measurable;
- ◇ the curve  $\mathbf{x}(\cdot) := (x_1(\cdot), \dots, x_N(\cdot))$  is the unique solution of the controlled dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_i^0, \quad (3)$$

with initial datum

$$x_i^0 := \frac{2i - N - 1}{N - 1} \quad \text{for each } i \in \{1, \dots, N\}.$$

The regularity required in  $(\mathcal{P}_N)$  for the controls is the standard one ensuring existence and uniqueness of the solution to (3), see e.g. (Clarke, 2013, Chapter 23).

For the infinite-dimensional problem, defining admissible controls is much more delicate, as one needs to ensure existence and uniqueness of the solution to (2). For this reason, we impose Lipschitz regularity of  $(t, x) \mapsto u(t, x)$  with respect to the space variable, together with measurability with respect to the time variable. For a thorough discussion of this issue, see e.g. Fornasier and Solombrino (2014); Bonnet and Rossi (2021). In this context, we study the following mean-field optimal control problem.

$(\mathcal{P}_\infty)$

Minimize the cost functional

$$\mathcal{C}_\infty(\mu^0, u) := \frac{1}{2} \int_0^T \int_{\mathbb{R}} u^2(t, x) d\mu(t)(x) dt - \frac{1}{2\lambda} \text{Var}(\mu(T)),$$

where

- ◇ the controls  $u : [0, T] \times \mathbb{R} \mapsto [-1, 1]$  are measurable in time and Lipschitz in space;
- ◇ the curve  $\mu(\cdot)$  is the unique solution of the controlled dynamics

$$\begin{cases} \partial_t \mu(t) + \nabla \cdot (u(t, \cdot) \mu(t)) = 0, \\ \mu(0) = \mu^0, \end{cases} \quad (4)$$

with initial datum  $\mu^0 := \frac{1}{2} \chi_{[-1, 1]}$ .

In what follows, we will describe precisely in which sense the problem  $(\mathcal{P}_\infty)$  is the limit of  $(\mathcal{P}_N)$  as  $N \rightarrow +\infty$ . For the moment, observe that to each vector  $\mathbf{x} \in \mathbb{R}^N$  of  $N$  agent positions, one can associate the *empirical measure*

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \quad \text{where } \mathbf{x} = (x_1, \dots, x_N).$$

Via this association, one can easily show that the discrete and continuous variances coincide, namely

$$\begin{aligned} \text{Var}(\mathbf{x}) &:= \frac{1}{N} \sum_{i=1}^N x_i^2 - \left( \frac{1}{N} \sum_{i=1}^N x_i \right)^2 \\ &= \int_{\mathbb{R}} x^2 d\mu_N(x) - \left( \int_{\mathbb{R}} x d\mu_N(x) \right)^2 = \text{Var}(\mu_N), \end{aligned}$$

and the initial data  $(\mu_N^0)$  associated with  $\mathbf{x}^0$  as in  $(\mathcal{P}_N)$  converge in the sense of measures (5) towards  $\frac{1}{2} \chi_{[-1, 1]}$ .

The terms involving the controls are more tricky, as some extra regularity is needed to ensure some sort of convergence between the discrete and continuous models. This is the crucial point of this article: we will show that the **optimizing induces sufficient regularity of the optimal control** for  $\lambda \in (-\infty, 0) \cup (T, +\infty)$ , while Lipschitz solutions to  $(\mathcal{P}_\infty)$  do not exist for  $\lambda \in (0, T]$ .

*Theorem 1.* (Main result). Let  $\lambda > T$  or  $\lambda < 0$ . Then, there exists a minimizer  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  of  $(\mathcal{P}_\infty)$  which is uniformly Lipschitz with respect to the space variable. Moreover, this minimizer is the limit of optimal controls  $u_i^*$  of  $(\mathcal{P}_N)$ , in the sense that  $|x_i^*(t) - x| \rightarrow 0$  implies  $|u_i^* - u(t, x)| \rightarrow 0$ . Instead, for  $\lambda \in (0, T]$  a Lipschitz-in-space minimizer for  $(\mathcal{P}_\infty)$  does not exist.

The dichotomy exposed above is paradigmatic of many mean-field optimal control problems, and is investigated in greater generality in Bonnet and Rossi (2021).

The structure of the article is the following. We first introduce the continuity equation and regularity issues for  $(\mathcal{P}_\infty)$  in Section 2. We then explicitly solve  $(\mathcal{P}_N)$  in Section 3, and proceed to rigorously study the limit  $(\mathcal{P}_N) \rightarrow (\mathcal{P}_\infty)$  in Section 4, proving Theorem 1. We finally draw some conclusions in Section 5.

## 2. TRANSPORT EQUATIONS AND MEAN-FIELD OPTIMAL CONTROL

In this section, we fix some notations and recall several results about Wasserstein distances, continuity equations and mean-field optimal control problems.

