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WEIGHTED ERDŐS-KAC THEOREM IN SHORT INTERVALS

KUI LIU & JIE WU

ABSTRACT. In this paper, we generalize Elliott's weighted Erdős-Kac theorem to the case of short intervals.

1. INTRODUCTION

Recently Elliott [3, 4] established a weighted central limit theorem for the Fourier coefficients of cusp form. For comparison, he also gave a weighted Erdős-Kac theorem on the value distribution of the function $\omega(n)$ that counts the number of distinct prime divisors of the positive integer n . As usual, denote by $d(n)$ the classic divisor function. For $\alpha \in \mathbb{R}$, define

$$(1.1) \quad D_\alpha(x) := \sum_{n \leq x} d(n)^\alpha.$$

Elliott's weighted Erdős-Kac theorem can be stated as follows (see [4, Theorem]): for each λ , we have

$$(1.2) \quad \frac{1}{D_\alpha(x)} \sum_{\substack{n \leq x \\ \omega(n) - 2^\alpha \log_2 x \leq \lambda(2^\alpha \log_2 x)^{1/2}}} d(n)^\alpha \rightarrow \Phi(\lambda)$$

as $x \rightarrow \infty$, where \log_k denotes the k -fold iterated logarithm and $\Phi(\lambda)$ is the Gaussian law defined by

$$(1.3) \quad \Phi(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\tau^2/2} d\tau.$$

The case of $\alpha = 0$ of (1.2) firstly was established by Erdős & Kac [5] in 1939. This problem has a long and rich history. The best actual result is due to Delange [2].

The aim of this paper is to generalize Elliott's result (1.2) to the case of short intervals. Similar to (1.1), write

$$(1.4) \quad D_\alpha(x, y) := \sum_{x < n \leq x+y} d(n)^\alpha.$$

Our result is as follows.

Theorem 1.1. (i) *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. Then for each real number λ , we have*

$$(1.5) \quad \frac{1}{D_\alpha(x, y)} \sum_{\substack{x < n \leq x+y \\ \omega(n) - 2^\alpha \log_2 x \leq \lambda(2^\alpha \log_2 x)^{1/2}}} d(n)^\alpha = \Phi(\lambda) + O_{\alpha, \varepsilon} \left(\frac{1}{\sqrt{\log_2 x}} \right)$$

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uniformly for $x \rightarrow \infty$ and $x^{7/12+\varepsilon} \leq y \leq x$, where the implied constant depends on α and ε only. The error term in (1.5) is optimal.

(ii) The same result also holds if the summation condition on $\omega(n)$ may be replaced by $\log d(n)/\log 2 - 2^\alpha \log_2 x \leq \lambda(2^\alpha \log_2 x)^{1/2}$.

Remark 1. The exponent $\frac{7}{12}$ in Theorem 1.1 comes from Huxley's zero-density bound for the Riemann ζ -function [6]. This constant can be reduced to $\frac{1}{2}$ if we assume the zero-density hypothesis.

In order to prove that the error term in (1.5) is optimal, we need to establish a weighted Landau prime number theorem in short intervals. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, define

$$(1.6) \quad \pi_{k,\alpha}(x, y) := \sum_{\substack{x < n \leq x+y \\ \omega(n)=k}} d(n)^\alpha.$$

We have the following result.

Theorem 1.2. *Let $\alpha \in \mathbb{R}$, $B > 0$ and $\varepsilon > 0$. Then we have*

$$(1.7) \quad \pi_{k,\alpha}(x, y) = \frac{y}{\log x} \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \left\{ \lambda_\alpha \left(\frac{k-1}{2^\alpha \log_2 x} \right) + O \left(\frac{\log_2 x}{k \log x} + \frac{k-1}{(\log_2 x)^2} \right) \right\}.$$

uniformly for $x \geq 3$, $x^{7/12+\varepsilon} \leq y \leq x$ and $1 \leq k \leq B2^\alpha \log_2 x$, where

$$(1.8) \quad \lambda_\alpha(z) = \frac{2^\alpha}{\Gamma(2^\alpha z + 1)} \prod_p \left(1 + \sum_{\nu \geq 1} \frac{(\nu+1)^\alpha z}{p^\nu} \right) \left(1 - \frac{1}{p} \right)^{2^\alpha z}$$

and the implied constant depend on α , B and ε only.

