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THE ARCSINE LAW ON DIVISORS IN ARITHMETIC PROGRESSIONS MODULAR PRIME POWERS

BIN FENG & JIE WU

ABSTRACT. Let $x \rightarrow \infty$ be a parameter. In 2016, Feng proved that Deshouillers-Dress-Tenenbaum's arcsine law on divisors of the integers less than x also holds in arithmetic progressions for non Siegel 'exceptional' modulus $q \leq \exp\{(\frac{1}{4} - \varepsilon)(\log_2 x)^2\}$, where ε is an arbitrarily small positive number. In this paper, we shall show that in the case of prime-power modulus ($q := \mathfrak{p}^\varpi$ with \mathfrak{p} a fixed odd prime and $\varpi \in \mathbb{N}$) the arcsine law on divisors holds in arithmetic progressions for $q \leq x^{15/52 - \varepsilon}$.

1. INTRODUCTION.

For each positive integer n , denote by $\tau(n)$ the number of divisors of n and define the random variable D_n to take the value $(\log d)/\log n$, as d runs through the set of the divisors of n , with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$(1.1) \quad F_n(t) := \text{Prob}(D_n \leq t) = \frac{1}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \quad (0 \leq t \leq 1).$$

Deshouillers, Dress and Tenenbaum ([4] or [10, Theorem II.6.7]) proved that the Cesàro means of F_n converges uniformly to the arcsine law. More precisely, the asymptotic formula

$$(1.2) \quad \frac{1}{x} \sum_{n \leq x} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right)$$

holds uniformly for $x \geq 2$ and $0 \leq t \leq 1$ and the error term in (1.2) is optimal. Various variants of (1.2) have been investigated by different authors. In particular, Cui & Wu [3] and Cui, Lü & Wu [2] considered generalisation of (1.2) to the short interval case; and Feng & Wu [6] showed that the average distribution of divisors over integers representable as sum of two squares converges to the beta law. Based on Cui-Wu's method [3], Feng [5] studied analogue of (1.2) for arithmetic progressions. His result can be stated as follows: Let a and q be integer such that $(a, q) = 1$, and suppose that q is not a Siegel 'exceptional' modulus. Then for any $\varepsilon \in (0, \frac{1}{4})$ we have

$$(1.3) \quad \frac{1}{(x/q)} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O_\varepsilon\left(\frac{e^{\sqrt{\log q}}}{\sqrt{\log x}}\right)$$

uniformly for $0 \leq t \leq 1$, $x \geq 2$ and $q \leq \exp\{(\frac{1}{4} - \varepsilon)(\log_2 x)^2\}$, where $\log_2 := \log \log$.

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The aim of this paper is to improve the result above in the case of prime power modulus. Our result is as follows.

Theorem 1. *Let $q := \mathfrak{p}^\varpi$ with \mathfrak{p} an odd prime and $\varpi \in \mathbb{N}$. Then for any $\varepsilon > 0$, we have*

$$(1.4) \quad \frac{1}{(x/q)} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O_{\mathfrak{p}, \varepsilon} \left(\frac{1}{\sqrt{\log x}} \right)$$

uniformly for $0 \leq t \leq 1$, $x \geq 2$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^$ such that $(a, q) = 1$, where the implied constant depends on \mathfrak{p} and ε at most.*

Our improvement is double. Firstly, with $q = \mathfrak{p}^\varpi$ any Siegel zero must occur for $L(s, \chi)$ where χ is a real character modulo \mathfrak{p} . Since the implied constant in Theorem 1 is allowed to depend on \mathfrak{p} , there is no Siegel zero for the modulus $q = \mathfrak{p}^\varpi$. These considerations allow to remove the assumption of Siegel zero in Feng's result for $q = \mathfrak{p}^\varpi$ with an implied constant in the error term depending on \mathfrak{p} . Alternatively, this follows from Feng's result and Corollary 3.4 of the Banks and Shparlinski paper [1] (cf. Lemma 2.3 below). Secondly the domain of q is extended significantly.

2. PRELIMINARY

Our first lemma is an effective Perron formula (cf. [10, Corollary II.2.2.1]).

Lemma 2.1. *Let $F(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with finite abscissa of absolute convergence σ_a . Suppose that there exist some real number $\alpha > 0$ and a non-decreasing function $B(n)$ such that:*

- (a) $\sum_{n=1}^{\infty} |a_n| n^{-\varsigma} \ll (\varsigma - \sigma_a)^{-\alpha} \quad (\varsigma > \sigma_a)$,
- (b) $|a_n| \leq B(n) \quad (n \geq 1)$.