We denote by  $\mathcal{P}_c(\mathbb{R}^d)$  the space of probability measures on  $\mathbb{R}^d$  with compact support, endowed with the standard *weak topology of measures*

$$\mu_n \xrightarrow{n \rightarrow +\infty} \mu \quad \text{if} \quad \int_{\mathbb{R}^d} f(x) d\mu_n(x) \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} f(x) d\mu(x), \quad (5)$$

for every  $f \in C_c^\infty(\mathbb{R}^d)$ . We also denote by  $\text{Lip}(f)$  the Lipschitz constant of a Lipschitz continuous function, i.e.

$$\text{Lip}(f) := \sup_{x, y \in \text{dom}(f), x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

We recall the definition of solution to continuity equations.

*Definition 1.* We say that  $\mu(\cdot) \in C^0([0, T], \mathcal{P}_c(\mathbb{R}^d))$  solves a *continuity equation* with initial condition  $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$  driven by a vector field  $w : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ , i.e. that it solves

$$\begin{cases} \partial_t \mu(t) + \nabla \cdot (w(t, \cdot) \mu(t)) = 0, \\ \mu(0) = \mu^0. \end{cases} \quad (6)$$

if the following distributional formulation holds

$$\int_0^T \int_{\mathbb{R}^d} \left( \partial_t \xi(t, x) + \langle \nabla_x \xi(t, x), w(t, x) \rangle \right) d\mu(t)(x) dt = 0, \quad (7)$$

for any  $\xi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ .

The connection between continuity equations in infinite dimension and ODEs in finite dimension is given by the following statement, which is colloquially known as the *method of characteristics*.

*Definition 2.* Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a Borel map. The *push-forward*  $f\#\mu$  of a measure  $\mu$  of  $\mathbb{R}^d$  is the measure satisfying

$$(f\#\mu)(E) := \mu(f^{-1}(E)),$$

for every  $E \subset \mathbb{R}^d$  such that  $f^{-1}(E)$  is  $\mu$ -measurable.

*Theorem 2.* (Method of characteristics). Let  $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$  and  $w : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a Carathéodory vector field condition that is locally Lipschitz and sublinear. Then, equation (6) admits a unique solution  $\mu(\cdot)$ , given by

$$\mu(t) = (\Phi_t^w)\#\mu^0 \quad \text{for each } t \in [0, T],$$

where  $x \in \mathbb{R}^d \mapsto \Phi_t^w(x) \in \mathbb{R}^d$  denotes the *flow map* of  $w$ .

*Remark 1.* It is known that weak solutions of continuity equations can exist in very low regularity contexts, see e.g. (Ambrosio et al., 2008, Chapter 8). However, such notions are not perfectly adapted to mean-field control, as they do not ensure well-posedness of (2) for arbitrary measures.

### 2.1 Wasserstein distance

We now recall the definition of Wasserstein distance, together with its connection with solutions of the continuity equation. We will only consider 1-Wasserstein distance, as this is sufficient for our goals in this article. For a thorough introduction, see e.g. (Ambrosio et al., 2008, Chapter 7).

*Definition 3.* Given  $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ , the 1-Wasserstein distance is defined by

$$W_1(\mu, \nu) := \sup_{\text{Lip}(f) \leq 1} \int_{\mathbb{R}^d} f(x) d(\mu - \nu)(x). \quad (8)$$

A first, fundamental property of the Wasserstein distance is that it metrises the weak convergence of measures (5) in the following sense.

*Proposition 3.* It holds  $\lim_{n \rightarrow +\infty} W_1(\mu_n, \mu) = 0$  if and only if

$$\mu_n \xrightarrow{n \rightarrow +\infty} \mu \quad \text{and} \quad \int_{\mathbb{R}^d} |x| d\mu_n(x) \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}^d} |x| d\mu(x).$$

We now describe the connection between solutions of the continuity equation, such as (2), and Wasserstein distance.

*Proposition 4.* Let  $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$  and  $w : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a vector field uniformly bounded, measurable in time and Lipschitz in space with constant equal to  $L \geq 0$ . Then for each  $t \in [0, T]$ , it holds

$$W_1((\Phi_t^w)\#\mu, (\Phi_t^w)\#\nu) \leq e^{Lt} W_1(\mu, \nu). \quad (9)$$

*Proof:* See e.g. Piccoli and Rossi (2013) or Bonnet and Frankowska (2021b) for a more general statement.  $\square$

## 3. SOLUTION OF $(\mathcal{P}_N)$

In this section, we explicitly compute the solutions of  $(\mathcal{P}_N)$ . We use bold notations  $(\mathbf{x}, \mathbf{r}) \in \mathbb{R}^{2N}$  to denote vectors in  $\mathbb{R}^N$ , as well as  $\mathbf{u} \in [-1, 1]^N$ . We now fix  $N \geq 2$  to be even, which ensures that  $x_i(0) \neq 0$ . This condition is not crucial for the result, but it simplifies the discussion.

