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ELLIOTT-HALBERSTAM CONJECTURE AND VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES

JIE WU

Abstract. Denote by \(\mathbb{P}\) the set of all primes and by \(P^+(n)\) the largest prime factor of integer \(n \geq 1\) with the convention \(P^+(1) = 1\). For each \(\eta > 1\), let \(c = c(\eta) > 1\) be some constant depending on \(\eta\) and

\[ P_{a,c,\eta} := \{ p \in \mathbb{P} : p = P^+(q - a) \text{ for some prime } q \text{ with } p^\eta < q \leq c(\eta)p^\eta \}. \]

In this paper, under the Elliott-Halberstam conjecture we prove, for \(y \to \infty\),

\[ \pi_{a,c,\eta}(x) = |\{1, x\} \cap P_{a,c,\eta}| \sim \pi(x) \quad \text{or} \quad \pi_{a,c,\eta}(x) \gg_{a,\eta} \pi(x) \]

according to values of \(\eta\). These complement for some results of Banks-Shparlinski [1], of Wu [12] and of Chen-Wu [2].

1. Introduction

Denote by \(\mathbb{P}\) the set of all prime numbers and by \(P^+(n)\) the largest prime factor of the positive integer \(n \geq 1\) with the convention \(P^+(1) = 1\). Banks & Shparlinski [1] proposed to estimate the number of primes \(p\) that occur as the largest prime factor of a shifted prime \(q - a\) when \(q \in \mathbb{P}\) lies in a certain interval determined by \(p\). This question has applications in theoretical computer science and has been considered by Vishnoi [10].

Let \(\mathbb{Z}^*\) be the set of non-zero integers. For \(a \in \mathbb{Z}^*, c > 1\) and \(\eta > 0\), we put

\[ P_{a,c,\eta} := \{ r \in \mathbb{P} : r = P^+(q - a) \text{ for some prime } q \text{ with } r^\eta < q \leq cr^\eta \} \]

and

\[ \pi_{a,c,\eta}(y) := |\{ r \leq y : r \in P_{a,c,\eta}\}|, \quad \pi(y) := |\{ r \leq y : r \in \mathbb{P}\}|. \]

Banks & Shparlinski [1, Theorem 1.1] proved that for each \(\eta \in (\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})\), there exists a constant \(c = c(\eta) > 1\) such that the asymptotic formula

\[ \pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\left(\frac{y}{(\log y)^A}\right) \quad (y \to \infty) \]

holds for every fixed non-zero integer \(a \in \mathbb{Z}^*\) and any constant \(A > 1\). Moreover for \(2 \leq \eta < 1 + \frac{3}{4}\sqrt{2} \approx 2.0606\), this estimate holds for any constant \(c > 1\). Very recently, Wu [12] extended Banks-Shparlinski’s interval \((\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})\) to \((\frac{32}{17}, \eta_0)\), where \(\eta_0 \approx 2.142\) is the unique solution of the equation \(\eta - 1 - 4\eta \log(\eta - 1) = 0\) in \((1, \infty)\). Banks & Shparlinski [1, page 144] also remarked that the asymptotic formula (1.1) holds for \(\eta \in (1, \frac{32}{17})\) if we assume the Elliott-Halberstam conjecture (see EH

\[ \text{prime}[\varepsilon] \]

below). Subsequently, Chen & Wu

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[2] further extended the domain of \( \eta \) at the price proportion positive instead of density 1. More precisely, they proved that

\[
\pi_{a,c,\eta}(y) \geq \left( \log \frac{\sqrt{2}}{2} \right) \frac{\eta - 1}{\eta} \left( 1 - 4 \log(\eta - 1) - \delta \frac{\log(\eta - 1)^8}{c - 1} \right) \pi(y),
\]

where \( \delta = \delta(c, \eta) \) is sufficiently small positive number. Clearly, (1.2) implies

\[
\pi_{a,c,\eta}(y) \gg \pi(y)
\]

provided \( \eta < 1 + \sqrt{e} \). This is complement for the results of Banks-Shparlinski and of Wu mentioned above. It seems rather natural to pose the following question.