Then for $x \geq 2$, $T \geq 2$, $\sigma \leq \sigma_a$ and $\kappa := \sigma_a - \sigma + 1/\log x$, we have

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa+iT}^{\kappa-iT} F(s+w) x^w \frac{dw}{w} + O \left(x^{\sigma_a - \sigma} \frac{(\log x)^\alpha}{T} + \frac{B(2x)}{x^\sigma} \left(1 + x \frac{\log T}{T} \right) \right).$$

Lemma 2.2. *Let $q > 2$ be an integer.*

- (i) *If χ is a Dirichlet character modulo q , then we have*

$$L(\sigma + i\tau, \chi) \ll q^{1-\sigma} (|\tau| + 1)^{1/6} \log(|\tau| + 1).$$

- (ii) *If χ is a non principal Dirichlet character modulo q , then for $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \leq \sigma \leq 1$, $|\tau| + 2 \leq T$, we have*

$$L(\sigma + i\tau, \chi) \ll_\varepsilon (q^{1/2} T)^{1-\sigma+\varepsilon}.$$

Proof. See [9, p.485, Theorem 1] and [11, Exercise 241]. □

The next lemma is due to Banks-Shparlinski [1, Corollary 3.4.], which will play a key role in the proof of Theorem 1.

Lemma 2.3. *Let $q = \mathfrak{p}^\varpi$ with \mathfrak{p} an odd prime and $\varpi \in \mathbb{N}$. For each constant $A > 0$, there is a constant $c_0 = c_0(A, \mathfrak{p}) > 0$ depending only on A and \mathfrak{p} such that for any character χ modulo q , the Dirichlet L -function has no zero in the region*

$$(2.1) \quad \sigma > 1 - \frac{c_0}{(\log q)^{2/3}(\log_2 q)^{1/3}} \quad \text{and} \quad |\tau| \leq q^A.$$

The following lemma is a key for the proof of Theorem 1.

Lemma 2.4. *Let $q := \mathfrak{p}^\varpi$ with \mathfrak{p} a prime and $\varpi \in \mathbb{N}$ and let χ_0 be the principal character to the modulus q . Then we have*

$$(2.2) \quad \sum_{n \leq x} \frac{\chi_0(n)}{\tau(nd)} = \frac{hx}{\sqrt{\pi \log x}} \left\{ g(d) + O\left(\frac{(3/4)^{\omega(d)}}{\log x}\right) \right\}$$

uniformly for $x \geq 2$, $1 \leq d \leq x$ and $\varpi \geq 1$, where the implied constant is absolute, $\omega(d)$ is the number of all distinct prime factors of d ,

$$(2.3) \quad h := \sqrt{1 - \mathfrak{p}^{-1}} \prod_{(p, \mathfrak{p})=1} \sqrt{1 - p^{-1}} \frac{\log(1 - p^{-1})}{-p^{-1}}$$

and

$$(2.4) \quad g(d) := \prod_{p^\alpha \parallel d} \left(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-1})^j}{j + \alpha + 1} \right) \frac{-\chi_0(p)p^{-1}}{\log(1 - \chi_0(p)p^{-1})}.$$

Proof. As usual, denote by $v_p(n)$ the p -adic valuation of n . By using the formula

$$(2.5) \quad \tau(dn) = \prod_p (v_p(n) + v_p(d) + 1),$$

we write for $\Re s > 1$

$$(2.6) \quad \begin{aligned} f_d(s, \chi_0) &:= \sum_{n=1}^{\infty} \frac{\chi_0(n)}{\tau(dn)} n^{-s} = \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + v_p(d) + 1} \\ &= \prod_{(p, d)=1} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \times \prod_{p^\alpha \parallel d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + \alpha + 1} \\ &= \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \times \prod_{p^\alpha \parallel d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + \alpha + 1} \left(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \right)^{-1} \\ &= L(s, \chi_0)^{1/2} G_d(s, \chi_0), \end{aligned}$$

where

$$G_d(s, \chi_0) := \prod_p \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \sqrt{1 - \chi_0(p)/p^s} \prod_{p^\alpha \parallel d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + \alpha + 1} \left(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \right)^{-1}$$

is a Dirichlet series that converges absolutely for $\Re s > \frac{1}{2}$.

We easily see that

$$\prod_{p^\alpha \parallel d} \sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + \alpha + 1} \left(\sum_{j=0}^{\infty} \frac{(\chi_0(p)p^{-s})^j}{j + 1} \right)^{-1} = \frac{1}{\alpha + 1} + O\left(\frac{1}{\sqrt{p}}\right).$$

for $\Re s \geq \frac{1}{2}$, where the implied constant is absolute. This implies that for any $\varepsilon > 0$,

$$(2.7) \quad G_d(s, \chi_0) \ll \prod_{p^\alpha \parallel d} \left\{ \frac{1}{\alpha + 1} + O\left(\frac{1}{\sqrt{p}}\right) \right\} \leq C_\varepsilon \left(\frac{3}{4}\right)^{\omega(d)}$$

for $\Re s \geq \frac{1}{2} + \varepsilon$, where $C_\varepsilon > 0$ is a constant depending on ε only.