Observe that the problem  $(\mathcal{P}_N)$  is regular both in the dynamics and the control. Moreover, optimal controls exist, as the set of admissible controls  $[-1, 1]^N$  is convex and compact (see e.g. (Clarke, 2013, Theorem 23.11)). Finally, the smoothness of the data allows us to compute optimal controls via the Pontryagin Maximum Principle, see e.g. (Clarke, 2013, Chapter 22).

Denoting by  $\mathbf{p}$  a costate variable associated to  $\mathbf{x}$ , the Hamiltonian function of  $(\mathcal{P}_N)$  writes

$$\mathcal{H}_N(\mathbf{x}, \mathbf{p}, \mathbf{u}) = \sum_{i=1}^N (p_i u_i - \frac{1}{2N} u_i^2). \quad (10)$$

Given an optimal trajectory-control pair  $(\mathbf{x}_N^*(\cdot), \mathbf{u}_N^*(\cdot))$ , the PMP provides the existence of a curve  $\mathbf{p}_N^*(\cdot)$  satisfying

$$\begin{cases} \dot{p}_i^*(t) &= 0 \\ p_i^*(T) &= \frac{1}{\lambda} \partial_{x_i} \text{Var}(\mathbf{x}_N^*(T)) = \frac{1}{N\lambda} (x_i^*(T) - \bar{\mathbf{x}}_N^*(T)), \\ u_i^*(t) &\in \text{argmax}_{v \in [-1, 1]} [p_i^*(t)v - \frac{1}{2N} v^2], \end{cases} \quad (11)$$

where  $\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N x_i$  is the mean value of the vector  $\mathbf{x}$ . Thus, the adjoint vector  $\mathbf{p}^*(\cdot)$  is constant and satisfies

$$p_i^*(t) = \frac{1}{\lambda N} (x_i^*(T) - \bar{\mathbf{x}}_N^*(T)),$$

for any  $i \in \{1, \dots, N\}$  and all times  $t \in [0, T]$ . As a consequence of the maximization condition, one can express the components of the optimal control  $\mathbf{u}^*(\cdot)$  explicitly as

$$u_i^*(t) = \pi\left(\frac{1}{\lambda} (x_i^*(T) - \bar{\mathbf{x}}_N^*(T))\right). \quad (12)$$

Here, we denoted by  $u \in \mathbb{R} \mapsto \pi(u) \in [-1, 1]$  the projection onto the set of admissible controls  $[-1, 1]$ , i.e.

$$\pi(u) := \begin{cases} 1 & \text{if } u \geq 1, \\ u & \text{if } u \in [-1, 1], \\ -1 & \text{if } u \leq -1. \end{cases}$$

Our goal now is to analytically solve  $(\mathcal{P}_N)$ . First remark that the optimal control  $\mathbf{u}_N^*(\cdot) \equiv \mathbf{u}_N^*$  does not depend on

time, and let us denote by  $\bar{u}_N^*$  its mean value. The mean value of the state then satisfies  $\dot{\bar{x}}_N^*(t) = \bar{u}_N^*$ , which implies

$$\bar{x}_N^*(T) = T\bar{u}_N^* \quad \text{since} \quad \bar{x}_N(0) = 0.$$

Then, one can show that  $\bar{u}_N^* = 0$ , e.g. by observing that choosing  $\tilde{u}_i := (u_i^* - \bar{u}_N^*)$  instead of  $u_i^*$  would have no impact on the final cost while reducing the control cost. Consequently  $\bar{x}^*(t) = \bar{x}(0) = 0$  for all  $t \in [0, T]$ , and since  $x_i^*(T) = x_i(0) + Tu_i^*$  equation (12) now reads as

$$u_i^* = \pi\left(\frac{1}{\lambda}(x_i(0) + Tu_i^*)\right). \quad (13)$$

Now, we need to study the problem in several sub-cases depending on the value of  $\lambda$ .

### 3.1 The case $\lambda > T$

Consider the right-hand side of (13) as a function of  $u_i$  and observe that it is increasing, with maximal slope  $T/\lambda < 1$ . Since the left-hand side is the identity, then (13) always admits a unique solution. See Figure 1.

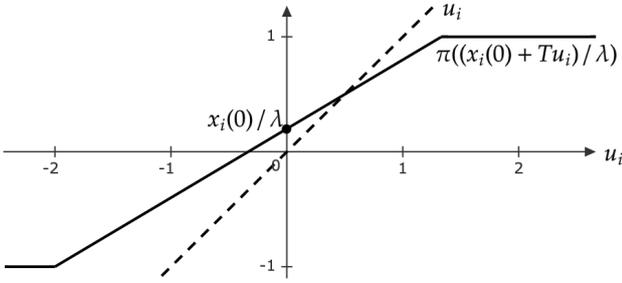


Figure 1. Solution of (13) for  $\lambda > T$ .