A principal tool for the proof of Theorems 1.1 and 1.2 is a rather general result of Cui, Lü and Wu [1] on the Selberg-Delange method in short intervals (see Lemma 2.1 below). A more result can be found in [8].

2. SOME PRELIMINARY LEMMAS

Let $f(n)$ be an arithmetic function and let its Dirichlet series be defined by

$$(2.1) \quad \mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}.$$

Let $z \in \mathbb{C}$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $C > 0$, $M > 0$ be some constants. We say that the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$ if the following conditions are verified:

(a) for any $\varepsilon > 0$ we have

$$(2.2) \quad |f(n)| \ll_\varepsilon M n^\varepsilon \quad (n \geq 1),$$

where the implied constant depends only on ε ;

(b) we have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \leq M(\sigma - 1)^{-\alpha} \quad (\sigma > 1);$$

(c) the Dirichlet series

$$(2.3) \quad \mathcal{G}(s; z, w) := \mathcal{F}(s) \zeta(s)^{-z} \zeta(2s)^{-w}$$

can be analytically continued to a holomorphic function in (some open set containing) $\sigma \geq \frac{1}{2}$ and, in this region, $\mathcal{G}(s; z, w)$ satisfies the bound

$$(2.4) \quad |\mathcal{G}(s; z, w)| \leq M(|\tau| + 1)^{\max\{\delta(1-\sigma), 0\}} \log^A(|\tau| + 1)$$

uniformly for $|z| \leq B$ and $|w| \leq C$, where and in the sequel we implicitly define the real numbers σ and τ by the relation $s = \sigma + i\tau$ and choose the principal value of the complex logarithm.

The following result is Corollary 1.2 of [1], which constitutes the key tool for the proof of Theorem 1.1.

Lemma 2.1. *If the Dirichlet series $\mathcal{F}(s)$ is of type $\mathcal{P}(z, w, \alpha, \delta, A, B, C, M)$, then for any $\varepsilon > 0$, we have*

$$(2.5) \quad \sum_{x < n \leq x+y} f(n) = y(\log x)^{z-1} \left\{ \lambda(z, w) + O\left(\frac{M}{\log x}\right) \right\}$$

uniformly for $x \geq 2$, $x^{(7+5\delta)/(12+5\delta)+\varepsilon} \leq y \leq x$, $|z| \leq B$ and $|w| \leq C$, where

$$\lambda(z, w) := \frac{\mathcal{G}(1; z, w) \zeta(2)^w}{\Gamma(z)}$$

and the implied constant in the O -term depends only on A, B, α, δ and ε .

Lemma 2.2. *Let $B > 0$ and $\varepsilon > 0$. Then we have*

$$(2.6) \quad \sum_{x < n \leq x+y} d(n)^\alpha z^{\omega(n)} = y(\log x)^{2^\alpha z - 1} \left\{ z \lambda_\alpha(z) + O_{B, \varepsilon} \left(\frac{1}{\log x} \right) \right\}$$

uniformly for $x \geq 2$, $x^{7/12+\varepsilon} \leq y \leq x$ and $|z| \leq B$, where

$$(2.7) \quad \lambda_\alpha(z) = \frac{2^\alpha}{\Gamma(2^\alpha z + 1)} \prod_p \left(1 + \sum_{\nu \geq 1} \frac{(\nu + 1)^\alpha z}{p^\nu} \right) \left(1 - \frac{1}{p} \right)^{2^\alpha z}.$$

In particular, we have

$$(2.8) \quad D_\alpha(x, y) = y(\log x)^{2^\alpha - 1} \left\{ \lambda_\alpha + O_\varepsilon \left(\frac{1}{\log x} \right) \right\}$$

uniformly for $x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$, where

$$(2.9) \quad \lambda_\alpha := \lambda_\alpha(1) = \frac{1}{\Gamma(2^\alpha)} \prod_p \left(1 + \sum_{\nu \geq 1} \frac{(\nu + 1)^\alpha}{p^\nu} \right) \left(1 - \frac{1}{p} \right)^{2^\alpha}.$$

