EQUIVARIANT TRIANGULATIONS OF TORI OF COMPACT LIE GROUPS AND HYPERBOLIC EXTENSION TO NON-CRYSTALLOGRAPHIC COXETER GROUPS

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Abstract. Given a simple connected compact Lie group K and a maximal torus T of K, the Weyl group $W = N_K(T)/T$ naturally acts on T.

First, we use the combinatorics of the (extended) affine Weyl group to provide an explicit W-equivariant triangulation of T. We describe the associated cellular homology chain complex and give a formula for the cup product on its dual cochain complex, making it a $\mathbb{Z}[W]$-dg-algebra.

Next, remarking that the combinatorics of this dg-algebra is still valid for Coxeter groups, we associate a closed compact manifold $\mathbf{T}(W)$ to any finite irreducible Coxeter group W, which coincides with a torus if W is a Weyl group and is hyperbolic in other cases. This relies on the choice of a suitable reflection in W. Of course, we focus our study on non-crystallographic groups, which are $I_2(m)$ with $m = 5$ or $m \geq 7$, H_3 and H_4.

The manifold $\mathbf{T}(W)$ comes with a W-action and an equivariant triangulation, whose related $\mathbb{Z}[W]$-dg-algebra is the one mentioned above. We finish by computing the homology of $\mathbf{T}(W)$, as a representation of W.

Contents

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0. Introduction

Given a simple compact connected Lie group K and a maximal torus T of K, the Weyl group $W := N_K(T)/T$ acts on T by conjugation by a representative element of $N_K(T)$. This is well-defined since T is abelian. As T is clearly a CW-complex, it is known that there exists a W-equivariant cellular structure on T. The first aim of this work is to provide an explicit W-triangulation and to describe the associated cellular homology cochain complex, as a $\mathbb{Z}[W]$-dg-algebra. This study is motivated by the research of equivariant cellular structures in Lie theory, and more precisely for flag manifolds and classifying spaces of tori.

Let $\mathfrak{t}$ denote the Lie algebra of T. The exponential map $\exp : \mathfrak{t} \to T$ induces a W-equivariant isomorphism of Lie groups $\mathfrak{t}/\Lambda \to T$, where $\Lambda \subset \mathfrak{t}$ is a W-lattice. The isomorphism type of the pair (K, T) is determined by the root system $\Phi \subset i\mathfrak{t}^*$ and the lattice Λ. This is equivalent to the *root datum* of (K, T) and this gives the suitable vocabulary to work with.

An important distinction comes from the fundamental group $\pi_1(K)$ of K. In the case where $\pi_1(K) = 1$, the combinatorics of the *affine Weyl group* W_{a} and alcoves easily give the desired CW-structure. Indeed, the group W_{a} is Coxeter with one additional generator, corresponding to the reflection of the highest root of Φ. Moreover, the fundamental alcove is a standard simplex and its triangulation provides a W-equivariant triangulation of T. We also give a formula for the cup product (see Corollary 2.2.3). We obtain the following result:

Theorem. *Let K be a simply-connected simple compact Lie group of rank n, $T < K$ be a maximal torus and $W = N_K(T)/T$ be the associated Weyl group. If $Q^\vee$ denotes the coroot lattice of the root system of (K, T), $W_{\mathrm{a}} := Q^\vee \rtimes W$ the affine Weyl group and $\pi : W_{\mathrm{a}} \twoheadrightarrow W$ the natural projection, then the torus T admits a W-equivariant triangulation whose associated cohomology W-dg-algebra $C^*_{\mathrm{cell}}(T, W; \mathbb{Z})$ has homogeneous components*

$$C^k_{\mathrm{cell}}(T, W; \mathbb{Z}) = \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}[\pi({}^I W_{\mathrm{a}})],$$

where $S_0 = \{0, 1, \dots, n\}$. For a subset $I \subset S_0$, denoting by $\{j_0 < \cdots < j_k\}$ the complement of I in S_0, the differential d^k is given, for all $w \in W_{\mathrm{a}}$, by

$$d^k(\pi({}^I w)) = \sum_{\substack{0 \le u \le k+1 \\ j_{u-1} < j < j_u}} (-1)^u \pi(\epsilon_I^{I \setminus \{j\}} w), \ \epsilon_I^J = \sum_{x \in {}^J_I W_{\mathrm{a}}} x$$

and product induced by the formula

$$\pi({}^I x) \cup \pi({}^J y) = \delta_{\max(I^{\complement}), \min(J^{\complement})} \times \begin{cases} \pi({}^{I \cap J}((xy^{-1})_J y)) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$H^\bullet(C^*_{\mathrm{cell}}(T, W; \mathbb{Z})) = H^\bullet(T, \mathbb{Z}) = \Lambda^\bullet(Q^\vee).$$

In the general case, the *extended affine Weyl group* $W_\Lambda := \Lambda \rtimes W$ is no longer a Coxeter group and the above combinatorics does not hold anymore. This comes from the non-trivial symmetries of the fundamental alcove in the group. However, it is enough to consider the barycentric subdivision of the fundamental alcove (see Theorem 3.2.3). Though heavy in computations, this has the advantage of giving a general statement for all cases at once. Moreover, this construction applied to the simply-connected case gives the same complex as the first one, up to $\mathbb{Z}[W]$-homotopy equivalence.

In the simply-connected case, the complex is described using minimal length coset representatives, but if we rewrite it using transitive sets, then the combinatorics makes sense for every finite group with a preferred element and set of generators. If moreover a notion of parabolic subgroups is available, then the cup product, as described in Theorem 2.2.2 makes sense as well. In particular, we can construct such a complex for every finite (irreducible) Coxeter group.

It is a natural question to ask if there still is any geometric information behind the above complex in the non-crystallographic Coxeter groups. Our main result is the following one, which gathers Theorem 4.3.3, Corollary 4.5.1 and Proposition 4.5.6:

Theorem. *Let (W, S) be a finite irreducible Coxeter system of rank n. Given a reflection $r \in W$, we consider the Coxeter system $(\widehat{W}, S \cup \{r\})$ whose diagram is the one of W, with the additional node corresponding to r and with associated edges given by the orders of sr for $s \in S$. Then there is a reflection $r_W \in W$ such that the extension $\widehat{W}$ is affine if W is a Weyl group and compact hyperbolic otherwise. If moreover $n > 2$, then the reflection r_W is unique with this property.*

If $\widehat{W}$ is such an extension, if we denote by $\widehat{\Sigma}$ the Coxeter complex of $\widehat{W}$ and $Q := \ker(\widehat{W} \twoheadrightarrow W)$, then $\mathbf{T}(W) := \widehat{\Sigma}/Q$ is a connected, orientable, compact, W-triangulated Riemannian W-manifold of dimension n such that,

- *if W is a Weyl group, then $\mathbf{T}(W)$ is W-isometric to a maximal torus of the simply-connected compact Lie group with root system that of W,*
- *otherwise, the manifold $\mathbf{T}(W)$ is hyperbolic.*

In the dihedral case, the surfaces $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g)$ and $\mathbf{T}(I_2(4g+2))$ are naturally Riemann surfaces of genus g, definable over $\overline{\mathbb{Q}}$ and rational elliptic curves if $g = 1$.

It should be mentioned that the manifolds $\mathbf{T}(H_3)$ and $\mathbf{T}(H_4)$ were already constructed respectively by Zimmermann ([Zim93]) and by Davis ([Dav85]), using a different method.

Our construction relies on the choice of a particular reflection of W, which is the one associated to the highest root in the root system of W, in all cases except H_3. We use this reflection to build an infinite Coxeter group $\widehat{W}$, whose Coxeter diagram has one more node than the one of W and in which W is a maximal parabolic subgroup. This imitates the construction of the affine Weyl group. Furthermore, our choice implies that $\widehat{W}$ is in fact a *compact hyperbolic* Coxeter group (see [Hum92, §6.8]).

This extension comes with a torsion-free normal subgroup $Q \trianglelefteq \widehat{W}$ such that $\widehat{W} = Q \rtimes W$. A key fact is that the action of Q (under the dual geometric representation of W) on the *Coxeter complex* $\Sigma(\widehat{W})$ (seen as a quotient of the Tits cone) is a covering space action and we naturally define $\mathbf{T}(W) := \Sigma(\widehat{W})/Q$.

It is clear from the construction that $\mathbf{T}(W)$ is equipped with a W-triangulation (which yields a *dessin d'enfant* when $W = I_2(m)$), whose associated W-dg-algebra has the same combinatorics as in the Weyl group case (see Theorem 5.1.1). We use the *Hopf trace formula* (see lemma 5.2.3) to describe the homology of $\mathbf{T}(W)$ as a representation of W. The following result summarizes 5.2.1, 5.2.2 and 5.2.8:

Theorem. *If W is a finite Coxeter group of rank n and $\mathbf{T}(W)$ is the W-manifold from the previous theorem, then we have*

$$H_0(\mathbf{T}(W), \mathbb{Z}) = \mathbb{1} \quad \text{and} \quad H_n(\mathbf{T}(W), \mathbb{Z}) = \varepsilon,$$

where ε is the signature representation of W.

The homology $H_*(\mathbf{T}(W), \mathbb{Z})$ is torsion-free and, in particular, the Betti numbers of $\mathbf{T}(W)$ are palindromic, meaning $b_i = b_{n-i}$ for all i.

Moreover, the geometric representation of W is a direct summand of $H_1(\mathbf{T}(W), \Bbbk)$, where $\Bbbk$ is a splitting field for W and $H_1(\mathbf{T}(W), \Bbbk)$ is irreducible if and only if W is crystallographic.

Finally, if $W(q)$ (resp. $\widehat{W}(q)$) is the Poincaré series of W (resp. of $\widehat{W}$), then

$$\chi(\mathbf{T}(W)) = \left.\frac{W(q)}{\widehat{W}(q)}\right|_{q=1}.$$

Finally, a GAP4 package[1] is provided to compute these complexes, along with the DeConcini-Salvetti complex of a finite Coxeter group (see [CS00]).

[1] `https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

As an appendix, we use GAP again to determine all the possible hyperbolic extensions of the finite Weyl groups that correspond to a reflection in the finite group. We notice that all of them (except for $G_2 = I_2(6)$) are non-compact and our construction doesn't apply; at least not immediately.

Part 1. Weyl-equivariant triangulations of tori of compact Lie groups and related W-dg-algebras

1. Prerequisites and notation

Let K be a simple compact Lie group, T a maximal torus of K, we denote by $\mathfrak{k}$ and $\mathfrak{t}$ their respective Lie algebras and by $\Phi \subset i\mathfrak{t}^* =: V$ the root system of $(\mathfrak{k}, \mathfrak{t})$. We consider the *character lattice* of T given by

$$X(T) = \{d\lambda : \mathfrak{t} \to i\mathbb{R} \ ; \ \lambda \in \mathrm{Hom}\,(T, \mathbb{S}^1)\} \subset i\mathfrak{t}^* = V$$

and its *cocharacter lattice* is $Y(T) := X(T)^\wedge \subset V^*$, so that $(X(T), \Phi, Y(T), \Phi^\vee)$ is a root datum. Since T is abelian, the elements of the Weyl group $W = W(\Phi) \simeq N_K(T)/T$ act on T by conjugation by a representative element in $N_K(T)$. By [KK05, Lemma 1], the normalized exponential map defines a W-isomorphism of Lie groups

$$V^*/Y(T) \xrightarrow{\sim} T. \tag{$\dagger$}$$

Moreover, we have the following isomorphisms

$$P/X(T) \simeq \pi_1(K) \ \text{ and } \ X(T)/Q \simeq Z(K).$$

This shows that we may reformulate the initial problem as follows: given an irreducible root datum $(X, \Phi, Y, \Phi^\vee)$ with Weyl group W and ambient space $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, find a W-equivariant triangulation of the torus V^*/Y. As mentioned above, this will depend on the *fundamental group* P/X of the root datum.

Notation. *Throughout the first part of the paper we fix, once and for all, an irreducible root datum $(X, \Phi, Y, \Phi^\vee)$ and rank n, with ambient space $V = \mathbb{Z}\Phi \otimes \mathbb{R}$, simple roots $\Pi \subset \Phi^+$, Weyl group $W = \langle s_\alpha, \ \alpha \in \Pi \rangle$, fundamental (co)weights $(\varpi_\alpha)_{\alpha\in\Pi}$ and $(\varpi_\alpha^\vee)_{\alpha\in\Pi}$, (co)root lattices Q and $Q^\vee$ and (co)weight lattices P and $P^\vee$. We index the set Π of simple roots by $\{\alpha_1, \dots, \alpha_n\}$ and the sets of fundamental (co)weights accordingly. Let also $\alpha_0 = \sum_{i=1}^n n_i\alpha_i \in \Phi^+$ be the highest root of Φ.*

Consider the affine transformation

$$s_0 := \mathrm{t}_{\alpha_0^\vee} s_{\alpha_0} : \lambda \longmapsto s_{\alpha_0}(\lambda) + \alpha_0^\vee = \lambda - (\langle \lambda, \alpha_0 \rangle - 1)\alpha_0^\vee.$$

Then, the group $W_{\mathrm{a}} := \langle s_0, s_1, \dots, s_n \rangle \leq \mathrm{Aff}(V^*)$ is a Coxeter group, called the *affine Weyl group*. It splits as $W_{\mathrm{a}} = Q^\vee \rtimes W$. For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we consider the affine hyperplanes $H_{\alpha,k} := \{\lambda \in V^* \ ; \ \langle \lambda, \alpha \rangle = k\}$ and we call *alcove* any connected component of $V^* \backslash \bigcup_{\alpha,k} H_{\alpha,k}$. The *fundamental alcove* is

$$\mathcal{A}_0 := \{\lambda \in V^* \ ; \ \forall \alpha \in \Phi^+, \ 0 < \langle \lambda, \alpha \rangle < 1\} = \{\lambda \in V^* \ ; \ \forall 1 \leq i \leq n, \ \langle \lambda, \alpha_i \rangle > 0, \ \langle \lambda, \alpha_0 \rangle < 1\}.$$

Then, by [Bou02, V, §2.2, Corollaire], its closure is a standard simplex

$$\overline{\mathcal{A}_0} = \mathrm{conv}\left(\{0\} \cup \left\{ \frac{\varpi_i^\vee}{n_i} \right\}_{1 \leq i \leq n} \right) \simeq \Delta^n$$

and by [Hum92, §4.5 and 4.8], $\overline{\mathcal{A}_0}$ is a fundamental domain for W_{a} in V^* and moreover, W_{a} acts simply transitively on the set of open alcoves.

Before going any further into our study, we shall give some reminders and notation on equivariant CW-complexes. A detailed treatment of these objects can be found in [tD87, II, §1].

Recall that, for a discrete group G, a G-space X is said to be a *G-CW-complex* if it has a CW-complex structure such that G acts on the k-cells of X for all k and, for any cell e of X and any $g \in G$, if $ge = e$ then $gx = x$ for evey $x \in e$.

For a given CW-complex X, we can consider its *cellular homology chain complex* $C_*^{\text{cell}}(X, \mathbb{Z})$, where each $C_n^{\text{cell}}(X, Z) = \bigoplus_{i \in I} \mathbb{Z}e_i$ with e_i the n-cells of X. If X is a G-CW-complex, then its cellular chain complex $C_*^{\text{cell}}(X, \mathbb{Z})$ is a chain complex of $\mathbb{Z}[G]$-modules, which we denote by $C_*^{\text{cell}}(X, G; \mathbb{Z})$ if the acting group G is ambiguous. Moreover, if $\mathcal{E}_n$ is the (possibly infinite) set of n-cells of X, with $n \in \mathbb{N}$, then G acts on $\mathcal{E}_n$ and the $\mathbb{Z}$-module $C_n^{\text{cell}}(X, \mathbb{Z})$ is free with basis $\mathcal{E}_n$, so that $C_n^{\text{cell}}(X, G; \mathbb{Z})$ is a *permutation module*. Furthermore, decomposing $\mathcal{E}_n = \bigsqcup_i G/H_i$ into orbits, we get

$$C_n^{\text{cell}}(X, G; \mathbb{Z}) \simeq \bigoplus_i \mathbb{Z}[G/H_i],$$

where H_i runs through a representative set of stabilizers of n-cells of X. Since the action of G on X is cellular, this implies that each H_i is in fact the stabilizer of any point of the corresponding cell.

We may describe the dual complex $C^*_{\text{cell}}(X, G; \mathbb{Z})$ in a similar way, but we have to take care of the dualisation when the number of cells is infinite. For an arbitrary set S, we denote by $\mathbb{Z}[[S]]$ the set of families $x = (x_s)_{s \in S}$ of integers, indexed by S. It will be convenient to prefer the formal notation $x = \sum_{s \in S} x_s s$. Notice that, for an arbitrary group G and $H \leq G$, we have a canonical isomorphism of right $\mathbb{Z}[G]$-modules

$$\begin{array}{ccc} \mathbb{Z}[G/H]^\vee \overset{\text{df}}{=} \text{Hom}\left(\mathbb{Z}[G/H], \mathbb{Z}\right) & \longrightarrow & \mathbb{Z}[[H\backslash G]] \\ (gH)^* & \longmapsto & Hg^{-1} \end{array}$$

and this yields an isomorphism $\mathbb{Z}[G/H]^\vee \to \mathbb{Z}[H\backslash G]$ in case H is of finite index. This allows to give a general description for the homogeneous components of the dual complex:

$$C^n_{\text{cell}}(X, G; \mathbb{Z}) = \prod_i \mathbb{Z}[[H_i\backslash G]]$$

where the H_i's are as above. This is indeed a right $\mathbb{Z}[G]$-module, but it is a permutation module only when the number of cells is finite.

