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An elementary approach to subdiffusion

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DYNAMICAL REGIMES AND FINITE TIME BEHAVIOR IN A TRAPPED RANDOM WALK: A DIRECT ITERATIVE APPROACH

E. FLORIANI, R. LIMA AND E. UGALDE

ABSTRACT. We consider a basic one-dimensional model of diffusion which allows to obtain a diversity of diffusive regimes whose speed depends on the moments of the per-site trapping time. This model is closely related to the continuous time random walks widely studied in the literature. The model we consider allows a detailed treatment, making it possible to study deviations from normality due to finite time effects.

1. INTRODUCTION

1.1. The random walks with trapping times we study in this paper can be seen as the natural discrete version of the continuous time random walk (CTRW), which has been treated in mathematical-physics literature since its introduction by Montroll and Weiss in 1965 [14]. A recent account on the subject can be found in [13], where CTRWs are considered as models for anomalous diffusion. A presentation of the relation between CTRWs and fractional diffusion can be found in [12]. The asymptotic behavior of continuous time random walks, and consequently of the model considered here, is well known and can be found in [2, 11]. In recent years two kind of generalizations of the continuous time random walk have been studied: on the one hand there are models including spatial inhomogeneities [6, 3], otherwise known as random environment; on the other hand, some models consider correlations between the space and time variables [4]. In both directions limit theorems have been obtained, and the relation to fractional dynamics and anomalous diffusion has been exposed. The aim of the present work is quite modest in comparison. We will treat in full detail, and from elementary grounds, the one-dimensional and discrete time version of the CTRW. We will be concerned in particular with the speed of convergence towards normal diffusion, and with the finite time deviations from normality which can be observed in the case of trapping time with infinite variance. Our main result, then, concerns the deviations from normality, and the emergence of a sub-diffusive behavior due to finite time effects, in the case of infinite variance. In the case of infinite mean trapping time, we study among other phenomena, the scaling behavior of the random walk and the deviations from this asymptotic scaling law at finite times.

1.2. In our model, a particle performs a random walk in a one-dimensional lattice in such a way that at each site it can stay trapped for an integer random time. The properties of the diffusion process depend on the distribution of this trapping time. As mentioned above, the model can be seen as a natural discrete version of the CTRW widely studied in the literature. We will recover, from elementary grounds, a classical limit theorem, as for instance the fact that for trapping times with finite mean, the particle performs a random walk consistent with a diffusive process with a diffusion coefficient decreasing with the expected trapping time. In the case of infinite mean

trapping time, a sub-diffusive behavior is obtained. In both cases we are able to determine the squared mean displacement at all times, and to evaluate its asymptotic behavior. Our formulas give quite precise estimates of the deviations from the asymptotic scaling at finite times. To the best of our knowledge, these are the first results of this kind. We are also able, in the case of finite mean trapping times, to estimate the speed of convergence towards normal diffusion, by using elementary probability inequalities. In this way we develop a direct approach to diffusion regimes leading to explicit estimates of rates of convergence and finite time deviations.

1.3. The paper is organized as follows: In the next section we define the model and give some basic general results, in particular, we establish a formula allowing us to compute the evolution of the mean squared displacement (MSD) of the random walk; in sections 3 and 4 we respectively examine the diffusive and sub-diffusive regimes, paying special attention to the finite time behavior of the MSD; finally, the last section is devoted to a summary of our results and some concluding remarks.

2. GENERALITIES

2.1. **The model as a Markov chain.** At each site $z \in \mathbb{Z}$, the diffusive particle gets trapped a random time $T_z \in \mathbb{N}_0$, after which it moves with equal probability to either of the two neighboring sites $z-1$ or $z+1$. We will consider the homogeneous case, for which the distribution of the random trapping time, $p_z(\tau) := \mathbb{P}(T_z = \tau)$, does not depend on the site $z \in \mathbb{Z}$. Hence, we model the process as a discrete-time Markov chain on the set $E = \mathbb{Z} \times \mathbb{N}_0$, where the first component represents the position of the random walker and the second one the trapping time. The transition probabilities are

$$(1) \quad \mathbb{P}((X, T)_{t+1} = (z', \tau') | (X, T)_t = (z, \tau)) = \begin{cases} p(\tau')/2 & \text{if } |z - z'| = 1 \text{ and } \tau = 0, \\ 1 & \text{if } z = z' \text{ and } \tau' = \tau - 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in the statistical behavior of the position X_t , as a function of the distribution $\{p(\tau) : \tau \in \mathbb{N}_0\}$. We will assume that the particle starts its random walk at the position $z = 0$, so that

$$(2) \quad \mathbb{P}((X, T)_t = (z, \tau)) = \begin{cases} p(\tau) & \text{if } z = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $t \leq 0$.

Notice that the restriction of the process to the variable T is also a Markov chain:

$$(3) \quad \mathbb{P}(T_{t+1} = \tau' | T_t = \tau) = \begin{cases} p(\tau') & \text{if } \tau = 0, \\ 1 & \text{if } \tau' = \tau - 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

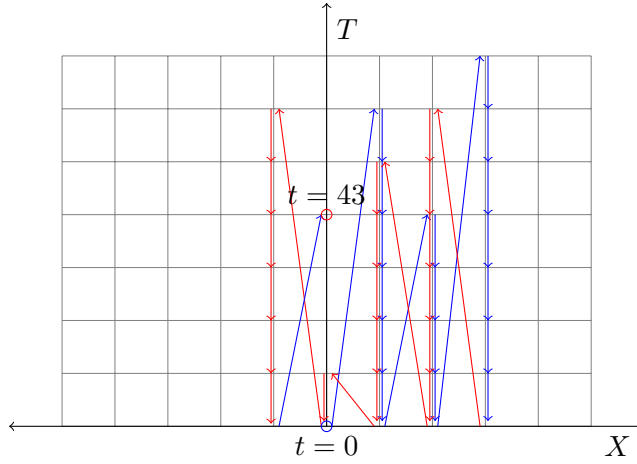


FIGURE 1. The trajectory in the (X, T) plane of a possible realization of the process. The trajectory starts at $(X, T) = (0, 0)$ and reaches $(X, T) = (0, 4)$ at time $t = 43$.

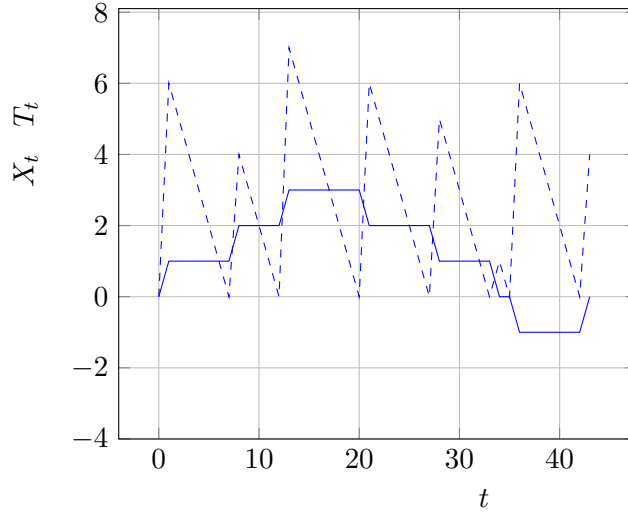


FIGURE 2. The sequence of positions X_t (solid line) and trapping times T_t (dashed line), corresponding to the trajectory depicted in Figure 1.

2.2. The model as a subordinated random walk. We can write the trapped random walk as a standard binary random walk subordinated to a monotonously increasing random variable governing the jump times. For this note that the space increments

$$\Delta_t := X_{t+1} - X_t \in \{-1, 0, 1\},$$

form a sequence of random variables satisfying $\mathbb{P}(\Delta_t = 1 | T_t = 0) = \mathbb{P}(\Delta_t = -1 | T_t = 0) = 1/2$, and $\mathbb{P}(\Delta_t = 0 | T_t > 0) = 1$. Here T_t denotes the trapping at time t of the random waker. A direct computation shows that for each finite collection of times $t_1 < t_2 < \dots < t_n$ and corresponding

increments $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in \{-1, 1\}$, we have

$$\mathbb{P}(\Delta_{t_1} = \epsilon_1, \dots, \Delta_{t_n} = \epsilon_n | T_{t_1} = \dots = T_{t_n} = 0) = \prod_{k=1}^n \mathbb{P}(\Delta_{t_k} = \epsilon_k | T_{t_k} = 0) = \frac{1}{2^n}.$$

Therefore the increments Δ_t are independent when conditioned to the escape level $E_0 := \{T = 0\}$ which is nothing but a renewal time. From this it readily follows that

$$X_{t+1} = X_t + \chi_{\{T_t=0\}} \Delta_t,$$

where $\chi_{\{T_t=0\}}$ is the characteristic function for the event $\{T_t = 0\} \subset E_0$, and Δ_t is a sequence of i.i.d. random variables such that $\mathbb{P}(\Delta_t = 1) = \mathbb{P}(\Delta_t = -1) = 1/2$. From this we obtain

$$X_{t+1} = \sum_{s=0}^t \chi_{\{T_s=0\}} \Delta_s = \sum_{n=0}^{N_t} \Delta_n,$$

where $N_t := \#\{0 \leq s \leq t : T_s = 0\}$ is nothing but the number of times the random walker visited the escape level up to time t . Since Δ_n and N_t are independent, then

$$\begin{aligned} \mathbb{P}(X_t = z) &= \sum_{n=0}^{t-1} \mathbb{P}(N_{t-1} = n) \mathbb{P}\left(\sum_{m=0}^n \Delta_m = z\right) \\ (4) \quad &= \sum_{n=0}^{t-1} 2^{-n} \mathbb{P}(N_{t-1} = n) \binom{n}{\frac{n+z}{2}}. \end{aligned}$$

2.3. A recurrence for the MSD. In order to quantify the diffusivity of the dynamics, we focus now on the mean squared displacement (MSD)

$$(5) \quad \sigma_t^2 := \mathbb{E}(X_t^2) = \sum_{z \in \mathbb{Z}} z^2 \mathbb{P}(X_t = z).$$

We have the following.

Proposition 1 (Recurrence relation). *Suppose that the trapped random walk $\{(X, T)_t\}_{t \in \mathbb{Z}}$ is at the origin at time $t = 0$, i.e., it satisfies the initial condition (2). Then, for each $t \in \mathbb{N}$ we have,*

$$\sigma_{t+1}^2 = \sum_{0 \leq \tau \leq t} p(\tau) (\sigma_{t-\tau}^2 + 1).$$

We will use this recurrence relation to quantify the diffusivity of the trapped random walk as a function of the trapping time distribution.