**Question 1.** Is the asymptotic formula (1.1) true for all \( \eta > 1 \)?

In this paper, we shall try to answer this question under the well-known Elliott-Halberstam conjecture. Firstly we state two versions of this conjecture for prime numbers.

**Conjecture 1** (Elliott–Halberstam). Let \( a \in \mathbb{Z}^* \) and \( \varepsilon \in (0, 1) \) be fixed constants.

(i) For any \( A > 0 \), the inequality

\[
\sum_{q \leq x^{1-\varepsilon}} \left| \sum_{p \leq x \atop p \equiv a \pmod{q}} 1 - \frac{\pi(x)}{\varphi(q)} \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}
\]

holds uniformly for all \( x \geq 3 \), where the letter \( p \) always denotes prime numbers, \( \varphi(q) \) is the Euler function and the implied constant depends on \( A, a \) and \( \varepsilon \).

(ii) Let \( \kappa_1(m) \) and \( \kappa_2(m) \) be the characteristic functions of the odd integers and of even integers, respectively. Then for any \( A > 0 \), we have

\[
\sum_{q \leq x^{1-\varepsilon}} \left| \sum_{n \leq x \atop n \equiv a \pmod{q}} \kappa_i(m) - \frac{1}{\varphi(q)} \sum_{m \leq x \atop (mq, 1)} \kappa_i(m) \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}
\]

uniformly for all \( x \geq 3 \), where the letter \( p \) always denotes prime numbers and the implied constant depends on \( A \) and \( a \).

**Remark 1.** According to the classical Bombieri-Vinogradov theorem and Proposition 2.2 of Wu [12], the Elliott-Halberstam conjectures \( \text{EH}_{\text{prime}}[\varepsilon] \) and \( \text{EH}_{\text{prime}}^*[\varepsilon] \) hold for all \( \varepsilon \in (\frac{1}{2}, 1) \).

Secondly we also need a version of this conjecture for friable numbers.

**Conjecture 2** (Elliott–Halberstam). Let \( a \in \mathbb{Z}^* \) and \( \varepsilon \in (0, 1) \) be fixed constants. For any \( A > 0 \), we have

\[
\sum_{q \leq x^{1-\varepsilon}} \left| \sum_{n \leq x \atop n \equiv a \pmod{q}, P^+(n) \leq y} 1 - \frac{1}{\varphi(q)} \sum_{n \leq x \atop (n,q) = 1, P^+(n) \leq y} 1 \right| \ll_{A,a,\varepsilon} \frac{x}{(\log x)^A}
\]

uniformly in \( x \geq y \geq 2 \).

**Remark 2.** According to Wolke’s work [11] (see also [4, Theorem 6]), \( \text{EH}_{\text{friable}}[\varepsilon] \) holds unconditionally for all \( \varepsilon \in (\frac{1}{2}, 1) \).
Our results are as follows.

**Theorem 1.** Let \( a \in \mathbb{Z}^* \) and \( c > 1 \) be fixed constants.

(i) Let \( \eta \in (1, \frac{2}{3}] \) and assume the Elliott-Halberstam conjecture \( \text{EH}_{\text{prime}}[\varepsilon] \) with \( \varepsilon = 1 - 1/\eta > 0 \). Then for any \( A > 1 \) we have

\[
\pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\left(\frac{y}{(\log y)^A}\right),
\]

as \( y \to \infty \).

(ii) Let \( \eta_1 \approx 2.3303 \) be the unique positive zero of the equation \( \eta - 1 - 2\eta \log(\eta - 1) = 0 \) in \((1, \infty)\). For each \( \eta \in [\eta_0, \eta_1) \), there is a sufficiently small positive number \( \varepsilon = \varepsilon(\eta) \) such that assuming the Elliott-Halberstam conjecture \( \text{EH}_{\text{prime}}[\varepsilon] \), for any \( A > 1 \) we have

\[
\pi_{a,c,\eta}(y) = \pi(y) + O_{A,a,c,\eta}\left(\frac{y}{(\log y)^A}\right),
\]

as \( y \to \infty \).