If  $|x_i(0) + Tu_i^*| \leq \lambda$ , then the projection coincides with the identity operator. In this case, equation (13) reads as  $\lambda u_i^* = x_i(0) + Tu_i^*$ . We can then write  $u_i^* = \frac{x_i(0)}{\lambda - T}$  whenever such formula yields a control  $u_i \in [-1, 1]$ , i.e. for  $x_i(0) \in [-(\lambda - T), \lambda - T]$ . For other indices, one can easily check that the unique solution for  $x_i(0) > \lambda - T$  is  $u_i^* = 1$ . Similarly for  $x_i(0) < -(\lambda - T)$ , the unique solution to (13) is  $u_i^* = -1$ . This gives the following optimal controls

$$u_i^* = \begin{cases} 1 & \text{if } x_i(0) > \lambda - T, \\ \frac{x_i(0)}{\lambda - T} & \text{if } x_i(0) \in [-(\lambda - T), \lambda - T], \\ -1 & \text{if } x_i(0) < -(\lambda - T). \end{cases} \quad (14)$$

### 3.2 The case $\lambda = T$

In this case, equation (13) has no solution whenever the projection coincides with the identity, as  $x_i(0) \neq 0$  for every  $i \in \{1, \dots, N\}$ . A simple computation shows that the only solution is the following

$$u_i^* = \begin{cases} 1 & \text{if } x_i(0) > 0, \\ -1 & \text{if } x_i(0) < 0. \end{cases} \quad (15)$$

### 3.3 The case $\lambda \in (0, T)$

In this case, equation (13) does not have a uniquely defined solution when the projection operator coincides with the

identity. For  $x_i(0) \in [-(T - \lambda), T - \lambda]$  one has that both  $\frac{x_i(0)}{\lambda - T}$  and  $\text{sign}(x_i(0))$  solve it. One can prove that this second choice provides the minimizer of the cost. The optimal control then coincides with (15).

### 3.4 The case $\lambda < 0$

In this case one aims at minimizing  $\text{Var}(\mathbf{x}(T))$ , together with the cost of the control. Moreover, the Pontryagin Maximum Principle still reads as (11), thus (13) holds too. In this case, the right-hand side is decreasing and bounded, then (13) always admits a unique solution. Direct computations, similar to the previous ones, allow to prove that the optimal controls are given by

$$u_i^* = \begin{cases} -1 & \text{if } x_i(0) > -(\lambda - T), \\ \frac{x_i(0)}{\lambda - T} & \text{if } x_i(0) \in [(\lambda - T), -(\lambda - T)], \\ 1 & \text{if } x_i(0) < \lambda - T. \end{cases} \quad (16)$$

### 3.5 Comparison

We now highlight two important features of the optimal controls written above. First, for each pair of parameters  $(\lambda, T)$ , the value of  $u_i^*$  only depends on  $x_i(0)$  and not on the actual number  $N \geq 1$  of agents. This will play a crucial role in the following Section 4.

Second, we aim to evaluate the following quantity, that can be seen as the Lipschitz constant of the optimal control:

$$L(t) := \max_{i \neq j} \frac{|u_i^* - u_j^*|}{|x_i^*(t) - x_j^*(t)|}.$$

By recalling that  $u_i^*, u_j^*$  are constant in time and that  $x_i^*(t) = x_i(0) + tu_i^*$ , we have the following scenarios.

- $\lambda > T$  : here  $|x_i^*(t) - x_j^*(t)|$  is increasing in time, thus

$$L(t) \leq L(0) \leq \frac{x_i(0) - x_j(0)}{(\lambda - T)(x_i(0) - x_j(0))} = \frac{1}{\lambda - T}.$$

Remark that the Lipschitz constant is bounded, uniformly with respect to  $N \geq 1$ .

- $\lambda \in (0, T]$  : in this case, one can easily see that the maximum is attained for  $i = \frac{N}{2}$  and  $j = i + 1$ , i.e. for the maximal negative initial position and the the minimal positive one. This implies

$$L(t) = \frac{|1 - (-1)|}{|(x_i(0) - t) - (x_j(0) + t)|} = \frac{N - 1}{1 + t(N - 1)}.$$

Contrary to the previous cases, this constant depends explicitly on  $N$  and can be arbitrarily large as  $N \rightarrow +\infty$  when  $t \in [0, T]$  is small.

- $\lambda < 0$  : The result is similar to the case  $\lambda > T$ . Here, the maximal value is attained for  $t = T$ , hence

$$L(t) \leq L(T) = \frac{|x_i(0) - x_j(0)|}{|(\lambda - T)(x_i(0) - x_j(0) + T(\frac{x_i(0) - x_j(0)}{\lambda - T}))|} = \frac{1}{|\lambda|}.$$

Also in this case, the Lipschitz constant is independent on  $N$  and bounded.

## 4. SOLUTION OF $(\mathcal{P}_\infty)$

In this section, we prove our main result Theorem 1. The interest of the proof is two-fold. First, it shows the

fundamental role of  $(\lambda - T)$  in existence of the solution of  $(\mathcal{P}_\infty)$ . Second, for the method of proof, that is indeed based on the discretised problem  $(\mathcal{P}_N)$  solved above.