Proof. For each $t \geq 1$ we have

$$\begin{aligned} \mathbb{P}(X_t = z) &= \sum_{0 \leq \tau} \mathbb{P}((X, T)_t = (z, \tau)) = \sum_{0 \leq \tau} \sum_{0 \leq \tau' \leq t-1} \frac{p(\tau + \tau')}{2} \mathbb{P}\{(X, T)_{t-\tau'-1} = (z \pm 1, 0)\} \\ &= \sum_{0 \leq \tau' \leq t-1} \mathbb{P}\{(X, T)_{t-\tau'-1} = (z \pm 1, 0)\} \sum_{0 \leq \tau} \frac{p(\tau + \tau')}{2} \\ &= \sum_{0 \leq \tau' \leq t-1} \frac{\mathbb{P}\{(X, T)_{t-\tau'-1} = (z \pm 1, 0)\}}{2} \mathbb{P}(T \geq \tau'). \end{aligned}$$

where $\mathbb{P}\{(X, T)_{t-\tau'-1} = (z \pm 1, 0)\} = \mathbb{P}\{(X, T)_{t-\tau'-1} = (z - 1, 0)\} + \mathbb{P}\{(X, T)_{t-\tau'-1} = (z + 1, 0)\}$. Hence, the distribution of X_t is determined by its distribution restricted to the escape level E_0 and the queue of the trapping time distribution $\tau \mapsto \mathbb{P}(T \geq \tau)$. The distribution of X_t restricted to E_0 satisfies the recurrence relation,

$$\mathbb{P}\{(X, T)_t = (z, 0)\} = \sum_{0 \leq \tau \leq t-1} \frac{p(\tau)}{2} \mathbb{P}\{(X, T)_{t-\tau-1} = (z \pm 1, 0)\}.$$

Using this we obtained

$$\begin{aligned} \sigma_t^2 &= \sum_{z \in \mathbb{Z}} z^2 \sum_{0 \leq \tau' \leq t-1} \frac{\mathbb{P}\{(X, T)_{t-\tau'-1} = (z \pm 1, 0)\}}{2} \mathbb{P}(T \geq \tau') \\ &= \sum_{z \in \mathbb{Z}} z^2 \sum_{0 \leq \tau' \leq t-1} \mathbb{P}(T \geq \tau') \sum_{0 \leq \tau < t-\tau'-1} \frac{p(\tau)}{2} \frac{\mathbb{P}\{(X, T)_{t-\tau'-\tau-2} = (z - 1 \pm 1, 0)\}}{2} \\ &\quad + \sum_{z \in \mathbb{Z}} z^2 \sum_{0 \leq \tau' \leq t-1} \mathbb{P}(T \geq \tau') \sum_{0 \leq \tau < t-\tau'-1} \frac{p(\tau)}{2} \frac{\mathbb{P}\{(X, T)_{t-\tau'-\tau-2} = (z + 1 \pm 1, 0)\}}{2}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \sigma_t^2 &= \sum_{0 \leq \tau \leq t-1} \frac{p(\tau)}{2} \sum_{0 \leq \tau' \leq t-\tau-2} \mathbb{P}(T \geq \tau') \sum_{z \in \mathbb{Z}} (z + 1)^2 \frac{\mathbb{P}\{(X, T)_{(t-\tau-1)-\tau'-1} = (z \pm 1, 0)\}}{2} \\ &\quad + \sum_{0 \leq \tau \leq t-1} \frac{p(\tau)}{2} \sum_{0 \leq \tau' \leq t-\tau-2} \mathbb{P}(T \geq \tau') \sum_{z \in \mathbb{Z}} (z - 1)^2 \frac{\mathbb{P}\{(X, T)_{(t-\tau-1)-\tau'-1} = (z \pm 1, 0)\}}{2} \\ &= \sum_{0 \leq \tau \leq t-1} \frac{p(\tau)}{2} (\sigma_{t-\tau-1}^2 + 2\mathbb{E}(X_{t-\tau-1}) + 1) + \sum_{0 \leq \tau \leq t-1} \frac{p(\tau)}{2} (\sigma_{t-\tau-1}^2 - 2\mathbb{E}(X_{t-\tau-1}) + 1) \\ &= \sum_{0 \leq \tau \leq t-1} p(\tau) (\sigma_{t-\tau-1}^2 + 1). \end{aligned}$$

□

The recurrence relation just obtained already ensures the unboundedness of the trapped random walk. Indeed, we have the following.

Proposition 2 (Unboundedness). *Suppose that the trapped random walk $\{(X, T)_t\}_{t \in \mathbb{Z}}$ satisfies the initial condition (2). Then the sequence $\{\sigma_t^2\}_{t \in \mathbb{N}}$ is increasing and unbounded.*

Proof. For the first claim it is enough to notice that for each $t \geq 0$, $\Delta\sigma_{t+1}^2 := \sigma_{t+1}^2 - \sigma_t^2$ satisfies the recurrence relation $\Delta\sigma_{t+1}^2 = p(t) + \sum_{0 \leq \tau \leq t} p(\tau) \Delta\sigma_{t-\tau}^2$, and since $\Delta\sigma_1^2 = p(0) \geq 0$, it follows by induction on t that $\Delta\sigma_t^2 \geq 0$ for all $t \in \mathbb{N}$.

For the other claim, if we suppose on the contrary that the sequence $\{\sigma_t^2\}_{t \in \mathbb{N}}$ is bounded, since it is monotonous, then it necessarily has a limit $\sigma_\infty^2 := \lim_{t \rightarrow \infty} \sigma_t^2$. This limit must satisfy

$$\sigma_\infty^2 = \lim_{t \rightarrow \infty} \sum_{0 \leq \tau \leq t} p(\tau) (\sigma_{t-\tau}^2 + 1),$$

but this is impossible since there exist $t_0 \in \mathbb{N}$ such that $\sigma_t^2 \geq \sigma_\infty^2 - 1/2$ for each $t \geq t_0$, and therefore

$$\lim_{t \rightarrow \infty} \sum_{0 \leq \tau \leq t} p(\tau) (\sigma_{t-\tau}^2 + 1) \geq (\sigma_\infty^2 + 1/2) \left(\lim_{t \rightarrow \infty} \sum_{0 \leq \tau \leq t-t_0} p(\tau) \right) > \sigma_\infty^2.$$

□

3. DIFFUSIVE REGIME

In this section we study the situation $\mathbb{E}(T^\alpha) < \infty$, for some $\alpha \geq 1$. In this case, X_t follows asymptotically a normal diffusion, and satisfies a central limit theorem. Nevertheless, the speed of convergence towards the normal behavior strongly depends on the trapping time distribution tail, and finite time deviations from normality appear. We will analyze the convergence towards normal diffusion, by first considering the behavior of the mean squared displacement, then we will prove a Central Limit Theorem.

3.1. Normal diffusion via MSD. In all cases when $\mathbb{E}(T) < \infty$, the mean squared displacement asymptotically follows a linear growth, with slope, or diffusion coefficient, $\mathcal{D} := (\mathbb{E}(T) + 1)^{-1}$. Nevertheless, the finite time deviations from this linear growth may give place to an apparent sub-linear growth. Our main result concerning this is the following.

Theorem 1 (Normal diffusion). *Let us suppose that $\mathbb{E}(T) < \infty$ and let $\mathcal{D} := (\mathbb{E}(T) + 1)^{-1}$. Then, the sequence $|\sigma_t^2 - \mathcal{D}t|$ is bounded if and only if $\mathbb{E}(T^2) < \infty$. Furthermore, $\lim_{t \rightarrow \infty} \mathcal{D}t/\sigma_t^2 = 1$.*

Proof. Let $R_t := \mathcal{D} \sum_{\tau \leq t} \sum_{\tau' > \tau} (\tau' - \tau) p(\tau')$ for each $t \geq 0$. We start by proving

$$(6) \quad e^{-q\mathbb{E}(T)} R_t - \mathbb{E}(T) \leq \sigma_t^2 - \mathcal{D}t \leq R_t,$$

with $q = -\log(p(0))/(1 - p(0))$. For this let $\epsilon_t = \sigma_t^2 - \mathcal{D}t$. According to Proposition 1 we have

$$(7) \quad \begin{aligned} \epsilon_{t+1} &= \sum_{0 \leq \tau \leq t} p(\tau) \epsilon_{t-\tau} + \mathbb{P}(T \leq t) - \mathcal{D} \left(t\mathbb{P}(T > t) + 1 + \sum_{0 \leq \tau \leq t} \tau p(\tau) \right), \\ &= \sum_{0 \leq \tau \leq t} p(\tau) \epsilon_{t-\tau} + \mathcal{D} \sum_{\tau > t} (\tau - t) p(\tau) - \mathbb{P}(T > t), \end{aligned}$$

which can be written as

$$(8) \quad \epsilon_{t+1} = \sum_{0 \leq \tau \leq t} \delta_\tau^+ Q_{t-\tau} - \sum_{0 \leq \tau \leq t} \delta_\tau^- Q_{t-\tau},$$

with $\delta_\tau^+ := \mathcal{D} \sum_{\tau > t} (\tau - t)p(\tau)$, $\delta_\tau^- := \mathbb{P}(T > t)$ and Q_τ defined recursively by the convolution $Q_{t+1} = \sum_{0 \leq \tau \leq t} Q_{t-\tau} p(\tau)$, starting with $Q_0 = 1$. Since

$$0 < \mathbb{P}(T \leq t) \min_{\tau \leq t} Q_\tau \leq Q_{t+1} \leq \mathbb{P}(T \leq t) \max_{\tau \leq t} Q_\tau \leq 1,$$

therefore

$$(9) \quad e^{-q\mathbb{E}(T)} = e^{-\alpha \sum_{\tau \geq 0} \mathbb{P}(T > \tau)} \leq \prod_{\tau \geq 0} \mathbb{P}(T \leq \tau) \leq Q_t \leq 1.$$

Finally, since $0 \leq \sum_{0 \leq \tau \leq t} \delta_\tau^- = \sum_{0 \leq \tau \leq t} \mathbb{P}(T < \tau) \leq \mathbb{E}(T)$, taking into account Equations (8) and (9), we obtain $e^{-q\mathbb{E}(T)} \left(\sum_{0 \leq \tau \leq t} \delta_\tau^+ \right) - \mathbb{E}(T) \leq \epsilon_{t+1} \leq \sum_{0 \leq \tau \leq t} \delta_\tau^+$, which is exactly (6) since $\sum_{0 \leq \tau \leq t} \delta_\tau^+ = R_t$.

To complete the proof, notice that

$$R_t := \mathcal{D} \sum_{\tau \leq t} \sum_{\tau' > \tau} (\tau' - \tau)p(\tau') = \mathcal{D} \sum_{\tau \leq t+1} p(\tau) \frac{\tau(\tau+1)}{2} + (t+1) \mathcal{D} \sum_{\tau > t+1} \left(\tau - \frac{t}{2} \right) p(\tau).$$

Hence, when $\mathbb{E}(T^2) < \infty$, we necessarily have $t \sum_{\tau \geq t} \tau p(\tau) \rightarrow 0$ when $t \rightarrow \infty$, and therefore $\lim_{t \rightarrow \infty} R_t = \mathcal{D} (\mathbb{E}(T^2) + \mathbb{E}(T)) / 2$, which implies $|\sigma_t^2 - \mathcal{D}t|$ bounded. If on the contrary $\mathbb{E}(T^2) = \infty$, we necessarily have

$$\liminf_{t \rightarrow \infty} \sigma_t^2 - \mathcal{D}t \geq \liminf_{t \rightarrow \infty} \mathcal{D} e^{-q\mathbb{E}(T)} \sum_{\tau \leq t+1} p(\tau) \frac{\tau(\tau+1)}{2} - \mathbb{E}(T) = \infty$$

In any case, the Inequalities (6) imply

$$e^{-q\mathbb{E}(T)} \limsup_{t \rightarrow \infty} R_t/t \leq \liminf_{t \rightarrow \infty} (\sigma_t^2/t - \mathcal{D}) \leq \limsup_{t \rightarrow \infty} (\sigma_t^2/t - \mathcal{D}) \leq \liminf_{t \rightarrow \infty} R_t/t.$$

Hence, $\lim_{t \rightarrow \infty} R_t/t$ exists, and since $\mathbb{E}(T) < \infty$, then we necessarily have