We can apply Lemma 2.1 with the choice of parameters $\sigma_a = 1$, $B(n) = 1$, $\alpha = \frac{1}{2}$ and $\sigma = 0$ to write

$$\sum_{n \leq x} \frac{\chi_0(n)}{\tau(nd)} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f_d(s, \chi_0) \frac{x^s}{s} ds + O_\varepsilon\left(\frac{x \log x}{T}\right),$$

where $b = 1 + 2/\log x$ and $100 \leq T \leq x$ such that $\zeta(\sigma + iT) \neq 0$ for $0 < \sigma < 1$.

Let \mathcal{M}_T be the boundary of the modified rectangle with vertices $(\frac{1}{2} + \varepsilon) \pm iT$ and $b \pm iT$ as follows:

- $\varepsilon > 0$ is a small constant chosen such that $\zeta(\frac{1}{2} + \varepsilon + i\gamma) \neq 0$ for $|\gamma| < T$;
- the zeros of $\zeta(s)$ of the form $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2} + \varepsilon$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$;
- the pole of $\zeta(s)$ at the points $s = 1$ is avoided by the truncated Hanke contour Γ (its upper part is made up of an arc surrounding the point $s = 1$ with radius $r := 1/\log x$ and a line segment joining $1 - r$ to $(\frac{1}{2} + \varepsilon)$).

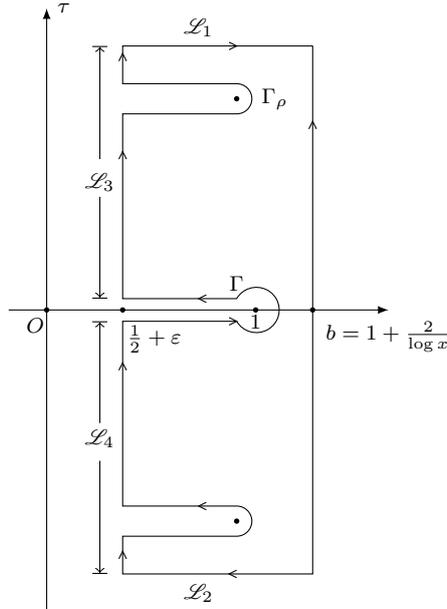


Figure 1 – Contour \mathcal{M}_T

Clearly the function $f_d(s, \chi_0)$ is analytic inside \mathcal{M}_T . By the residue theorem, we can write

$$(2.8) \quad \sum_{n \leq x} \frac{\chi_0(n)}{\tau(nd)} = I + \frac{1}{2\pi i} \left(I_1 + \cdots + I_4 + \sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} I_\rho \right) + O_\varepsilon \left(\frac{x \log x}{T} \right),$$

where

$$I := \frac{1}{2\pi i} \int_{\Gamma} f_d(s, \chi_0) \frac{x^s}{s} ds, \quad I_\rho := \int_{\Gamma_\rho} f_d(s, \chi_0) \frac{x^s}{s} ds, \quad I_j := \int_{\mathcal{L}_j} f_d(s, \chi_0) \frac{x^s}{s} ds.$$

A. *Evaluation of I .*

Let $0 < c < \frac{1}{10}$ be a small constant. Since $G_d(s, \chi_0)((s-1)\zeta(s))^{1/2}(1-\mathfrak{p}^{-s})^{1/2}$ is holomorphic and $O((3/4)^{\omega(d)})$ in the disc $|s-1| \leq c$ thanks to (2.7), the Cauchy formula allows us to write

$$G_d(s, \chi_0)((s-1)\zeta(s))^{1/2}(1-\mathfrak{p}^{-s})^{1/2} = G_d(1, \chi_0)(1-\mathfrak{p}^{-1})^{1/2} + O((3/4)^{\omega(d)}|s-1|)$$

for $|s-1| \leq \frac{1}{2}c$. In view of

$$L(s, \chi_0) = \zeta(s)(1-\mathfrak{p}^{-s}) \quad \text{and} \quad G_d(1, \chi_0)(1-\mathfrak{p}^{-1})^{1/2} = hg(d),$$

it follows that

$$f_d(s, \chi_0) = hg(d)(s-1)^{-1/2} + O((3/4)^{\omega(d)}|s-1|^{1/2})$$

for $|s-1| \leq \frac{1}{2}c$. So we have

$$(2.9) \quad I = hg(d)M(x) + O((3/4)^{\omega(d)}E_0(x)),$$

where

$$M(x) := \frac{1}{2\pi i} \int_{\Gamma} (s-1)^{-1/2} x^s ds, \quad E_0(x) := \int_{\Gamma} |(s-1)^{1/2} x^s| |ds|.$$

Firstly we evaluate $M(x)$. By using [10, Corollary II.5.2.1], we have

$$(2.10) \quad M(x) := \frac{x}{\sqrt{\log x}} \left\{ \frac{1}{\Gamma(\frac{1}{2})} + O(x^{-c/2}) \right\}.$$