Proof. Since the function $n \mapsto d(n)^\alpha z^{\omega(n)}$ is multiplicative, for $\Re s > 1$ we can write

$$\begin{aligned} \mathcal{F}_{\alpha, z}(s) &:= \sum_{n \geq 1} \frac{d(n)^\alpha z^{\omega(n)}}{n^s} \\ &= \prod_p \left(1 + \sum_{\nu \geq 1} \frac{(\nu + 1)^\alpha z}{p^{\nu s}} \right) \\ &= \zeta(s)^{z^\alpha} \zeta(2s)^{w_\alpha} G(s; z_\alpha, w_\alpha), \end{aligned}$$

where $z_\alpha := 2^\alpha z$, $w_\alpha := -2^{2\alpha-1}z^2 - (2^{\alpha-1} - 3^\alpha)z$ and the Euler product

$$(2.10) \quad \mathcal{G}(s; z_\alpha, w_\alpha) := \prod_p \left(1 + \sum_{\nu \geq 1} \frac{(\nu+1)^\alpha z^\nu}{p^{\nu s}} \right) \left(1 - \frac{1}{p^s} \right)^{z_\alpha} \left(1 - \frac{1}{p^{2s}} \right)^{w_\alpha}.$$

This Euler product is expandable as a Dirichlet series

$$\mathcal{G}(s; z_\alpha, w_\alpha) = \sum_{n \geq 1} b(n) n^{-s},$$

where $n \mapsto b(n)$ is the multiplicative function whose values on prime powers are given by the identity

$$1 + \sum_{\nu \geq 1} b(p^\nu) \xi^\nu = \left(1 + \sum_{\nu \geq 1} (\nu+1)^\alpha z^\nu \xi^\nu \right) (1 - \xi)^{z_\alpha} (1 - \xi^2)^{w_\alpha} \quad (|\xi| < 1).$$

In particular, we have

$$(2.11) \quad b(p) = b(p^2) = 0 \quad \text{for all primes } p$$

and

$$(2.12) \quad |b(p^\nu)| \leq M_1(B) 2^{\nu/6} \quad (\nu \geq 3, |z| \leq B)$$

with

$$M_1(B) := \max_{|z| \leq B} \max_{|\xi| \leq 2^{-1/6}} \left| \left(1 + \sum_{\nu \geq 1} (\nu+1)^\alpha z^\nu \xi^\nu \right) (1 - \xi)^{z_\alpha} (1 - \xi^2)^{w_\alpha} \right|.$$

With the help of (2.11) and (2.12), for $\sigma > \frac{1}{3}$ and $|z| \leq B$ we easily deduce that

$$\begin{aligned} \sum_p \sum_{\nu \geq 1} |b(p^\nu)| p^{-\nu\sigma} &\leq M_1(B) \sum_p \sum_{\nu \geq 3} (2^{-1/6} p^\sigma)^{-\nu} \\ &= M_1(B) \sum_p \frac{(2^{-1/6} p^\sigma)^{-3}}{1 - (2^{-1/6} p^\sigma)^{-1}} \\ &\leq \frac{2^{1/2} M_1(B)}{1 - 2^{-1/6}} \sum_p \frac{1}{p^{3\sigma}}. \end{aligned}$$

This shows that the Euler product $\mathcal{G}(s; z_\alpha, w_\alpha)$ converges absolutely for $\sigma > \frac{1}{3}$ and

$$(2.13) \quad |\mathcal{G}(s; z_\alpha, w_\alpha)| \leq M(B) \quad (\sigma \geq \tfrac{1}{2}, |z| \leq B)$$

with

$$M(B) := \exp \left(\frac{2^{1/2} M_1(B)}{1 - 2^{-1/6}} \sum_p \frac{1}{p^{3/2}} \right).$$

Consequently, $\mathcal{F}_{\alpha,z}(s)$ is a Dirichlet series of type

$$\mathcal{P}(z_\alpha, w_\alpha, |z_\alpha|, 0, 0, 2^\alpha B, 2^{2\alpha-1} B^2 + (3^\alpha - 2^{\alpha-1}) B, M(B)).$$

Applying Lemma 2.1, we get the required asymptotic formula (2.8). □

The third lemma is the Berry-Esseen inequality (see [7, Theorem II.7.14]).