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow \infty} \frac{R_t}{t} \leq \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau \leq t+1} \tau^2 p(\tau) \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau \leq \sqrt{t}} \tau^2 p(\tau) + \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\sqrt{t} < \tau \leq t} \tau^2 p(\tau) \\ &\leq \frac{1}{2} \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \mathbb{E}(T) + \frac{1}{2} \lim_{t \rightarrow \infty} \sum_{\sqrt{t} < \tau < \infty} \tau p(\tau) = 0 \end{aligned}$$

and therefore $\lim_{t \rightarrow \infty} \sigma_t^2/t = \mathcal{D}$. □

The diffusive case is the one where the T -Markov chain (3) admits a stationary distribution $\pi(\tau)$. Indeed, the stationarity condition is

$$\pi(\tau') = \sum_{\tau=0}^{\infty} \pi(\tau) \mathbb{P}(T_{t+1} = \tau' | T_t = \tau) = \pi(0)p(\tau') + \pi(\tau' + 1)$$

so that

$$\pi(\tau) = \pi(0) \sum_{k=\tau}^{\infty} p(k)$$

The normalization condition

$$1 = \sum_{\tau=0}^{\infty} \pi(\tau) = \pi(0) \sum_{\tau=0}^{\infty} \sum_{k=\tau}^{\infty} p(k) = \pi(0) \sum_{\tau=0}^{\infty} (\tau+1)p(\tau) = \pi(0)(\mathbb{E}(T) + 1)$$

implies that the stationary distribution exist if and only if $\mathbb{E}(T)$ is finite. In this case

$$\pi(\tau) = \frac{1}{\mathbb{E}(T) + 1} \sum_{k=\tau}^{\infty} p(k) = \frac{1}{\mathcal{D}} \sum_{k=\tau}^{\infty} p(k).$$

Exponential trapping time. A nice example is supplied by the exponential trapping time distribution. Let us assume that $p(\tau) = (1-\lambda)\lambda^\tau$ for each τ . In this case

$$\mathbb{E}(T) + 1 = (1-\lambda) \sum_{\tau=1}^{\infty} \tau \lambda^\tau + 1 = \frac{\lambda}{1-\lambda} + 1 = \frac{1}{1-\lambda},$$

hence, $\sigma_t^2 \sim (1-\lambda)t$ is the expected asymptotic behavior for the MSD. If $\sigma_t^2 = (1-\lambda)t + \epsilon_t$, then, according to Equation (7) we have

$$\begin{aligned} \epsilon_{t+1} &= (1-\lambda) \sum_{0 \leq \tau \leq t} \lambda^\tau \epsilon_{t-\tau} + (1-\lambda)^2 \sum_{\tau > t} \lambda^\tau (\tau - t) - (1-\lambda) \sum_{\tau > t} \lambda^\tau \\ &= (1-\lambda) \sum_{0 \leq \tau \leq t} \lambda^\tau \epsilon_{t-\tau} + (1-\lambda)^2 \lambda^t \sum_{\tau \geq 0} \lambda^\tau \tau - \lambda^{t+1} = (1-\lambda) \sum_{0 \leq \tau \leq t} \lambda^\tau \epsilon_{t-\tau}. \end{aligned}$$

Since $\epsilon_0 = 0$, it can be easily checked, by induction for instance, that the previous recursion has solution $\epsilon_t = 0$ for all $t \in \mathbb{N}_0$ and in this case

$$\sigma_t^2 = (1-\lambda)t.$$

We have therefore an exact diffusive behavior for all times.

Power-law trapping time. Another interesting example is given by power-law distributed trapping times. Let us assume that $p(\tau) = (\tau+1)^{-q}/\zeta(q)$ for each τ . For $2 < q \leq 3$ we have

$$\mathbb{E}(T) = \zeta(q-1)/\zeta(q) - 1 < \infty$$

but $\mathbb{E}(T^2) = \infty$. In this case we have

$$\begin{aligned} R_t &:= \frac{\sum_{\tau \leq t+1} p(\tau) \tau(\tau+1)/2}{\mathbb{E}(T) + 1} + (t+1) \frac{\sum_{\tau > t+1} (\tau - t/2) p(\tau)}{\mathbb{E}(T) + 1} \\ &= \frac{\sum_{\tau \leq t+1} ((\tau+1)^{-q+2} - (\tau+1)^{-q+1})}{2\zeta(q-1)} + \frac{(t+1) \sum_{\tau > t+1} ((\tau+1)^{-q+1} - (t/2+1)(\tau+1)^{-q})}{\zeta(q-1)}. \end{aligned}$$

From here it is easy to deduce that $R_t \geq c(t+2)^{3-q} - c_0$ for each $t \geq 0$ and constants c, c_0 which depend on q . Taking this into account, according to Theorem 1 we have

$$\sigma_t^2 \geq \frac{\zeta(q)t}{\zeta(q-1)} + c(t+2)^{3-q} - c_0$$

for each $t \geq 0$. Hence, in this case the diffusive behavior suffers an increasingly diverging sub-linear deviation of the order of t^{3-q} . In this case, although σ_t^2 is asymptotically dominated by a linear behavior, the sub-linear deviation on σ_t^2 is enough to produce an effective sub-diffusion at all finite times. Indeed, using the recurrence established in Proposition 1 we computed σ_t^2 for $1 \leq t \leq 2^{17}$, and fitted it to a power-law $\sigma_t^2 = \mathcal{O}(t^\beta)$, with exponent $\beta = \beta_N(q)$ depending on the time interval chosen for the fit. As expected, $\beta(q) \rightarrow 1$ as the length of the fitting time interval increases. We chose the time interval $10 \leq t \leq N$, with $N = 2^{13}, 2^{15}$ and 2^{17} , to compute the power-law approximation. The so obtained exponents $\beta_N(q)$ are plotted in Figure 3. For each N , the behavior of $\beta_N(q)$ can be very well fitted by a sigmoidal function $\beta(q) = 2/(1 + e^{|q-3|^\eta})$.

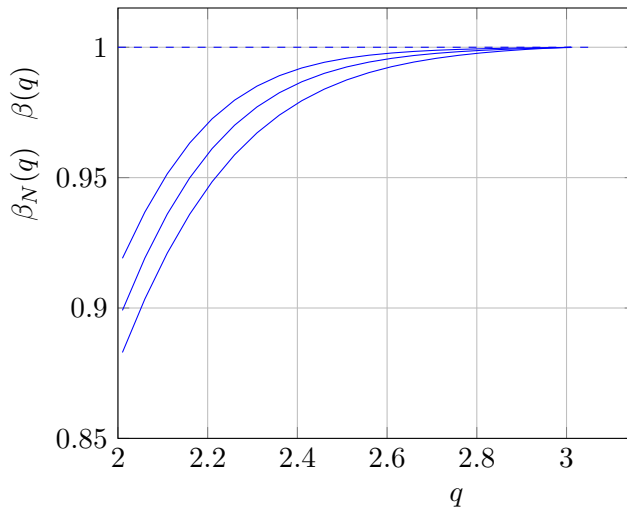


FIGURE 3. The curves $q \mapsto \beta_N(q)$ correspond to the exponent of the approximated power-law behavior of the MSD as a function of the trapping time distribution's exponent. We show these curves for total observation times $N = 2^{13}, 2^{15}$ and 2^{17} . The curves approaches, as $N \rightarrow \infty$, the asymptotic exponent $q \mapsto \beta(q) = 1$ (dashed line).

3.2. Central Limit Theorem. Let us now focus on the distribution $\mathbb{P}(X_t = z)$. According to what we deduced in subsection 2.2,

$$\mathbb{P}(X_t = z) = \sum_{n=0}^{t-1} \mathbb{P}(N_t = n) \mathbb{P}(S_n = z) = \sum_{n=0}^{t-1} 2^{-n} \mathbb{P}(N_t = n) \binom{n}{\frac{n+z}{2}}.$$

On the other hand, N_t can also be related to a sum of i.i.d. trapping times. Let us remind that, $N_t := \#\{0 \leq s \leq t : T_s = 0\}$. It can be easily verified that, for any finite collection $t_1 < t_2 < \dots < t_n$ and corresponding trapping times $\tau_1, \tau_1, \dots, \tau_n \geq 0$, we have

$$\mathbb{P}(T_{t_1+1} = \tau_1, \dots, T_{t_n+1} = \tau_n | T_{t_1} = \dots = T_{t_n} = 0) = \prod_{k=1}^n \mathbb{P}(T_{t_k+1} = \tau_k | T_{t_k} = 0) = \prod_{k=1}^n p(\tau_k).$$

From this it follows that

$$(10) \quad N_t = \max \left\{ n \geq 0 : \sum_{k=1}^n (T_k + 1) \leq t, \right\}$$

where the random variables T_k are independent copies of the trapping time T , which satisfies $\mathbb{P}(T = \tau) = p(\tau)$. Equation (4) suggests that the shape of the probability distribution $\mathbb{P}(X_t = z)$ strongly depends on the behavior of the distribution of the sum $\sum_{k=0}^n (T_k + 1)$. We now consider the case where $\mathbb{E}(T^\alpha) < \infty$ for some $\alpha > 1$. In this case the distribution of N_t concentrates around its mean, and this ensures the following.

Theorem 2 (Central Limit Theorem for $\alpha > 1$). *Let us assume that $\mathbb{E}(T^\alpha) < \infty$ for some $\alpha \in (1, 2]$ and define $C_T := \mathbb{E}(|T - \mathbb{E}(T)|^\alpha)$ and $\mu = \mathbb{E}(T) + 1$. For each $C > 8C_T/\mu + \sqrt{2\mu/\pi}$, there exists $t^* = t^*(C) \in \mathbb{N}$ such that for each bounded interval $I \subset \mathbb{R}$ and each $t \geq t^*$, we have*

$$\left| \mathbb{P} \left(X_t \in \sqrt{\frac{t}{\mu}} I \right) - \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2} dx \right| \leq C t^{\frac{1-\alpha}{1+\alpha}}$$

The proof of this result relies on two classical inequalities. On one hand the Berry-Esseen inequality (as it appears in [9]), which gives the rate of convergence in the de Moivre-Laplace theorem. According to it,

$$(11) \quad \sup_{x \in \mathbb{R}} \left| 2^{-n} \sum_{z \leq x} \binom{n}{\frac{n+z}{2}} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{n}} e^{-z^2/2} dz \right| < \frac{1}{\sqrt{2\pi} n}$$

for all $n \in \mathbb{N}$. On the other hand, we use a generalization of Chebyshev's inequality, due to von Bahr and Esseen. It states that if X_k , $1 \leq k \leq n$ are i.i.d. random variables such that $\mathbb{E}(X_k) = 0$ and $\mathbb{E}(|X_k|^\alpha) := C_\alpha < \infty$, for some $\alpha \in [1, 2]$, then

$$(12) \quad \mathbb{P} \left(\left| \sum_{k=0}^n X_k \right| > \epsilon \right) \leq \frac{2n C_\alpha}{\epsilon^\alpha}.$$

The inequality was established in [15].

Proof. It follows from Eq. (10) that

$$(13) \quad \mathbb{P}(N_t \leq n) = \mathbb{P} \left(\sum_{k=1}^{n+1} (T_k + 1) \geq t + 1 \right),$$

$$\mathbb{P}(N_t \geq n) = 1 - \mathbb{P}(N_t \leq n - 1)$$

$$(14) \quad = 1 - \mathbb{P} \left(\sum_{k=1}^n (T_k + 1) \geq t + 1 \right) = \mathbb{P} \left(\sum_{k=1}^n (T_k + 1) \leq t \right).$$

for each $n \in \mathbb{N}$.