Next we deduce that

$$(2.11) \quad \begin{aligned} E_0(x) &\ll \int_{1/2+\varepsilon}^{1-1/\log x} (1-\sigma)^{1/2} x^\sigma d\sigma + \frac{x}{(\log x)^{3/2}} \\ &\ll \frac{x}{(\log x)^{3/2}} \left(\int_1^\infty t^{1/2} e^{-t} dt + 1 \right) \ll \frac{x}{(\log x)^{3/2}}. \end{aligned}$$

Inserting (2.10) and (2.11) into (2.9) and noticing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we find that

$$(2.12) \quad I = \frac{x}{\sqrt{\pi \log x}} \left\{ hg(d) + O_\varepsilon \left(\frac{(3/4)^{\omega(d)}}{\log x} \right) \right\}.$$

B. *Estimations of I_1 and I_2 .*

It is well known that (cf. [10, Corollary II.3.5.2])

$$(2.13) \quad |\zeta(\sigma + i\tau)| \ll |\tau|^{(1-\sigma)/3} \log |\tau| \quad \left(\frac{1}{2} \leq \sigma \leq 1 + \log^{-1} |\tau|, |\tau| \geq 3 \right).$$

Noticing that $q := \mathfrak{p}^\varpi$, it follows that

$$(2.14) \quad L(s, \chi_0) = \zeta(s)(1-\mathfrak{p}^{-s}) \ll |\tau|^{(1-\sigma)/3} \log |\tau|$$

for $\frac{1}{2} \leq \sigma \leq 1 + \log^{-1}(|\tau| + 3)$ and $|\tau| \geq 3$. From (2.6), (2.7) and (2.14), we derive that

$$(2.15) \quad \begin{aligned} |I_1| + |I_2| &\ll_{\varepsilon} (3/4)^{\omega(d)} \int_{1/2+\varepsilon}^{1+2/\log x} T^{(1-\sigma)/6} (\log T) \frac{x^{\sigma}}{T} d\sigma \\ &\ll_{\varepsilon} (3/4)^{\omega(d)} \frac{x}{T} \log T. \end{aligned}$$

C. *Estimations of I_3 and I_4 .*

As before, (2.6) and (2.14) allow us to deduce

$$(2.16) \quad \begin{aligned} |I_3| + |I_4| &\ll_{\varepsilon} (3/4)^{\omega(d)} \int_1^T (|\tau| + 1)^{1/12} \log(|\tau| + 1) \frac{x^{1/2+\varepsilon}}{|(\frac{1}{2} + \varepsilon) + i\tau|} d\tau \\ &\ll_{\varepsilon} (3/4)^{\omega(d)} x^{1/2+\varepsilon} \int_1^T (\tau + 1)^{-1+1/12} d\tau \ll_{\varepsilon} (3/4)^{\omega(d)} x^{1/2+\varepsilon} T^{1/12}. \end{aligned}$$

D. *Estimation of I_{ρ} .*

With the help of (2.14) and (2.7), we can derive that for $s = \sigma + i\gamma$ with

$$(2.17) \quad I_{\rho} \ll_{\varepsilon} (3/4)^{\omega(d)} \int_{1/2+\varepsilon}^{\beta} |\gamma|^{(1-\sigma)/6} (\log |\gamma|)^{1/2} \frac{x^{\sigma}}{|\sigma + i\gamma|} d\sigma.$$

Denote by $N(\alpha, T)$ the number of zeros of $\zeta(s)$ in the region $\Re s \geq \alpha$ and $|\Im s| \leq T$ and define $\sigma(\tau) := c \log^{-2/3}(|\tau| + 3) \log_2^{-1/3}(|\tau| + 3)$ ($c > 0$ absolute constant). Summing (2.17) over $|\gamma| < T$ and interchanging the summations and noticing that $\beta < 1 - \sigma(T_1)$ (the Korobov-Vinogradov zero free region), we have

$$\begin{aligned} \sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} |I_{\rho}| &\ll (3/4)^{\omega(d)} (\log T) \max_{T_1 \leq T} \sum_{\beta > \frac{1}{2} + \varepsilon, T_1/2 < |\gamma| < T_1} |I_{\rho}| \\ &\ll_{\varepsilon} (3/4)^{\omega(d)} (\log T) \max_{T_1 \leq T} \int_{1/2+\varepsilon}^{1-\sigma(T_1)} T_1^{(1-\sigma)/6} \cdot \frac{x^{\sigma}}{T_1} \cdot N(\sigma, T_1) d\sigma. \end{aligned}$$

According to [7], it is well known that

$$(2.18) \quad N(\sigma, T) \ll T^{(12/5)(1-\sigma)} (\log T)^{44}$$

for $\frac{1}{2} + \varepsilon \leq \sigma \leq 1$, and $T \geq 2$. Thus

$$(2.19) \quad \begin{aligned} \sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} |I_{\rho}| &\ll (3/4)^{\omega(d)} (\log T)^{45} \max_{T_1 \leq T} \int_{1/2+\varepsilon}^{1-\sigma(T_1)} T_1^{(1-\sigma)/6} \frac{x^{\sigma}}{T_1} T_1^{(12/5)(1-\sigma)} d\sigma \\ &\ll x (\log T)^{45} \max_{T_1 \leq T} \int_{1/2+\varepsilon}^{1-\sigma(T_1)} \left(\frac{T_1^{17/30}}{x} \right)^{1-\sigma} d\sigma \\ &\ll x (\log T)^{45} \max_{T_1 \leq T} \left(\frac{T_1^{17/30}}{x} \right)^{\sigma(T_1)} \\ &\ll x (\log T)^{45} \left(\frac{T^{17/30}}{x} \right)^{\sigma(T)}. \end{aligned}$$