**Theorem 2.** Let \( a \in \mathbb{Z}^* \) and \( c > 1 \) be fixed constants. For every \( \eta \in [2, \infty) \), there is a sufficiently small positive number \( \varepsilon = \varepsilon(\eta) \) such that under the Elliott-Halberstam conjecture \( \text{EH}_{\text{friable}}[\varepsilon] \), we have

\[
\pi_{a,c,\eta}(y) \geq (\log \sqrt{2}) \frac{\eta - 1}{\eta} \pi(y) \left\{ 1 + O_{a,c,\eta}\left(\frac{1}{\sqrt{\log y}} + \varepsilon\right) \right\},
\]

as \( y \to \infty \).

**Remark 3.** (a) For comparison, we have

\[
1 + \frac{3}{4} \sqrt{2} \approx 2.060, \quad \eta_0 \approx 2.142, \quad \eta_1 \approx 2.330, \quad 1 + \sqrt{e} \approx 2.284, \quad 1 + \sqrt{\gamma} \approx 2.648.
\]

(b) The first assertion of Theorem 1 is due to Banks & Shparlinski [1, page 144]. Here we give a proof for convenience of the reader. By combining Theorem 1 with results of [12], we see that the asymptotic formula (1.1) holds for \( 1 < \eta < \eta_1 \) under the Elliott-Halberstam conjecture.

(c) Theorem 2 improves significantly Chen-Wu’s result (1.2) in two aspects: result and method. Firstly the proportion in (1.5) increases from \( \log 2 \) to \( \log \sqrt{2} \) when \( \eta \) runs over \([2, \infty)\), and that in (1.2) tends toward 0 as \( \eta \to 1 + \sqrt{e} \).Secondly our proof of (1.5) needs information of prime numbers in “arithmetic progressions” \( \{a + mq\}_{m \text{ friable}} \) with friable indice. For \((a, q) = 1 \) and \( x \geq y > 2 \), define the counting function

\[
\pi(x, y; q, a) := \sum_{\substack{p \leq x \mod q \in \mathbb{Z}^* \mod q \mod \eta \leq y}} 1.
\]

A systematic study on the asymptotic behaviour has been done by Liu, Wu & Xi [8], recently. We need a theorem of Bombieri-Vinogradov type and an inequality of Brun-Titchmarsh for this new counting function (see Lemmas 2.2–2.3 below).
2. SOME PRELIMINARY LEMMAS

In this section, we present three lemmas, which will be useful later.

2.1. The Rosser-Iwaniec linear sieve.

The first lemma is due to Iwaniec [6, 7].

Lemma 2.1. Let $D \geq 2$ and let $\mu(n)$ be the M"obius function. Then there are two sequences $\{\lambda_d^\pm\}_{d \geq 1}$, vanishing for $d > D$ or $\mu(d) = 0$, satisfying $|\lambda_d^\pm| \leq 1$, such that

\[
\sum_{d \mid n} \lambda_d^- \leq \sum_{d \mid n} \mu(d) \leq \sum_{d \mid n} \lambda_d^+ \quad (n \geq 1)
\]

and

\[
\sum_{d \mid P_p(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{p \leq z} \left(1 - \frac{w(p)}{p}\right) \left\{F(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt{\log D}}\right)\right\}
\]

\[
\sum_{d \mid P_p(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{p \leq z} \left(1 - \frac{w(p)}{p}\right) \left\{f(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt{\log D}}\right)\right\}
\]

for any $z \in [2, D]$, $s = (\log D) / \log z$, set of prime numbers $P$ and multiplicative function $w$ satisfying

\[
0 < w(p) < p \quad (p \in P),
\]

\[
\prod_{u < p \leq v, p \nmid P_p(z)} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v),
\]

where $P_p(z) := \prod_{p \leq z, p \nmid P_p(z)} p$ and the implied $O$-constants are absolute. Here $F, f$ are defined by the continuous solutions to the system

\[
\begin{cases}
    sF(s) = 2e^\gamma & (1 \leq s \leq 2) \\
    sf(s) = 0 & (0 < s \leq 2) \\
    (sF(s))' = f(s-1) & (s > 2) \\
    (sf(s))' = F(s-1) & (s > 2)
\end{cases}
\]

where $\gamma$ is the Euler constant.