Lemma 2.3. *Let F, G be two distribution functions with respective characteristic functions f and g . Suppose that G is differentiable and that G' is bounded on \mathbb{R} . Then we have*

$$\|F - G\|_\infty \leq 16 \frac{\|G'\|_\infty}{T} + 6 \int_{-T}^T \left| \frac{f(\tau) - g(\tau)}{\tau} \right| d\tau$$

for all $T > 0$, where $\|H\|_\infty := \sup_{\lambda \in \mathbb{R}} |H(\lambda)|$ for any real-valued function H defined on the real numbers.

3. PROOF OF THEOREM 1.2

Recall $\pi_{k,\alpha}(x, y)$ defined as in (1.6). Noticing that

$$\sum_{x < n \leq x+y} d(n)^\alpha z^{\omega(n)} = \sum_k \pi_{k,\alpha}(x, y) z^k,$$

we can apply the Cauchy formula to write, with $r := k/(2^\alpha \log_2 x)$,

$$\pi_{k,\alpha}(x, y) = \frac{1}{2\pi i} \oint_{|z|=r} \left(\sum_{x < n \leq x+y} d(n)^\alpha z^{\omega(n)} \right) \frac{dz}{z^{k+1}}.$$

By Lemma 2.2, it follows that

$$(3.1) \quad \pi_{k,\alpha}(x, y) = \frac{y}{\log x} \cdot I_{k,\alpha}(x; r) + O_{\alpha, B, \varepsilon} \left(\frac{y}{(\log x)^2} \cdot \frac{(2^\alpha \log_2 x)^k}{k!} \right),$$

uniformly for $x \geq 2$, $x^{7/12+\varepsilon} \leq y \leq x$ and $k \leq 2^\alpha B \log_2 x$, where

$$I_{k,\alpha}(x; r) := \frac{1}{2\pi i} \oint_{|z|=r} \frac{(\log x)^{2^\alpha z} \lambda_\alpha(z)}{z^k} dz$$

and we have used the following estimations

$$\begin{aligned} & \ll \oint_{|z|=r} \frac{(\log x)^{2^\alpha \Re z}}{|z|^{k+1}} |dz| \\ & \ll \left(\frac{2^\alpha \log_2 x}{k} \right)^k \int_0^{2\pi} e^{k \cos \theta} d\theta \\ & \ll \left(\frac{2^\alpha \log_2 x}{k} \right)^k \left(\int_0^{\pi/2} e^{k \cos \theta} d\theta + 1 \right) \quad (t = k(1 - \cos \theta)) \\ & \ll \left(\frac{2^\alpha \log_2 x}{k} \right)^k \left(\frac{e^k}{\sqrt{k}} \int_0^k e^{-t} t^{-1/2} dt + 1 \right) \\ & \ll \frac{(2^\alpha \log_2 x)^k}{k!}, \end{aligned}$$

thanks to the Stirling formula.

It remains to evaluate $I_{k,\alpha}(x; r)$. We shall discuss two cases: $k = 1$ or $k \geq 2$. Since $z \mapsto \lambda_\alpha(z)$ is analytic for $|z| \leq B$, we have

$$I_{1,\alpha}(x; r) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{e^{z 2^\alpha \log_2 x} \lambda_\alpha(z)}{z} dz = \lambda_\alpha(0) = 2^\alpha.$$

Inserting it into (3.1), we obtain that

$$\pi_{1,\alpha}(x, y) = \frac{2^\alpha y}{\log x} \left\{ 1 + O_{\alpha,\varepsilon} \left(\frac{\log_2 x}{\log x} \right) \right\}.$$

This proves (1.7) for $k = 1$.