Inserting (2.12), (2.15), (2.16) and (2.19) into (2.8), we find that

$$\sum_{n \leq x} \frac{\chi_0(n)}{\tau(nd)} = \frac{x}{\sqrt{\pi \log x}} \left\{ hg(d) + O_\varepsilon \left(\frac{(3/4)^{\omega(d)}}{\log x} \right) \right\} + O_\varepsilon(R_{d,T}(x)),$$

where

$$R_{d,T}(x) := \left(\frac{3}{4} \right)^{\omega(d)} \left\{ \frac{x}{T} \log T + x^{1/2+\varepsilon} T^{1/12} + x(\log T)^{45} \left(\frac{T^{17/30}}{x} \right)^{\sigma(T)} \right\} + \frac{x \log x}{T}.$$

Taking $T = x$ and $\varepsilon = 10^{-3}$ and noticing that $\omega(d) \ll (\log x)/\log_2 x$ for $d \leq x$, it is easy to verify that $R_{d,T}(x) \ll (3/4)^{\omega(d)} x / (\log x)^{3/2}$ for $d \leq x$. This completes the proof. \square

Lemma 2.5. *Under the notation in Lemma 2.4, we have*

$$(2.20) \quad h \sum_{d \leq x} \chi_0(d) g(d) = \frac{(\varphi(q)/q)x}{\sqrt{\pi \log x}} \left\{ 1 + O \left(\frac{1}{\log x} \right) \right\},$$

where the implied constant is absolute.

Proof. According to (2.4), it is easy to see that $g(d)$ is a multiplicative function and

$$(2.21) \quad \begin{aligned} g(p^\nu) &= \sum_{j \geq 0} \frac{(\chi_0(p)p^{-1})^j}{j + \nu + 1} \left(\sum_{k \geq 0} \frac{(\chi_0(p)p^{-1})^k}{k + 1} \right)^{-1} \\ &= \frac{-\chi_0(p)p^{-1}}{\log(1 - \chi_0(p)p^{-1})} \sum_{j \geq 0} \frac{(\chi_0(p)p^{-1})^j}{j + \nu + 1}. \end{aligned}$$

For $\sigma > 1$, we can write

$$\sum_{n \geq 1} \chi_0(n) g(n) n^{-s} = L(s, \chi_0)^{1/2} \sum_{n \geq 1} \beta(n) n^{-s} = \zeta(s)^{1/2} (1 - \mathfrak{p}^{-s})^{1/2} \sum_{n \geq 1} \beta(n) n^{-s},$$

where $\beta(n)$ is a multiplicative function determined by

$$(2.22) \quad \sum_{\nu \geq 1} \beta(p^\nu) \xi^\nu = (1 - \chi_0(p)\xi)^{1/2} \sum_{\nu \geq 0} \chi_0(p) g(p^\nu) \xi^\nu \quad (|\xi| < 1).$$

Since $|g(p^\nu)| \leq 1$, the right-hand side is holomorphic for $|\xi| < 1$ and we have $\beta(p^\nu) \ll \left(\frac{3}{2}\right)^\nu$ ($\nu = 1, 2, \dots$). In addition, $\beta(p) = \chi_0(p)(g(p) - 1/2) = O(1/p)$. These imply the absolute convergence of $\sum \beta(n) n^{-s}$ for $\sigma > \frac{1}{2}$ and $\sum \beta(n) n^{-s} \ll_\varepsilon 1$ for $\sigma \geq \frac{1}{2} + \varepsilon$.

Applying Theorem II. 5.3 of [10], we have

$$\sum_{n \leq x} \chi_0(n) g(n) = \frac{x}{\sqrt{\log x}} \left\{ \lambda_0\left(\frac{1}{2}\right) + O \left(\frac{1}{\log x} \right) \right\},$$

where we have

$$\lambda_0\left(\frac{1}{2}\right) := \frac{(1 - \mathfrak{p}^{-1})^{1/2}}{\Gamma\left(\frac{1}{2}\right)} \prod_p (1 - \chi_0(p)p^{-1})^{1/2} \sum_{\nu \geq 0} \frac{\chi_0(p)^\nu g(p^\nu)}{p^\nu},$$

thanks to (2.21) and (2.22). In view of (2.21), it follows, with the notation $\xi = \chi_0(p)p^{-1}$,

$$\begin{aligned} \sum_{\nu \geq 0} \frac{\chi_0(p)^\nu g(p^\nu)}{p^\nu} (1 - \chi_0(p)p^{-1}) &= \left(\sum_{j \geq 0} \frac{\xi^j}{j+1} \right)^{-1} (1 - \xi) \sum_{\nu \geq 0} \sum_{j \geq 0} \frac{\xi^{j+\nu}}{j+\nu+1} \\ &= \left(\sum_{j \geq 0} \frac{\xi^j}{j+1} \right)^{-1} (1 - \xi) \sum_{k \geq 0} \xi^k \\ &= \frac{-\chi_0(p)p^{-1}}{\log(1 - \chi_0(p)p^{-1})}. \end{aligned}$$