2.2. Bombieri-Vinogradov theorem of for $\pi(x,y; q, a)$.

The second lemma is a theorem of Bombieri-Vinogradov type for the counting function $\pi(x,y; q, a)$ defined as in (1.6) (see [8, Theorem 2]).

Lemma 2.2. Let $a \in \mathbb{Z}^*$, $A > 0$ and $\kappa$ a non-negative arithmetic function. Assuming the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\epsilon]$, the following estimate

\[
\sum_{\substack{q < Q \leq \exp\{(\log x)^2/5+\epsilon\} \leq y \leq x \text{ and } Q \leq \min\{y, \sqrt{x}\}, \text{ where } \rho(u) \text{ is the Dickman function.}}}
\]
2.3. Brun-Titchmarsh inequality for $\pi(x, y; q, a)$.

The lemma below is a variant of [8, Theorem 1].

Lemma 2.3. Let $a \in \mathbb{Z}^+$ and $c > 1$ be fixed constants. For any $\varepsilon > 0$, we have

$$\pi(cx, y; q, a) - \pi(x, y; q, a) \leq \frac{4(c - 1)x}{\varphi(q) \log(x/q)} \rho\left(\frac{\log(x/q)}{\log y}\right) \left\{ 1 + O_{a,c,\varepsilon}\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\}$$

(2.6)

uniformly in $\exp\{\log \log x\}^{5/3 + \varepsilon} \leq y \leq x$ and $1 \leq q \leq \min\{y, \sqrt{x}\}$ with $(a, q) = 1$.

Proof. Denote by $S$ the quantity on the left-hand side of (2.6). Without loss of generality, we can assume $q$ is even and $a$ odd. Put $P_{2a}(z) := \prod_{p \leq z, p \nmid 2a} p$. By the Möbius inversion, we can write

$$S = \sum_{x < a + mq \leq cx} 1 + O(x^\varepsilon)$$

$$= \sum_{x < a + mq \leq cx} \sum_{d|(a + mq, P_{2a}(\sqrt{cx}))} \mu(d) + O(x^\varepsilon),$$

where $\{\lambda^+_d\}_{d \geq 1}$ is an upper bound sieve of level $D$ as in Lemma 2.1 and $q'$ is the inverse of $q$ modulo $d$ (i.e. $qq' \equiv 1 \pmod{d}$). To apply the Elliott-Halberstam conjecture $EH_{\text{friable}}[\varepsilon]$, we would like to remove the restriction $(a, m) = 1$ by Möbius inversion, so that

$$S \leq \sum_{x < a + mq \leq cx} \sum_{d|(a + mq, P_{2a}(\sqrt{cx}))} \lambda^+_d + O(x^\varepsilon)$$

$$= \sum_{d \leq D} \lambda^+_d \sum_{|d|P_{2a}(\sqrt{cx})} \sum_{(x-a)/q < \ell \leq (x-a)/q \atop m \equiv aq'(mod d)} \sum_{\ell|m, P^+(m) \leq y} 1 + O(x^\varepsilon),$$

We are now in a good position to employ the Elliott-Halberstam conjecture $EH_{\text{friable}}[\varepsilon]$ with $D = (x/q)^{1-\varepsilon}$, getting

$$S \leq S^* + O\left(\frac{x}{q(\log x)^A}\right),$$

(2.7)
where

\[ S^+ := \sum_{\ell|a} \mu(\ell) \sum_{d \leq D} \frac{\lambda_d^+}{\varphi(d)} \sum_{x/\ell q < m \leq cx/\ell q} \frac{1}{(d,m)=1, P^+(m) \leq y} \]

and we have used the trivial bound

\[ \sum_{\ell|a} \sum_{d \leq D} \frac{|\lambda_d^+|}{\varphi(d)} \sum_{x/\ell q < m \leq cx/\ell q} \frac{1}{(d,m)=1, P^+(m) \leq y} \ll a \log x. \]

Here we removed the restriction that \( P^+(\ell) \leq y \) since we henceforth assume \( y > |a| \).