Recall that if G acts on a set X and if $N \trianglelefteq G$, then we may consider the *deflation* of X: $\text{Def}^G_{G/N}(X) := X/N$, with the induced action of G/N. On another hand, if $\pi : G \twoheadrightarrow G/N$ is the projection map, then we have a canonical isomorphism of G/N-sets $\text{Def}^G_{G/N}(G/H) \simeq \pi(G)/\pi(H)$. This gives a functor $\text{Def}^G_{G/N} : G\text{-}\mathbf{Set} \to G/N\text{-}\mathbf{Set}$ and linearizing it gives the usual linear deflation

$$\begin{array}{rccc} \text{Def}^G_{G/N} \quad : & \mathbb{Z}[G]\text{-}\mathbf{Mod} & \longrightarrow & \mathbb{Z}[G/N]\text{-}\mathbf{Mod} \\ & U & \longmapsto & U_N := U/\left\langle nu - u \right\rangle \end{array}$$

and we may extend this functor to (co)chain complex categories. We have the following straightforward result:

Lemma 1.0.1. *Let G be a discrete group, written as a semi-direct product $G = N \rtimes H$ and X be an G-CW-complex. Denote by $p : X \twoheadrightarrow X/N$ and by $\pi : G \twoheadrightarrow H$ the natural projections. If the quotient space X/N is Hausdorff, then it is an H-CW-complex such that, for all $k \in \mathbb{N}$,*

$$\mathcal{E}_k(X/N) = \{p(e),\ e \in \mathcal{E}_k(X)\}$$

and the map π induces a natural isomorphism

$$C_*^{\text{cell}}(X/N, H; \mathbb{Z}) \simeq \text{Def}^G_H\left(C_*^{\text{cell}}(X, G; \mathbb{Z})\right).$$

2. The simply-connected case

2.1. The W_{a}-triangulation of V^* associated to the fundamental alcove.

As observed above, the alcove $\overline{\mathcal{A}_0}$ is a fundamental domain for W_{a} acting on V^*, so it suffices to have a triangulation of $\overline{\mathcal{A}_0}$, which is compatible with the action of W_{a} in the sense that if a face is fixed globally by some $w \in W_{\mathrm{a}}$, then w induces the identity on this face.

Given a polytope $\mathcal{P} \subset \mathbb{R}^n$ and an integer $k \geq -1$, we denote by $F_k(\mathcal{P})$ the set of k-dimensional faces of $\mathcal{P}$. In particular, we have $F_{-1}(\mathcal{P}) = \{\emptyset\}$, $F_{\dim(\mathcal{P})}(\mathcal{P}) = \{\mathcal{P}\}$, $F_{\dim(\mathcal{P})-1}(\mathcal{P})$ is the set of facets of $\mathcal{P}$ and $F_0(\mathcal{P}) = \mathrm{vert}(\mathcal{P})$ is its set of vertices. Moreover, we let $F(\mathcal{P}) := \bigcup_k F_k(\mathcal{P})$ be the *face lattice* of $\mathcal{P}$. It is indeed a lattice for the inclusion relation.

Resuming to root data, for each $i \in S := \{1, \dots, n\}$ we consider the hyperplane

$$H_i := H_{\alpha_i,0} = \{\lambda \in V^* \ ; \ \langle \lambda, \alpha_i \rangle = 0\}$$

and

$$H_0 := H_{\alpha_0,1} = \{\lambda \in V^* \ ; \ \langle \lambda, \alpha_0 \rangle = 1\}$$

with $\alpha_0 = \sum_i n_i \alpha_i$ the highest root. We also take the following notation for the vertices of $\overline{\mathcal{A}_0}$, where $i \in S$,

$$v_i := \frac{\varpi_i^\vee}{n_i} \quad \text{and} \quad v_0 := 0 \quad \text{so that} \quad \mathrm{vert}(\overline{\mathcal{A}_0}) = \{v_0, v_1, \dots, v_n\}.$$

The hyperplanes H_i for $i \in S_0 := S \cup \{0\}$ give a complete set of bounding hyperplanes for the n-simplex $\overline{\mathcal{A}_0}$. Furthermore, by definition, for every face $f \in F_k(\overline{\mathcal{A}_0})$ there exists a subset $I \subseteq S_0$ of cardinality $|I| = \mathrm{codim}_{\overline{\mathcal{A}_0}}(f) = n - k$ such that

$$f = f_I := \overline{\mathcal{A}_0} \cap \bigcap_{i \in I} H_i$$

and we readily have

$$\mathrm{vert}(f_I) = \{v_i \ ; \ i \in S_0 \setminus I\}.$$

For $I \subseteq S_0$, we may consider the *(standard) parabolic subgroup* $(W_{\mathrm{a}})_I$ of W_{a} generated by the subset $\{s_i, \ i \in I\}$. If $0 \notin I$, then $(W_{\mathrm{a}})_I$ is in fact a parabolic subgroup of W.

Lemma 2.1.1. *Let $0 \leq k \leq n$ and $I \subseteq S_0$ with $|I| = n - k$. Then the stabilizer of the face $f_I \in F_k(\overline{\mathcal{A}_0})$ is the parabolic subgroup of W_{a} associated to I. In other words,*

$$(W_{\mathrm{a}})_{f_I} = (W_{\mathrm{a}})_I.$$

Proof. As $\mathrm{vert}(f_I) = \{v_i, \ i \notin I\}$ is $(W_{\mathrm{a}})_{f_I}$-stable, the Theorem from [Hum92, §4.8] ensures that

$$(W_{\mathrm{a}})_{f_I} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{v_i}.$$

Moreover, each group $(W_{\mathrm{a}})_{v_j}$ is generated by the reflections it contains, so that $v_j \in H_i$. A reflection s_i fixes 0 if and only if it is linear, so $(W_{\mathrm{a}})_{v_0} = \langle s_i \ ; \ i \neq 0 \rangle = (W_{\mathrm{a}})_S = W$. Let now $j \in S$. Since

$$\{v_j\} = \left\{ \frac{\varpi_j^\vee}{n_j} \right\} = \bigcap_{j \neq i \in S_0} H_i,$$

we have that $s_i(v_j) = v_j$ if and only if $i \in S_0 \setminus \{j\}$ and hence, for every $j \in S_0$, we have

$$(W_{\mathrm{a}})_{v_j} = (W_{\mathrm{a}})_{S_0 \setminus \{j\}}$$

and thus

$$(W_{\mathrm{a}})_{f_I} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{v_i} = \bigcap_{i \in S_0 \setminus I} (W_{\mathrm{a}})_{S_0 \setminus \{i\}} = (W_{\mathrm{a}})_{\bigcap_{i \notin I} S_0 \setminus \{i\}} = (W_{\mathrm{a}})_I.$$

□

Therefore, we have a triangulation

$$V^* = \coprod_{\substack{f\in F(\overline{\mathcal{A}_0})\\ \widetilde{w}\in W_{\mathrm{a}}/(W_{\mathrm{a}})_f}} \widetilde{w}\cdot f$$

which is W_{a}-equivariant and following the notation from the first section, we have $\mathcal{E}_k(V^*)/W_{\mathrm{a}} = F_k(\overline{\mathcal{A}_0})$ for all k. Therefore, we get isomorphisms of $\mathbb{Z}[W_{\mathrm{a}}]$-modules

$$C_k^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \simeq \bigoplus_{f\in F_k(\overline{\mathcal{A}_0})} \mathbb{Z}[W_{\mathrm{a}}/(W_{\mathrm{a}})_f] = \bigoplus_{\substack{I\subset S_0\\ |I|=n-k}} \mathbb{Z}[W_{\mathrm{a}}/(W_{\mathrm{a}})_I].$$

We have to fix an orientation of the cells in V^* and determine their boundary. But each one of them is a simplex, so its orientation is determined by an orientation on its vertices. We choose to orient them as the index set $(S_0, \leq)$. For $I \subseteq S_0$ with corresponding k-face $f_I = \mathrm{conv}(\{v_i \ ; \ i \in S_0 \setminus I\})$, we write

$$f_I = [v_{j_1}, \ldots, v_{j_{k+1}}] \ \text{ with } \ \{j_1 < j_2 < \ldots < j_{k+1}\} = S_0 \setminus I$$

to make its orientation explicit. The oriented boundary of f_I is then simply given by the formula

$$\partial_k(f_I) = \sum_{u=1}^{k+1} (-1)^u \underbrace{[v_{j_1}, \ldots, \widehat{v_{j_u}}, \ldots, v_{j_{k+1}}]}_{=\mathrm{conv}(\{v_j \ ; \ j_u \neq j \in S_0\setminus I\})} = \sum_{u=1}^{k+1} (-1)^u f_{I\cup\{j_u\}}$$

We have thus obtained the following result:

Theorem 2.1.2. *The face lattice of the n-simplex $\overline{\mathcal{A}_0}$ induces a W_{a}-equivariant triangulation of V^*, whose cellular complex $C_*^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ is given (in homogeneous degrees k and $k-1$) by*

$$\cdots \longrightarrow \bigoplus_{\substack{I\subset S_0\\ |I|=n-k}} \mathbb{Z}\left[W_{\mathrm{a}}^I\right] \xrightarrow{\partial_k} \bigoplus_{\substack{I\subset S_0\\ |I|=n-k+1}} \mathbb{Z}\left[W_{\mathrm{a}}^I\right] \longrightarrow \cdots$$

where $W_{\mathrm{a}}^I \approx W_{\mathrm{a}}/(W_{\mathrm{a}})_I$ is the W_{a}-set of minimal length left coset representatives, modulo the parabolic subgroup $(W_{\mathrm{a}})_I$ and boundaries are defined as follows: for $k \in \mathbb{N}$ and $I \subset S_0$, letting $\{j_1 < \cdots < j_{k+1}\} := S_0 \setminus I$,

$$(\partial_k)_{|\mathbb{Z}[W_{\mathrm{a}}^I]} = \sum_{u=1}^{k+1} (-1)^u p^I_{I\cup\{j_u\}},$$

where, for $I \subset J$, p^I_J denotes the projection

$$p^I_J : W_{\mathrm{a}}^I = W_{\mathrm{a}}/(W_{\mathrm{a}})_I \twoheadrightarrow W_{\mathrm{a}}/(W_{\mathrm{a}})_J = W_{\mathrm{a}}^J.$$

Example 2.1.3. *We look at the case of the group $SU(3)$ in type A_2. We denote by $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ a root system of type A_2, with simple system $\Pi = \{\alpha, \beta\}$. The Figure 1 depicts the (dual) root system of type A_2 and its fundamental alcove. The chain complex $C_*^{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ is readily given by*

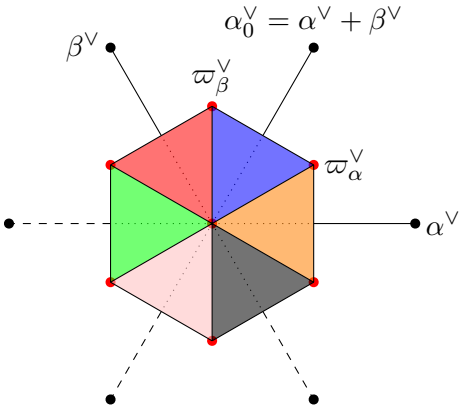
$$\mathbb{Z}[W_{\mathrm{a}}] \xrightarrow{\partial_2} \mathbb{Z}[W_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_0\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha\rangle] \xrightarrow{\partial_1} \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha, s_\beta\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\beta, s_0\rangle] \oplus \mathbb{Z}[W_{\mathrm{a}}/\langle s_\alpha, s_0\rangle],$$

where the boundaries are

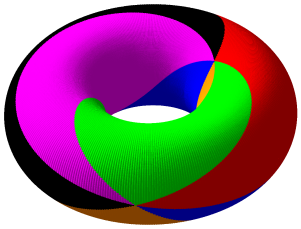
$$\partial_2 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}, \ \partial_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Applying the deflation functor $\mathrm{Def}_W^{W_\mathrm{a}}$*, we obtain the complex* $C_*^{\mathrm{cell}}(T, W; \mathbb{Z})$ *where* $T = S(U(1)^3) \leq SU(3)$ *as*

$$\mathbb{Z}[W] \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}} \mathbb{Z}[W/\langle s_\beta \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta s_\alpha \rangle] \oplus \mathbb{Z}[W/\langle s_\alpha \rangle] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3 .$$



(A) The fundamental alcove $\overline{\mathcal{A}_0}$ (in blue) in type A_2, and its $\mathfrak{S}_3$-translates.



(B) The resulting $\mathfrak{S}_3$-equivariant triangulation of $S(U(1)^3) \simeq (\mathbb{S}^1)^2$.

FIGURE 1. Triangulation of the torus $S(U(1)^3)$ of $SU(3)$ from the fundamental alcove.

2.2. The W-dg-algebra structure.

We now make the cup product on $C_*^{\mathrm{cell}}(V^*, W_\mathrm{a}; \mathbb{Z})$ more explicit, in terms of parabolic double cosets. We write

$$C_{\mathrm{cell}}^k(V^*, W_\mathrm{a}; \mathbb{Z}) = \prod_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[W_\mathrm{a}/(W_\mathrm{a})_I\right]^\vee \simeq \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\left[{}^I W_\mathrm{a}\right]\right],$$

where

$${}^I W_\mathrm{a} \overset{\mathrm{df}}{=} \{w \in W_\mathrm{a} \ ; \ \ell(s_i w) > \ell(w), \ \forall i \in I\} \approx (W_\mathrm{a})_I \backslash W_\mathrm{a}$$

is the set of minimal length right coset representatives. Recall the following general result about double cosets:

Lemma 2.2.1 ([BKP+16, §3, Proposition 2 and Corollary 3])**.** *Let* (W, S) *be a Coxeter system and* $I, J \subset S$*. Denote as usual*

$$W^I := \{w \in W \ ; \ \ell(ws) > \ell(w), \ \forall s \in I\} \approx W/W_I,$$
$${}^I W := \{w \in W \ ; \ \ell(sw) > \ell(w), \ \forall s \in I\} \approx W_I \backslash W$$

and for $J \subset I$*,*

$$W_I^J := \{w \in W_I \ ; \ \ell(ws) > \ell(w), \ \forall s \in J\} \approx W_I/W_J.$$

(1) Each double coset in $W_I \backslash W / W_J$ *has a unique element of minimal length.*
(2) An element $w \in W$ *is of minimal length in its double coset if and only if* $w \in {}^I W \cap W^J$*. In particular, we have a bijection*

$$W_I \backslash W / W_J \approx {}^I W \cap W^J.$$

(3) As a consequence, if $w \in {}^I W \cap W^J$ *and* $x \in W_I$*, then* $xw \in W^J$ *if and only if* $x \in W_I^{I \cap {}^w J}$*. Hence, we have the following property:*

$$\forall x \in W_I w W_J,\ \exists! (u,v) \in W_I^{I \cap {}^w J} \times W_J\ ;\ \begin{cases} x = uwv, \\ \ell(x) = \ell(u) + \ell(w) + \ell(v). \end{cases}$$

We can now formulate the main result:

Theorem 2.2.2. *The* $\mathbb{Z}[W_{\mathrm{a}}]$*-cochain complex* $C^*_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$ *associated to the* W_{a}*-triangulation of* V^* *is a* $\mathbb{Z}[W_{\mathrm{a}}]$*-dg-algebra with homogenous components*

$$\forall 0 \leq k \leq n,\ C^k_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) = \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[[(W_{\mathrm{a}})_I \backslash W_{\mathrm{a}}]\right] \simeq \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right]$$

and differentials defined, for any $I \subset S_0$ *and* $w \in W_{\mathrm{a}}$*, by*

$$d^k({}^I w) = \sum_{\substack{0 \leq u \leq k+1 \\ j_{u-1} < j < j_u}} (-1)^u \epsilon_I^{I \setminus \{j\}} w$$

where $\{j_0 < \cdots < j_k\} := S_0 \setminus I$ *and, by convention,* $j_{-1} = -1$*,* $j_{k+1} = n+1$ *and for* $J \subset I$*,*

$${}^J_I W_{\mathrm{a}} := \{w \in (W_{\mathrm{a}})_I\ ;\ \ell(s_j w) > \ell(w),\ \forall j \in J\} \quad \text{and} \quad \epsilon_I^J := \sum_{x \in {}^J_I W_{\mathrm{a}}} x \in \mathbb{Z}\left[{}^J W_{\mathrm{a}}\right].$$