The idea of the proof is the following. For the cases in which a Lipschitz minimizer exists (i.e.  $\lambda > T$  or  $\lambda < 0$ ), we explicitly build the minimizer for  $(\mathcal{P}_\infty)$ , based on our previous knowledge of minimizers of  $(\mathcal{P}_N)$ . Similarly, in the case of non-existence of a Lipschitz minimizer (i.e.  $\lambda \in (0, T]$ ), we use the limit of the discretised version  $(\mathcal{P}_N)$  to contradict the optimality of any candidate Lipschitz minimizer for  $(\mathcal{P}_\infty)$ .

With this goal in mind, we explicitly build controlled vector fields that will be optimal for  $(\mathcal{P}_N)$  and possibly for  $(\mathcal{P}_\infty)$ . We will define a candidate optimal vector field  $u^* : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  by requiring that

$$u^*(t, x_i^*(t)) = u_i^*(t), \quad (17)$$

for every  $i \in \{1, \dots, N\}$  and almost every  $t \in [0, T]$ .

Therefore, we need to consider the three following cases.

- $\lambda > T$  : the optimal controls  $u_i^*(\cdot)$  satisfy (14), and we first consider the case  $x_i(0) > \lambda - T$ . By setting

$$u^*(t, y) = 1, \quad y \geq \lambda - T + t,$$

one then has  $u^*(t, x_i(0) + t) = 1$ , so that (17) holds. The case  $x_i(0) < -(\lambda - T)$  is completely similar.

If  $x_i(0) \in [-(\lambda - T), \lambda - T]$ , then (17) reads as

$$u^* \left( t, x_i(0) + t \frac{x_i(0)}{\lambda - T} \right) = \frac{x_i(0)}{\lambda - T}.$$

This condition is verified by choosing the velocity field

$$u^*(t, y) := \frac{y}{\lambda - T + t}, \quad y \in [-(\lambda - T + t), \lambda - T + t].$$

These three cases can be then merged into the formula

$$u^*(t, y) = \pi \left( \frac{y}{\lambda - T + t} \right). \quad (18)$$

- $\lambda \in (0, T]$  : we now start from (15), and set

$$u^*(t, y) = \text{sign}(y), \quad y \in (-\infty, -t) \cup (t, +\infty). \quad (19)$$

We do not define  $u^*(t, y)$  for  $y \in [-t, t]$ .

- $\lambda < 0$  : starting from (16), we easily show that

$$u^*(t, y) = \pi \left( \frac{y}{\lambda - T + t} \right),$$

satisfies (17).

The optimal agent trajectories  $\mathbf{x}_N^*(\cdot)$  along with the velocity field  $u^*(\cdot, \cdot)$  are illustrated for each scenario in Figure 2. In the sequel, we will need the following useful lemma.

*Lemma 5.* Let  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be measurable in time, Lipschitz in space and bounded. For each initial data  $\mathbf{x}_N^0$  of  $(\mathcal{P}_N)$ , define the control  $\mathbf{u}_N(\cdot)$  via  $u_i(t) := u(t, x_i(t))$ . If  $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$  and the empirical measures  $\mu_N^0 := \frac{1}{N} \sum_{i=1}^N \delta_{x_i^0}$  have uniformly compact support, then  $\mu_N^0 \xrightarrow{N \rightarrow +\infty} \mu^0$  implies  $\mathcal{C}_N(\mathbf{x}_N^0, \mathbf{u}_N) \xrightarrow{N \rightarrow +\infty} \mathcal{C}_\infty(\mu^0, u)$ .

*Proof:* Consider the trajectory  $\mu_N(t) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i(t)}$ , where  $\mathbf{x}_N(\cdot)$  is the unique solution of (3). By definition of the controls  $\mathbf{u}_N(\cdot)$ , we know that  $\mu_N(\cdot)$  solves

$$\begin{cases} \partial_t \mu(t) + \nabla \cdot (u(t, \cdot) \mu(t)) = 0, \\ \mu(0) = \mu_N^0, \end{cases}$$