Switching summations, it follows that

\[ S^+ = \sum_{\ell|a} \mu(\ell) \sum_{x/\ell q < m \leq cx/\ell q} \frac{\lambda_d^+}{\varphi(d)} \sum_{d \leq D} \frac{1}{(d,m)=1, P^+(m) \leq y} \sum_{d \in \mathcal{P}_{2amq}(\sqrt{cx})} a d \phi(d). \]

From Lemma 2.1, we deduce

\[ S^+ \leq \frac{\log \sqrt{x}}{\log(x/q)} \left\{ 2e^\gamma + O \left( \frac{1}{\sqrt{\log x}} \right) \right\} \prod_{x/q < m \leq cx/q} \left( 1 - \frac{1}{p-1} \right) \prod_{p < \sqrt{x}} \left( \frac{1}{p-1} \right) \sum_{x/q < m \leq cx/q} H(m), \]

(2.8)

where \( H(m) \) is the multiplicative function, defined by

\[ H(p^\nu) = \begin{cases} 1 & \text{if } p | 2q \text{ or } p > x^{1/2} \\ \frac{p-1}{p^2-2} & \text{if } p \nmid 2q \text{ and } p \leq x^{1/2} \end{cases} \]

for all \( \nu \geq 1 \). According to [8, (3.5)], we have

\[ \sum_{m \in S(x/q,y)} H(m) = \Phi \left( \frac{x}{q}, y \right) \frac{\varphi(a)}{a} \prod_{p < \sqrt{x}} \left( 1 + \frac{1}{p(p-2)} \right) \left\{ 1 + O \left( \frac{(\log \log x)^2}{\log y} \right) \right\}, \]

where \( S(x,y) := \{ n \leq x : P^+(n) \leq y \} \) and \( \Phi(x,y) := |S(x,y)| \). Combining this with (2.8), we find that

\[ S^+ \leq \frac{\log x}{\log(x/q)} \left\{ 2e^\gamma + O \left( \frac{1}{\sqrt{\log x}} \right) \right\} \frac{\varphi(a)}{a} \left\{ \Phi \left( \frac{cx}{q}, y \right) - \Phi \left( \frac{x}{q}, y \right) \right\} \prod_{p < \sqrt{x}} \left( 1 - \frac{1}{p} \right). \]
By the Mertens formula, it follows that
\[
\prod_{p \leq x^{1/2}} \left( 1 - \frac{1}{p} \right) = \frac{a q}{\varphi(a) \varphi(q)} \cdot \frac{2e^{-\gamma}}{\log x} \left\{ 1 + O\left( \frac{1}{\log x} \right) \right\}.
\]

On the other hand, according to [5, Theorem 1], we have
\[
\Psi(x, y) = x \rho \left( \frac{\log x}{\log y} \right) \left\{ 1 + O\left( \frac{\log \left( \frac{\log x}{\log y + 1} \right)}{\log y} \right) \right\}
\]
uniformly for \( x \geq 3 \) and \( \exp \left\{ \left( \frac{\log \log x}{\log y} \right)^{5/3 + \varepsilon} \right\} \leq y \leq x \). Combining these with (2.9) and (2.7), we can get the required inequality (2.6).