Moreover, the cup product

$$C^p_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \otimes C^q_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z}) \xrightarrow{\cup} C^{p+q}_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})$$

is induced by the unique map

$$\mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right] \otimes \mathbb{Z}\left[\left[{}^J W_{\mathrm{a}}\right]\right] \longrightarrow \mathbb{Z}\left[\left[{}^{I \cap J} W_{\mathrm{a}}\right]\right].$$

satisfying the formula

$${}^I x \cup {}^J y = \delta_{\max(I^{\complement}), \min(J^{\complement})} \times \begin{cases} {}^{I \cap J}((xy^{-1})_J y) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J, \\ 0 & \text{otherwise,} \end{cases}$$

where $\complement$ *denotes the complementary of a subset in* S_0 *and, given* $w \in W_{\mathrm{a}}$*, we denote by* ${}^{I \cap J} w \in {}^{I \cap J} W_{\mathrm{a}}$ *its minimal length right coset representative and if* $(W_{\mathrm{a}})_I w (W_{\mathrm{a}})_J = (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J$*, we let* w_J *be the unique element* $v \in (W_{\mathrm{a}})_J$ *such that* $w = uv$*, with* $u \in (W_{\mathrm{a}})_I^{I \cap J}$ *and* $\ell(w) = \ell(u) + \ell(v)$*.*

Proof. Take a k-simplex $\sigma = [j_0, \ldots, j_k] \subset \overline{\mathcal{A}_0}$ with $j_u \in S_0$ and set $j_{-1} := -1$ and $j_{k+1} := n+1$. By definition of the cochain differential d^k, we have

$$d^k(\sigma^*)_{|\overline{\mathcal{A}_0}} = \sum_{u=0}^{k+1} \sum_{j_{u-1} < j < j_u} (-1)^u [j_0, \ldots, j_{u-1}, j, j_u, \ldots, j_k]^*.$$

Letting $I := S_0 \setminus \{j_0, \ldots, j_k\}$, we have $(W_{\mathrm{a}})_{\sigma^*} = (W_{\mathrm{a}})_I$ and the above formula reads

$$d^k((W_{\mathrm{a}})_I \cdot 1)_{|\overline{\mathcal{A}_0}} = \sum_{u=0}^{k+1} \sum_{j_{u-1} < j < j_u} (-1)^u ((W_{\mathrm{a}})_{I \setminus \{j\}} \cdot 1).$$

Therefore, as $\overline{\mathcal{A}_0}$ is a fundamental domain for W_{a} in V^*, this yields

$$d^k((W_{\mathrm{a}})_I \cdot 1) = \sum_{\substack{0 \leq u \leq n+1-|I| \\ j_{u-1} < j < j_u}} \left(\sum_{w \in {}^{I \setminus \{j\}}_I W_{\mathrm{a}}} (-1)^u ((W_{\mathrm{a}})_{I \setminus \{j\}} \cdot w) \right),$$

which leads to the stated formula.

To compute the cup product, using the bijection ${}^I W_{\mathrm{a}} \approx (W_{\mathrm{a}})_I \backslash W_{\mathrm{a}}$, the stated formula is

$$(W_{\mathrm{a}})_I x \cup (W_{\mathrm{a}})_J y = \delta_{\max I^{\complement}, \min J^{\complement}} \delta_{(W_{\mathrm{a}})_I x y^{-1} (W_{\mathrm{a}})_J, (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J} (W_{\mathrm{a}})_{I \cap J}((xy^{-1})_J y).$$

Let $x, y \in W_{\mathrm{a}}$. As W_{a} acts simplicially on $\overline{\mathcal{A}_0}$, we have

$$(W_{\mathrm{a}})_I x \cup (W_{\mathrm{a}})_J y = ((W_{\mathrm{a}})_I x y^{-1} \cup (W_{\mathrm{a}})_J) y,$$

hence we may assume that $y = 1$ and we just have to compute $(W_{\mathrm{a}})_I w \cup (W_{\mathrm{a}})_J$.

First, we compute $\sigma^* \cup \tau^*$ for $\sigma, \tau \subset \overline{\mathcal{A}_0}$. As $\overline{\mathcal{A}_0} \simeq \Delta^r$ is a simplex, we may write $\sigma = [i_0, \dots, i_a]$ with $a = \dim \sigma$ and $I^{\complement} := \{i_0, \dots, i_a\} \subset \mathrm{vert}(\overline{\mathcal{A}_0}) \simeq S_0$. Write similarly $\tau = [j_0, \dots, j_b]$. We have $(W_{\mathrm{a}})_\sigma = (W_{\mathrm{a}})_{S_0 \setminus \{i_0, \dots, i_a\}} = (W_{\mathrm{a}})_I$, $(W_{\mathrm{a}})_\tau = (W_{\mathrm{a}})_J$ and

$$\sigma^* \cup \tau^* = \delta_{i_a, j_0} [i_0, \dots, i_a, j_1, \dots, j_b]^*$$

and the stabilizer in W_{a} of this last dual cell is $(W_{\mathrm{a}})_{S_0 \setminus \{i_0, \dots, i_a, j_1, \dots, j_b\}} = (W_{\mathrm{a}})_{I \cap J}$. Moreover, if $\sigma^* \cup \tau^* \neq 0$ then we must have $i_a = j_0$, that is, $\max(I^{\complement}) = \min(J^{\complement})$. We make this assumption for the rest of this proof and we have indeed

$$\sigma^* \cup \tau^* = (W_{\mathrm{a}})_I \cup (W_{\mathrm{a}})_J = (W_{\mathrm{a}})_{I \cap J}.$$

Claim: For $\tau \subset \overline{\mathcal{A}_0}$ a simplex and $P \in F_k(V^*)$ a k-cell of V^*, if $\tau \subset P$ then $P \in (W_{\mathrm{a}})_\tau \cdot F_k(\overline{\mathcal{A}_0})$.

Indeed, we may assume that $\dim P = n = \dim \mathcal{A}_0$ so that there is some $z \in W_{\mathrm{a}}$ such that $P = z(\overline{\mathcal{A}_0})$ and so $\tau \subset \overline{\mathcal{A}_0} \cap z(\overline{\mathcal{A}_0})$, thus $z \in (W_{\mathrm{a}})_\tau$ (see [Hum92, §4.8]).

We are left to compute $\sigma^* w \cup \tau^*$ for $w \in W_{\mathrm{a}}$. If $\sigma^* w \cup \tau^* \neq 0$, then $w^{-1}\sigma$ and τ are included is some common simplex $P \in F_k(V^*)$ and by the claim we may choose $w_\tau \in (W_{\mathrm{a}})_\tau = (W_{\mathrm{a}})_J$ such that $w^{-1}\sigma \subset w_\tau(\overline{\mathcal{A}_0})$. But then $\sigma \subset \overline{\mathcal{A}_0} \cap w w_\tau(\overline{\mathcal{A}_0})$ and so $w_\tau \sigma = w^{-1}\sigma$. This yields

$$\sigma^* w \cup \tau^* = \sigma^* w_\tau^{-1} \cup \tau^* = \sigma^* w_\tau^{-1} \cup \tau^* w_\tau^{-1} = (\sigma^* \cup \tau^*) w_\tau^{-1} = (W_{\mathrm{a}})_{I \cap J} \cdot w_\tau^{-1}.$$

Furthermore, if $\sigma^* w \cup \tau^* \neq 0$ then we must have $w w_\tau \in (W_{\mathrm{a}})_I$, so $w \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J$. In this case, the parabolic double coset decomposition from Lemma 2.2.1 applied to the trivial double coset $(W_{\mathrm{a}})_I w (W_{\mathrm{a}})_J$ allows one to write uniquely w as $w = u w_J$ with $u \in (W_{\mathrm{a}})_I^{I \cap J}$ and $w_J \in (W_{\mathrm{a}})_J$ such that $\ell(w) = \ell(u) + \ell(w_J)$. We obtain $w_J w_\tau \in (W_{\mathrm{a}})_J$ as well as $w_J w_\tau = u^{-1} w w_\tau \in (W_{\mathrm{a}})_I$. Hence $w_J w_\tau \in (W_{\mathrm{a}})_I \cap (W_{\mathrm{a}})_J = (W_{\mathrm{a}})_{I \cap J}$ and

$$\sigma^* w \cup \tau^* = (W_{\mathrm{a}})_{I \cap J} \cdot w_\tau^{-1} = (W_{\mathrm{a}})_{I \cap J} \cdot w_J.$$

The only thing remaining to be proved is that the formula

$${}^I x \cup {}^J y = \delta_{\max(I^{\complement}), \min(J^{\complement})} \times \begin{cases} {}^{I \cap J}((xy^{-1})_J y) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J, \\ 0 & \text{otherwise,} \end{cases}$$

indeed induces a well-defined map $\mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right] \otimes \mathbb{Z}\left[\left[{}^J W_{\mathrm{a}}\right]\right] \to \mathbb{Z}\left[\left[{}^{I \cap J} W_{\mathrm{a}}\right]\right]$. To see this, we show that for a given $z \in {}^{I \cap J} W_{\mathrm{a}}$, there are only finitely many pairs $({}^I x, {}^J y)$ for which $z = {}^I x \cup {}^J y$. Indeed, given $x, y \in W_{\mathrm{a}}$, if $x', y' \in W_{\mathrm{a}}$ are such that ${}^I x \cup {}^J y = {}^I x' \cup {}^J y'$, then $(xy^{-1})_J y$ and $(x'y'^{-1})_J y'$ are in the same class modulo $(W_{\mathrm{a}})_{I \cap J}$, hence in the same class modulo $(W_{\mathrm{a}})_J$ and therefore ${}^J y = {}^J y'$. Since $(W_{\mathrm{a}})_J$ is finite, there are only finitely many possibilities for y' and the same goes for $x' \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J y'$. Therefore, if $a = \sum_{x \in {}^I W_{\mathrm{a}}} a_x x$ and $b = \sum_{y \in {}^J W_{\mathrm{a}}} b_y y$ with $a_x, b_y \in \mathbb{Z}$ (we use the formal series notation for simplicity), we can define

$$a \cup b := \sum_{z \in {}^{I \cap J} W_{\mathrm{a}}} \left(\sum_{\substack{(x,y) \in {}^I W_{\mathrm{a}} \times {}^J W_{\mathrm{a}} \\ x \cup y = z}} a_x b_y \right) z.$$

It is obvious that this is the only way of defining a bilinear map $\mathbb{Z}\left[\left[{}^I W_{\mathrm{a}}\right]\right] \times \mathbb{Z}\left[\left[{}^J W_{\mathrm{a}}\right]\right] \to \mathbb{Z}\left[\left[{}^{I \cap J} W_{\mathrm{a}}\right]\right]$ satisfying the stated formula. □

Corollary 2.2.3. *The $\mathbb{Z}[W]$-cochain complex $C^*_{\mathrm{cell}}(T, W; \mathbb{Z})$ associated to the W-triangulation of $T = V^*/Q^\vee$ induced by the W_{a}-triangulation of V^* is given by*

$$C^*_{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}_W^{W_{\mathrm{a}}}\left(C^*_{\mathrm{cell}}(V^*, W_{\mathrm{a}}; \mathbb{Z})\right).$$

In other words, if $\pi : W_{\mathrm{a}} \twoheadrightarrow W$ is the projection, then

$$C^k_{\mathrm{cell}}(T, W; \mathbb{Z}) = \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\pi\left({}^I W_{\mathrm{a}}\right)\right] \simeq \bigoplus_{\substack{I \subset S_0 \\ |I| = n-k}} \mathbb{Z}\left[\pi((W_{\mathrm{a}})_I)\backslash W\right],$$

with differentials given, for any $I \subset S_0$ and $w \in W_{\mathrm{a}}$, by

$$d^k(\pi({}^I w)) = \sum_{\substack{0 \le u \le k+1 \\ j_{u-1} < j < j_u}} (-1)^u \pi\left(\epsilon_I^{I \setminus \{j\}} w\right), \ \epsilon_I^J = \sum_{x \in {}^J_I W_{\mathrm{a}}} x$$

where $\{j_0 < \cdots j_k\} := S_0 \setminus I$. Its product induced by the formula

$$\pi({}^I x) \cup \pi({}^J y) = \delta_{\max({}^I \mathfrak{C}), \min({}^J \mathfrak{C})} \times \begin{cases} \pi\left({}^{I \cap J}((xy^{-1})_J y)\right) & \text{if } xy^{-1} \in (W_{\mathrm{a}})_I (W_{\mathrm{a}})_J \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have

$$H^\bullet(C^*_{\mathrm{cell}}(T, W; \mathbb{Z})) = H^\bullet(T, \mathbb{Z}) = \Lambda^\bullet(Q^\vee).$$

3. The general case

3.1. **The fundamental group as symmetries of an alcove.** The *extended affine Weyl group* $\widehat{W_{\mathrm{a}}} := P^\vee \rtimes W$ acts on alcoves (transitively since $W_{\mathrm{a}} \trianglelefteq \widehat{W_{\mathrm{a}}}$ does) but not simply-transitively. We introduce the stabilizer

$$\Omega := \{\widehat{w} \in \widehat{W_{\mathrm{a}}} \ ; \ \widehat{w}(\mathcal{A}_0) = \mathcal{A}_0\}$$

and we see that we have a decomposition $\widehat{W_{\mathrm{a}}} \simeq W_{\mathrm{a}} \rtimes \Omega$ and in particular,

$$\Omega \simeq \widehat{W_{\mathrm{a}}}/W_{\mathrm{a}} \simeq P^\vee/Q^\vee \simeq P/Q.$$

Thus, Ω is a finite abelian group. The following table details the fundamental groups of the irreducible root systems:

Type	$\Omega \simeq P/Q$
A_n $(n \ge 1)$	$\mathbb{Z}/(n+1)\mathbb{Z}$
B_n $(n \ge 2)$	$\mathbb{Z}/2\mathbb{Z}$
C_n $(n \ge 3)$	$\mathbb{Z}/2\mathbb{Z}$
D_{2n} $(n \ge 2)$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
D_{2n+1} $(n \ge 2)$	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	1
F_4	1
G_2	1

Table 1. Fundamental groups of irreducible root systems

The description of Ω given in [Bou02, VI, §2.3] is useful. Recall that a fundamental weight ϖ_i is called *minuscule* if $n_i = 1$ and that minuscule weights form a set of representatives of the non-trivial classes in P/Q (see [Bou02, Chapter VI, Exercise 24]). Dually, we have the same notion and result for *minuscule coweights*.

Proposition-Definition 3.1.1 ([Bou02, VI, §2.3, Proposition 6])**.** *Define $M := \{i \in S \ ; \ n_i = 1\}$ and let $w_0 \in W$ be the longest element. For $i \in S$, denote by $W_i \le W$ the Weyl group of the subsystem of Φ generated by $\{\alpha_j\}_{j \ne i} \subset \Pi$. For $i \in M$, let $w_0^i \in W_i$ be*

the longest element of W_i *and* $w_i := w_0^i w_0$. *Then the element* $\mathrm{t}_{\varpi_i^\vee} w_i \in \widehat{W_{\mathrm{a}}}$ *is in* Ω *and the map*

$$\begin{array}{ccc} M & \longrightarrow & \Omega \setminus \{1\} \\ i & \longmapsto & \omega_i := \mathrm{t}_{\varpi_i^\vee} w_i \end{array}$$

is a bijection.

We now have to see what happens if the W-lattice Y is such that $Q^\vee \subsetneq Y \subsetneq P^\vee$. To simplify notation of this section, we identify a lattice $\Lambda \subset V^*$ with its translation group $\mathrm{t}(\Lambda) \subset \mathrm{Aff}(V^*)$ and for such a lattice Λ, we define the *intermediate affine Weyl group* $W_\Lambda := \Lambda \rtimes W$. There is a correspondence between W-lattices $Q^\vee \subseteq \Lambda \subseteq P^\vee$ and the subgroups of Ω. In order to state this correspondence properly, we temporarily drop the letter Y and we work in the root system Φ only. Though straightforward, the following result is key:

Proposition 3.1.2. *Recall that* $\widehat{W_{\mathrm{a}}} \simeq W_{\mathrm{a}} \rtimes \Omega$ *and denote by*

$$\pi : \widehat{W_{\mathrm{a}}} \twoheadrightarrow \Omega$$

the natural projection. We have a bijective correspondence

$$\begin{array}{ccc} \{\Lambda \ ; \ Q^\vee \subseteq \Lambda \subseteq P^\vee \text{ is a } W\text{-lattice}\} & \overset{1-1}{\longleftrightarrow} & \{H \leq \Omega\} \\ \Lambda & \longmapsto & \Omega_\Lambda := \pi(W_\Lambda) \\ \pi^{-1}(H) \cap P^\vee =: \Lambda(H) & \longleftarrow & H \end{array}$$

3.2. **A $\widehat{W_{\mathrm{a}}}$-triangulation of V^* from the barycentric subdivision of an alcove.** In order to obtain a W_Y-triangulation of the torus V^*/Y, we just have to exhibit an Ω_Y-triangulation of the alcove $\overline{\mathcal{A}_0}$. As the group Ω_Y acts by affine automorphisms of $\overline{\mathcal{A}_0}$, the construction follows from the next easy result about simplicial subdivisions.