Let $n_1 := (t + 1 - d_1)/\mu - 1$ and $n_2 := (t + d_2)/\mu - 1$ be integer numbers, with d_1, d_2 yet to be specified. Let $d = \min(d_1, d_2)$, then, taking (13) and (14) into account, we obtain

$$\begin{aligned} \mathbb{P}(N_t \notin (n_1, n_2)) &\leq \mathbb{P}\left(\left|\sum_{k=1}^{n_1+1} (T_k + 1) - (n_1 + 1)\mu\right| \geq d_1\right) + \mathbb{P}\left(\left|\sum_{k=1}^{n_2+1} (T_k + 1) - (n_2 + 1)\mu\right| \geq d_2\right) \\ &\leq \frac{2(n_1 + n_2 + 2) C_T}{\mu d^\alpha} = \frac{2(2t + 1 + d_2 - d_1) C_T}{\mu d^\alpha} \end{aligned}$$

for each $t \in \mathbb{N}$. From now on we will suppose that $t > (1 + \mu)^{1/\beta} 2^{-1/\beta}$, which ensures that $n_1 < n_2$. Now, by taking $\beta \in (1/\alpha, 1)$ and d_1, d_2 such that $|t^\beta - d_1|, |t^\beta - d_2| \leq \mu$, we obtain

$$\mathbb{P}(N_t \notin (n_1, n_2)) \leq \frac{2(2 + (1 + 2\mu)t^{-1}) C_T}{\mu (1 - \mu t^{-\beta})^\alpha} t^{1-\alpha\beta}.$$

From now on, $g_1(t) := (1 + 1/2(1 + 2\mu)t^{-1})/(1 - \mu t^{-\beta})^\alpha$. From the previous inequality and Berry-Esseen's, it follows

$$\begin{aligned} \left| \mathbb{P}\left(X_t \in \sqrt{\frac{t}{\mu}} I\right) - \frac{1}{\sqrt{2\pi}} \sum_{n=n_1}^{n_2} \mathbb{P}(N_t = n) \int_{\sqrt{\frac{t}{\mu n}} I} e^{-z^2/2} dz \right| &\leq g_1(t) \frac{4C_T}{\mu} t^{1-\alpha\beta} + \frac{2}{\sqrt{2\pi n_1}} \\ (15) \qquad \qquad \qquad &= g_1(t) \frac{4C_T}{\mu} t^{1-\alpha\beta} + g_2(t) \sqrt{\frac{2\mu}{\pi t}} \end{aligned}$$

where $g_2(t) := 1/\sqrt{1 - t^{\beta-1} - (2\mu - 1)t^{-1}}$.

For $n \in (n_1, n_2)$ we have

$$\begin{aligned} \left| \int_{\sqrt{\frac{t}{\mu n}} I} e^{-z^2/2} dz - \int_I e^{-z^2/2} dz \right| &\leq \int_{\sqrt{\frac{t}{\mu n}} I \Delta I} e^{-z^2/2} dz \leq \sqrt{\frac{t}{n_1 \mu}} \left(1 - \sqrt{\frac{n_1}{n_2}}\right) \\ &\leq \frac{1}{\sqrt{1 - t^{\beta-1} - (2\mu - 1)t^{-1}}} \left(1 - \sqrt{\frac{1 - t^{\beta-1} - (2\mu - 1)t^{-1}}{1 + t^{\beta-1} + 2\mu t^{-1}}}\right) \\ &\leq 2(t^{\beta-1} + (4\mu - 1)t^{-1}), \end{aligned}$$

provided $t \geq t_1 := \min\{t \in \mathbb{N} : (t^{\beta-1} + (4\mu - 1)t^{-1})(t^{\beta-1} + (2\mu - 1)t^{-1}) \leq 1\}$. Finally, using this in (15), and taking into account that $\mathbb{P}(N_t \notin (n_1, n_2)) \leq g_1(t) (4C_T/\mu) t^{1-\alpha\beta}$, we finally obtain

$$\left| \mathbb{P}\left(X_t \in \sqrt{\frac{t}{\mu}} I\right) - \int_I e^{-z^2/2} dz \right| \leq g_1(t) \frac{8C_T}{\mu} t^{1-\alpha\beta} + g_2(t) \sqrt{\frac{2\mu}{\pi t}} + g_3(t) \sqrt{\frac{2}{\pi}} t^{\beta-1},$$

with $g_3(t) := (1 + (4\mu - 1)t^{-\beta})$. Now, optimizing $\beta \in (1/\alpha, 1)$ we obtain

$$\left| \mathbb{P}\left(X_t \in \sqrt{\frac{t}{\mu}} I\right) - \int_I e^{-z^2/2} dz \right| \leq \left(g_1(t) \frac{8C_T}{\mu} + g_2(t) \sqrt{\frac{2\mu}{\pi}} t^{-\frac{3-\alpha}{2(\alpha+1)}} + g_3(t) \sqrt{\frac{2}{\pi}} \right) t^{\frac{1-\alpha}{1+\alpha}}.$$

The theorem follows by taking into account that $\max(g_1(t), g_2(t), g_3(t)) \rightarrow \infty$ when $t \rightarrow \infty$. \square

Remark 1. If the distribution of trapping times has exponential moments, i.e., if $\mathbb{E}(\exp(qT)) < \infty$ for some $q \geq 0$, then the upper bound on the rate of convergence can be improved by replacing van Bahr-Esseen's inequality by a tighter inequality, using a Chernoff's bound. In that case we can take $|\mu n_1 - t|$ and $|\mu n_2 - t|$ of order $\sqrt{t \log(t)}$, which gives a rate of convergence of the order of $1/\sqrt{t}$. This is the case of the exponential trapping time distribution $p(\tau) = (1 - \lambda) \lambda^\tau$.

The case $\mathbb{E}(T) < \infty$ and $\mathbb{E}(T^\alpha) = \infty$ for each $\alpha > 1$ has to be treated separately. In this case we are unable to give a general explicit bound for the rate of convergence towards normality, since we can no longer use an explicit concentration inequality. In this case our proof relies on the attractiveness of the Cauchy distribution and applies only to trapping time distributions of the kind $p(\tau) = \tau^{-2} L(\tau)$ with $L(\tau)$ a slowly varying function.

Theorem 3 (CTL for $\alpha = 1$). *Let $\mathbb{E}(T) < \infty$ and define $\mu = \mathbb{E}(T) + 1$ as above. If in addition $\lim_{\tau \rightarrow \infty} \mathbb{P}(T \geq t\tau)/\mathbb{P}(T \geq \tau) = t^{-1}$ for each $t \in \mathbb{R}^+$, then*

$$\lim_{t \rightarrow \infty} \left| \mathbb{P} \left(X_t \in \sqrt{\frac{t}{\mu}} I \right) - \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2} dx \right| = 0.$$

Remark 2. Notice that $\lim_{\tau \rightarrow \infty} \mathbb{P}(T \geq t\tau)/\mathbb{P}(T \geq \tau) = t^{-1}$ for all $t \in \mathbb{R}^+$ implies that $\mathbb{E}(T^\alpha) = \infty$ for each $\alpha > 1$. For this it is enough to remark that if $\sum_{\tau \in \mathbb{N}} \tau^{1+\epsilon} p(\tau) < \infty$ for some $\epsilon > 0$, then $\sum_{\tau \in \mathbb{N}} \tau^\epsilon \mathbb{P}(T \geq \tau) < \infty$, and necessarily

$$\liminf_{\tau \rightarrow \infty} \frac{(t\tau)^\epsilon \mathbb{P}(T \geq t\tau)}{\tau^\epsilon \mathbb{P}(T \geq \tau)} = \liminf_{\tau \rightarrow \infty} \frac{t^\epsilon \mathbb{P}(T \geq t\tau)}{\mathbb{P}(T \geq \tau)} = c \in (0, 1),$$

for each $t \in \mathbb{R}^+$, and therefore $\liminf_{\tau \rightarrow \infty} \mathbb{P}(T \geq t\tau)/\mathbb{P}(T \geq \tau) = ct^{-\epsilon}$ for all $t \in \mathbb{R}^+$.

Proof. As mentioned before, the proof of this result relies on the attractiveness of the Cauchy distribution. Indeed, the condition $\lim_{n \rightarrow \infty} \mathbb{P}(T \geq tn)/\mathbb{P}(T \geq n) = t^{-1}$ ensures that T belongs to the domain of attraction of a stable distribution with index $\alpha = 1$. Furthermore, since $T > 0$, then the stable limiting distribution, which necessarily has characteristic function $t \mapsto \exp(-|t|)$, is nothing but the Cauchy distribution, $x \mapsto \pi/(1+x^2)$ (see [8] for instance). Therefore, according to Gnedenko's Theorem [7], there are sequences $\{a_n\}_{n \in \mathbb{N}}$ in \mathbb{N} and $\{b_n\}_{n \in \mathbb{N}}$ in \mathbb{R} such that $\sum_{k=1}^n T_k/a_n - b_n$ converges in law to $x \mapsto \pi^{-1} \int_{t \leq x} dt/(1+t^2)$. Furthermore, we can take $a_n := \min\{\tau \in \mathbb{N} : \mathbb{P}(T > \tau) \leq 1/n\}$ and $b_n = n(\sum_{\tau \leq a_n} \tau p(\tau))/a_n$ (see for instance Theorem 3.7.4 in [5]). It follows that there exists a positive sequence $\{\epsilon_n\}_{n \in \mathbb{N}}$ converging to zero, such that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\mathbb{P} \left(\frac{\sum_{k=1}^n T_k}{a_n} - b_n \leq t \right) \leq \frac{1}{\pi} \int_{-\infty}^t \frac{dx}{1+x^2} + \epsilon_n.$$

Let $t \mapsto d(t)$ be a positive diverging function, yet to be specified, and define

$$n_1 := \max\{n \leq t/\mu : t > n + a_n(b_n + d(t))\}, \quad n_2 := \min\{n \geq t/\mu : t < n + a_{n+1}(b_{n+1} - d(t))\}.$$

With this we have

$$\begin{aligned}
\mathbb{P}(N_t \leq n_1) &= \mathbb{P}\left(\sum_{k=1}^{n_1} T_k \geq t - n_1\right) \leq \frac{1}{\pi} \int_{d(t)}^{\infty} \frac{dx}{1+x^2} + \epsilon_{n_1}, \\
&= \frac{1}{2} - \frac{\arctan(d(t))}{\pi} + \epsilon_{n_1} \\
\mathbb{P}(N_t \geq n_2) &= \mathbb{P}\left(\sum_{k=1}^{n_2+1} T_k \leq t - n_2\right) \leq \frac{1}{\pi} \int_{-\infty}^{d(t)} \frac{dx}{1+x^2} + \epsilon_{n_2+1} \\
&= \frac{1}{\pi} \int_{d(t)}^{\infty} \frac{dx}{1+x^2} + \epsilon_{n_2+1} \\
&= \frac{1}{2} - \frac{\arctan(d(t))}{\pi} + \epsilon_{n_2+1}.
\end{aligned}$$

Therefore

$$(16) \quad \mathbb{P}(N_t \notin (n_1, n_2)) \leq 2 \left(\epsilon_{n_1} + \frac{1}{2} - \frac{\arctan(d(t))}{\pi} \right) \leq 2 \left(\epsilon_{n_1} + \frac{2}{d(t)} \right),$$

provided $d(t)$ is sufficiently large. Following exactly the same computations as in the proof of Theorem 2, we obtain

$$\left| \mathbb{P}\left(X_t \in \sqrt{\frac{t}{\mu}} I\right) - \int_I e^{-z^2/2} dz \right| \leq 4 \left(\epsilon_{n_1} + \frac{2}{d(t)} \right) + \sqrt{\frac{t}{n_1 \mu}} \left(1 - \sqrt{\frac{n_1}{n_2}} \right) + \frac{2}{\sqrt{2\pi n_1}}$$

provided $\mu n_1 \leq t \leq \mu n_2$.