Thus

$$\lambda_0\left(\frac{1}{2}\right) = \frac{(1 - \mathfrak{p}^{-1})^{1/2}}{\sqrt{\pi}} \prod_p (1 - \chi_0(p)p^{-1})^{-1/2} \frac{-\chi_0(p)p^{-1}}{\log(1 - \chi_0(p)p^{-1})}$$

and $h\lambda_0\left(\frac{1}{2}\right) = (1 - \mathfrak{p}^{-1})/\sqrt{\pi} = (\varphi(q)/q)/\sqrt{\pi}$, which concludes the proof of (2.20). \square

Lemma 2.6. *Let $q = \mathfrak{p}^\varpi$ with \mathfrak{p} an odd prime and $\varpi \in \mathbb{N}$. For any $\varepsilon > 0$, there is a positive constant $c_1(\varepsilon) > 0$ depending on ε such that we have*

$$(2.23) \quad \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(d) \sum_{n \leq x} \frac{\chi(n)}{\tau(nd)} \ll x e^{-c_1(\varepsilon)(\log x)^{1/3}(\log_2 x)^{-1/3}}$$

uniformly for $d \geq 1$, $x \geq 2$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a, q) = 1$.

Proof. Since the proof is rather close to that of Lemma 2.4, we only mention the principal points. As before, by (2.5), we can write for $\sigma := \Re s > 1$

$$(2.24) \quad f_d(s, \chi) := \sum_{n=1}^{\infty} \chi(n)\tau(dn)^{-1}n^{-s} = L(s, \chi)^{1/2}G_d(s, \chi),$$

where

$$G_d(s, \chi) := \prod_p \sum_{j=0}^{\infty} \frac{(\chi(p)p^{-s})^j}{j+1} (1 - \chi(p)p^{-s})^{1/2} \prod_{p^\alpha \parallel d} \sum_{j=0}^{\infty} \frac{(\chi(p)p^{-s})^j}{j+\alpha+1} \left(\sum_{j=0}^{\infty} \frac{(\chi(p)p^{-s})^j}{j+1} \right)^{-1}$$

is a Dirichlet series that converges absolutely for $\sigma > \frac{1}{2}$ and verifies $|G_d(s, \chi)| \leq C_\varepsilon \left(\frac{3}{4}\right)^{\omega(d)}$ for $\sigma \geq \frac{1}{2} + \varepsilon$ and $d \geq 1$, where ε is an arbitrarily small positive constant and $C_\varepsilon > 0$ is a constant depending only on ε .

We apply Lemma 2.1 with $\sigma_a = 1$, $B(n) = 1$, $\alpha = \frac{1}{2}$ and $\sigma = 0$ to write

$$\sum_{n \leq x} \frac{\chi(n)}{\tau(nd)} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f_d(s, \chi) \frac{x^s}{s} ds + O\left(\frac{x \log x}{T}\right),$$

where $b = 1 + 2/\log x$ and $100 \leq T \leq x$ such that $L(\sigma + iT, \chi) \neq 0$ for $0 < \sigma < 1$.

Let \mathcal{M}_T be the boundary of the modified rectangle with vertices $(\frac{1}{2} + \varepsilon) \pm iT$ and $b \pm iT$ as follows:

- $\varepsilon > 0$ is a small constant chosen such that $L(\frac{1}{2} + \varepsilon + i\gamma, \chi) \neq 0$ for $|\gamma| < T$;
- the zeros of $L(s, \chi)$ of the form $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$.

Clearly the function $f_d(s, \chi)$ is analytic inside \mathcal{M}_T . By the Cauchy residue theorem, we can write

$$(2.25) \quad \sum_{n \leq x} \frac{\chi(n)}{\tau(nd)} = I_1 + \cdots + I_4 + \sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} I_\rho + O\left(\frac{x \log x}{T}\right),$$

where

$$I_j := \frac{1}{2\pi i} \int_{\mathcal{L}_j} f_d(s, \chi) \frac{x^s}{s} ds, \quad I_\rho := \frac{1}{2\pi i} \int_{\Gamma_\rho} f_d(s, \chi) \frac{x^s}{s} ds$$

and \mathcal{L}_j and Γ_ρ are as in Figure 1.