Recall that, given a polytope $\mathcal{P}$, its *barycentric subdivision* is the simplicial complex $\mathrm{Sd}(\mathcal{P})$ whose k-simplices are increasing chains of non-empty faces of $\mathcal{P}$ of length $k+1$. A k-simplex $(f_0, f_1, \dots, f_k)$ of $\mathrm{Sd}(\mathcal{P})$ may be geometrically realized as $\mathrm{conv}(\mathrm{bar}(f_0), \dots, \mathrm{bar}(f_k))$, where $\mathrm{bar}(f_i)$ stands for the barycenter of the face f.

Lemma 3.2.1. *If* $\mathcal{P}$ *is a polytope, then* $\mathrm{Sd}(\mathcal{P})$ *is an* $\mathrm{Aut}(\mathcal{P})$*-triangulation of* $\mathcal{P}$.

Proof. It is well-known that $\mathrm{Sd}(\mathcal{P})$ triangulates $\mathcal{P}$ and it is clear that $\Gamma := \mathrm{Aut}(\mathcal{P})$ permutes the simplices of $\mathrm{Sd}(\mathcal{P})$. We have to prove that, for a simplex $\sigma = (f_0, \dots, f_k)$ of $\mathrm{Sd}(\mathcal{P})$ and $\gamma \in \Gamma$, if $\gamma\sigma = \sigma$, then $\gamma x = x$ for each $x \in |\sigma|$.

Take $0 \leq i \leq k$. The point $\mathrm{bar}(f_i)$ is taken by γ to some $\mathrm{bar}(f_j)$ and since the barycenter of a polytope lies in its relative interior, we have $\gamma(\mathring{f}_i) \cap \mathring{f}_j \neq \emptyset$ (where $\mathring{\ }$ is the relative interior) and as γ acts as an automorphism of $\mathcal{P}$, this forces $\gamma(f_i) = f_j$ and $\dim(f_i) = \dim(\gamma(f_i)) = \dim(f_j)$. But the sequence $(\dim f_0, \dots, \dim f_k)$ is increasing, so $f_i = f_j$ and $\mathrm{bar}(f_i) = \mathrm{bar}(f_j) = \gamma(\mathrm{bar}(f_i))$. The conclusion now follows from the equality $|\sigma| = \mathrm{conv}(\mathrm{bar}(f_0), \dots, \mathrm{bar}(f_k))$. □

From this we deduce that $W_{\mathrm{a}} \cdot \mathrm{Sd}(\overline{\mathcal{A}_0})$ is a W_Y-triangulation of V^* for all $Q^\vee \subset Y \subset P^\vee$ at once. There is a bijection $\mathrm{vert}(\overline{\mathcal{A}_0}) \approx S_0 = \{0, \dots, n\}$ and $\overline{\mathcal{A}_0} \simeq \Delta^n$, so that the face lattice of $\overline{\mathcal{A}_0}$ is $F(\overline{\mathcal{A}_0}) \simeq (\mathscr{P}(S_0), \subset)$. This gives a description of the face lattice of $\mathrm{Sd}(\overline{\mathcal{A}_0})$: for $0 \leq d \leq n$, we have

$$F_d(\mathrm{Sd}(\overline{\mathcal{A}_0})) = \{Z_\bullet = (Z_0, Z_1, \dots, Z_d) \ ; \ \forall i, \ \emptyset \neq Z_i \subset S_0, \ Z_i \subsetneq Z_{i+1}\}$$

and $Z_\bullet \subset Z'_\bullet$ if $Z_\bullet$ is a subsequence of $Z'_\bullet$.

Lemma 3.2.2. *The group* Ω_Y *acts on* $\overline{\mathcal{A}_0}$ *and this induces an action on* S_0. *The resulting action on* $F(\mathrm{Sd}(\overline{\mathcal{A}_0}))$ *corresponds to the action of* Ω_Y *on* $|\mathrm{Sd}(\overline{\mathcal{A}_0})| = \overline{\mathcal{A}_0}$. *Moreover, for* $Z_\bullet \in F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))$, *the stabilizer of* $Z_\bullet$ *in* W_Y *decomposes as*

$$(W_Y)_{Z_\bullet} = (W_{\mathrm{a}})_{Z_\bullet} \rtimes (\Omega_Y)_{Z_\bullet} = (W_{\mathrm{a}})_{S_0 \setminus Z_d} \rtimes (\Omega_Y)_{Z_\bullet} \quad \text{and} \quad (\Omega_Y)_{Z_\bullet} = \bigcap_{i=0}^{d} \Omega_{Z_i}.$$

Proof. The first statement is obvious. Write $Z_\bullet = (Z_0 \subsetneq \cdots \subsetneq Z_d)$ and let $\widehat{w} := w\omega_j \in (W_Y)_{Z_\bullet}$ with $w \in W_{\mathrm{a}}$ and $\omega_j \in \Omega_Y$. Then, for every $x \in |Z_\bullet|$, we have $\widehat{w}(x) = w(\omega_j(x)) = x$ and $\omega_j(x) \in \overline{\mathcal{A}_0}$ so $x = \omega_j(x)$ and $\omega_j \in (\Omega_Y)_{Z_\bullet}$. On another hand we get $w(x) = x$ so $w \in (W_{\mathrm{a}})_{Z_\bullet}$.

Now, an element $w \in W_{\mathrm{a}}$ fixes $Z_\bullet$ if and only if it fixes the maximal face of $Z_\bullet$, i.e. Z_d. This is indeed the parabolic subgroup $(W_{\mathrm{a}})_{S_0 \backslash Z_d}$. □

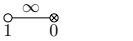
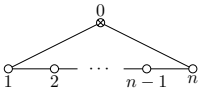
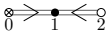
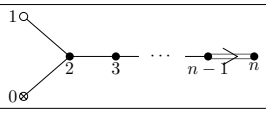
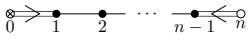
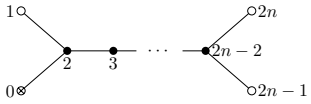
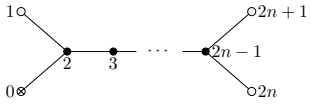
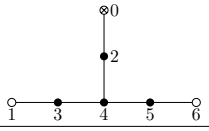
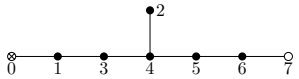
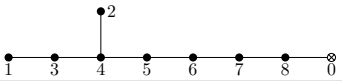
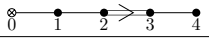
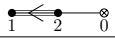
Type	Extended Dynkin diagram	Fundamental group $\Omega \leq \mathrm{Aut}(\mathcal{D}ynkin_0)$
$\widetilde{A_1}$		$\omega_1 = (0,1)$
$\widetilde{A_n}$ $(n \geq 2)$		$\omega_1 = (0,1,2,\cdots,n)$ $\omega_i = (\omega_1)^i,\ 0 \leq i \leq n$
$\widetilde{B_2} = \widetilde{C_2}$		$\omega_1 = (0,2)$
$\widetilde{B_n}$ $(n \geq 3)$		$\omega_1 = (0,1)$
$\widetilde{C_n}$ $(n \geq 3)$		$\omega_n = (0,n)\prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor}(i,n-i)$
$\widetilde{D_{2n}}$ $(n \geq 2)$		$\omega_1 = (0,1)(2n-1,2n)$ $\omega_{2n-1} = (0,2n-1)(1,2n)\prod_{i=2}^{n-1}(i,2n-i)$ $\omega_{2n} = (0,2n)(1,2n-1)\prod_{i=2}^{n-1}(i,2n-i) = \omega_1\omega_{2n-1}$
$\widetilde{D_{2n+1}}$ $(n \geq 2)$		$\omega_1 = (0,1)(2n,2n+1)$ $\omega_{2n} = (0,2n,1,2n+1)\prod_{i=2}^{n}(i,2n+1-i)$ $\omega_{2n+1} = (0,2n+1,1,2n)\prod_{i=2}^{n}(i,2n+1-i)$
$\widetilde{E_6}$		$\omega_1 = (0,1,6)(2,3,5)$ $\omega_6 = (1,0,6)(3,2,5) = \omega_1^{-1}$
$\widetilde{E_7}$		$\omega_7 = (0,7)(1,6)(3,5)$
$\widetilde{E_8}$		$\varnothing$
$\widetilde{F_4}$		$\varnothing$
$\widetilde{G_2}$		$\varnothing$

TABLE 2. Extended Dynkin diagrams and fundamental groups elements, represented as permutations of the nodes.

To avoid too many choices, we fix a total ordering $\prec$ on $F(\mathrm{Sd}(\overline{\mathcal{A}_0}))$. For instance, the lexicographical order $<_{lex}$ induced by the order on $\mathscr{P}(S_0) = 2^{S_0}$ inherited from the natural order on S_0. As the barycentric subdivision of $\overline{\mathcal{A}_0}$ is simplicial, the boundaries of the complex and the cup product are easily determined and lead to the following result:

Theorem 3.2.3. *For $0 \leq d \leq n$, decompose the Ω_Y-set $F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))$ into orbits*

$$F_d(\mathrm{Sd}(\overline{\mathcal{A}_0}))/\Omega_Y \approx \{Z_{d,1} \prec \cdots \prec Z_{d,k_d}\}, \quad \textit{where} \quad Z_{d,i} = \min_{\prec}(\Omega_Y \cdot Z_{d,i}).$$

Denote further, for $0 \leq p \leq d$ and $1 \leq i \leq k_d$,

$$Z_{d,i}^{(p)} := ((Z_{d,i})_0, \ldots, \widehat{(Z_{d,i})_p}, \ldots, (Z_{d,i})_d).$$

Then the complex $C_*^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z})$ *is given by*

$$C_d^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z}) = \bigoplus_{i=1}^{k_d} \mathbb{Z}\left[W_Y/(W_Y)_{Z_{d,i}}\right],$$

with

$$(W_Y)_{Z_{d,i}} = (W_{\mathrm{a}})_{(Z_{d,i})_d^{\complement}} \rtimes \bigcap_{j=0}^{d} (\Omega_Y)_{(Z_{d,i})_j}.$$

The boundaries are given by

$$\partial_d(Z_{d,i}) = \sum_{p=0}^{d} (-1)^p \omega_{p,i}(Z_{d-1,u_i}), \text{ where } u_i \in S_0 \ ; \ Z_{d-1,u_i} = \min_{\prec}(\Omega_Y \cdot Z_{d,i}^{(p)}) \text{ and } \omega_{p,i}(Z_{d-1,u_i}) = Z_{d,i}^{(p)}.$$

Moreover, the dual complex $C^*_{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ *is a* $\mathbb{Z}[W_Y]$*-dg-algebra with product*

$$Z^*_{d,i} \cup Z^*_{e,j} = \delta_{(Z_{d,i})_d, (Z_{e,j})_0} \omega(Z_{d+e,k})^*,$$

where

$$Z_{d+e,k} = \min_{\prec}(\Omega_Y \cdot ((Z_{d,i})_0, \dots, (Z_{d,i})_d, (Z_{e,j})_0, \dots, (Z_{e,j})_e)) \text{ and } \omega(Z_{d+e,k}) = ((Z_{d,i})_0, \dots, (Z_{e,j})_e).$$

Finally, the complex for the torus V^*/Y *is given by*

$$C_*^{\mathrm{cell}}(V^*/Y, W; \mathbb{Z}) = \mathrm{Def}_W^{W_Y}(C_*^{\mathrm{cell}}(V^*, W_Y; \mathbb{Z})).$$

Example 3.2.4. *Continuing the Example 2.1.3, we treat the extended type* A_2*, which is fairly computable by hand. We have* $S_0 = \{0, 1, 2\} = J$ *and*

$$\Omega = \Omega_{P^\vee} = \{1, \underbrace{t_{\varpi_\alpha^\vee} s_\alpha s_\beta}_{\omega_\alpha}, \underbrace{t_{\varpi_\beta^\vee} s_\beta s_\alpha}_{\omega_\beta}\} \simeq \mathbb{Z}/3\mathbb{Z}.$$

In this case, $W_{P^\vee} = \widehat{W}_{\mathrm{a}}$ *is the classical extended affine Weyl group. Geometrically, the element* ω_α *acts as the rotation with angle* $2\pi/3$ *around the barycenter of* $\overline{\mathcal{A}_0} = \mathrm{conv}(0, \varpi_\alpha^\vee, \varpi_\beta^\vee) =: [0, 1, 2] \simeq \Delta^2$*. The situation can be visualized in Figure 2.*

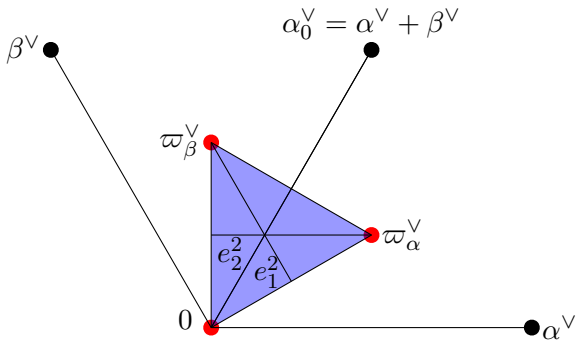


FIGURE 2. Barycentric subdivision $|\mathrm{Sd}(\overline{\mathcal{A}_0})|$ of the fundamental alcove $\overline{\mathcal{A}_0}$.

There are three $\widehat{W}_{\mathrm{a}}$*-orbits of points in* $|\mathrm{Sd}(\overline{\mathcal{A}_0})|$ *and we represent them by the points*

$$e_1^0 := (\{0\}) = 0, \ e_2^0 := (\{0, 1\}) = \frac{\varpi_\alpha^\vee}{2}, \ e_3^0 := (\{0, 1, 2\}) = \frac{\varpi_\alpha^\vee + \varpi_\beta^\vee}{3}.$$

Remember that we order $\mathscr{P}(S_0)$ *lexicographically and these are lex-minimal in their orbits. There are also four orbits of 1-cells represented by*

$$e_1^1 := (\{0\}, \{0, 1\}), \ e_2^1 := (\{0\}, \{0, 2\}), \ e_3^1 := (\{0\}, \{0, 1, 2\}), \ e_4^1 := (\{0, 1\}, \{0, 1, 2\}).$$

Finally, there are two orbits of 2-cells represented by

$$e_1^2 := (\{0\},\{0,1\},\{0,1,2\}),\ e_2^2 := (\{0\},\{0,2\},\{0,1,2\}).$$

Now, we have

$$\forall e \in \{e_1^0, e_2^0, e_i^1, e_j^2\},\ \Omega_e = 1 \quad \textit{and} \quad \Omega_{e_3^0} = \Omega$$

and we obtain the non-trivial stabilizers in $\widehat{W}_{\mathrm{a}}$*:*

$$(\widehat{W}_{\mathrm{a}})_{e_3^0} = \Omega,\ (\widehat{W}_{\mathrm{a}})_{e_1^0} = W,\ (\widehat{W}_{\mathrm{a}})_{e_2^0} = (\widehat{W}_{\mathrm{a}})_{e_1^1} = \langle s_\beta \rangle, (\widehat{W}_{\mathrm{a}})_{e_2^1} = \langle s_\alpha \rangle .$$

The boundaries are readily computed, with for instance

$$\partial_2(e_2^2) = -e_2^1 + e_3^1 - (\{0,2\},\{0,1,2\}) = -e_2^1 + e_3^1 - \omega_\beta e_4^1.$$

Therefore, the complex $C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ *is given by*

$$\mathbb{Z}[\widehat{W}_{\mathrm{a}}]^2 \xrightarrow{\partial_2} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\alpha\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}]^2 \xrightarrow{\partial_1} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/W] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\Omega],$$

with

$$\partial_2 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -\omega_\beta \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & \omega_\beta & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Moreover, the root datum $(P, \Phi, P^\vee, \Phi^\vee)$ *may be realized by the Lie group* $PSU(3) = SU(3)/\mu_3$ *with torus* $T = T_0/\mu_3 \simeq V^*/P^\vee$*, where* $T_0 = S(U(1)^3)$ *is the standard torus consisting of diagonal matrices of* $SU(3)$*. The complex*

$$C_*^{\mathrm{cell}}(T, W; \mathbb{Z}) = \mathrm{Def}_W^{\widehat{W}_{\mathrm{a}}}(C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z}))$$

then becomes

$$\mathbb{Z}[W]^2 \xrightarrow{\overline{\partial_2}} \mathbb{Z}[W/\langle s_\beta\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha\rangle] \oplus \mathbb{Z}[W]^2 \xrightarrow{\overline{\partial_1}} \mathbb{Z} \oplus \mathbb{Z}[W/\langle s_\beta\rangle] \oplus \mathbb{Z}[W/\langle s_\alpha s_\beta\rangle],$$

with

$$\overline{\partial_2} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -s_\beta s_\alpha \end{pmatrix}, \quad \overline{\partial_1} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & s_\beta s_\alpha & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The complexes $C_*^{\mathrm{cell}}(T_0, W; \mathbb{Z})$ *and* $C_*^{\mathrm{cell}}(T, W; \mathbb{Z})$ *may be obtained using the commands* `ComplexForFiniteCoxeterGroup("A",2)` *and* `CellularComplexT("A",2,[0,1,2])` *provided by the package* `Salvetti-and-tori-complexes`[2]*.*