To complete the proof we need to find a diverging function $t \mapsto d(t)$ such that $n_2/n_1 \rightarrow 1$ as $t \rightarrow \infty$. For this, consider $g(\tau) := \tau \mathbb{P}(\tau)$. Since $g(\tau) \leq \sum_{\tau' \geq \tau} \tau p(\tau')$ and $\mathbb{E}(T) < \infty$, then $\lim_{\tau \rightarrow \infty} g(\tau) = 0$. On the other hand, by the hypothesis on $\mathbb{P}(T \geq \tau)$, for each $\epsilon > 0$, the function $\tau \mapsto g_\epsilon(\tau) := \tau^\epsilon g(\tau)$ is regularly varying of order ϵ , i.e. $\lim_{\tau \rightarrow \infty} g_\epsilon(t\tau)/g_\epsilon(\tau) = t^\epsilon$ for each $t \in \mathbb{R}^+$. Therefore $g_\epsilon(\tau)$ diverges for all $\epsilon > 0$, which proves that $g(\tau)$ converges to zero slower than any power-law. Now, since by definition $g(n) \geq a_n/n \geq g(n+1) - \mathbb{P}(T \geq a_n+1)$, then $\{n/a_n\}_{n \in \mathbb{N}}$ is a diverging sequence growing slower than any power-law, and indeed $n \mapsto n/a_n$ is a slowly varying function, i.e. a regularly varying function of order zero. Taking all this into account, it follows that the function

$$n \mapsto h(n) := \min \left\{ \frac{\tau}{a_\tau} : \tau \geq n \right\},$$

diverges monotonously and slower than any power-law. Furthermore, it is a slowly varying function. Finally, let $d(t) := \sqrt{h([t/\mu])}$, which is monotonously diverging and slow varying as well. To conclude, notice that from the definition of n_1 , n_2 , and $h(t)$, we have on the one hand,

$$n_1 + 1 \geq \frac{t}{\mu} - \frac{a_{n_1+1} d(t)}{\mu} \geq \frac{t}{\mu} - \frac{a_{[t/\mu]+1} d(t)}{\mu} \geq \frac{t}{\mu} - \frac{(t/\mu + 1) d(t)}{\mu h([t/\mu] + 1)} \geq \frac{t}{\mu} - \frac{(t/\mu + 1)}{\mu d(t)}$$

and on the other hand

$$\begin{aligned} n_2 - 1 &\leq \frac{t}{\mu} + \frac{a_{n_2} d(t)}{\mu} + \frac{n_2 \sum_{\tau > n_2} \tau p(\tau)}{\mu} \leq \frac{t}{\mu} + \frac{n_2 d(t)}{\mu h(n_2)} + \frac{n_2 \sum_{\tau > n_2} \tau p(\tau)}{\mu} \\ &\leq \frac{t}{\mu} + \frac{n_2 d(t)}{\mu h(\lceil t/\mu \rceil)} + \frac{n_2 \sum_{\tau > t/\mu} \tau p(\tau)}{\mu}, \end{aligned}$$

and therefore $n_2 \left(1 - (\mu d(t))^{-1} - \mu^{-1} \sum_{\tau > t/\mu} \tau p(\tau)\right) - 1 \leq t/\mu$. Hence,

$$1 \leq \frac{n_2}{n_1} \leq \frac{(t/\mu + 1) \left(1 - (\mu d(t))^{-1} - \mu^{-1} \sum_{\tau > t/\mu} \tau p(\tau)\right)^{-1}}{t/\mu \left(1 - (1 + \mu t^{-1}) (d(t) \mu)^{-1}\right)} \rightarrow 1, \text{ as } t \rightarrow \infty,$$

and the proof is done. \square

4. SUB-DIFFUSIVE REGIME

In this section we study trapping time distributions leading to a sub-normal diffusion. First, we will analyze the sub-linear growth of the MSD in the case of a trapping time distribution with diverging first moment. In the particular case of power-law distributed trapping times, we are able to determine the behavior of this sub-linear growth, which turns out to be a power-law as well. Then, we study the concentration of the number of steps made by the trapped random walker, for the larger class of distributions for which $\mathbb{E}(T^\alpha) < \infty$, for some $\alpha < 1$.

4.1. Sub-diffusion via MSD. Let us consider power-law decaying probability distributions for which the mean trapping time diverges. In this case we loose the asymptotic normal behavior since the MSD now follows a sub-linear power-law growth. Let us start by establishing the sub-linearity of the MSD with respect to time, which holds in all cases when $\mathbb{E}(T) = \infty$.

Proposition 3 (Sub-diffusion). *Suppose that the trapped random walk satisfies the initial condition (2). If the trapping-time has infinite mean, then $\lim_{t \rightarrow \infty} \sigma_t^2/t = 0$.*

Proof. Let us assume that $\sigma_t^2 = \alpha t + \epsilon_t$ for some $\alpha > 0$ and each $t \geq 0$. Using (7) we obtain $\epsilon_{t+1} = \sum_{0 \leq \tau \leq t} \epsilon_{t-\tau} p(\tau) + \delta_t$, with

$$\delta_t = \mathbb{P}(T \leq t) - \alpha \left((t+1) \mathbb{P}(T > t) + \sum_{0 \leq \tau \leq t} (\tau+1) p(\tau) \right).$$

Following the same reasoning as in the proof of Theorem 1 we obtain $\epsilon_{t+1} = \sum_{0 \leq \tau \leq t} Q_{t-\tau} \delta_\tau$, with Q_τ recursively defined by $Q_{t+1} = \sum_{0 \leq \tau \leq t} Q_{t-\tau} p(\tau)$, and such that $Q_0 = 1$. Taking this into account we have,

$$\begin{aligned} \epsilon_{t+1} &= \sum_{0 \leq \tau \leq t} Q_{t-\tau} \delta_\tau \geq -\alpha \left(\sum_{0 \leq \tau \leq t} (\tau+1) \mathbb{P}(T > \tau) + \sum_{0 \leq \tau \leq t} \sum_{0 \leq \tau' \leq \tau} (\tau'+1) p(\tau') \right) \\ &\geq -\alpha (t+1) \left(\frac{t+2}{2} \mathbb{P}(T \geq t+1) + \sum_{0 \leq \tau \leq t} (\tau+1) p(\tau) \right), \end{aligned}$$

for all $t \in \mathbb{N}$. Since $\mathbb{E}(T) = \infty$, then $\lim_{t \rightarrow \infty} \delta_t = -\infty$ which implies that $\delta_t < 0$ eventually for all t . Hence, there exists $t_1 \in \mathbb{N}$ such that $\epsilon_{t+1} = \sum_{0 \leq \tau \leq t} Q_{t-\tau} \delta_\tau \leq \sum_{0 \leq \tau < t_1} |\delta_\tau|$ for all $t \in \mathbb{N}$. Thus, we have the inequalities

$$-\alpha(t+1) \left(\frac{t+2}{2} \mathbb{P}(T \geq t+1) + \sum_{0 \leq \tau \leq t} (\tau+1) p(\tau) \right) \leq \epsilon_{t+1} \leq \sum_{0 \leq \tau < t_1} |\delta_\tau|$$

for all $t \in \mathbb{N}$, from which it follows that

$$-\infty \leq \limsup_{t \rightarrow \infty} \frac{\sigma_t}{t} = \alpha + \limsup_{t \rightarrow \infty} \frac{\epsilon_t}{t} \leq \alpha.$$

The proposition follows from the fact that $\sigma_t^2 > 0$ for each $t \geq 0$, taking into account that $\alpha > 0$ can be taken arbitrarily small. \square

Variation slower than any power-law. In the case of trapping time distributions decaying as a power-law, the growth of the MSD is dominated by a power-law sub-linear growth. The exponent of the power-law governing the MSD growth directly depends on the exponent of the power-law decay of the trapping time distribution. To be more precise we need the following definition. We will say that a strictly positive function $\tau \mapsto g(\tau)$ varies slower than any power-law if $\lim_{t \rightarrow \infty} g(t) t^{-\epsilon} = 0$ and $\lim_{t \rightarrow \infty} g(t) t^\epsilon = \infty$ for any $\epsilon > 0$. In particular, any regularly varying function of order zero varies slower than any power-law.

In the following we will need some basic properties satisfied by functions varying slower than any power-law, the proof of which is postponed to the Appendix A.

Claim 1. *Let $t \mapsto g(t)$ and $t \mapsto h(t)$ be two functions varying slower than any power-law. Then the following are functions varying slower than any power-law.*

- a) $t \mapsto \lambda g(t)$ with $\lambda > 0$, b) $t \mapsto g(t) + h(t)$, c) $t \mapsto g(t) h(t)$,
d) $t \mapsto 1/g(t)$, e) $t \mapsto \min_{\mu t \leq \tau \leq \lambda t} g(\tau)$, f) $t \mapsto \max_{\mu t \leq \tau \leq \lambda t} g(\tau)$ with $0 \leq \mu < \lambda$.

Furthermore, if $g(t) \leq f(t) \leq h(t)$ for each $t \in \mathbb{N}$, then

g) $t \mapsto f(t)$ varies slower than any power-law,

and, if $\tau \mapsto P(\tau) \geq 0$ is such that $0 < \sum_{s \geq 1} P(s) (s+1)^{\epsilon_0} < \infty$ for some $\epsilon_0 > 0$, then

h) $t \mapsto \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau)$, varies slower than any power-law as well.

We are now able to prove the following.

Theorem 4. *Let suppose that $p(\tau) = (\tau+1)^{-q} g(\tau)$, with $1 < q \leq 2$ and $\tau \mapsto g(\tau)$ varying slower than any power-law. Then there exists $\tau \mapsto h(\tau)$, varying slower than any power-law, such that $\sigma_t^2 = h(t) t^{q-1}$ for all $t \in \mathbb{N}$.*

Proof. Let $\beta = \beta(q) := \sup\{b \geq 0 : \limsup_{n \rightarrow \infty} \sigma_t^2 / t^b = \infty\}$. Proposition 2 ensures that $\lim_{t \rightarrow \infty} \sigma_t^2 = \infty$, which implies $\beta(q) \geq 0$. It is easy to verify that $\lim_{t \rightarrow \infty} \sigma_t^2 / t^b = \infty$ for each $0 \leq b < \beta$ and $\lim_{t \rightarrow \infty} \sigma_t^2 / t^b = 0$ for each $b > \beta$.