3. Proof of Theorem 2

For each prime \( r \in (\frac{1}{2} y, y] \), consider
\[
Q_r(y) := \sum_{x < q \leq cx \atop P^+(q-a) = r} 1.
\]
Noticing that \( P^+(q-a) = r \Leftrightarrow q \equiv a \pmod{r} \) and \( P^+(q-a) \leq r \),
we can write
\[
\sum_{y < r \leq 2y} Q_r(y) \geq \sum_{y < r \leq 2y} \sum_{x < q \leq cx \atop q \equiv a \pmod{r}, P^+(q-a) \leq y} 1
\]
\[
= \sum_{y < r \leq 2y} \left( \pi(cx, y; r, a) - \pi(x, y; r, a) \right)
\]
\[
= \mathcal{M} + \mathcal{E},
\]
where
\[
\mathcal{M} := \sum_{y < r \leq 2y} \left( \frac{\pi(cx)}{\varphi(r)} \rho \left( \frac{\log(cx/r)}{\log y} \right) - \frac{\pi(x)}{\varphi(r)} \rho \left( \frac{\log(x/r)}{\log y} \right) \right),
\]
\[
\mathcal{E} := \sum_{y < r \leq 2y} \left( \mathcal{E}(cx, y; r, a) - \mathcal{E}(x, y; r, a) \right),
\]
and
\[
\mathcal{E}(x, y; r, a) := \pi(x, y; r, a) - \frac{\pi(x)}{\varphi(r)} \rho \left( \frac{\log(x/r)}{\log y} \right).
\]

Since \( \eta \geq 2 \), we have \( y = x^{1/\eta} \leq x^{1/2} \) and \( Q = \min(y, \sqrt{x}) = y \). Using Lemma 2.2 with the characteristic function of prime numbers in \((y, 2y]\) in place of \( \kappa(q) \), we easily derive, under the conjecture of Elliott-Halberstam EH\( \varepsilon \)\( \prime \), that
\[
|\mathcal{E}| \leq \sum_{y < r \leq 2y} \left( |\mathcal{E}(cx, y; r, a)| + |\mathcal{E}(x, y; r, a)| \right)
\]
\[
\ll_a \frac{x}{(\log x)^2} + \varepsilon \eta \frac{\pi(x)}{\log y} \ll_a, \eta \varepsilon \frac{\pi(x)}{\log y}
\]
for all $x \geq x_0(\varepsilon)$, where we have used the following bound
\begin{equation}
\sum_{y < r < 2y} \frac{1}{\varphi(r)} = \sum_{y < r < 2y} \frac{1}{r} \left\{ 1 + O\left( \frac{1}{y} \right) \right\} = \frac{\log 2}{\log y} \left\{ 1 + O\left( \frac{1}{\log y} \right) \right\}
\end{equation}
and the implied constant depends on $a, \eta$ at most.

According to [9, Corollary III.5.8.3], we have $|\rho'(u)| \ll \rho(u) \log u$ ($u > 1$). Thus for all $r \in (y, 2y]$, we have
\[ \rho\left( \frac{\log(x/r)}{\log y} \right) = \rho(\eta - 1) \left\{ 1 + O_{\eta} \left( \frac{1}{\log y} \right) \right\}. \]

From this and (3.4), we derive
\begin{equation}
M = (\log 2)(c - 1)\rho(\eta - 1) \frac{\pi(x)}{\log y} \left\{ 1 + O_{a,c,\eta} \left( \frac{1}{\log y} + \varepsilon \right) \right\}.
\end{equation}

On the other hand, the Brun-Titchmarsh inequality (2.6) give us
\[ Q_r(y) \leq 4(c - 1) \frac{\rho(\eta - 1)}{\eta - 1} \cdot \frac{x}{y \log y} \left\{ 1 + O_{a,\eta} \left( \frac{1}{\sqrt{\log x}} \right) \right\} \]
for all primes $r \in (y, 2y]$. This implies that
\begin{equation}
\sum_{y < r < 2y} Q_r(y) \leq \rho(\eta - 1) \frac{4(c - 1)}{\eta - 1} \cdot \frac{x}{y \log y} \left\{ 1 + O_{a,\eta} \left( \frac{1}{\sqrt{\log x}} \right) \right\} \sum_{y < r < 2y, Q_r(y) \neq 0} 1.
\end{equation}

Combining (3.6) and (3.7), it follows that
\[ \sum_{y < r < 2y, Q_r(y) \neq 0} 1 \geq (\log \sqrt{2}) \cdot \frac{\eta - 1}{\eta} \pi(y) \left\{ 1 + O_{a,c,\eta} \left( \frac{1}{\sqrt{\log x}} + \varepsilon \right) \right\}. \]

This completes the proof of Theorem 2.