Remark 3.2.5. *The complex* $C_*^{\mathrm{cell}}(V^*, \widehat{W}_{\mathrm{a}}; \mathbb{Z})$ *in the previous example can be reduced. Indeed, we can take* $e^2 := e_1^2 \cup_{e_3^1} e_2^2$ *as 2-cell. This deletes the 1-cell* e_3^1 *and the complex reduces to*

$$\mathbb{Z}[\widehat{W}_{\mathrm{a}}] \xrightarrow{{}^t\begin{pmatrix} 1 \\ -1 \\ 1-\omega_\beta \end{pmatrix}} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\alpha\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}] \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & \omega_\beta & 0 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z}[\widehat{W}_{\mathrm{a}}/W] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\langle s_\beta\rangle] \oplus \mathbb{Z}[\widehat{W}_{\mathrm{a}}/\Omega].$$

We recognize the closure $\overline{e^2} = \mathrm{conv}(e_1^0, e_2^0, \omega_\beta e_2^0, e_3^0)$ *as the fundamental polytope* $F_{P^\vee}$ *for* $\widehat{W}_{\mathrm{a}}$ *acting on* V^**.*

More generally, Komrakov and Premet have proved in [KP84] *that a fundamental polytope for the action of the extended affine Weyl group* $\widehat{W}_{\mathrm{a}}$ *is given by*

$$\begin{aligned} F_{P^\vee} :&= \{\lambda \in \overline{\mathcal{A}_0}\ ;\ \langle \lambda, \alpha + \alpha_0\rangle \le 1,\ \forall \alpha \in \Pi\ ;\ n_\alpha = 1\} \\ &= \{\lambda \in V^*\ ;\ \langle \lambda, \alpha_0\rangle \le 1\ \forall \alpha \in \Pi,\ \langle \lambda, \alpha\rangle \ge 0 \textit{ and } n_\alpha = 1 \implies \langle \lambda, \alpha + \alpha_0\rangle \le 1\}. \end{aligned}$$

[2] `https://github.com/arthur-garnier/Salvetti-and-tori-complexes`

where we have written the highest root α_0 as $\alpha_0 = \sum_{\alpha\in\Pi} n_\alpha \alpha$. It would be nice to obtain a cell decomposition from this polytope. However, this approach fails in general. Take the example of type C_3, whose positive coroots are depicted in Figure 3; denote by $\Pi = \{\alpha, \beta, \gamma\}$ a simple system and by $\varpi_\alpha^\vee = \alpha^\vee + \beta^\vee + \gamma^\vee$, $\varpi_\beta^\vee = \alpha^\vee + 2\beta^\vee + 2\gamma^\vee$, $\varpi_\gamma^\vee = \frac{1}{2}\alpha^\vee + \beta^\vee + \frac{3}{2}\gamma^\vee$ the corresponding coweights. The highest root is $\alpha_0 = 2\alpha + 2\beta + \gamma$ and from the Table 2, the fundamental group is

$$\Omega = \{1, \underbrace{\mathrm{t}_{\varpi_\gamma^\vee}(s_\gamma s_\alpha)^{s_\beta s_\gamma}}_{\omega_\gamma}\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the fundamental alcove is $\overline{\mathcal{A}_0} = \mathrm{conv}(0, \varpi_\alpha^\vee/2, \varpi_\beta^\vee/2, \varpi_\gamma^\vee)$ and from this we see that the non trivial element $\omega_\gamma \in \Omega$ acts on the vertices of $\overline{\mathcal{A}_0}$ by exchanging $\varpi_\alpha^\vee/2$ and $\varpi_\beta^\vee/2$, as well as 0 and $\varpi_\gamma^\vee$. Therefore, the affine facet of $F_{P^\vee} = \mathrm{conv}(0, \varpi_\alpha^\vee/2, \varpi_\beta^\vee/2, \varpi_\gamma^\vee/2)$ given by

$$F_{P^\vee} \cap \{\lambda \in V^* \ ; \ 1 = \langle \lambda, \alpha_0 + \gamma\rangle = 2\langle \lambda, \alpha + \beta + \gamma\rangle\} = \frac{1}{2}\mathrm{conv}(\varpi_\alpha^\vee, \varpi_\beta^\vee, \varpi_\gamma^\vee)$$

is taken to itself by ω_γ but is not fixed pointwise since two of its three vertices are exchanged. Therefore, the triangulation of V^* induced by translating the simplex $F_{P^\vee}$ is not *$\widehat{W}_{\mathrm{a}}$-equivariant.*

From the Table 2, we see that the same issue occurs for $C_{n\geq 3}$, $D_{n\geq 4}$, E_6 and E_7. We have tried to find a pattern for the non-pointwise fixed by some element of the fundamental group. However, after computations, we haven't found a general description for this. In some examples, no facet presents a problem, but some faces with higher codimension do. Hence, even though the barycentric subdivision yields many simplices, it works for any lattice and is rather simple to implement.

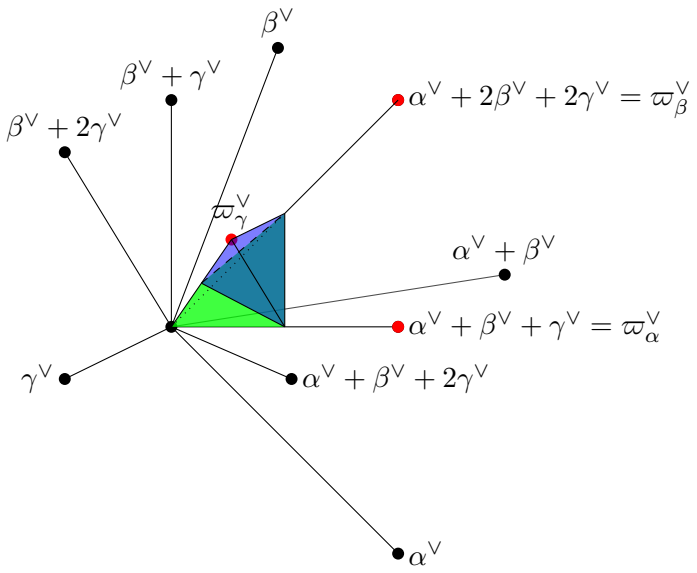


FIGURE 3. The Komrakov-Premet polytope $F_{P^\vee}$ (in green) inside the fundamental alcove $\overline{\mathcal{A}_0}$ (in blue) in type C_3.

Part 2. Hyperbolic tori for non-crystallographic Coxeter groups

The goal of this part is to construct a smooth manifold affording a dg-algebra with a similar combinatorics as the one in Theorem 2.2.2 and playing the role of a torus for non-crystallographic Coxeter groups. First, we will define *compact hyperbolic extensions* of non-crystallographic finite Coxeter groups and the desired manifold will then be constructed as an orbit space of the *Coxeter complex* of the hyperbolic extension.

4. Construction of the hyperbolic extensions and the hyperbolic torus

The non-crystallographic finite irreducible Coxeter groups are listed in the following table:

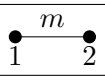
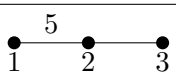
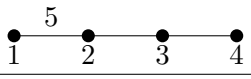
Type	Coxeter diagram
$I_2(m)$ $(5 \leq m \neq 6)$	
H_3	
H_4	

Table 3. Coxeter diagrams of finite non-crystallographic Coxeter systems.

Although we shall focus on the non-crystallographic case, what follows applies to all finite irreducible Coxeter groups. In particular, in the $I_2(m)$ case, we only assume that $m \geq 3$.

4.1. Compact hyperbolic extensions of $I_2(m)$, H_3 and H_4.

Let us first recall some basic terminology concerning Coxeter groups. For more detailed discussions, the reader is referred to [Bou02] and [Hum92].

Let (W, S) be an irreducible Coxeter system of rank n. We write

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \rangle ,$$

with $M = (m_{i,j})_{1 \leq i,j \leq n}$ the Coxeter matrix of (W, S). Recall ([Bou02, V, §4] or [Hum92, Chap. 5]) that on the formal vector space $V := \mathrm{span}_{\mathbb{R}}(\alpha_i,\ 1 \leq i \leq n)$ we may define a symmetric bilinear form by

$$B(\alpha_i, \alpha_j) := -\cos\left(\frac{\pi}{m_{i,j}}\right)$$

as well as the linear mappings

$$\forall 1 \leq i \leq n,\ \sigma_i := (v \mapsto v - 2B(\alpha_i, v)\alpha_i).$$

Then the assignment $s_i \mapsto \sigma_i$ extends uniquely to a faithful irreducible representation

$$\sigma : W \longrightarrow GL(V),$$

known as the *geometric representation* of W.

Moreover, W is finite (resp. affine) if and only if the form B is positive definite (resp. positive semidefinite) (see [Bou02, V, §4.8 and 4.9]).

Proposition-Definition 4.1.1 ([Hum92, §6.8])**.** *The followings are equivalent*

(i) The form B has signature $(n-1, 1)$ and $B(\lambda, \lambda) < 0$ for $\lambda \in C$,

(ii) The form B is non-degenerate but not positive and the graph obtained by removing any vertex from the graph of W is of non-negative type (i.e. its group is finite or affine).

If these conditions occur, then W is said to be hyperbolic*. If the second condition is enhanced by requiring that any such sub-graph is of positive definite type (i.e. its group is finite), then W is said to be* compact hyperbolic*.*

Remark 4.1.2. *As mentioned in* [Hum92]*, the terminology comes from the fact that the homogeneous space $O(V, B)/W$, equipped with the induced measure coming from the Haar measure on $O(V, B)$, is of finite volume if and only if W is finite or hyperbolic and, in the hyperbolic case, a component of $\{\lambda \in V\ ;\ B(\lambda, \lambda) = -1\}$ gives a model for the hyperbolic $(n-1)$-space $\mathbb{H}^n$. Moreover, the space $O(V, B)/W$ is compact if and only if W is compact hyperbolic. Moreover, W is compact hyperbolic if and only if B is non-degenerate non-positive and every proper parabolic subgroup of W is finite.*

We are ready to define the compact hyperbolic extensions of non-crystallographic groups. From now on, we let (W, S) be a finite irreducible Coxeter system of rank n with Coxeter matrix M. The notation in the following result is the same as in Table 4.

Proposition-Definition 4.1.3. *Let W be non-crystallographic and choose $r_W \in W$ to be the following reflection in W:*

$$r_W := \begin{cases} (s_1 s_2)^{\left\lfloor \frac{m-1}{2} \right\rfloor} s_1 & \text{if } W = I_2(m),\ m \geq 3, \\ s_3^{(s_2 s_1)^2} & \text{if } W = H_3, \\ s_4^{\left(s_1^{s_2 s_3} (s_1 s_2)^2 s_3 s_4\right)^2} & \text{if } W = H_4. \end{cases}$$

Define

$$\widehat{W} := \left\langle \widehat{s}_0, \widehat{s}_1, \dots, \widehat{s}_r \;\middle|\; \forall i, j \geq 1,\ (\widehat{s}_i \widehat{s}_j)^{m_{i,j}} = (\widehat{s}_0 \widehat{s}_i)^{o(r_W s_i)} = \widehat{s}_0^2 = 1 \right\rangle,$$

where $o(x)$ is the order of the element x and $\widehat{S} := \{\widehat{s}_0, \dots, \widehat{s}_n\}$. Then the pair $(\widehat{W}, \widehat{S})$ is a compact hyperbolic Coxeter system, whose Coxeter graph is as in the following table:

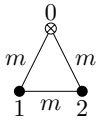
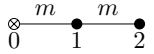
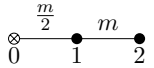
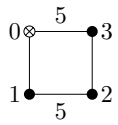
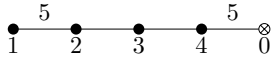
Extension	*Coxeter graph*
$\widehat{I_2(m)}$ $(m \equiv 1[2])$	
$\widehat{I_2(m)}$ $(m \equiv 0[4])$	
$\widehat{I_2(m)}$ $(m \equiv 2[4])$	
$\widehat{H_3}$	
$\widehat{H_4}$	

TABLE 4. Compact hyperbolic extensions of $I_2(m)$, H_3 and H_4.

Moreover, in type H, the reflection r_W is the only one for which the resulting group $\widehat{W}$ is compact hyperbolic.

Proof. The expression we give for r_W indicates that r_W indeed is a reflection of W. As r_W has order 2, the matrix $\widehat{M} := (\widehat{m}_{i,j})_{0 \leq i,j \leq n}$ defined by

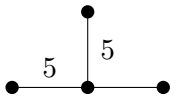
$$\forall i, j \geq 1,\ \widehat{m}_{i,j} = m_{i,j},\ \widehat{m}_{0,i} = \widehat{m}_{i,0} := o(r_W s_i),\ \widehat{m}_{0,0} := 1$$

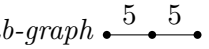
is indeed a Coxeter matrix and $\widehat{W}$ is the associated Coxeter group. Moreover, we may compute the integers $o(r_W s_i)$ directly and find the above Coxeter graphs and these are indeed graphs of compact hyperbolic groups, as all those graphs are well-known, see [Che69, Appendice].

The second statement comes from a tedious, but elementary verification on the 15 (resp. 60) reflections of H_3 (resp. H_4): only the reflection r_W from the statement gives a graph which appears in the table of [Che69]. □

Remark 4.1.4. *A (non-crystallographic) root system Φ may be associated to W. More precisely, Φ is the orbit under W of the vectors α_i spanning V. Then Φ forms a (non-Euclidean) root system in V, which is non-crystallographic in the sense that the condition $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$ does no longer hold. We still may choose a highest root in Φ. If $W \neq H_3$, then the reflection associated to this highest root is indeed r_W.*

The extension of H_3 with r_W the highest reflection has been considered in [PT02]*. It has the following Coxeter graph*



However, the sub-graph 

is of negative type, hence this extension is neither affine or hyperbolic and the sequel does not apply.

Using the very definitions of W and $\widehat{W}$ as finitely presented groups, we obtain the following result:

Corollary 4.1.5. *The assignment*

$$\begin{cases} \widehat{s}_0 & \longmapsto & r_W \\ \widehat{s}_i & \longmapsto & s_i \end{cases}$$

extends (uniquely) to a surjective reflection-preserving group homomorphism

$$\widehat{W} \xrightarrow{\pi} \!\!\!\!\rightarrow W.$$

Moreover, if $r_W = s_{i_1} \cdots s_{i_k}$ is a reduced expression of r_W, then the element $\widehat{r_W} = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k} \in \widehat{W}$ is well-defined and we have

$$\ker \pi = \left\langle (\widehat{s}_0 \widehat{r_W})^{\widehat{W}} \right\rangle,$$

that is, $\ker(\pi)$ *is the normal closure of $\widehat{s}_0 \widehat{r_W}$ in $\widehat{W}$.*

Proof. In every reduced expression as in the statement, we have $i_j \geq 1$ so that the element $\widehat{r_W} = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k}$ is in the parabolic subgroup $\widehat{W}_{\{1,\dots,n\}} \simeq W$ and thus $\widehat{r_W}$ doesn't depend on the chosen reduced expression for r_W. We have $\pi(\widehat{s}_0 \widehat{r_W}) = r_W^2 = 1$ so that the subgroup $N := \left\langle (\widehat{s}_0 \widehat{r_W})^{\widehat{W}} \right\rangle$ is certainly contained in $\ker(\pi)$. Furthermore, we easily find a presentation of $\widehat{W}/N$ by adding the relation $\widehat{s}_0 = \widehat{s}_{i_1} \cdots \widehat{s}_{i_k}$ for $r_W = s_{i_1} \cdots s_{i_k}$ as above to the already known relations for $\widehat{W}$. The composite

$$\langle \overline{s}_0, \overline{s}_1, \dots, \overline{s}_n \mid \forall i,j \geq 1,\ (\overline{s}_i \overline{s}_j)^{m_{i,j}} = 1,\ \overline{s}_0 = \overline{s}_{i_1} \cdots \overline{s}_{i_k} \rangle \simeq \widehat{W}/N \longrightarrow\!\!\!\!\rightarrow \widehat{W}/\ker \pi = W$$

maps $\overline{s}_i$ to s_i and is an isomorphism. In particular, this yields an isomorphism of $\widehat{W}$-sets

$$\widehat{W}/N \simeq \widehat{W}/\ker \pi,$$

forcing $\ker(\pi)$ and N to be conjugate in $\widehat{W}$, hence equal. □

Definition 4.1.6. *We denote the kernel of the projection from the previous Corollary by*

$$Q := \ker \pi = \left\langle (\widehat{s}_0 \widehat{r_W})^{\widehat{W}} \right\rangle.$$