For β defined as above, the function $t \mapsto h(t) := \sigma_t^2/t^\beta$ varies slower than any power-law. This follows from Claim 1-h, and the fact that both $h_1(t) := \min_{0 \leq \tau \leq t} \sigma_\tau^2/\tau^\beta$ and $h_2(t) := \max_{0 \leq \tau \leq t} \sigma_\tau^2/\tau^\beta$ vary slower than any power-law. Indeed, for each $\epsilon > 0$ we have

$$\begin{aligned} \lim_{t \rightarrow \infty} h_1(t) t^{-\epsilon} &= \lim_{t \rightarrow \infty} \left(\min_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^\beta} \right\} \min_{1 \leq \tau \leq t} \{\tau^{-\epsilon}\} \right) \leq \lim_{t \rightarrow \infty} \left(\min_{1 \leq \tau \leq t} \frac{\sigma_\tau^2}{\tau^{\beta+\epsilon}} \right) \leq \lim_{t \rightarrow \infty} \frac{\sigma_t^2}{t^{\beta+\epsilon}} = 0, \\ \lim_{t \rightarrow \infty} h_2(t) t^\epsilon &= \lim_{t \rightarrow \infty} \left(\max_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^\beta} \right\} \max_{1 \leq \tau \leq t} \{\tau^\epsilon\} \right) \geq \lim_{t \rightarrow \infty} \left(\max_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^{\beta-\epsilon}} \right\} \right) \geq \lim_{t \rightarrow \infty} \frac{\sigma_t^2}{t^{\beta-\epsilon}} = \infty. \end{aligned}$$

On the other hand, taking into account that $\inf_{t \in \mathbb{N}} \sigma_t^2/t^b > 0$ for each $b < \beta$, and $\sup_{t \in \mathbb{N}} \sigma_t^2/t^b < \infty$ for each $b > \beta$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} h_1(t) t^\epsilon &= \lim_{t \rightarrow \infty} \left(\min_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^\beta} \right\} t^\epsilon \right) \geq \lim_{t \rightarrow \infty} \left(\min_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^{\beta-\epsilon/2}} \right\} t^{\epsilon/2} \right) \\ &\geq \left(\inf_{t \in \mathbb{N}} \frac{\sigma_t^2}{t^{\beta-\epsilon/2}} \right) \lim_{t \rightarrow \infty} t^{\epsilon/2} = \infty, \\ \lim_{t \rightarrow \infty} h_2(t) t^{-\epsilon} &= \lim_{t \rightarrow \infty} \left(\max_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^\beta} \right\} t^{-\epsilon} \right) \leq \lim_{t \rightarrow \infty} \left(\max_{1 \leq \tau \leq t} \left\{ \frac{\sigma_\tau^2}{\tau^{\beta+\epsilon/2}} \right\} t^{-\epsilon/2} \right) \\ &\leq \left(\sup_{t \in \mathbb{N}} \frac{\sigma_t^2}{t^{\beta+\epsilon/2}} \right) \lim_{t \rightarrow \infty} t^{-\epsilon/2} = 0. \end{aligned}$$

To conclude the proof verify that $\beta(q) = q - 1$. For this, according to Proposition 2, $\sigma_s^2 - \sigma_t^2 \geq 0$ whenever $s \geq t$. Then, using this and Proposition 1 we obtain

$$(17) \quad \mathbb{P}(T > t) \sigma_t^2 = \mathbb{P}(T \leq t) - \sum_{\tau \leq t} (\sigma_t^2 - \sigma_{t-\tau}^2) p(\tau) - (\sigma_{t+1}^2 - \sigma_t^2).$$

for each $t \geq 1$. This implies that $\sigma_t^2 \leq 1/\mathbb{P}(T > t)$ and therefore, for each $b > q - 1$ and $t \in \mathbb{N}$ we have

$$\begin{aligned} \frac{\sigma_t^2}{t^b} &\leq \frac{t^{-b+q-1}}{\sum_{\tau > t} g(\tau) t^{q-1} (\tau+1)^{-q}} \leq \frac{t^{-b+q-1}}{\left(\sum_{t < \tau \leq 2t} g(\tau) (\tau/t+1)^{-q} \right) t^{-1}} \\ &\leq \frac{t^{-b+q-1}}{\min_{t < \tau \leq 2t} g(\tau) \left(\sum_{t < \tau \leq 2t} (\tau/t+1)^{-q} \right) t^{-1}} \leq \frac{t^{-b+q-1}}{3^{-q} \min_{t < \tau \leq 2t} g(\tau)} \end{aligned}$$

Since by Claim 1-e, $t \mapsto \min_{t < \tau \leq 2t} g(\tau)$ varies slower than any power-law, then $\lim \sigma_t^2/t^b = 0$ for each $b > q - 1$, hence $\beta(q) \leq q - 1$.

Let us assume that $q < 2$. In this case, using $\sigma_t^2 = h(t)t^\beta$ in Equation (17), and taking into account that $\beta \leq q - 1$ and $p(\tau) = g(\tau)(\tau + 1)^{-q}$, we obtain

$$\begin{aligned}
(18) \quad \mathbb{P}(T > t) \sigma_t^2 &\geq \mathbb{P}(T \leq t) - \max_{\tau \leq t} h(\tau) \left(\sum_{\tau \leq t} \left(1 - (1 - \tau/t)^\beta \right) p(\tau) + \left(1 - (1 - 1/t)^\beta \right) \right) t^\beta \\
&\geq \mathbb{P}(T \leq t) - \max_{\tau \leq t} h(\tau) \left(\sum_{\tau \leq t} \tau p(\tau) + 1 \right) t^{\beta-1} \\
&\geq \mathbb{P}(T \leq t) - \max_{\tau \leq t} h(\tau) \left(\max_{\tau \leq t} g(\tau) \left(1 + \frac{(t+1)^{2-q}}{2-q} \right) + 1 \right) t^{\beta-1} \\
&\geq \mathbb{P}(T \leq t) - \max_{\tau \leq t} h(\tau) \left(\max_{\tau \leq t} g(\tau) \left(1 + \frac{1}{2-q} \right) + 1 \right) (t+1)^{\beta+1-q}.
\end{aligned}$$

If $\beta = q - 1$, then we have nothing to prove. Otherwise, if $\beta < q - 1$, since by Claim 1-*f*, $t \mapsto \max_{\tau \leq t} h(\tau)$ and $t \mapsto \max_{\tau \leq t} g(\tau)$ both vary slower than any power-law, then there exists t_0 such that for all $t \geq t_0$ we have $2\sigma_t^2 \geq 1/\mathbb{P}(T > t)$. Hence, for each $b < q - 1$ and $t \geq t_0$ we have

$$\frac{\sigma_t^2}{t^b} \geq \frac{t^{-b+q-1}}{\sum_{s \geq 1} s^{-q} \left(\sum_{s t < \tau \leq (s+1)t} g(\tau) \right) t^{-1}} \geq \frac{t^{-b+q-1}}{\sum_{s \geq 1} s^{-q} \max_{s t < \tau \leq (s+1)t} g(\tau)}.$$

Claim 1-*h* ensures that $t \mapsto \sum_{s \geq 1} s^{-q} \max_{s t < \tau \leq (s+1)t} g(\tau)$ varies slower than any power-law, therefore $\lim \sigma_t^2/t^b = \infty$ for each $b < q - 1$, hence $\beta(q) \geq q - 1$. For the remaining case, $q = 2$, Inequality (18) implies

$$\mathbb{P}(T > t) \sigma_t^2 \geq \mathbb{P}(T \leq t) - \max_{\tau \leq t} h(\tau) \left(\max_{\tau \leq t} g(\tau) (1 + \log(t+1)) + 1 \right) t^{\beta-1},$$

and since $t \mapsto \max_{\tau \leq t} h(\tau)$, $t \mapsto \max_{\tau \leq t} g(\tau)$ and $t \mapsto \log(t+1)$ vary slower than any power-law, then $2\sigma_t^2 \geq 1/\mathbb{P}(T > t)$ for all t sufficiently large, and from there we can proceed as in the previous case, concluding that $\beta(2) \geq 1$. And with this we finish the proof. \square

Power-law trapping time. For power-law distributed trapping times with exponent $q \leq 2$, the finite-time behavior of the MSD can be reasonably well fitted by a power-law. Using Proposition 1 we computed σ_t^2 for $1 \leq t \leq 2^{17}$ and fitted the behavior to a power-law, $\sigma_t^2 = \mathcal{O}(t^\beta)$, with an exponent $\beta_N(q)$, which deviates from the expected exponent $\beta(q) = q - 1$. We plot this effective exponent in Figure 4 for time intervals $10 \leq t \leq N$, with $N = 2^{13}, 2^{15}$ and 2^{17} . For each N , the behavior of $\beta_N(q)$ can be very well fitted by a sigmoidal function $\beta(q) = c + 2(1 - c)/(1 + e^{r|d-3|^\eta})$.

4.2. Sub-diffusion via X_t . In this subsection we will consider trapping time probability distributions satisfying $\mathbb{E}(T^\alpha) < \infty$ and $\mathbb{E}(T^\beta) = \infty$, for some $\alpha \in (0, 1)$ and all $\beta > \alpha$. This case is consistent with a sub-linear growth of the MSD dominated by a power-law of order α . We will show that the expected number N_t of steps the random walker makes up to time t grows like t^α .

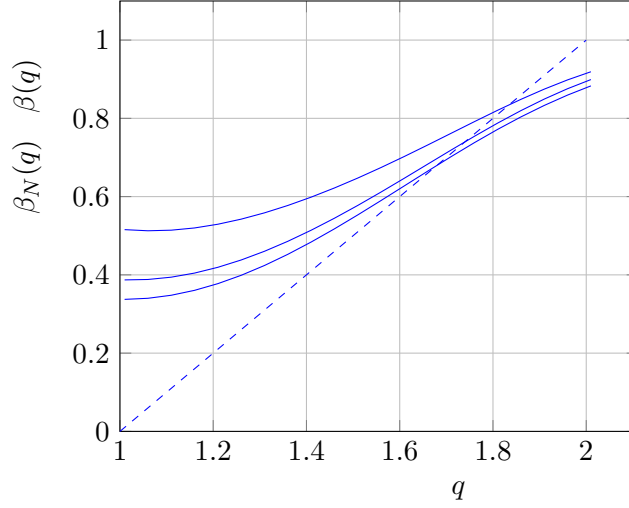


FIGURE 4. The curves $q \mapsto \beta_N(q)$ correspond to the exponent of the approximated power-law behavior of the MSD as a function of the trapping time distribution's exponent. We show these curves for total observation times $N = 2^{13}, 2^{15}$ and 2^{17} . The curves approach, as $N \rightarrow \infty$, the asymptotic exponent $q \mapsto \beta(q) = q - 1$ (dashed line).

Proposition 4 (N_t concentration for $\alpha < 1$). *Let us assume that $\mathbb{E}(T^\alpha) < \infty$ and $\mathbb{E}(T^\beta) = \infty$, for some $\alpha \in (0, 1)$ and all $\beta > \alpha$. Then there exists a function $t \mapsto h(t)$ varying slower than any power-law and converging to 0, and a constant $C > 0$ such that*

$$\mathbb{P}(N_t \notin t^\alpha [h(t), h^{-2}(t)]) \leq C h(t)$$

for each $t \in \mathbb{N}$.