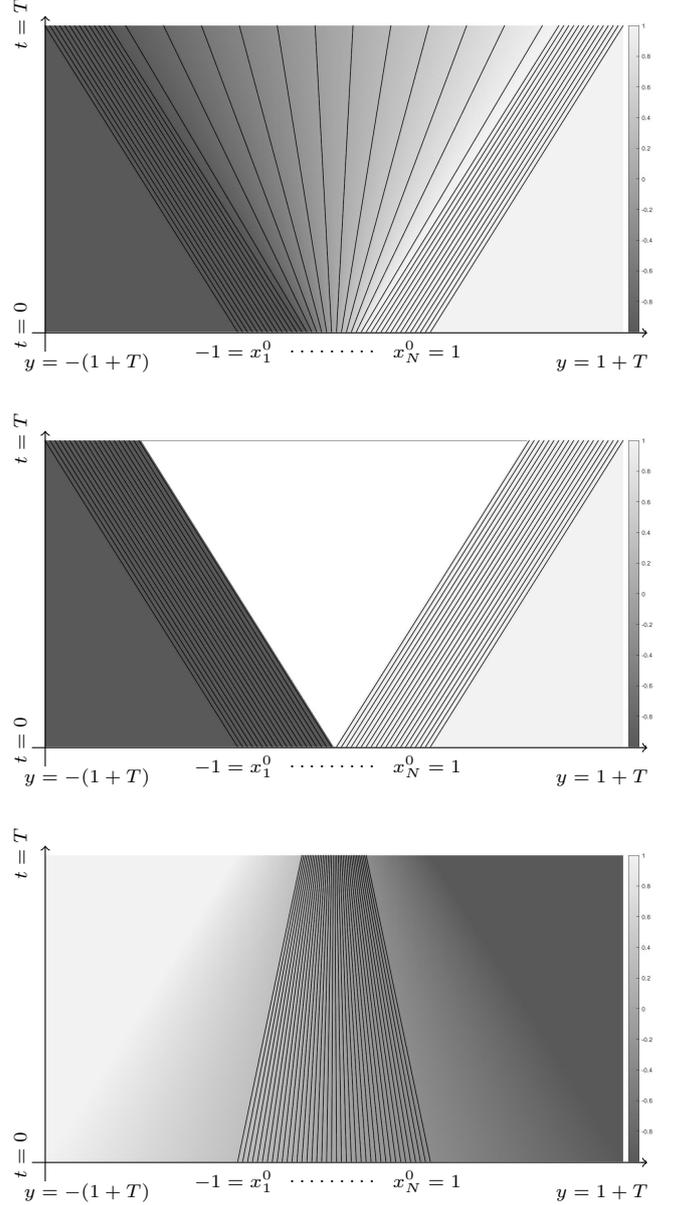


Figure 2. Trajectories of  $N = 40$  agents (foreground) with the magnitude of  $u^*(t, y)$  (background) in the case where  $\lambda > T$  (top),  $\lambda \in (0, T]$  (middle) and  $\lambda < 0$  (bottom).

and the regularity of  $u(t, \cdot)$  ensures that such a solution is unique. By a direct computation, one can check that

$$\mathcal{C}_N(\mathbf{x}_N^0, \mathbf{u}_N) = \mathcal{C}_\infty(\mu_N^0, u). \quad (20)$$

Denote now by  $\mu(\cdot)$  the unique solution of (4) with control  $(t, x) \mapsto u(t, x)$  and initial datum  $\mu_0$ . By the Lipschitz regularity of  $u(t, \cdot)$  and Proposition 4, it then holds

$$W_1(\mu_N(t), \mu(t)) \leq e^{LT} W_1(\mu_N(0), \mu(0)), \quad (21)$$

for all times  $t \in [0, T]$ . Since the  $(\mu_N(0))$  have uniformly bounded support and  $u(\cdot, \cdot)$  is bounded, then  $\mu_N(t)$  and  $\mu(t)$  have uniformly bounded support. Recall that  $\text{Var}(\mu) = \int_{\mathbb{R}} x^2 d\mu(x) - (\int_{\mathbb{R}} x d\mu(x))^2$ , and observe that the function  $x \mapsto x^2$  can be replaced by a map  $\phi(\cdot)$  with Lipschitz constant  $L_2$ , that coincides with the latter on the supports of  $(\mu_N(t)), \mu(t)$ . Notice now that the function  $x \in \mathbb{R} \mapsto \phi(x)/L_2$  is 1-Lipschitz, hence

$$\left| \int_{\mathbb{R}} x^2 d(\mu_N(T) - \mu(T))(x) \right| \leq L_2 W_1(\mu_N(T), \mu(T)).$$

By using the fact that  $u(t, \cdot)$  is Lipschitz and bounded, so is  $u^2(t, \cdot)$  with some constant  $L_u$ . We can also estimate  $(\int_{\mathbb{R}} x d\mu(x))^2$  with similar computation, since the integrand is 1-Lipschitz. Then, direct computations show

$$\begin{aligned} \left| \mathcal{C}_{\infty}(\mu_N^0, u) - \mathcal{C}_{\infty}(\mu^0, u) \right| &\leq \int_0^T L_u W_1(\mu_N(t), \mu(t)) \\ &+ \frac{1}{|\lambda|} \left( L_2 W_1(\mu_N(T), \mu(T)) + W_1(\mu_N(T), \mu(T))^2 \right). \end{aligned}$$

By combining (20)-(21), we conclude the proof.  $\square$

We are now ready to prove our main result Theorem 1.

*Proof of Theorem 1.* We study the three cases separately.