Corollary 4.1.7. *With the notation of the above theorem, we have*

$$\widehat{W} = Q \rtimes W.$$

Remark 4.1.8. *Let Φ denote the (non-crystallographic) root system of W and $\widetilde{\alpha} \in \Phi^+$ be the (positive) root associated to the reflection r_W, i.e. such that $r_W = s_{\widetilde{\alpha}^\vee}$. If $W \neq H_3$, then $\widetilde{\alpha} = \alpha_0$ is the highest root of Φ. Denote by $\mathrm{t}_{\widetilde{\alpha}^\vee}$ the translation by $\widetilde{\alpha}^\vee$ and by $\sigma^* : W \to GL(V^*)$ the dual of the geometric representation of W. We can define a homomorphism*

$$\widehat{W} \xrightarrow{\mathbf{a}} \mathrm{Aff}(V^*)$$

by sending $\widehat{s}_i$ *to* $\sigma^*(s_i)$ *for* $i \geq 1$ *and* $\mathbf{a}(\widehat{s}_0) := \mathrm{t}_{\widetilde{\alpha}^\vee}\sigma^*(r_W)$. *If* W *is a Weyl group, then* $\mathbf{a}$ *is injective and identifies* $\widehat{W}$ *with* $W_{\mathrm{a}} \leq \mathrm{Aff}(V^*)$. *Moreover, in this case we have*

$$Q \simeq \mathbf{a}(Q) = \mathbf{a}\left(\left\langle (\widehat{s}_0\widehat{s_{\alpha_0^\vee}})^{\widehat{W}}\right\rangle\right) = \left\langle (\mathbf{a}(\widehat{s}_0)\mathbf{a}(\widehat{s_{\alpha_0^\vee}}))^{\mathbf{a}(\widehat{W})}\right\rangle$$
$$= \left\langle (t_{\alpha_0^\vee})^{W_{\mathrm{a}}}\right\rangle = \langle t_{\alpha^\vee},\ \alpha \in \Phi\rangle \simeq \mathbb{Z}\Phi^\vee \overset{df}{=} Q^\vee \simeq \mathbb{Z}^n.$$

This is the coroot lattice of Φ *and in particular, the group* Q *is abelian.*

However, a relatively recent result ([Qi07, Corollary 1.6]*) states that an irreducible, infinite Coxeter group is affine if and only if it contains an abelian subgroup of finite index and, as* $[\widehat{W} : Q] = |W| < \infty$, *the group* Q *cannot be abelian in the hyperbolic case.*

Moreover, in the non-crystallographic case, the image of $\mathbf{a}$ *is no longer discrete because* $\mathbb{Z}\Phi^\vee \subset V^*$ *is dense in* V^* *and also, the morphism* $\mathbf{a}$ *has no reason to be injective, because we cannot relate the length function on* $\widehat{W}$ *with separating reflection hyperplanes in* V *any longer.*

4.2. **A key property of the subgroup** Q**.**

The following result will be crucial in the sequel.

Lemma 4.2.1. *The normal subgroup* Q *trivially intersects every proper parabolic subgroup of* $\widehat{W}$, *i.e.*

$$\forall I \subsetneq \widehat{S},\ Q \cap \widehat{W}_I = 1.$$

Proof. Recall the projection $\pi : \widehat{W} \twoheadrightarrow W$. The statement may be rephrased as follows:

$$\forall s \in \widehat{S},\ \ker\left(\widehat{W}_{\widehat{S}\setminus\{s\}} \xrightarrow{\pi} W\right) = 1.$$

For $s = \widehat{s}_0$, this is obvious since $\widehat{W}_{\widehat{S}\setminus\{\widehat{s}_0\}} \xrightarrow{\pi} W$ is an isomorphism.

Let $s \in \widehat{S}\setminus\{\widehat{s}_0\}$. Since $\widehat{W}$ is compact hyperbolic, the parabolic subgroup $\widehat{W}_{\widehat{S}\setminus\{s\}}$ is finite. Hence, to prove that the morphism

$$\widehat{W}_{\widehat{S}\setminus\{s\}} \xrightarrow{\pi} \pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right)$$

is injective, it suffices to prove that

$$(\star_s) \qquad \left|\pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right)\right| = \left|\widehat{W}_{\widehat{S}\setminus\{s\}}\right|.$$

The right-hand side is easily computed using the Coxeter diagram of $\widehat{W}$ (see Table 4). To compute the left-hand side, we proceed by a case-by-case analysis. For H_4, we will need the following trick:

Denote by

$$R := \bigcup_{w\in W} wSw^{-1} = \bigcup_{w\in W} S^w$$

the set of reflections of W and

$$\forall w \in W,\ N(w) := \{r \in R\ ;\ \ell(rw) < \ell(w)\}.$$

If $H \leq W$ is a reflection subgroup of W (i.e. if $H = \langle H \cap R\rangle$), then the set

$$D(H) := \{r \in R\ ;\ N(r) \cap H = \{r\}\}$$

is a set of Coxeter generators of H (see [Dye90, Theorem 3.3]). In our situation, we find the Coxeter generators $D(\pi(\widehat{W}_{\widehat{S}\setminus\{s\}}))$ and determine the resulting Coxeter diagram, giving the order of $\pi(\widehat{W}_{\widehat{S}\setminus\{s\}})$.

- $W = I_2(m)$ with $m = 2k+1$. We have defined $r_W = (s_1s_2)^k s_1$ and we readily compute $s_2 = {s_1}^{r_W}$ and $s_1 = {s_2}^{r_W}$ so that $\pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_1}) = \pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_2}) = W$. On the other hand, we get from the diagram $|\widehat{W}_{\widehat{s}_0,\widehat{s}_1}| = |\widehat{W}_{\widehat{s}_0,\widehat{s}_1}| = 2m = |W|$. This proves $(\star_s)$ for $s = \widehat{s}_1, \widehat{s}_2$.

- $W = I_2(m)$ with $m = 4k$. In this case we have $r_W = (s_1s_2)^{2k-1}s_1$ and since $s_2 = (s_1r_W)^{2k-1}s_1$, we also have $\langle r_W, s_1\rangle = W$ and $(\star_s)$ is thus true for $s = \widehat{s}_2$ as $\widehat{W}_{\widehat{s}_0,\widehat{s}_1} \simeq W$. Because $s_2r_W = r_Ws_2$, we have $\langle s_2, r_W\rangle = A_1 \times A_1$ and $\widehat{W}_{\widehat{s}_0,\widehat{s}_2} \simeq A_1 \times A_1$ so $(\star_s)$ also holds for $s = \widehat{s}_2$.
- $W = I_2(m)$ with $m = 4k + 2$. Here, $r_W = (s_1s_2)^{2k}s_1$ and we compute $r_Ws_1r_W = (s_1s_2)^{4k}s_1 = s_2s_1s_2 = s_1^{s_2}$. In the same way, we get $(s_1(s_1^{s_2}))^k s_1 = (s_1s_2)^{2k}s_1 = r_W$. This implies $\langle s_1, r_W\rangle = \langle s_1, s_1^{s_2}\rangle \simeq I_2(2k + 1) \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1}$. In fact, we have $D(\langle s_1, r_W\rangle) = \{s_1, s_1^{s_2}\}$. Now, as above we have $s_2r_W = r_Ws_2$ and $\widehat{W}_{\widehat{s}_0,\widehat{s}_2} \simeq A_1 \times A_1 \simeq \langle s_2, r_W\rangle$.
- $W = H_3$. Special relations among reflections occur in this case. Namely

  $$r_W = s_3^{(s_2s_1)^2},\ s_3 = r_W^{(s_1s_2)^2},\ s_2 = s_3(r_Ws_3s_1)^2r_Ws_3,\ s_1 = (r_Ws_3s_2)^2r_Ws_3r_W.$$

  Hence, for $s \in \widehat{S}$, we have $\pi\left(\widehat{W}_{\widehat{S}\setminus\{s\}}\right) = W \simeq \widehat{W}_{\widehat{S}\setminus\{s\}}$, this last isomorphism being given by the diagram of $\widehat{H_3}$. Therefore, all the relations $(\star_s)$ hold in this case.
- $W = H_4$. The additional reflection is

  $$r_W = s_4^{(s_3s_2s_1s_2s_3(s_1s_2)^2s_3s_4)^2}.$$

  We notice the following relation

  $$s_1 = s_2s_3(s_4r_W)^2(s_3s_4r_Ws_2(s_3s_4r_W)^2s_2)^3s_3s_4r_Ws_4s_3s_2.$$

  This proves that $s_1 \in \langle r_W, s_2, s_3, s_4\rangle$ so $\pi(\widehat{W}_{\widehat{s}_0,\widehat{s}_2,\widehat{s}_3,\widehat{s}_4}) = W \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_2,\widehat{s}_3,\widehat{s}_4}$. We treat the remaining cases by determining the Dyer generators of the reflection subgroups. Calculations can be done on the sixty reflections of H_4 (though easier using [GAP21]). We obtain

  $$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_3,\widehat{s}_4}\right) = \langle r_W, s_1, s_3, s_4\rangle = \left\langle s_1^{s_2s_3(s_1s_2)^2}, s_1, s_3, s_4\right\rangle \simeq A_1 \times H_3 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_3,\widehat{s}_4}$$

  and

  $$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_4}\right) = \langle r_W, s_1, s_2, s_4\rangle = \left\langle s_3^{s_4s_2s_1s_2s_3(s_1s_2)^2s_3}, s_1, s_2, s_4\right\rangle \simeq I_2(5)^2 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_4}$$

  and finally,

  $$\pi\left(\widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3}\right) = \langle r_W, s_1, s_2, s_3\rangle \simeq H_3 \times A_1 \simeq \widehat{W}_{\widehat{s}_0,\widehat{s}_1,\widehat{s}_2,\widehat{s}_3}.$$

  This establishes the relations $(\star_s)$ for $W = H_4$, finishing the proof.

□

Corollary 4.2.2. *The group Q is torsion-free.*

Proof. Let $q \in Q$ be of finite order. By a theorem of Tits (see [Qi07, Theorem 3.10]), there are $w \in \widehat{W}$ and $J \subset \widehat{S}$ such that $q \in w\widehat{W}_Jw^{-1}$ and $\widehat{W}_J$ is finite. This last condition implies $J \neq \widehat{S}$ and since Q is normal in $\widehat{W}$, we get $q^w \in Q \cap \widehat{W}_J = 1$, so $q = 1$. □

4.3. **The hyperbolic torus $\mathbf{T}(W)$ of W and its first properties.**

Before defining the manifold $\mathbf{T}(W)$, we have to study the action of the subgroup $Q \trianglelefteq \widehat{W}$ on the Tits cone of $\widehat{W}$. Recall some notation: define $\widehat{V} := \mathrm{span}_{\mathbb{R}}(\alpha_s,\ s \in \widehat{S})$ and the bilinear form $\widehat{B}$ given by

$$\widehat{B}(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{\widehat{m}_{s,t}}\right),$$

with $(\widehat{m}_{s,t})_{s,t\in\widehat{S}}$ the Coxeter matrix of $(\widehat{W}, \widehat{S})$. As $\widehat{W}$ is hyperbolic, the form $\widehat{B}$ has signature $(n-1, 1)$. We also have the geometric representation $\widehat{\sigma} : \widehat{W} \hookrightarrow O(\widehat{V}, \widehat{B})$ and consider its contragredient representation

$$(\circledast) \qquad \widehat{\sigma}^* : \widehat{W} \hookrightarrow GL(\widehat{V}^*)$$

and define $(\alpha_s^\vee)_{s\in\widehat{S}}$ to be the dual basis of $\widehat{V}^*$ associated to $(\alpha_s)_{s\in\widehat{S}}$. We have $\widehat{\sigma}^*(w) = {}^t\widehat{\sigma}(w^{-1})$, that is

$$\forall s,t\in\widehat{S},\ \widehat{\sigma}^*(s)(\alpha_t^\vee) = \alpha_t^\vee - 2\delta_{s,t}\widehat{B}(-,\alpha_s).$$

The duality pairing of $\widehat{V}$ is denoted $\langle\cdot,\cdot\rangle$ as usual. For $s\in\widehat{S}$, let moreover

$$H_s := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle = 0\}\ \text{ and }\ A_s := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle > 0\}$$

and consider the respective *fundamental chamber* and *Tits cone*

$$C := \{\lambda\in\widehat{V}^*\ ;\ \langle\lambda,\alpha_s\rangle > 0,\ \forall s\in\widehat{S}\} = \bigcap_{s\in\widehat{S}} A_s\ \text{ and }\ X := \bigcup_{w\in\widehat{W}} w(\overline{C}).$$

This is indeed a convex cone and $\overline{C}$ is a fundamental domain for the action of $\widehat{W}$ on X. Finally, for $I\subseteq\widehat{S}$ we let

$$C_I := \left(\bigcap_{s\in I} H_s\right)\cap\left(\bigcap_{s\notin I} A_s\right)\subset\overline{C},$$

in particular $C_\emptyset = C$, $C_{\widehat{S}} = \{0\}$ and we have $\overline{C} = \bigsqcup_{I\subseteq\widehat{S}} C_I$.

In this context, we have the *Coxeter complex*

$$\widehat{\Sigma} := \Sigma(\widehat{W},\widehat{S}) = (X\setminus\{0\})/\mathbb{R}_+^*.$$

This is a $\widehat{W}$-pseudomanifold and we have a decomposition

$$\widehat{\Sigma} = \bigcup_{\substack{w\in\widehat{W}\\ I\subsetneq\widehat{S}}} \mathbb{R}_+^* w(\overline{C_I})$$

which is in fact a $\widehat{W}$-triangulation since $\mathbb{R}_+^* w(\overline{C_I})$ may be identified with the standard $(n-|I|)$-simplex: $\mathbb{R}_+^* w(\overline{C_I})\simeq\Delta^{n-|I|}$. Moreover, since $\widehat{W}$ is infinite, $\widehat{\Sigma}$ is contractible and by [Bro89, III, §2, Corollary 3], as every proper parabolic subgroup of $\widehat{W}$ is finite, the pseudomanifold $\widehat{\Sigma}$ is in fact a smooth n-manifold.

Remark 4.3.1. *The construction of the Coxeter complex makes sense for any Coxeter group. If the group is finite, then its Coxeter complex is homeomorphic to a sphere.*

We can give a natural simplicial structure to the Coxeter complex (see [BR04, Corollary 2.6]). Consider the set of parabolic cosets of $\widehat{W}$

$$P(\widehat{W},\widehat{S}) := \{w\widehat{W}_I\ ;\ w\in\widehat{W},\ I\subsetneq\widehat{S}\}.$$

We partially order this set as follows:

$$w\widehat{W}_I \preceq w'\widehat{W}_J \overset{\text{df}}{\iff} w\widehat{W}_I\supseteq w'\widehat{W}_J.$$

Notice that $w\widehat{W}_I\preceq w'\widehat{W}_J$ implies $w\widehat{W}_I = w'\widehat{W}_I$ and $J\subset I$. We define the simplicial complex $\Delta(\widehat{W},\widehat{S})$ as the nerve of this poset:

$$\Delta(\widehat{W},\widehat{S}) := \mathcal{N}(P(\widehat{W},\widehat{S}),\preceq).$$

If we denote by $\mathcal{P}(\widehat{\Sigma})$ the poset of faces of $\widehat{\Sigma}$ with respect to the triangulation described above. Then we have an isomorphism of posets

$$\begin{array}{ccc}(P(\widehat{W},\widehat{S}),\preceq) & \xrightarrow{\sim} & (\mathcal{P}(\widehat{\Sigma}),\subseteq)\\ w\widehat{W}_I & \longmapsto & \mathbb{R}_+^* w(\overline{C_I})\end{array}$$

and this yields a $\widehat{W}$-equivariant homeomorphism

$$|\Delta(\widehat{W},\widehat{S})|\xrightarrow{\sim}\widehat{\Sigma}.$$

Now, recall that an action of a group G on a space Z is said to be *properly discontinuous* (or a *covering space action*, see [Hat02, §1.3]) if every point $z\in Z$ has an open neighbourhood

$z \in U \subset Z$ such that if $g \in G$ is such that $gU \cap U \neq \emptyset$, then $g = 1$. In other words, such that

$$O_G(U) := \{g \in G \ ; \ g(U) \cap U \neq \emptyset\} = \{1\}.$$

Lemma 4.3.2. *Recall from (⊛) the representation $\widehat{\sigma}^*$. The action of the discrete subgroup $\widehat{\sigma}^*(Q) \leq GL(\widehat{V}^*)$ on the Coxeter complex $\widehat{\Sigma}$ is free and properly discontinuous.*

Proof. Of course, we identify the group $\widehat{W}$ with $\widehat{\sigma}^*(\widehat{W})$. Let $\overline{x} \in \widehat{\Sigma}$ (with $x \in X \setminus \{0\}$). First, we prove that $q(\overline{x}) \neq \overline{x}$ for $q \in Q \setminus \{1\}$. To say that $q(\overline{x}) = \overline{x}$ amounts to say that $q(x) = ax$ for some $a \in \mathbb{R}_+^*$ and we may assume that $x \in \overline{C} \setminus \{0\}$ since $Q \trianglelefteq \widehat{W}$. There is some $I \subsetneq \widehat{S}$ such that $x \in C_I$. Because C_I is a cone, we have $ax \in C_I \cap q(C_I) \neq \emptyset$ and by [Bou02, V, §4, Proposition 5], we obtain $q(C_I) = C_I$ so $q \in \widehat{W}_I \cap Q = 1$ by Lemma 4.2.1.