A. *Estimations of I_1 and I_2 .*

In view of (2.24) and Lemma 2.2, we have

$$(2.26) \quad \begin{aligned} |I_1| + |I_2| &\ll \int_{1/2+\varepsilon}^{1+2/\log x} (q^{1/2}T)^{\frac{1}{2}(1-\sigma)+\varepsilon} \cdot \frac{x^\sigma}{T} d\sigma \\ &\ll \frac{x}{T} \int_{1/2+\varepsilon}^{1+2/\log x} \left(\frac{q^{1/4}T^{1/2}}{x}\right)^{1-\sigma} d\sigma \ll \frac{x}{T}. \end{aligned}$$

B. *Estimations of I_3 and I_4 .*

By (2.24) and Lemma 2.2, we have

$$(2.27) \quad \begin{aligned} |I_3| + |I_4| &\ll \int_1^T q^{1/4}(|\tau| + 1)^{1/12} \frac{x^{1/2+\varepsilon}}{|(\frac{1}{2} + \varepsilon) + i\tau|} d\tau \\ &\ll x^{1/2+\varepsilon} q^{1/4} T^{1/12}. \end{aligned}$$

C. *Estimation of I_ρ .*

With the help of (2.24) and Lemma 2.5, we have

$$(2.28) \quad I_\rho \ll \int_{1/2+\varepsilon}^\beta (q^{\frac{1-\sigma}{2}} |\gamma|^{1/12+\varepsilon}) \frac{x^\sigma}{|\sigma + i\gamma|} d\sigma.$$

Denote by $N(\sigma, T, \chi)$ the number of zeros of $L(s, \chi)$ in the region $\Re s \geq \sigma$ and $|\Im s| \leq T$. Summing (2.28) over $|\gamma| < T$ and interchanging the summations, we have

$$\sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} |I_\rho| \ll (\log T) \max_{T_1 \leq T} \int_{1/2+\varepsilon}^{1-\sigma(T_1; q)} q^{\frac{1}{2}(1-\sigma)} T_1^{1/12+\varepsilon} \frac{x^\sigma}{T_1} N(\sigma, T_1, \chi) d\sigma.$$

where $\sigma(\tau; q) := C \log^{-2/3}(q|\tau| + 3q) \log_2^{-1/3}(q|\tau| + 3q)$ ($C = C(\mathfrak{p})$ is a positive constant depending on \mathfrak{p}) and we have used Lemma 2.3.

It is well-known that (cf. [8, Theorem 12.1] and [7])

$$N(\sigma, T, q) := \sum_{\chi \pmod{q}} N(\sigma, T, \chi) \ll (qT)^{\frac{12}{5}(1-\sigma)} \log^9(qT).$$

Thus

$$\begin{aligned}
(2.29) \quad \sum_{\chi \neq \chi_0} \sum_{\beta > \frac{1}{2} + \varepsilon, |\gamma| < T} |I_\rho| &\ll \log^{10}(qT) \max_{T_1 \leq T} \int_{1/2 + \varepsilon}^{1 - \sigma(T_1; q)} q^{\frac{1-\sigma}{2}} T_1^{1/12 + \varepsilon} \frac{x^\sigma}{T_1} (qT_1)^{\frac{12}{5}(1-\sigma)} d\sigma \\
&\ll x \log^{10}(qT) \max_{T_1 \leq T} \int_{1/2 + \varepsilon}^{1 - \sigma(T_1; q)} \left(\frac{q^{87/30} T_1^{17/30}}{x} \right)^{1-\sigma} d\sigma \\
&\ll x \log^{10}(qT) \max_{T_1 \leq T} \left(\frac{q^{87/30} T_1^{17/30}}{x} \right)^{\sigma(T_1; q)} \\
&\ll x \log^{10}(qT) \left(\frac{q^{87/30} T^{17/30}}{x} \right)^{\sigma(T; q)}.
\end{aligned}$$

provided $q^{87/30} T^{17/30} \leq x$. Inserting (2.26), (2.27) and (2.29) into (2.25), we find that

$$\begin{aligned}
\sum_{\chi \neq \chi_0} \bar{\chi}(a) \chi(d) \sum_{n \leq x} \frac{\chi(n)}{\tau(nd)} &\ll \frac{qx \log x}{T} + x^{1/2} q^{5/4} T^{1/12 + \varepsilon} + x \log^{10}(qT) \left(\frac{q^{87/30} T^{17/30}}{x} \right)^{\sigma(T; q)} \\
&\ll (x^{-13} q^{104})^{1/17 + \varepsilon} + (x^{33} q^{42})^{1/51 + \varepsilon} + x (\log x)^{10} x^{-\varepsilon \sigma(T; q)/195}
\end{aligned}$$

thanks to the choice of $T = (x^{30(1-\varepsilon)} q^{-87})^{1/17}$. This implies the required result. \square

3. PROOF OF THEOREM 1

Firstly we write

$$(3.1) \quad S(x, t; q, a) := \frac{1}{(x/q)} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} F_n(t).$$

In view of the symmetry of the divisors of n about \sqrt{n} , it follows that

$$F_n(t) = \text{Prob}(D_n \geq 1 - t) = 1 - \text{Prob}(D_n < 1 - t) = 1 - F_n(1 - t) + O(\tau(n)^{-1}).$$