Next we suppose that $k \geq 2$. Since $z \mapsto \lambda_\alpha(z)$ is analytic for $|z| \leq B$, we have $I_{k,\alpha}(x; r) = I_{k,\alpha}(x; r_0)$ with $r_0 := (k-1)/(2^\alpha \log_2 x)$. Writing the Taylor expansion of $\lambda_\alpha(z)$ at $z = r_0$:

$$(3.2) \quad \lambda_\alpha(z) = \lambda_\alpha(r_0) + \lambda'_\alpha(r_0)(z - r_0) + (z - r_0)^2 \int_0^1 (1-t) \lambda''_\alpha(r_0 + t(z - r_0)) dt,$$

we shall estimate the contributions of three terms on the right-hand side of (3.2) to $I_{k,\alpha}(x; r_0)$. Firstly those of the first two terms are, respectively,

$$(3.3) \quad \frac{\lambda_\alpha(r_0)}{2\pi i} \oint_{|z|=r} \frac{e^{z 2^\alpha \log_2 x}}{z^k} dz = \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \lambda_\alpha \left(\frac{k-1}{2^\alpha \log_2 x} \right)$$

and

$$(3.4) \quad \frac{\lambda'_\alpha(r_0)}{2\pi i} \oint_{|z|=r} \frac{e^{z 2^\alpha \log_2 x} (z - r_0)}{z^k} dz = \lambda'_\alpha(r_0) \left(\frac{(2^\alpha \log_2 x)^{k-2}}{(k-2)!} - r_0 \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \right) = 0.$$

For $0 \leq t \leq 1$ and $|z| = r_0$, we have

$$|r_0 + t(z - r_0)| = |r_0(1-t) + tz| \leq r_0(1-t) + t|z| = r_0.$$

Since $z \mapsto \lambda_\alpha(z)$ is analytic for $|z| \leq B$, there is a positive constant C_α such that $|\lambda''_\alpha(z)| \leq C_\alpha$ for $|z| \leq B$. Thus the contribution of the third term on the right-hand side of (3.2) to $I_{k,\alpha}(x; r_0)$ is

$$(3.5) \quad \begin{aligned} & \ll_\alpha \int_0^{2\pi} e^{(k-1)\cos\theta} r_0^{-(k-3)} |e^{i\theta} - 1|^2 d\theta \\ & \ll_\alpha r_0^{-(k-3)} \left(\int_0^{\pi/2} e^{(k-1)\cos\theta} (1 - \cos\theta) d\theta + \pi \right) \\ & \ll_\alpha r_0^{-(k-3)} e^{k-1} (k-1)^{-3/2} \left(\int_0^{k-1} e^{-t} t^{1/2} dt + \pi \right) \\ & \ll_\alpha \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \cdot \frac{k-1}{(2^\alpha \log_2 x)^2}. \end{aligned}$$

Inserting (3.3), (3.4) and (3.5) into (3.1), we find that

$$\begin{aligned} \pi_{k,\alpha}(x, y) &= \frac{y}{\log x} \cdot \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \lambda_\alpha \left(\frac{k-1}{2^\alpha \log_2 x} \right) \\ &\quad + O_{\alpha,B,\varepsilon} \left(\frac{y}{(\log x)^2} \cdot \frac{(\log_2 x)^k}{k!} + \frac{y}{\log x} \cdot \frac{(2^\alpha \log_2 x)^{k-1}}{(k-1)!} \cdot \frac{k-1}{(\log_2 x)^2} \right), \end{aligned}$$

which is equivalent to (1.7).

4. PROOF OF THEOREM 1.1

Denote by $F_{x,y}(\lambda)$ the left-hand side of (1.5) and by $\varphi_{x,y}(\tau)$ its characteristic function, i.e.,

$$\begin{aligned}
 \varphi_{x,y}(\tau) &:= \int_{-\infty}^{+\infty} e^{i\tau\lambda} dF_{x,y}(\lambda) \\
 (4.1) \quad &= \frac{1}{D_\alpha(x,y)} \sum_{x < n \leq x+y} d(n)^\alpha \exp \left\{ i\tau \frac{\omega(n) - 2^\alpha \log_2 x}{\sqrt{2^\alpha \log_2 x}} \right\} \\
 &= \frac{e^{-i\tau T}}{D_\alpha(x,y)} \sum_{x < n \leq x+y} d(n)^\alpha e^{i(\tau/T)\omega(n)},
 \end{aligned}$$

where $T = \sqrt{2^\alpha \log_2 x}$. By using Lemma 2.3 with $(F, G) = (F_{x,y}, \Phi)$, it follows that

$$\|F_{x,y} - \Phi\|_\infty \leq \frac{16}{\sqrt{2\pi}T} + 6 \int_{-T}^T \left| \frac{\varphi_{x,y}(\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau.$$

Thus it suffices to show that

$$(4.2) \quad \int_{-T}^T \left| \frac{\varphi_{x,y}(\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau \ll \frac{1}{T}$$

uniformly for $x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$.