4. Proof of Theorem 1

As in [12], the letters $p$, $q$, $r$ and $\ell$ are always used to denote prime numbers, and $d$, $m$, and $n$ always denote positive integers. In what follows, let $a \in \mathbb{Z}^*$ and $\eta \in (1, \frac{32}{17}] \cup [\eta_0, \eta_1)$. Let $\delta$ be a sufficiently small positive constant and let $c > 1$ be a parameter to be chosen later. Let $x_0(A, a, c, \eta, \delta)$ be a large constant depending on $A, a, c, \eta, \delta$ at most. For $x \geq x_0(A, a, c, \eta, \delta)$ and $r \in (\frac{1}{2} y, y]$, put $x := r^\eta$. As usual, for $(a, d) = 1$ define
\[ \pi(x; d, a) := \sum_{p \leq x \atop p \equiv a (\mod d)} 1. \]
4.1. The case of $\eta \in (1, \frac{32}{17}]$.

For $\eta > 1$, $c > 1$, $y \geq 3$ and $x = y^n$, put

$$\mathcal{R}_b'(y) := \left\{ y < r \leq 2y : \left| \pi(x; r, a) - \pi(x) \varphi(r) \right| \geq \delta \pi(x) \varphi(r) \right\},$$

$$\mathcal{R}_b''(y) := \left\{ y < r \leq 2y : \left| \pi(cx; r, a) - \pi(cx) \varphi(r) \right| \geq \delta \pi(cx) \varphi(r) \right\}.$$

Noticing that $y = x^{1/n} = x^{1-(1-1/\eta)}$, the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}[\epsilon]$ with $\epsilon = 1 - 1/\eta$ allows us to deduce that

$$\delta \frac{\pi(x)}{y} |\mathcal{R}_b'(y)| \leq \sum_{y < r \leq 2y} \left| \pi(x; r, a) - \pi(x) \varphi(r) \right| \leq \frac{1}{A_{a,c,\delta,\eta}} \frac{x}{(\log x)^{A+1}},$$

which gives immediately

$$|\mathcal{R}_b'(y)| \ll A_{a,c,\delta,\eta} \frac{y}{(\log y)^A}. \tag{4.1}$$

Similarly

$$|\mathcal{R}_b''(y)| \ll A_{a,c,\delta,\eta} \frac{y}{(\log y)^A}. \tag{4.2}$$

Define

$$\mathcal{R}_g'(y) := \left\{ y < r \leq 2y : \pi(x; r, a) \leq (1 + \delta) \frac{\pi(x)}{\varphi(r)} \right\},$$

$$\mathcal{R}_g''(y) := \left\{ y < r \leq 2y : \pi(cx; r, a) \geq (1 - \delta) \frac{\pi(cx)}{\varphi(r)} \right\},$$

and

$$\mathcal{R}_g(y) := \mathcal{R}_g'(y) \cap \mathcal{R}_g''(y).$$

Clearly

$$\mathcal{R}_g(y) \subset \mathcal{P} \cap (y, 2y] \subset \mathcal{R}_b'(x) \cup \mathcal{R}_b''(x) \cup \mathcal{R}_g(y).$$

Thus the estimations (4.1) and (4.2) imply that

$$|\mathcal{R}_g(y)| = \pi(2y) - \pi(y) + O_{A_{a,c,\delta,\eta}} \left( \frac{y}{(\log y)^A} \right) \quad (y \geq 2). \tag{4.3}$$