Summing over $n \leq x$ with $n \equiv a \pmod{q}$, we have

$$S(x, t; q, a) + S(x, 1 - t; q, a) = \frac{1}{(x/q)} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \{1 + O(\tau(n)^{-1})\} = 1 + O\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $x \geq 3$, $q \leq x^{15/52 - \varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a, q) = 1$, where we have used the orthogonality and Lemmas 2.4 and 2.6 with $d = 1$ to deduce that

$$\begin{aligned}
\frac{1}{(x/q)} \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \frac{1}{\tau(n)} &= \frac{q}{x\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(a) \sum_{n \leq x} \frac{\chi(n)}{\tau(n)} \\
&\ll \frac{(q/\varphi(q))}{e^{c_1(\varepsilon)} (\log x)^{1/3} (\log_2 x)^{-1/3}} \ll \frac{1}{\sqrt{\log x}}.
\end{aligned}$$

On the other hand, we have the identity

$$\frac{2}{\pi} \arcsin \sqrt{t} + \frac{2}{\pi} \arcsin \sqrt{1 - t} = 1 \quad (0 \leq t \leq 1).$$

Therefore it is sufficient to prove (1.3) for $0 \leq t \leq \frac{1}{2}$.

For $0 \leq t \leq \frac{1}{2}$, we can write

$$\begin{aligned}
(3.2) \quad S(x, t; q, a) &= \frac{q}{x\varphi(q)} \sum_{n \leq x} \sum_{\chi \pmod{q}} \frac{\bar{\chi}(a)\chi(n)}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \quad (n = dm) \\
&= \frac{q}{x\varphi(q)} \sum_{d \leq x^t} \sum_{\chi \pmod{q}} \bar{\chi}(a)\chi(d) \sum_{d^{1/t-1} \leq m \leq x/d} \frac{\chi(m)}{\tau(md)} \\
&= \frac{q}{x\varphi(q)} (S_1 - S_2 + S_3 - S_4),
\end{aligned}$$

where

$$\begin{aligned}
S_1 &:= \sum_{d \leq x^t} \bar{\chi}_0(a)\chi_0(d) \sum_{m \leq x/d} \frac{\chi_0(m)}{\tau(md)}, \\
S_2 &:= \sum_{d \leq x^t} \bar{\chi}_0(a)\chi_0(d) \sum_{m \leq d^{1/t-1}} \frac{\chi_0(m)}{\tau(md)}, \\
S_3 &:= \sum_{d \leq x^t} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(d) \sum_{m \leq x/d} \frac{\chi(m)}{\tau(md)}, \\
S_4 &:= \sum_{d \leq x^t} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(d) \sum_{m \leq d^{1/t-1}} \frac{\chi(m)}{\tau(md)}.
\end{aligned}$$

For S_1 , we apply Lemmas 2.4 and 2.5 to write

$$\begin{aligned}
(3.3) \quad S_1 &= \frac{h}{\sqrt{\pi}} \sum_{d \leq x^t} \frac{\chi_0(d)}{d\sqrt{\log(x/d)}} \left\{ g(d) + O\left(\frac{(3/4)^{\omega(d)}}{\log x}\right) \right\} \\
&= \frac{\varphi(q)}{q} x \left\{ \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right) \right\}.
\end{aligned}$$

For S_2 , we have

$$(3.4) \quad S_2 \leq \sum_{d \leq x^t} \sum_{m < d^{1/t-1}} \frac{1}{\tau(m)} \ll \sum_{d \leq x^t} \frac{d^{1/t-1}}{\sqrt{1 + \log d^{1/t-1}}} \ll \frac{x}{\sqrt{1 + \log x^{1-t}}} \ll \frac{x}{\sqrt{\log x}}.$$

By Lemma 2.6, we have

$$(3.5) \quad S_3 \ll \sum_{d \leq x^t} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(d) \sum_{m \leq x/d} \frac{\chi(m)}{\tau(md)} \ll x e^{-c_2(\varepsilon)} \sqrt[3]{(\log x)/\log_2 x}$$

$$(3.6) \quad S_4 \ll \sum_{d \leq x^t} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\chi(d) \sum_{m \leq d^{1/t-1}} \frac{\chi(m)}{\tau(md)} \ll x e^{-c_2(\varepsilon)} \sqrt[3]{(\log x)\log_2 x}$$

uniformly for $x \geq 3$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a, q) = 1$.

Inserting (3.3)–(3.6) into (3.2), we find that

$$S(x, t; q, a) = \frac{2}{\pi} \arcsin \sqrt{t} + O_{p,\varepsilon} \left(\frac{1}{\sqrt{\log x}} \right)$$

uniformly for $0 \leq t \leq \frac{1}{2}$, $x \geq 3$, $q \leq x^{15/52-\varepsilon}$ and $a \in \mathbb{Z}^*$ such that $(a, q) = 1$.

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