Applying Lemma 2.2 with $z = e^{it}$, we have

$$\frac{1}{D_\alpha(x,y)} \sum_{x < n \leq x+y} d(n)^\alpha e^{it\omega(n)} = (\log x)^{2^\alpha(e^{it}-1)} A(e^{it}) + O_\varepsilon\left(\frac{1}{\log x}\right)$$

uniformly for $t \in \mathbb{R}$, $x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$, where $\lambda_\alpha(z)$ and λ_α are defined as in (2.7) and (2.9) respectively, and $A(z) := z\lambda_\alpha(z)/\lambda_\alpha$ is an entire function of z such that $A(1) = 1$. Taking $t = \tau/T$, the preceding asymptotic formula implies that

$$(4.3) \quad \varphi_{x,y}(\tau) = (\log x)^{2^\alpha(e^{i(\tau/T)}-1)} A(e^{i(\tau/T)}) e^{-i\tau T} + O_\varepsilon((\log x)^{-1})$$

uniformly for $x \geq 2$, $x^{7/12+\varepsilon} \leq y \leq x$ and $|\tau| \leq T$.

In view of the inequality $\cos t - 1 \leq -2(t/\pi)^2$ ($|t| \leq 1$), we have

$$|(\log x)^{2^\alpha(e^{i(\tau/T)}-1)}| = e^{(\cos(\tau/T)-1)T^2} \leq e^{-2(\tau/\pi)^2},$$

from which we deduce that $\varphi_{x,y}(\tau) \ll_\varepsilon e^{-2(\tau/\pi)^2}$ uniformly for $x \geq 2$, $x^{7/12+\varepsilon} \leq y \leq x$ and $|\tau| \leq T$. Thus

$$(4.4) \quad \int_{\pm T^{1/3}}^{\pm T} \left| \frac{\varphi_{x,y}(\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau \ll \int_{T^{1/3}}^T e^{-2(\tau/\pi)^2} d\tau \ll \frac{1}{T}.$$

With the help of the Taylor developments

$$A(e^{i(\tau/T)}) = 1 + O(\tau/T), \quad e^{i(\tau/T)} - 1 = i(\tau/T) - \frac{1}{2}(\tau/T)^2 + O((\tau/T)^3),$$

we deduce that

$$\begin{aligned}
 (\log x)^{2^\alpha(e^{i(\tau/T)}-1)} A(e^{i(\tau/T)}) e^{-i\tau T} &= e^{-\tau^2/2 + O(\tau^3/T)} \left\{ 1 + O\left(\frac{|\tau|}{T}\right) \right\} \\
 &= e^{-\tau^2/2} \left\{ 1 + O\left(\frac{|\tau| + |\tau|^3}{T}\right) \right\}
 \end{aligned}$$

for $|\tau| \leq T^{1/3}$. Inserting into (4.3), it follows that

$$\varphi_{x,y}(\tau) = e^{-\tau^2/2} \left\{ 1 + O\left(\frac{|\tau| + |\tau|^3}{T}\right) \right\} + O_\varepsilon\left(\frac{1}{\log x}\right)$$

uniformly for $x \geq 2$, $x^{7/12+\varepsilon} \leq y \leq x$ and $|\tau| \leq T^{1/3}$. With the help of this evaluation, we can deduce that

$$(4.5) \quad \begin{aligned} \int_{\pm 1/\log x}^{\pm T^{1/3}} \left| \frac{\varphi_{x,y}(\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau &\ll \int_{1/\log x}^{T^{1/3}} \left(e^{-\tau^2/2} \frac{1 + \tau^2}{T} + \frac{1}{\tau \log x} \right) d\tau \\ &\ll \frac{1}{T} + \frac{\log_2 x}{\log x} \\ &\ll \frac{1}{T}. \end{aligned}$$