We first consider the situation in which  $\lambda > T$ , and prove that the Lipschitz-in-space control  $u^*(\cdot, \cdot)$  given by (18) is optimal for  $(\mathcal{P}_{\infty})$ . By contradiction, assume that there exists another Lipschitz control  $\tilde{u}(\cdot, \cdot)$  such that  $\mathcal{C}_{\infty}(\mu^0, \tilde{u}) < \mathcal{C}_{\infty}(\mu^0, u^*)$ , and define  $\tilde{\mathbf{u}}_N(\cdot)$  as  $\tilde{u}_i(t) := \tilde{u}(t, \hat{x}_i(t))$ . Then by Lemma 5, there exists  $N \geq 1$  large enough such that

$\mathcal{C}_N(\mathbf{x}_N^0, \tilde{\mathbf{u}}_N) = \mathcal{C}_{\infty}(\mu_N^0, \tilde{u}) < \mathcal{C}_{\infty}(\mu_N^0, u^*) = \mathcal{C}_N(\mathbf{x}_N^0, \mathbf{u}_N^*)$ , which contradicts the fact that  $u^*(\cdot, \cdot)$  is optimal for  $(\mathcal{P}_N)$ , as proven in Section 3. The case  $\lambda < 0$  is analogous.

We now sketch the proof for the case  $\lambda \in (0, T]$ . By contradiction, assume that  $\tilde{u}(\cdot, \cdot)$  is optimal for  $(\mathcal{P}_{\infty})$  and that  $u(t, \cdot)$  is Lipschitz with constant  $L > 0$ . By Lemma 5, for each  $\varepsilon > 0$  there exists  $N_{\varepsilon} \geq 1$  such that

$$|\mathcal{C}_N(\mathbf{x}(0), \tilde{\mathbf{u}}_N) - \mathcal{C}_{\infty}(\mu_0, \tilde{u})| < \varepsilon, \quad (22)$$

where  $\tilde{\mathbf{u}}_N(\cdot)$  is defined as before by  $\tilde{u}_i(t) := \tilde{u}(t, x_i(t))$  for almost every  $t \in [0, T]$ . Our goal is now to compute

$$c_N := \mathcal{C}_N(\mathbf{x}_N^0, \tilde{\mathbf{u}}_N) - \mathcal{C}_N(\mathbf{x}_N^0, \mathbf{u}_N^*),$$

where  $\mathbf{u}$  is given by (15). Since  $u^*(\cdot)$  is optimal for  $(\mathcal{P}_N)$ , it necessarily holds  $c_N \geq 0$ . Then, we have two cases.

- If  $c_N \geq d$  for some  $d > 0$  and all  $N \geq 1$ , there exists a Lipschitz approximation  $\hat{u}(t, \cdot)$  of (19) such that

$$\mathcal{C}_N(\mathbf{x}_N^0, \tilde{\mathbf{u}}_N) - \mathcal{C}_N(\mathbf{x}_N^0, \hat{\mathbf{u}}_N) > d/2,$$

independently of  $N$ . By letting  $N \rightarrow \infty$  in (22), and thus  $\varepsilon \rightarrow 0$ , it holds

$$\mathcal{C}_{\infty}(\mu_0, \hat{u}) \leq \mathcal{C}_{\infty}(\mu_0, \tilde{u}) - d/2,$$

which contradicts the optimality of  $\tilde{u}(\cdot, \cdot)$ .

- If  $(c_N)$  is not bounded from below by a positive constant, there exists a subsequence (that we do not relabel) such that  $c_N \rightarrow 0$ , i.e. the cost  $\mathcal{C}_N(\mathbf{x}_N^0, \tilde{\mathbf{u}}_N)$  is arbitrarily close to the optimal value  $\mathcal{C}_N(\mathbf{x}_N^0, \mathbf{u}_N^*)$ . However, a direct computation of solutions of (3) with any  $L$ -Lipschitz control shows that the cost does not converge, which leads to a contradiction.

## 5. CONCLUSIONS

In this article, we showed that the problem of either maximizing or minimizing the variance for multi-agent optimal control problems and for their mean-field approximation as  $N \rightarrow +\infty$  exhibit very different behaviors, depending on the relative weight between the quadratic control penalization and the variance. When minimizing the variance functional, or maximizing it with a sufficiently small time

horizon, the optimal controls for  $(\mathcal{P}_N)$  allow to build an optimal control for  $(\mathcal{P}_{\infty})$  that is Lipschitz. On the contrary when the time horizon is large in the variance maximization problem, the discrete optimal controls cannot give rise to a regular vector field, and this allows to prove that there exist no Lipschitz solutions to  $(\mathcal{P}_{\infty})$ .

This interplay between Lipschitz solutions for the macroscopic problem and the construction of suitable Lipschitz feedbacks at the microscopic level has been investigated in greater generality in Bonnet and Rossi (2021).