To prove that the action is properly discontinuous at $\overline{x}$, we have to find an open neighbourhood U of $\overline{x}$ in $\widehat{\Sigma}$ such that, for $1 \neq q \in Q$, we have $U \cap q(U) = \emptyset$, i.e. $O_Q(U) = \{1\}$. By definition of the topology on the Coxeter complex, it suffices to prove the statement for $X \setminus \{0\}$.

First, we show that the action of $\widehat{W}$ is *wandering* at x, that is, we can find an open neighbourhood A of x such that $O_{\widehat{W}}(A)$ is finite. We may assume that $x \in \overline{C} \setminus \{0\}$, say $x \in C_I$ with $I \subsetneq \widehat{S}$. Define A to be the interior in $X \setminus \{0\}$ of the subset $\bigcup_{v \in \widehat{W}_I} v(\overline{C})$. We prove that there are only finitely many $w \in \widehat{W}$ such that $A \cap w(A) \neq \emptyset$. Suppose that $w \in O_{\widehat{W}}(A)$ and choose $a \in A$ with $w(a) \in A$. Notice that we have

$$A \subseteq \bigcup_{u \in \widehat{W}_I} u(C) \cup \bigcup_{\substack{v \in \widehat{W}_I \\ s \in I}} v(H_s \cap \partial\overline{C}).$$

Thus, we distinguish four cases:

- As $\widehat{W}$ acts on $X \setminus \bigcup_{s \in \widehat{S}} H_s$, we cannot have $a \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$ and $w(a) \in \bigcup_u u(C)$.
- Similarly, we cannot have $a \in \bigcup_u u(C)$ and $w(a) \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$.
- Suppose that $a \in \bigcup_v v(C)$ and $w(a) \in \bigcup_v v(C)$, say $a \in u(C)$ and $w(a) \in v(C)$. This implies $u^{-1}(a) \in C$ and $v^{-1}w(a) = v^{-1}wu(u^{-1}(a)) \in C$, thus $uv^{-1}w(C) \cap C \neq \emptyset$ and so $w = vu^{-1} \in \widehat{W}_I$ by Tits' lemma.
- Suppose now that we have $a \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$ and $w(a) \in \bigcup_{v,s} v(H_s \cap \partial\overline{C})$, say $a \in u(H_s \cap \partial\overline{C})$ and $w(a) \in v(H_t \cap \partial\overline{C})$. This implies $u^{-1}(a) \in \overline{C}$ and $v^{-1}wu(u^{-1}(a)) = v^{-1}w(a) \in \overline{C}$ and by [Bou02, V, §4, Proposition 6] we get $v^{-1}w(a) = u^{-1}(a)$ and thus $uv^{-1}w \in (\widehat{W})_a = u\widehat{W}_J u^{-1}$ for some $J \subsetneq \widehat{S}$ (in fact, J is defined by the condition $\widehat{W}_J = (\widehat{W})_{u^{-1}(a)}$). Therefore, we have $w \in v\widehat{W}_J u^{-1}$.

In any case, we have

$$O_{\widehat{W}}(A) \stackrel{\text{df}}{=} \{w \in \widehat{W} \ ; \ w(A) \cap A \neq \emptyset\} \subset \bigcup_{\substack{u,v \in \widehat{W}_I \\ J \subsetneq \widehat{S}}} u\widehat{W}_J v.$$

However, as $\widehat{W}$ is compact, any proper parabolic subgroup is finite and so this last subset is finite and $O_{\widehat{W}}(A)$ is then finite as well.

The rest of the proof is very standard. For each $w \in O_{\widehat{W}}(A) \setminus \widehat{W}_I$ we have $w(x) \neq x$ and we may choose an open subset A_w such that $x \in A_w \subset A$ and $w(A_w) \cap A_w = \emptyset$ and define

$$B := \bigcap_{w \in O_{\widehat{W}}(A) \setminus \widehat{W}_I} A_w \subset A.$$

Because $O_{\widehat{W}}(A)$ is finite, B is open and let $w' \in O_{\widehat{W}}(B) \subset O_{\widehat{W}}(A)$. We must have $w' \in \widehat{W}_I$ because otherwise, $\emptyset \neq B \cap w'(B) \subset A_{w'} \cap w'(A_{w'}) = \emptyset$ and thus $O_{\widehat{W}}(B) \subset \widehat{W}_I$.

Consider the open subset

$$U := \bigcap_{w \in \widehat{W}_I} w(B) \subset B.$$

We have $O_{\widehat{W}}(U) \subset O_{\widehat{W}}(B) \subset \widehat{W}_I$ and U is $\widehat{W}_I$-stable (i.e. U is a $\widehat{W}$-*slice* at x). In particular, if $q \in Q \setminus \{1\}$, then $q \notin \widehat{W}_I$ by Lemma 4.2.1 and thus $q \notin O_{\widehat{W}}(U)$. □

We arrive then to the main result of this section. Remark that the Tits form $\widehat{B}$ induces a Riemannian metric on the Coxeter complex $\widehat{\Sigma}$.

Theorem 4.3.3. *Let (W, S) be a finite irreducible Coxeter group of rank n and $(\widehat{W}, \widehat{S})$ be either the affine Weyl group associated to W if W is crystallographic, or the extension constructed above otherwise, with $Q := \ker(\widehat{W} \twoheadrightarrow W)$. If $\widehat{\sigma}^*$ denotes the contragredient geometric representation (as in (⊛)), then the orbit space*

$$\mathbf{T}(W) := \widehat{\Sigma}/\widehat{\sigma}^*(Q)$$

is a closed, connected, orientable, compact smooth W-manifold of dimension n.

If W is a Weyl group, then we have a diffeomorphism $\widehat{\Sigma} \simeq \mathbb{R}^n$ and the manifold $\mathbf{T}(W)$ is W-diffeomorphic to a maximal torus of the simply-connected compact Lie group with root system that of W. Otherwise, the Riemannian manifold $\widehat{\Sigma}$ is isometric to the hyperbolic n-space $\mathbb{H}^n$ and $\mathbf{T}(W) \simeq \mathbb{H}^n/Q$ is a hyperbolic W-manifold.

Furthermore, the canonical projection yields a covering space

$$Q \hookrightarrow \widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$$

and the quotient simplicial complex $\Delta(\widehat{W}, \widehat{S})/Q$ is a regular W-triangulation of $\mathbf{T}(W)$.

Proof. Since $\widehat{\Sigma}$ is a closed smooth manifold and the action $\widehat{\sigma}^*(Q) \curvearrowright \widehat{\Sigma}$ is properly discontinous by Lemma 4.3.2, the quotient manifold theorem ensures that $\mathbf{T}(W)$ is indeed a closed smooth manifold and by [Hat02, Proposition 1.40], the projection $\widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$ is a covering map. Moreover, $\mathbf{T}(W)$ is connected since the Coxeter complex is and, as $(\overline{C} \setminus \{0\})/\mathbb{R}_+^* \simeq \overline{C} \cap \mathbb{S}^n$ is a $\widehat{W}$-fundamental domain on the Coxeter complex, its projection onto $\mathbf{T}(W)$ is a W-fundamental domain, hence $\mathbf{T}(W)$ is compact (W is finite). Since Q is normally generated by $\widehat{s}_0\widehat{r_W}$ and because $\ell(\widehat{r_W})$ is odd, we have $\varepsilon(\widehat{s}_0\widehat{r_W}) = 1$ and so $Q \leq \ker(\varepsilon)$. This proves that the action of Q on $\widehat{\Sigma}$ preserves the orientation, ensuring the orientability of $\mathbf{T}(W)$. The comparison with a torus of a Lie group follows directly from the Remark 4.1.8.

In the non-crystallographic case, let $v^* \in V^*$ be a normalized eigenvector for the negative eigenvalue of $\widehat{B}$. Then the subset $\mathcal{H} := \{\lambda \in \widehat{V}^* \ ; \ \widehat{B}(\lambda, \lambda) = -1, \ \widehat{B}(v^*, \lambda) < 0\}$, together with the metric induced by the restriction of $\widehat{B}$ is a Riemannian manifold isometric to the hyperbolic space $\mathbb{H}^n$. We have $\mathbf{T}(W) = \widehat{\Sigma}/Q \simeq \mathcal{H}/Q \simeq \mathbb{H}^n/Q$ and since Q preserves the form $\widehat{B}$, the manifold $\mathbf{T}(W)$ naturally inherits a hyperbolic Riemannian metric. □

Remark 4.3.4. *After we did this work, we realized that the manifolds $\mathbf{T}(H_3)$ and $\mathbf{T}(H_4)$ have already been discovered in* [Zim93] *and* [Dav85]*. Zimmermann and Davis construct them by taking the orbit under the action of Q (which is defined slightly differently) of hyperbolic polyhedra. However, our approach has the advantages of being more systematic and to work with any finite Coxeter group. The Zimmermann manifold $\mathbf{T}(H_3)$ has the particularity of being* maximally symmetric *among hyperbolic 3-manifolds with Heegaard genus 11, in the sense of* [Zim92]*. On the other hand, the Davis manifold $\mathbf{T}(H_4)$ has a* spin structure *(equivalently, its second Stiefel-Whitney class w_2 vanishes) and seems to be the only closed 4-manifold for which the intersection form has been explicitly determined, see* [RT01] *and* [Mar15]*.*

Recall that, as $\widehat{W}$ is infinite, the Coxeter complex is contractible.

Corollary 4.3.5. *The covering space*

$$Q \longhookrightarrow \widehat{\Sigma} \twoheadrightarrow \mathbf{T}(W)$$

is a universal principal Q-bundle. In particular, $\mathbf{T}(W)$ is a classifying space for Q and an Eilenberg-MacLane space

$$\mathbf{T}(W) \simeq B_Q \simeq K(Q,1).$$

4.4. **Presentation on the fundamental group of $\mathbf{T}(W)$.**

In this section, we use Poincaré's fundamental polyhedron theorem (see [Rat06, Theorem 11.2.2]) to derive a presentation of the group $\pi_1(\mathbf{T}(W)) \simeq Q \trianglelefteq \widehat{W}$ in the non-crystallographic case.

The tessellation $\Delta(\widehat{W}, \widehat{S})$ of $\widehat{\Sigma} \simeq \mathbb{H}^n$ is indeed exact and yields a fundamental polyhedron for Q acting on $\widehat{\Sigma}$. Choose $v^* \in V^*$ a normalized eigenvector of the Tits form $\widehat{B}$ for its unique negative eigenvalue and consider the subset

$$\mathcal{H} := \{\lambda \in V^* \ ; \ \widehat{B}(\lambda,\lambda) = -1, \ \widehat{B}(v^*,\lambda) < 0\} \subset V^*.$$

As already noted in the proof of Theorem 4.3.3, the form $\widehat{B}$ induces a Riemannian metric on $\mathcal{H}$ and we have an isometry $\mathcal{H} \simeq \mathbb{H}^n$, where $\mathbb{H}^n$ is the standard hyperbolic n-space. By Remark 4.1.2, the fundamental chamber C is included in the subset $\{\lambda \ ; \ \widehat{B}(\lambda,\lambda) < 0\}$, hence we can project the punctured Tits cone $X \setminus \{0\}$ on the sheet $\mathcal{H}$ of the hyperbola $\{\lambda \ ; \ \widehat{B}(\lambda,\lambda) = -1\}$ and we get $\widehat{\Sigma} \simeq X \cap \mathcal{H}$. Consider the n-simplex

$$\Delta_0 := (\overline{C} \setminus \{0\})/\mathbb{R}_+^* \simeq \overline{C} \cap \mathcal{H} \subset \widehat{\Sigma}.$$

Recall that we have denoted $H_s := \{\lambda \ ; \ \langle \lambda, \alpha_s \rangle = 0\}$ for $s \in \widehat{S}$. As the subset $L_0 := \overline{C} \cap \bigcap_{s \neq \widehat{s}_0} H_s$ is a line, its intersection with $\mathcal{H}$ is a vertex of the tessellation $\Delta(\widehat{W}, \widehat{S})$ and we may consider its star

$$\Delta := \mathrm{St}\,(L_0 \cap \mathcal{H}) \overset{\mathrm{df}}{=} \bigcup_{\substack{\sigma \in F_n(\Delta(\widehat{W},\widehat{S})) \\ L_0 \cap \mathcal{H} \subset \sigma}} \sigma = \bigcup_{w \in W} w(\Delta_0).$$

We will describe the generators and relations for $\pi_1(\mathbf{T}(W))$ in terms of side-pairing and cycle relations, as in [Rat06, §6.8]. It is easy to see that the facets of Δ are the W-translates of the facet

$$\sigma_0 := H_{\widehat{s}_0} \cap \Delta \in F_{n-1}(\Delta),$$

in other words, $\partial\Delta = \bigcup_w w(\sigma_0)$. By [Rat06, Theorem 6.8.3], the group $Q = \pi_1(\mathbf{T}(W))$ is generated by the set

$$\Psi := \{q \in Q \ ; \ \Delta \cap q\Delta \in F_{n-1}(\Delta)\}.$$

Lemma 4.4.1. *The set Ψ of generators of Q is given by the W-conjugates of the normal generator of Q. In other words, if $r_W \in W$ is the chosen reflection and if $q_0 := \widehat{s}_0 r_W \in \widehat{W}$ then we have*

$$\Psi = \{{}^w q_0, \ w \in W\} = \{w q_0 w^{-1}, \ \overline{w} \in W/C_W(q_0)\}.$$

Proof. Let $1 \neq q \in Q$ be such that $\Delta \cap q\Delta$ is a facet of Δ, say $w(\sigma_0)$ for some $w \in W$. We have

$$w(\sigma_0) = \Delta \cap q\Delta = \bigcup_{u,v \in W} u(\Delta_0) \cap qv(\Delta_0) = \bigcup_{u,v \in W} u(\Delta_0 \cap u^{-1}qv(\Delta_0)).$$

Since any term of the last union is (empty or) a closed simplex, this means that one of them has to be the whole of $w(\sigma_0)$, so we can find $u, v \in W$ such that

$$u^{-1}w(\sigma_0) = \Delta_0 \cap u^{-1}qv(\Delta_0).$$

In particular, we have $u^{-1}w(\sigma_0) \subset \Delta_0$ and since any $\widehat{W}$-orbit meets Δ_0 in only one point, this implies that $u^{-1}w(\sigma_0) = \sigma_0$ and so $u^{-1}w \in \widehat{W}_{\sigma_0} = \langle \widehat{s}_0 \rangle$ but as $u^{-1}w \in W$, this is possible only when $u = w$. Hence we get

$$\sigma_0 = \Delta_0 \cap u^{-1}qv(\Delta_0).$$

This implies in turn that $u^{-1}qv \in \langle \widehat{s}_0 \rangle$ and since $q \neq 1$, we must have $u^{-1}qv = \widehat{s}_0$, i.e. $q = u\widehat{s}_0v^{-1}$. Finally, because $q \in Q$, applying the projection $\pi : \widehat{W} \twoheadrightarrow W$ to this equality yields $1 = ur_Wv^{-1}$, so $v = ur_W$ and $q = u\widehat{s}_0v^{-1} = uq_0u^{-1}$. □

We can formulate the side-pairing and cycle relations using the combinatorics of W. To do this, we need a technical lemma on the centralizer of q_0.

Lemma 4.4.2. *The centralizer of $q_0 = \widehat{s}_0r_W$ in W is given by*

$$C_W(q_0) = C_W(\widehat{s}_0) = \langle s \in S \ ; \ s\widehat{s}_0 = \widehat{s}_0s \rangle .$$

In particular, this is (standard) parabolic.

Proof. First, we borrow an argument due to Sebastian Schoennenbeck[3] to prove the second equality above. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of an element $w \in C_W(\widehat{s}_0)$. To show that w is in the parabolic subgroup of the statement, since the elements of $C_W(\widehat{s}_0)$ of length 1 are the simple reflections of W commuting with $\widehat{s}_0$, by induction it is enough to show that $\widehat{s}_0s_{i_r} = s_{i_r}\widehat{s}_0$. We have $\ell(w\widehat{s}_0) = \ell(w) + 1$ and $\ell(w\widehat{s}_0w^{-1}) = \ell(\widehat{s}_0) = 1$, so $\ell(w\widehat{s}_0s_{i_r}) = \ell(w\widehat{s}_0w^{-1}ws_{i_r}) \leq 1 + \ell(ws_{i_r}) = \ell(w)$ and thus $\ell(w\widehat{s}_0s_{i_r}) = \ell(w)$. Thus, by the exchange condition, there is a reduced expression $w\widehat{s}_0 = s_{j_1} \cdots s_{j_r}s_{i_r}$ for $w\widehat{s}_0$ and since $s_{i_1} \cdots s_{i_r}\widehat{s}_0$ is already a reduced expression, by Matsumoto's lemma, there is a finite series of braid-moves from the second to the first. The expression $s_{i_1} \cdots s_{i_r}\widehat{s}_0$ satisfies the property

(∗) The expression contains only one occurrence of $\widehat{s}_0$ and there is no simple reflection appearing on the right of $\widehat{s}_0$ that does not commute with it.