For $|\tau| \leq (\log x)^{-1}$, we have trivially

$$|\tau(\omega(n) - 2^\alpha \log_2 x) / \sqrt{2^\alpha \log_2 x}| \ll (|\tau| \log x) / T.$$

Thus we can write

$$\exp \left\{ i\tau \frac{\omega(n) - 2^\alpha \log_2 x}{\sqrt{2^\alpha \log_2 x}} \right\} = 1 + O\left(\frac{|\tau| \log x}{T}\right).$$

Inserting into (4.1), it follows that

$$\varphi_{x,y}(\tau) = 1 + O\left(\frac{|\tau| \log x}{T}\right).$$

From this and the relation $e^{-\tau^2/2} = 1 + O(\tau^2)$, we derive that

$$(4.6) \quad \int_{-1/\log x}^{1/\log x} \left| \frac{\varphi_{x,y}(\tau) - e^{-\tau^2/2}}{\tau} \right| d\tau \ll \int_{-1/\log x}^{1/\log x} \left(\frac{\log x}{T} + |\tau| \right) d\tau \ll \frac{1}{T}.$$

Now (4.2) follows from (4.4), (4.5) and (4.6) immediately.

Finally we prove that the error term in (1.5) is optimal. Define

$$R_\lambda(x, y) := \frac{1}{D_\alpha(x, y)} \sum_{\substack{x < n \leq x+y \\ \omega(n) - 2^\alpha \log_2 x \leq \lambda(2^\alpha \log_2 x)^{1/2}}} d(n)^\alpha - \Phi(\lambda), \quad R(x, y) := \sup_{\lambda \in \mathbb{R}} |R_\lambda(x, y)|.$$

Let $k := [2^\alpha \log_2 x]$ and $\theta := k - 2^\alpha \log_2 x$. Then we have

$$(4.7) \quad \begin{aligned} \frac{\pi_{k,\alpha}(x, y)}{D_\alpha(x, y)} &= F_{x,y}\left(\frac{\theta}{\sqrt{2^\alpha \log_2 x}}\right) - F_{x,y}\left(\frac{\theta - \frac{1}{2\sqrt{2\pi}}}{\sqrt{2^\alpha \log_2 x}}\right) \\ &\leq \Phi\left(\frac{\theta}{\sqrt{2^\alpha \log_2 x}}\right) - \Phi\left(\frac{\theta - \frac{1}{2\sqrt{2\pi}}}{\sqrt{2^\alpha \log_2 x}}\right) + 2R(x, y) \\ &= \int_{(\theta - \frac{1}{2\sqrt{2\pi}})/\sqrt{2^\alpha \log_2 x}}^{\theta/\sqrt{2^\alpha \log_2 x}} e^{-\tau^2/2} d\tau + 2R(x, y) \\ &\leq \frac{1}{2\sqrt{\pi 2^{\alpha+1} \log_2 x}} + 2R(x, y). \end{aligned}$$

On the other hand, Theorem 1.2 and (2.8) imply that

$$(4.8) \quad \frac{\pi_{k,\alpha}(x, y)}{D_\alpha(x, y)} \sim \frac{(2^\alpha \log_2 x)^{k-1}}{(\log x)^{2^\alpha} (k-1)!} \sim \frac{1}{\sqrt{\pi 2^{\alpha+1} \log_2 x}}$$

thanks to the Stirling formula. From (4.7) and (4.8), we derive that

$$\begin{aligned} R(x, y) &\geq \frac{1 + o(1)}{2\sqrt{\pi 2^{\alpha+1} \log_2 x}} - \frac{1}{4\sqrt{\pi 2^{\alpha+1} \log_2 x}} \\ &= \frac{1 + o(1)}{4\sqrt{\pi 2^{\alpha+1} \log_2 x}} \end{aligned}$$

uniformly for $x \geq 2$ and $x^{7/12+\varepsilon} \leq y \leq x$.

This completes the proof of Theorem 1.1.

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