Let $r \in \mathcal{R}_g(y)$ and let $\mathcal{Q}_r(y)$ be defined as in (3.1). When $\eta \in (1, \frac{32}{17}]$, we have $r > y = x^{1/\eta} \geq x^{17/32} > (cx)^{1/2}$. Thus the definition of $\mathcal{R}_g(y)$ allows us to write

$$\mathcal{Q}_r(y) = \pi(cx; r, a) - \pi(x; r, a) \geq (c - 1 - 3\delta) \frac{\pi(x)}{\varphi(r)} > 0, \tag{4.4}$$

where we have used the inequality $\pi(cx) \geq (c - \delta) \pi(x)$ for $x \geq x_0(a, c, \delta)$. By the definition of $\mathcal{P}_{a,c,\eta}$ and (4.4), it is easy to see that $\mathcal{R}_g(y) \subset \mathcal{P}_{a,c,\eta} \cap [y, 2y]$. In view of (4.3), we find that

$$\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A_{a,c,\eta}} \left( \frac{y}{(\log y)^A} \right).$$

This implies the first assertion of Theorem 1, thanks to standard dyadic split.
4.2. The case of $\eta \in [\eta_0, \eta_1)$.

In this case, for every prime $r \in \mathcal{R}_\eta(y)$, we can write
\[
Q_r(y) = \pi(cx; r, a) - \pi(x; r, a) - \mathcal{Q}_r(y)
\]
(4.5)
\[
\geq (c - 1 - 3\delta)\frac{\pi(x)}{\varphi(r)} - \mathcal{Q}_r(y).
\]
for $x \geq x_0(a, c, \delta)$, where
\[
\mathcal{Q}_r(y) := \sum_{x < q \leq cx, q \equiv a \pmod{r}, P(q-a) > r} 1.
\]

Similar to [12, Proposition 2.1], we can prove
\[
Q_r(y) \leq (c - 1 + 2\delta)\frac{2\eta \log(\eta - 1)}{\eta - 1} \cdot \frac{\pi(y)}{\varphi(r)} \left\{ 1 + O_{a,c,\delta,\eta,\epsilon}\left(\frac{1}{\sqrt[3]{\log r}}\right) \right\}
\]
for $y \geq 3$, $r \in (y, 2y]$ and $\eta \geq 2$.

Inserting (4.7) into (4.4) and taking $c = 1 + 2\sqrt{\delta}$, we can find that
\[
Q_r(y) \geq 2\sqrt{\delta} - \delta \left( \eta - 1 - 2\eta \log(\eta - 1) \cdot \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} \right) \frac{\pi(y)}{\varphi(r)}
\]
\[
= \left\{ G(\eta) + O(\sqrt{\delta}) \right\} 2\sqrt{\delta} \frac{1 - \sqrt{\delta}}{\eta - 1} \cdot \frac{\pi(y)}{\varphi(r)},
\]
where
\[
G(\eta) := \eta - 1 - 2\eta \log(\eta - 1).
\]

It is easy to see that $G(\eta)$ is decreasing on $[2, \infty)$ and $G(2) = 1$. Therefore there is a unique real number $\eta_1 \in (2, \infty)$ such that $G(\eta_1) = 0$ and for $\eta \in [2, \eta_1)$ we have the inequality
\[
Q_r(y) \gg_{A,a,\delta,\eta} \frac{\pi(y)}{\varphi(r)}
\]
for $y \geq y_0(A, a, \delta, \eta)$. As before, (4.9) allows us to deduce that $\mathcal{R}_\eta(y) \subseteq \mathcal{P}_{a,c,\eta} \cap (y, 2y]$. Combining this with (4.3) leads to
\[
\pi_{a,c,\eta}(2y) - \pi_{a,c,\eta}(y) = \pi(2y) - \pi(y) + O_{A,a,c,\delta,\eta}\left(\frac{y}{(\log y)^B}\right).
\]
This implies the required asymptotic formula (1.4).

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*The proof is identical and the only difference is that we can take $z = (y/q)^{(1-\varepsilon)/2}$ and $D = z^2$ thanks to the Elliott-Halberstam conjecture $\text{EH}_{\text{prime}}^\varepsilon$, instead of $z = (y/q)^{1/4}/(\log y)^B$ and $D = z^2$.*
ELLIOTT-HALBERSTAM CONJECTURE AND SHIFTED PRIMES

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