Proof. We will start by observing that $\mathbb{P}(T \geq \tau) = h(\tau) (\tau + 1)^{-\alpha}$ for some function $\tau \mapsto h(\tau)$ varying slower than any power-law and converging to zero. For this notice that if $q \leq 1 - \alpha$, then

$$\sum_{0 \leq \tau} (\tau + 1)^{-q} \mathbb{P}(T \geq \tau) = \sum_{0 \leq \tau} p(\tau) \left(\sum_{0 \leq \tau' \leq \tau} (\tau + 1)^{-q} \right) \leq \frac{\mathbb{E}((T + 1)^{1-q})}{1 - q} < \infty.$$

On the other hand, if $q > 1 - \alpha$, then

$$\sum_{0 \leq \tau} (\tau + 1)^{-q} \mathbb{P}(T \geq \tau) \geq \frac{\mathbb{E}((T + 1)^{1-q}) - 1}{1 - q} = \infty.$$

Hence, $\lim_{\tau \rightarrow \infty} \mathbb{P}(T \geq \tau) (\tau + 1)^\beta = 0$ for all $\beta \leq \alpha$ while $\lim_{\tau \rightarrow \infty} \mathbb{P}(T \geq \tau) (\tau + 1)^\beta = \infty$ for $\beta > \alpha$. Now, for $n \geq t^\alpha h(t)^{-2}$ we have

$$\begin{aligned} \mathbb{P}(N_t \geq n) &= \mathbb{P}\left(\sum_{1 \leq k \leq n+1} T_k \leq t - n\right) \leq (1 - \mathbb{P}(T \leq t - n))^n \leq (1 - \mathbb{P}(T \leq t))^n \\ &= (1 - h(t) (t + 1)^{-\alpha})^n = \left(1 - \frac{h(t)}{(t + 1)^\alpha}\right)^{t^\alpha h(t)^{-2}} \\ &\leq \exp\left(-\frac{t^\alpha}{(t + 1)^\alpha h(t)}\right) \leq \left(-\frac{t^\alpha}{(t + 1)^\alpha h(t)}\right) \leq \frac{h(t)}{(1 + t^{-1})^\alpha}, \end{aligned}$$

provided $h(t) \leq (1 + t^{-1})^\alpha$. By taking $t_0 := \min\{t \geq (2^{1/\alpha} - 1)^{-1} : h(\tau) \leq (1 + \tau^{-1})^\alpha \forall \tau \geq t\}$ and $h_0 = \min_{0 \leq t \leq t_0} h(t)$ we have $\mathbb{P}(N_t \geq t^\alpha h(t)^{-2}) \leq \max(h_0^{-1}, 2) h(t)$ for all $t \in \mathbb{N}$.

For the other inequality, suppose that $n - 1 \leq t^\alpha h(t) \leq n$. Using Eq. 10 and Bahr-Esseen's inequality we have

$$\mathbb{P}(N_t \leq n) = \mathbb{P}\left(\sum_{1 \leq k \leq n} T_k \geq t - n\right) \leq \frac{n \mathbb{E}(T^\alpha)}{(t - n)^\alpha} \leq \frac{\mathbb{E}(T^\alpha)}{g(t) (1 - t^{\alpha-1} h(t) - t^{-1})^\alpha}.$$

Let $t_1 = \min\{t \in \mathbb{N} : 1 - \tau^{\alpha-1} h(\tau) - \tau^{-1} < 2^{-1/\alpha} \forall \tau \geq t\}$ and $h_1 = \min_{0 \leq t \leq t_1} h(t)$, then $\mathbb{P}(N_t \leq t^\alpha h(t)) \leq \max(h_1^{-1}, 2 \mathbb{E}(T^\alpha)) h(t)$ for all $t \in \mathbb{N}$.

The proof is done by taking $C := \max(h_0^{-1}, h_1^{-1}, 2, 2\mathbb{E}(T^\alpha))$. \square

Remark 3. According to Gnedenko's Theorem, for $\mathbb{P}(T \geq t) = (t + 1)^{-\alpha} h(t)$ with $t \mapsto h(t)$ a slowly varying function, the sequence of distributions $x \mapsto \mathbb{P}(n^{-1/\alpha} \sum_{k=1}^n T_k \leq x)$ converges in law to $x \mapsto \int_{-\infty}^x F_\alpha(z) dz$, where F_α is an α -stable distribution with support in \mathbb{R}^+ . This result and Barry-Esseen Inequality allow us to derive an analogous to the Central Limit Theorem. We have the following.

Theorem 5. *Let us assume that $\mathbb{P}(T \geq t) = (t + 1)^{-\alpha} h(t)$ with $t \mapsto h(t)$ a slowly varying function. Then*

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(\frac{X_t}{t^{\alpha/2}} \in I\right) = \int_{\mathbb{R}^+} F_\alpha(z) dz \frac{1}{\sqrt{2\pi} z} \int_I \exp\left(-\frac{x^2}{2z}\right) dx.$$

Remark 4. Under the hypotheses of this theorem, $\mathbb{E}(T^\gamma) < \infty$ and $\mathbb{E}(T^\beta) = \infty$, for all $\gamma < \alpha < \beta$. In this case it is not possible, in general, to explicitly determine F_α or the speed of convergence, but still some features of the limit behavior can be derived from this expression, for instance the fact that it is a symmetric distribution and that all absolute moments of order larger than α diverge. The scaling indicates that the support of the measure is concentrated around t^α , but this was already clear from Proposition 4.

Proof. For each $\delta > 0$ and $w \in \mathbb{R}^+$ we have

$$\begin{aligned} \mathbb{P}\left(w \leq \frac{N_t}{t^\alpha} \leq w + \delta\right) &= \mathbb{P}\left(\sum_{k=1}^{\lfloor t^\alpha w \rfloor} T_k \leq t - t^\alpha w\right) - \mathbb{P}\left(\sum_{k=1}^{\lfloor t^\alpha (w+\delta) \rfloor + 1} T_k \leq t - t^\alpha (w + \delta)\right) \\ &= \mathbb{P}\left(\frac{\sum_{k=1}^{\lfloor t^\alpha w \rfloor} T_k}{\lfloor t^\alpha w \rfloor^{1/\alpha}} \leq \frac{t - t^\alpha w}{\lfloor t^\alpha w \rfloor^{1/\alpha}}\right) \\ &\quad - \mathbb{P}\left(\frac{\sum_{k=1}^{\lfloor t^\alpha (w+\delta) \rfloor + 1} T_k}{(\lfloor t^\alpha (w + \delta) \rfloor + 1)^{1/\alpha}} \leq \frac{t - t^\alpha (w + \delta)}{(\lfloor t^\alpha (w + \delta) \rfloor + 1)^{1/\alpha}}\right). \end{aligned}$$

Hence, by virtue of Theorems 4.2.1 and 4.2.2 in [10], we obtain

$$\left| \mathbb{P}(N_t/t^\alpha \in [w, w + \delta]) - \int_{(w+\delta)^{-1/\alpha}}^{w^{-1/\alpha}} F_\alpha(z) dz \right| \leq g_1(t w^{1/\alpha}),$$

where $s \mapsto g_1(s) > 0$ is monotonously decreasing, such that $\lim_{s \rightarrow \infty} g_1(s) = 0$. On the other hand, for $m > n$ we have

$$\left| \int_{\sqrt{t^\alpha/n} I} e^{-x^2/2} dx - \int_{\sqrt{t^\alpha/m} I} e^{-x^2/2} dx \right| \leq e^{-1/2} \left(\sqrt{\frac{m}{n}} - 1 \right).$$

Hence, taking into account Equation (10), Barry-Esseen's inequality and Proposition 4, we obtain

$$\left| \mathbb{P}\left(\frac{X_t}{t^{\alpha/2}} \in I\right) - \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\lceil (h^{-2}(t)-h(t))/\delta \rceil} \int_{(h(t)+k\delta)^{-1/\alpha}}^{(h(t)+(k-1)\delta)^{-1/\alpha}} F_\alpha(z) \int_{I/\sqrt{z}} e^{-x^2/2} dx dz \right| \leq g_2(t, \delta),$$

with $g_2(t, \delta) = g_1(t h(t)^{1/\alpha}) + 2/\sqrt{2\pi t^\alpha h(t)} + e^{-1/2}(\sqrt{(h(t) + \delta)/h(t)} - 1)$. Finally, by taking $\delta = \delta(t) := h^2(t)$ we obtain

$$\left| \mathbb{P}\left(\frac{X_t}{t^{\alpha/2}} \in I\right) - \int_{h^{2/\alpha}(t)}^{h(t)^{-1/\alpha}} F_\alpha(z) \frac{1}{\sqrt{2\pi} z} \int_I \exp\left(\frac{-x^2}{2z}\right) dx dz \right| \leq g_3(t),$$

with $g_3(t) = g_2(t, h^2(t)) + h(t)^{2/\alpha+4}/\alpha$, and the result follows. \square

5. SUMMARY AND FINAL REMARKS

5.1. Summary.

A. Thanks to the recurrence relation established in Proposition 1, we were able to characterize the dynamical regimes of the trapped random walk. According to Proposition 2, regardless of the trapping time distribution, the MSD of the walk will always diverge. Hence, dynamical confinement is impossible in this model. Depending on the characteristics of the trapping time distribution, the random walk can display a diffusive or sub-diffusive dynamics.

- B. The diffusive behavior, which takes place when the mean trapping time is finite, can be observed at finite time if the second moment of the trapping time is finite, as established in Theorem 1. On the contrary, for trapping time distributions with finite mean and diverging second moment, the diffusive behavior takes place asymptotically, and for any finite time we can measure an effective sub-diffusive behavior. To illustrate this sub-diffusive behavior we have computed the MSD as a function of time using a collection of power-law trapping time distributions (see Figure 3).
- C. We consider a class of trapping time distributions consisting of a leading power-law behavior with fluctuations around this leading behavior, varying slower than any power law. As established in Theorem 4, in this case the MSD of the walk grows following a power-law directly related to the leading term of the trapping time distribution. This power-law behavior holds asymptotically, but at any finite time a deviation of smaller order can be observed. We illustrate this deviation with numerical computations from a collection of power-law distributions (see Figure 4).

In the next table we summarize the behavior of the MSD as a function of the trapping time distribution, for power-law trapping time distributions of the type $p(\tau) = \tau^{-q}/\zeta(q)$.

Exponent	MSD leading term	Finite time deviation
$q > 3$	$\mathcal{D} t$	$\mathcal{O}(1)$
$2 < q \leq 3$	$\mathcal{D} t$	$\mathcal{O}(t^{3-q})$
$1 < q \leq 2$	t^{q-1}	$h(t) t^{q-1}$

Here $t \mapsto h(t)$ is a function varying slower than any power-law and converging to 0.

In Figure 5.1 we depict the exponent of the approximated power-law behavior for different finite observation times for the same family of trapping time distributions.

- D. If the mean trapping time is finite, then the trapped random walk satisfies a Central Limit Theorem. The normalization required as well as the speed of convergence towards the normal distribution both depend on the characteristics of the trapping time distribution. In Theorem 2 we treat the case where the trapping time has a finite fractional moment strictly larger than one, for which we prove a power-law convergence towards the limit normal distribution. In the case of a trapping time distribution belonging to the domain of attraction of the Cauchy distribution, Theorem 3, the convergence we found towards the limit normal law is slower than any power-law.
- E. Finally, when the mean trapping time diverges, any trapping time distribution belonging to the domain of attraction of a stable law leads to a limiting distribution for the trapped random walk, once the length is properly re-normalized. The limit distribution is a convex combination, governed by the corresponding stable law, of normal distributions (see Theorem 5).

In the next table we present the behavior of the required normalization and the speed of convergence towards the normal distribution as a function of the trapping time leading exponent, for trapping time distributions of the type $p(\tau) = \tau^{-q}/\zeta(q)$.