## REFERENCES

- Ambrosio, L., Gigli, N., and Savaré, G. (2008). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag.
- Berman, S., Halasz, A., Hsieh, M., and Kumar, V. (2009). Optimized Stochastic Policies for Task Allocation in Swarms of Robots. *IEEE Trans. Rob.*, 25(4), 927–937.
- Bongini, M., Fornasier, M., Rossi, F., and Solombrino, F. (2017). Mean-field Pontryagin maximum principle. *J. Optim. Theory Appl.*, 175(1), 1–38.
- Bonnet, B. (2019). A Pontryagin Maximum Principle in Wasserstein Spaces for Constrained Optimal Control Problems. *ESAIM COCV*, 25(52).
- Bonnet, B. and Frankowska, H. (2021a). Necessary Optimality Conditions for Optimal Control Problems in Wasserstein Spaces. *Applied Mathematics and Optimization*, to appear.
- Bonnet, B. and Frankowska, H. (2021b). Differential inclusions in wasserstein spaces: The cauchy-lipschitz framework. *Journal of Differential Equations*, 271, 594–637.
- Bonnet, B. and Rossi, F. (2019). The Pontryagin maximum principle in the Wasserstein space. *Calc. Var. Partial Differential Equations*, 58:11.
- Bonnet, B. and Rossi, F. (2021). Intrinsic lipschitz regularity of mean-field optimal controls. *SIAM J. Control Optim.*, to appear.
- Bullo, F., Cortés, J., and Martinez, S. (2009). *Distributed Control of Robotic Networks*. App. Math. Princeton University Press.
- Burger, M., Di Francesco, M., Markowich, P., and Wolfram, M.T. (2014). Mean field games with nonlinear mobilities in pedestrian dynamics. *Discrete Contin. Dyn. Syst. Ser. B*, 19, 1311–1333.
- Camazine, S., Deneubourg, J.L., Franks, N.R., Sneyd, J., Theraulaz, G., and Bonabeau, E. (2001). *Self-Organization in Biological Systems*. Princeton University Press.
- Caponigro, M., Fornasier, M., Piccoli, B., and Trélat, E. (2015). Sparse Stabilization and Control of Alignment Models. *Math. Mod. Meth. Appl. Sci.*, 25 (3), 521–564.
- Carrillo, J., Fornasier, M., Rosado, J., and Toscani, G. (2010). Asymptotic Flocking for the Kinetic Cucker-Smale Model. *SIAM Journal on Mathematical Analysis*, 42(1), 218–236.
- Cavagnari, G., Marigonda, A., and Piccoli, B. (2020). Generalized Dynamic Programming Principle and Sparse Mean-Field Control Problems. *Journal of Mathematical Analysis and Applications*, 481(1), 123437.
- Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Springer.
- Cristiani, E., Piccoli, B., and Tosin, A. (2011). Multiscale modeling of granular flows with application to crowd dynamics. *Multiscale Modeling & Simulation*, 9(1), 155–182.
- Cucker, F. and Smale, S. (2007). Emergent Behavior in Flocks. *IEEE Trans. Automat. Control*, 52(5), 852–862.
- Duprez, M., Morancey, M., and Rossi, F. (2019). Approximate and Exact Controllability of the Continuity Equation with a Localized Vector Field. *SIAM Journal on Control and Optimization*, 57(2), 1284–1311.
- Duprez, M., Morancey, M., and Rossi, F. (2020). Minimal time problem for crowd models with a localized vector field. *Journal of Differential Equations*, 269(1), 82–124.

- Elamvazhuthi, K. and Berman, S. (2019). Mean-field models in swarm robotics: A survey. *Bioinspiration & Biomimetics*, 15(1), 015001.
- Elamvazhuthi, K., Kuiper, H., and Berman, S. (2018). Pde-based optimization for stochastic mapping and coverage strategies using robotic ensembles. *Automatica*, 95, 356–367.
- Fornasier, M., Lisini, S., Orrieri, C., and Savaré, G. (2019). Mean-Field Optimal Control as Gamma-Limit of Finite Agent Controls. *Europ. Journ. of App. Math.*, 1–34.
- Fornasier, M. and Solombrino, F. (2014). Mean Field Optimal Control. *Esaim COCV*, 20(4), 1123–1152.
- Fornasier, M., Piccoli, B., and Rossi, F. (2014). Mean-field sparse optimal control. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2028), 20130400, 21.
- Golse, F. (2016). *On the Dynamics of Large Particle Systems in the Mean Field Limit*, 1–144. Springer International Publishing.
- Ha, S. and Liu, J. (2009). A Simple Proof of the Cucker-Smale Flocking Dynamics and Mean-Field Limit. *Comm. Math. Sci.*, 7(2), 297–325.
- Leonard, N. (2013). Multi-agent system dynamics: Bifurcation and behavior of animal groups. *Plenary paper IFAC Symposium on Nonlinear Control Systems, Toulouse, France*.
- Piccoli, B. and Rossi, F. (2013). Transport Equation with Nonlocal Velocity in Wasserstein Spaces : Convergence of Numerical Schemes. *Acta App. Math.*, 124(1), 73–105.
- Piccoli, B., Rossi, F., and Trélat, E. (2015). Control to flocking of the kinetic Cucker-Smale model. *SIAM J. Math. Anal.*, 47(6), 4685–4719.
- Pogodaev, N. (2016). Optimal Control of Continuity Equations. *NoDEA*, 23, 21.