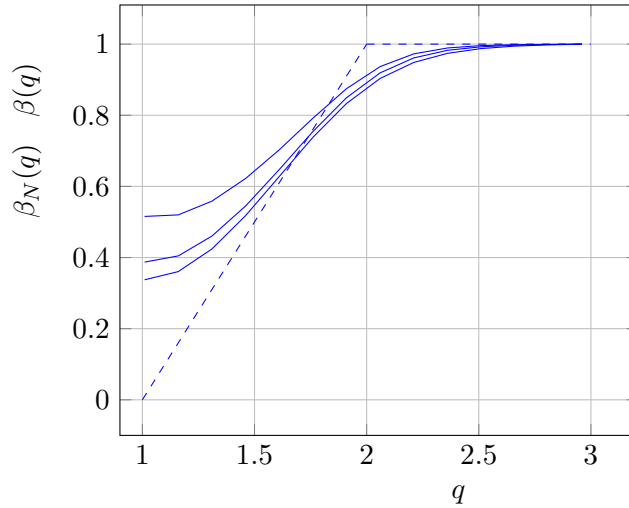


FIGURE 5. The curves $q \mapsto \beta_N(q)$ correspond to the exponent of the approximated power-law behavior of MSD as a function of the trapping time distribution's exponent. We show these curves for total observation times $N = 2^{13}, 2^{15}$ and 2^{17} . The curves approach, as $N \rightarrow \infty$, the asymptotic exponent $q \mapsto \beta(q) = \min(1, q - 1)$ (dashed line).

Exponent	Normalization	Speed of convergence
$q > 2$	$(t/\mu)^{1/2}$	$\mathcal{O}(t^{2/q-1})$
$1 < q \leq 2$	$t^{\alpha/2}$	Slower than any power-law.

In Figure 6 we depict several possible trajectories of the random walk in the diffusive regime. In the figure we also depict the behavior of the standard deviation $\sigma_t \propto \sqrt{t}$. According to Theorems 2 and 3, the random walk spreads around the vertical line $X_t = 0$, with oscillations contained inside the curves $t \mapsto \pm\sigma_t$. With high probability, the number of steps made by the random walker lies between $n_1 \approx (t/\mu)(1 - 1/d(t))$ and $n_2 \approx (t/\mu)(1 + 1/d(t))$, with $t \mapsto d(t)$ a diverging function which can be either a sub-linear power-law (in the case $\mathbb{E}(T^\alpha) < \infty$ for some $\alpha > 1$) or a function varying slower than any power-law (when $\mathbb{E}(T^\alpha) = \infty$ for each $\alpha > 1$). The image in the case of the sub-diffusive regime $\sigma_t \sim t^{\alpha/2}$, is qualitatively the same. In this case, the spread of the random walk is bounded, with high probability, between the curves $h(t)t^\alpha$ and $h^{-2}(t)t^\alpha$, where $h(t)$ is a positive function converging to zero slower than any power law, which can be computed from the trapping time distribution.

5.2. Final Remarks.

- A. In the case of finite mean trapping time we have two scenarios: either it exists $\alpha > 1$ such that $\mathbb{E}(T^\alpha) < \infty$, or $\mathbb{E}(T^\alpha) = \infty$ for each $\alpha > 1$. In the first case the behavior of the system is controlled by the exponent

$$\gamma = \sup\{\alpha > 1 : \mathbb{E}(T) < \infty\} = \inf\{\alpha > 1 : \mathbb{E}(T) = \infty\}.$$

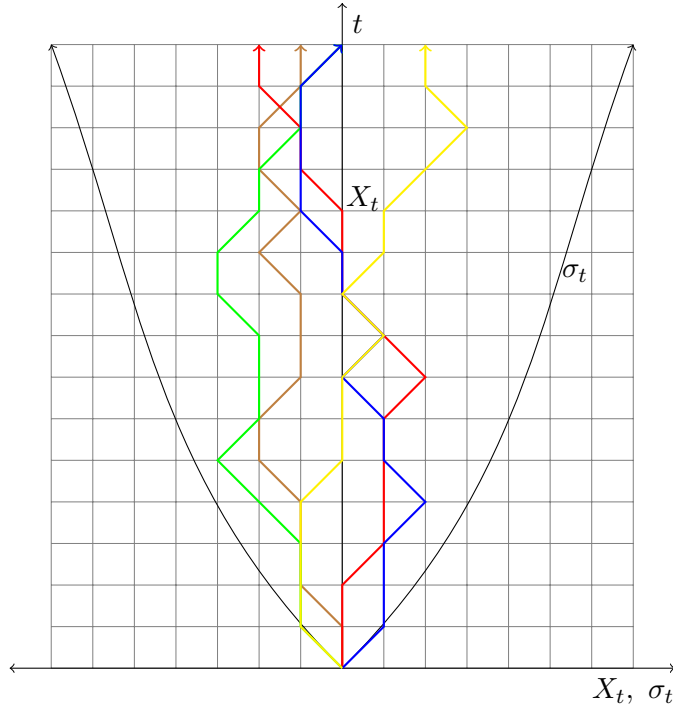


FIGURE 6. Five possible trajectories for the random walk. The spread of the random walk around the vertical line $X_t = 0$ is concentrated around σ_t .

In this case $\sqrt{\mu/t} X_t$ converges in law to the normal distribution, with a speed of the order $(1 - \gamma)/(1 + \gamma)$. This fact can be easily derived from Theorem 2, taking into account that all constants involved in upper bounds vary continuously with the exponents. In the case where $\mathbb{E}(T^\alpha) = \infty$ for each $\alpha > 1$, then in order to ensure convergence to the Gaussian, we have to restrict ourselves to trapping times in the domain of attraction of Cauchy's distribution (Theorem 3). The same kind of restriction has to be assumed when dealing with distributions with infinite mean trapping time (Theorem 5). It remains to determine whether these restrictions are of an essential nature or can be weakened and replaced by a hypothesis concerning only the value of the exponent of the largest finite moment as in Theorem 2.

- B. In Theorem 4 and Proposition 4 we consider trapping time distributions with a leading power-law behavior, modified by a term varying slower than any power-law. This is required to ensure the power-law growth of the MSD. Nevertheless, the convergence towards a limit law for the corresponding random walk requires a little more control on the decay of the distribution: to insure convergence, the distribution has to be regularly varying, which is a strictly stronger condition. In this case, the condition cannot be weakened since regular variation is a necessary condition for being in the domain of attraction of a stable law.

6. ACKNOWLEDGMENTS

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APPENDIX A. VARIATION SLOWER THAN ANY POWER-LAW

A strictly positive function $g : \mathbb{N} \rightarrow (0, \infty)$ varies slower than any power-law if $\lim_{t \rightarrow \infty} g(t) t^{-\epsilon} = 0$ and $\lim_{t \rightarrow \infty} g(t) t^\epsilon = \infty$ for any $\epsilon > 0$. In particular, any regularly varying function of order zero varies slower than any power-law. We have the following.

Claim 1. *Let $t \mapsto g(t)$ and $t \mapsto h(t)$ be two functions varying slower than any power-law. Then the following are functions varying slower than any power-law.*

- a) $t \mapsto \lambda g(t)$ with $\lambda > 0$, b) $t \mapsto g(t) + h(t)$, c) $t \mapsto g(t) h(t)$,
d) $t \mapsto 1/g(t)$, e) $t \mapsto \min_{\mu t \leq \tau \leq \lambda t} g(\tau)$, f) $t \mapsto \max_{\mu t \leq \tau \leq \lambda t} g(\tau)$ with $0 \leq \mu < \lambda$.

Furthermore, if $g(t) \leq f(t) \leq h(t)$ for each $t \in \mathbb{N}$, then

g) $t \mapsto f(t)$ varies slower than any power-law,

and, if $\tau \mapsto P(\tau) \geq 0$ is such that $0 < \sum_{s \geq 1} P(s) (s+1)^{\epsilon_0} < \infty$ for some $\epsilon_0 > 0$, then

h) $t \mapsto \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau)$, varies slower than any power-law as well.

Proof. Items a) to d) are easily proved and are left to the reader.

For e) let $\ell_t = \max\{\mu t \leq \tau \leq \lambda t : g(\tau) = \min_{\mu t \leq s \leq \lambda t} g(s)\}$. Supposing that $\lim_{t \rightarrow \infty} \ell_t = \infty$, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-\epsilon} \min_{\mu t \leq \tau \leq \lambda t} g(\tau) &= \lim_{t \rightarrow \infty} t^{-\epsilon} g(\ell_t) \leq \lambda^\epsilon \lim_{t \rightarrow \infty} \ell_t^{-\epsilon} g(\ell_t) = 0, \\ \lim_{t \rightarrow \infty} t^\epsilon \min_{\mu t \leq \tau \leq \lambda t} g(\tau) &= \lim_{t \rightarrow \infty} t^\epsilon g(\ell_t) \geq \lambda^{-\epsilon} \lim_{t \rightarrow \infty} \ell_t^\epsilon g(\ell_t) = \infty. \end{aligned}$$

Now, if $\lim_{t \rightarrow \infty} \ell_t = \ell < \infty$, in which case $\mu = 0$, then $g(\ell) \leq \min_{\tau \leq \lambda t} g(t) \leq \max_{\tau \leq \ell} g(\tau)$ for each $t \in \mathbb{N}$, and therefore $t \mapsto \min_{\tau \leq \lambda t} g(t)$ varies slower than any power-law.

Item f) directly follows from e) and d) by noticing that $\max_{\mu t \leq \tau \leq \lambda t} g(\tau) = (\min_{\mu t \leq \tau \leq \lambda t} 1/g(\tau))^{-1}$.

For g), it is enough to notice that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{-\epsilon} f(t) &\leq \lim_{t \rightarrow \infty} t^{-\epsilon} h(t) = 0, \\ \lim_{t \rightarrow \infty} t^\epsilon f(t) &\geq \lim_{t \rightarrow \infty} t^\epsilon g(t) = \infty. \end{aligned}$$

for each $\epsilon > 0$.

For h), let $u_{s,t} = \max \{s t < \tau \leq (s+1)t : g(\tau) = \max_{\mu t \leq s \leq \lambda t} g(s)\}$. For each $\epsilon \leq 2\epsilon_0$ we have

$$\begin{aligned} t^{-\epsilon} \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau) &= t^{-\epsilon/2} \sum_{s \geq 1} P(s) g(u_{s,t}) u_{s,t}^{-\epsilon/2} \left(\frac{t}{u_{s,t}} \right)^{-\epsilon/2} \\ &\leq t^{-\epsilon/2} \sum_{s \geq 1} P(s) (s+1)^{\epsilon/2} g(u_{s,t}) u_{s,t}^{-\epsilon/2}, \\ t^\epsilon \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau) &= t^{\epsilon/2} \sum_{s \geq 1} P(s) g(u_{s,t}) u_{s,t}^{\epsilon/2} \left(\frac{t}{u_{s,t}} \right)^{\epsilon/2} \\ &\geq t^{\epsilon/2} \sum_{s \geq 1} P(s) (s+1)^{-\epsilon/2} g(u_{s,t}) u_{s,t}^{\epsilon/2}. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} g(t) t^{-\epsilon/2} = 0$ then $g(t) t^{-\epsilon/2}$ is bounded, and therefore

$$\lim_{t \rightarrow \infty} t^{-\epsilon} \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau) \leq \left(\max_{s \geq 1} g(s) s^{-\epsilon/2} \right) \left(\sum_{s \geq 1} P(s) (s+1)^{\epsilon/2} \right) \lim_{t \rightarrow \infty} t^{-\epsilon/2} = 0.$$

On the other hand, since $g(t) t^{\epsilon/2}$ diverges, then

$$\lim_{t \rightarrow \infty} t^\epsilon \sum_{s \geq 1} P(s) \max_{s t < \tau \leq (s+1)t} g(\tau) \geq \left(\sum_{s \geq 1} P(s) (s+1)^{-\epsilon/2} \right) \lim_{t \rightarrow \infty} t^{\epsilon/2} = \infty.$$

The same obviously holds for $\epsilon > 2\epsilon_0$. □

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