Consider a braid relation $sts \cdots = tst \cdots$ connecting the two expressions of $w\widehat{s}_0$, with m factors on each side and suppose that we apply it to a reduced expression of $w\widehat{s}_0$ verifying (∗). If $s, t \neq \widehat{s}_0$, then the resulting expression still satisfies (∗). Now, if $s = \widehat{s}_0$ say, then t has to commute with $\widehat{s}_0$. Indeed, if not, then the left-hand side of the braid relation contains at least two occurrences of $\widehat{s}_0$ (one on each side of t) and, in the right-hand side there is at least one occurrence of t on the right of $\widehat{s}_0$, but none of these occur in the considered reduced expression. Therefore, the reduced expression resulting from the application of the braid move still verifies (∗). In particular, the expression $s_{j_1} \cdots s_{j_r}s_{i_r}$ satisfies (∗) and thus, every simple reflection appearing on the right of $\widehat{s}_0$ must commute with it. In particular, this is the case of s_{i_r}, as required.

We now prove that $C_W(q_0) = C_W(\widehat{s}_0)$. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of an element $w \in C_W(q_0)$. Since $wq_0 = q_0w$, we get $\widehat{s}_0w\widehat{s}_0 = r_Wwr_W \in W$. Let $\widehat{s}_0w\widehat{s}_0 = s_{j_1} \cdots s_{j_k}$ be a reduced expression in W. Since $\ell(w\widehat{s}_0) = \ell(w) + 1 = \ell(\widehat{s}_0w)$, we have $\ell(\widehat{s}_0w\widehat{s}_0) \in \{\ell(w), \ell(w) + 2\}$. But taking length in the equality $\widehat{s}_0s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_k}\widehat{s}_0$ gives $k = r$, that is $\ell(\widehat{s}_0w\widehat{s}_0) = \ell(w)$. In particular, $\ell(\widehat{s}_0w\widehat{s}_0) < \ell(w\widehat{s}_0)$ and by the exchange condition, there is a reduced expression $\widehat{s}_0w\widehat{s}_0 = s_{i_1} \cdots \widehat{s_{i_l}} \cdots s_{i_r}\widehat{s}_0$ (the reflection s_{i_l} is omitted) and since this last expression is in W, we must have $s_{i_l} = \widehat{s}_0$, thus $\widehat{s}_0w\widehat{s}_0 = s_{i_1} \cdots s_{i_r} = w$ and $w \in C_W(\widehat{s}_0)$. The reverse inclusion can be directly checked case by case using the parabolic description of $C_W(\widehat{s}_0)$. □

Remark 4.4.3. *From the diagrams of the hyperbolic extensions we get therefore*

$$C_{I_2(2g+1)}(q_0) = 1, \ C_{I_2(4g+2)}(q_0) = \langle s_2 \rangle , \ C_{I_2(4g)}(q_0) = \langle s_2 \rangle ,$$
$$C_{H_3}(q_0) = \langle s_2 \rangle , \ C_{H_4}(q_0) = \langle s_1, s_2, s_3 \rangle \simeq H_3.$$

[3] https://mathoverflow.net/questions/200433/centralizers-of-reflections-in-special-subgroups-of-coxeter-groups

Theorem 4.4.4. *Let W be non-crystallographic and $U := \{w \in W \ ; \ \ell(ws) > \ell(w), \ \forall s \in S \ ; \ s\widehat{s_0} = \widehat{s}_0 s\} \approx W/C_W(q_0)$ be the set of minimal length coset representatives modulo the parabolic subgroup $C_W(q_0)$ of W. The transitive action of W on $W/C_W(q_0)$ induces an action of W on U. Then the fundamental group $\pi_1(\mathbf{T}(W)) \simeq Q$ admits the following presentation*

$$\pi_1(\mathbf{T}(W)) = \langle q_u, \ u \in U \mid R_{side} \cup R_{cycle} \rangle \, ,$$

where

$$R_{side} = \{q_u q_v, \ u, v \in U \ ; \ u^{-1} v r_W \in C_W(\widehat{s}_0)\}$$

and

$$R_{cycle} = \Bigg\{ q_{w(u_1)} q_{w(u_2)} \cdots q_{w(u_r)}, \ w \in W, \ u_0, u_1, \ldots, u_r, u_{r+1} \in U \text{ such that } u_0 = u_{r+1} = 1$$
$$\text{and, for } i > 0, \ \left\langle \widehat{s}_0, \widehat{s}_0^{u_{i+1}^{-1} u_i r_W} \right\rangle \text{ and } \left\langle \widehat{s}_0, \widehat{s}_0^{r_W u_{i-1}^{-1} u_i} \right\rangle \text{ are conjugate under } C_W(\widehat{s}_0) \Bigg\} .$$

Proof. Drop the presentation notation and, for $u \in U$, denote $q_u := {}^u q_0 = u q_0 u^{-1}$, $\sigma_u := u(\sigma_0) = \Delta \cap q_u(\Delta)$ and $\sigma_v := q_u^{-1}(\sigma_u) = u q_0^{-1}(\sigma_0)$. To say that for some $u, v \in U$ we have $q_u q_v = 1$ amounts to say that ${}^u q_0 = {}^v q_0^{-1} = {}^{v r_W} q_0$, i.e. $u^{-1} v r_W \in C_W(q_0)$.

For the cycle relations, we follow the method detailed in [Rat06, §6.8]. First notice that each facet of σ_0 is of the form σ_u for some $u \in U$ (see [Rat06, Theorem 6.7.5]). Choose $\sigma \in F_{n-1}(\Delta)$ and $\tau \in F_{n-2}(\sigma) \subset F_{n-2}(\Delta)$. Recursively define a sequence of facets $\{\sigma_{u_j}\}_{j \in \mathbb{N}^*}$ as follows

- let $\sigma_{u_1} := \sigma$,
- let σ_{u_2} be the facet of σ adjacent to $\sigma'_{u_1} := q_{u_1}^{-1}(\sigma)$ such that $q_{u_1}(\sigma'_{u_1} \cap \sigma_{u_2}) = \tau$,
- for $i > 1$, let $\sigma_{u_{i+1}} \in F_{n-2}(\sigma)$ be the facet adjacent to $\sigma'_{u_i} := q_{u_i}^{-1}(\sigma_{u_i})$ such that $q_{u_i}(\sigma'_{u_i} \cap \sigma_{u_{i+1}}) = \sigma'_{u_{i-1}} \cap \sigma_{u_i}$.

By [Rat06, Theorem 6.8.7], there exists a least integer $k \in \mathbb{N}^*$ such that $\sigma_{u_{i+k}} = \sigma_{u_i}$ for all i and we have $q_{u_1} \cdots q_{u_k} = 1$. Moreover, by the Poincaré theorem [Rat06, Theorem 11.2.2], the set of all such relations (for $\sigma \in F_{n-1}(\Delta)$ and $\tau \in F_{n-2}(\sigma)$), together with the side-pairing relations described above, form a complete set of relations for Q.

Choose $\sigma \in F_{n-1}(\Delta) \subset W \cdot \sigma_0$ and $\tau \in F_{n-2}(\sigma) \subset F_{n-2}(\Delta)$ and let $\{\sigma_{u_j}\}_{j \in \mathbb{N}^*}$ denote the associated cycle of sides, with period ℓ, say. We have the relation $q_{u_1} \cdots q_{u_\ell} = 1$. Up to conjugation by an element of W, we may assume that $\sigma = \sigma_0$ and so $q_{u_1} = q_0$. Let $i > 1$ be such that we have some relation

$$q_{u_i}(\sigma'_{u_i} \cap \sigma_{u_{i+1}}) = \sigma'_{u_{i-1}} \cap \sigma_{u_i} \neq \emptyset.$$

We write

$$\begin{aligned}
q_{u_i}(\sigma'_{u_i} \cap \sigma_{u_{i+1}}) = \sigma_{u_i} \cap \sigma'_{u_{i-1}} &\iff \sigma_{u_i} \cap q_{u_i}(\sigma_{u_{i+1}}) = \sigma_{u_i} \cap q_{u_{i-1}}^{-1}(\sigma_{u_{i-1}}) \\
&\iff u_i(\sigma_0) \cap q_{u_i} u_{i+1}(\sigma_0) = u_i(\sigma_0) \cap q_{u_{i-1}}^{-1} u_{i-1}(\sigma_0) \\
&\iff u_i(\sigma_0 \cap u_i^{-1} q_{u_i} u_{i+1}(\sigma_0)) = u_i(\sigma_0 \cap u_i^{-1} u_{i-1} q_0^{-1}(\sigma_0)) \\
&\iff \sigma_0 \cap q_0 u_i^{-1} u_{i+1}(\sigma_0) = \sigma_0 \cap u_i^{-1} u_{i-1} r_W(\sigma_0),
\end{aligned}$$

and the two sides of the last equality are simplices of the tessellation $\Delta(\widehat{W}, \widehat{S})$, whose face lattice is the lattice of standard parabolic subgroups of $\widehat{W}$. Hence these two coincides if and only if their stabilizers in $\widehat{W}$ are equal. Though this condition depends on the choice of the elements of U, it is straightforward to check that different choices give conjugate stabilizers in $C_W(q_0)$. □

Corollary 4.4.5. *The group $\pi_1(\mathbf{T}(H_3))$ (resp. $\mathbf{T}(H_4)$) admits a presentation with 11 (resp. 24) generators , all of whose relations are products of commutators. In particular, we have*

$$H_1(\mathbf{T}(H_3), \mathbb{Z}) = \pi_1(\mathbf{T}(H_3))^{\mathrm{ab}} \simeq \mathbb{Z}^{11} \quad \text{and} \quad H_1(\mathbf{T}(H_4), \mathbb{Z}) = \pi_1(\mathbf{T}(H_4))^{\mathrm{ab}} \simeq \mathbb{Z}^{24}.$$

Proof. We apply the above result. For H_3, beside the side-pairing relations (which we can immediately simplify by removing half of the $[H_3 : C_{H_3}(q_0)] = [H_3 : \langle s_2 \rangle] = 60$ generators), we find only one primitive cycle relation (*primitive* meaning starting by q_0) of length 3 and one of length 5. Taking the H_3-conjugates of these gives 120 relations of length 3 and 120 relations of length 5. But the inverse of each of these relations appears so we can simplify them. We can also remove any cyclic permutation of these relations, which finally yields a presentation for $\pi_1(\mathbf{T}(H_3))$ with 30 generators, 20 relations of length 3 and 12 relations of length 5.

We do the same for H_4, where there is only one primitive cycle relation of length 5, which gives a presentation for $\pi_1(\mathbf{T}(H_4))$ with $\frac{1}{2}[H_4 : C_{H_3}(q_0)] = 60$ generators and 144 relations of length 5.

Using the relations, we can check that some of the generators are superfluous and that the simplified presentation has the stated number of generators (all among the original generators) and that the relations become trivial, once abelianized. □

Remark 4.4.6. *The intermediate presentations of $\pi_1(\mathbf{T}(H_3))$ and $\pi_1(\mathbf{T}(H_4))$ (with 30 generators and 32 relations for H_3 and 60 generators and 144 relations for H_4) are precisely (up to relabelling) the presentations given in* [Zim93] *and* [RT01].

4.5. **The manifolds $\mathbf{T}(I_2(m))$ as Riemann surfaces.**

A little bit more can be said about the case of the surfaces $\mathbf{T}(I_2(m))$. Recall that by a theorem of Gauss (see [Jos02, Theorem 3.11.1]), any Riemannian metric on an oriented 2-manifold M induces a complex structure on M (making M a Riemann surface), called the *conformal structure* induced by the metric.

Corollary 4.5.1. *For any $g \in \mathbb{N}^*$ the surfaces $\mathbf{T}(I_2(2g+1))$, $\mathbf{T}(I_2(4g))$ or $\mathbf{T}(I_2(4g+2))$ are closed compact Riemann surfaces of genus g. In particular, we have homeomorphisms*

$$\mathbf{T}(I_2(2g+1)) \simeq \mathbf{T}(I_2(4g)) \simeq \mathbf{T}(I_2(4g+2)).$$

Proof. Since the surfaces are orientable, the Riemannian metric induced by the one on the Coxeter complex induces a conformal structure on them. To obtain the genus, we only have to compute the Euler characteristic.

Let m be either $2g+1$, $4g$ or $4g+2$ and

$$W := I_2(m) = \left\langle s, t \mid s^2 = t^2 = (st)^m = 1 \right\rangle.$$

We will detail the combinatorics of the W-triangulation $\Delta(\widehat{W}, \widehat{S})/Q$ in the next section, however we only have to compute the Euler characteristic χ and very little information is needed. The rational chain complex associated to the simplicial complex $\Delta(\widehat{W}, \widehat{S})$ has the following shape:

$$\mathbb{Q}[\widehat{W}] \longrightarrow \mathbb{Q}[\widehat{W}/\langle s \rangle] \oplus \mathbb{Q}[\widehat{W}/\langle t \rangle] \oplus \mathbb{Q}[\widehat{W}/\langle \widehat{s}_0 \rangle] \longrightarrow \mathbb{Q}[\widehat{W}/\langle s,t \rangle] \oplus \mathbb{Q}[\widehat{W}/\langle s, \widehat{s}_0 \rangle] \oplus \mathbb{Q}[\widehat{W}/\langle t, \widehat{s}_0 \rangle]$$

Now, by Lemma 1.0.1, the complex for the surface $\mathbf{T}(I_2(m))$ is the image of the previous one by the deflation functor $\mathrm{Def}_W^{\widehat{W}}$. Thus, it is of the form

$$\mathbb{Q}[W] \longrightarrow \mathbb{Q}[W/\langle s \rangle] \oplus \mathbb{Q}[W/\langle t \rangle] \oplus \mathbb{Q}[W/\langle r \rangle] \longrightarrow \mathbb{Q} \oplus \mathbb{Q}[W/\langle s, r \rangle] \oplus \mathbb{Q}[W/\langle t, r \rangle],$$

where $r = r_W := (st)^{\lfloor (m-1)/2 \rfloor} s \in W$. Therefore the Euler characteristic is given by

$$\chi(\mathbf{T}_g) = 1 + [W : \langle s, r \rangle] + [W : \langle t, r \rangle] - [W : \langle s \rangle] - [W : \langle t \rangle] - [W : \langle r \rangle] + |W|$$
$$= 1 + [W : \langle s, r \rangle] + [W : \langle t, r \rangle] - 3[W : \langle s \rangle] + 2m = 1 - m + [W : \langle s, r \rangle] + [W : \langle t, r \rangle].$$

It is routine to compute that

$$[I_2(m) : \langle s, r \rangle] = \begin{cases} 2 & \text{if } m = 4g+2, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad [I_2(m) : \langle t, r \rangle] = \begin{cases} 1 & \text{if } m \text{ is odd}, \\ m/2 & \text{otherwise} \end{cases}$$

thus, the Euler characteristic is given by

$$\chi(\mathbf{T}(I_2(m))) = \begin{cases} 3-m & \text{if} \quad m = 2g+1, \\ 3-m/2 & \text{if} \quad m = 4g+2, \\ 2-m/2 & \text{if} \quad m = 4g, \end{cases}$$

in other words,

$$\chi(\mathbf{T}(I_2(m))) = 2-2g$$

and the genus of $\mathbf{T}(I_2(m))$ is indeed g for $m \in \{2g+1, 4g+2, 4g\}$. □

As the fundamental group of a Riemann surface of genus $g \geq 1$ is well-known (see [Hat02, §1.2]), we obtain a presentation for the group Q in the dihedral case.

Corollary 4.5.2. *Let $g \in \mathbb{N}^*$ and m be either $2g+1$, $4g$ or $4g+2$. Let also Q be the subgroup of $\widehat{I_2(m)}$ constructed in the previous section (see Definition 4.1.6). Then we have*

$$Q \simeq \pi_1(\mathbf{T}(I_2(m))) \simeq \langle x_1, \dots, x_g, y_1, \dots, y_g \mid [x_1, y_1] \cdots [x_g, y_g] = 1 \rangle$$

and in particular, $Q^{\mathrm{ab}} \simeq \mathbb{Z}^{2g}$.

In the cases where $g = 1$ that is, if $I_2(m)$ is one of the Weyl groups $I_2(3) = A_2$, $I_2(4) = B_2$ or $I_2(6) = G_2$, then $\mathbf{T}(I_2(m))$ is naturally an *elliptic curve*. More precisely, recalling the notation of the previous section, we have a preferred point

$$v_0 := \overline{C} \cap \mathcal{H} \cap \bigcap_{s \neq \widehat{s}_0} H_s \in \widehat{\Sigma},$$

and the pair $(\mathbf{T}(I_2(m)), [v_0])$ is a Riemann surface of genus 1 with a marked point, hence an elliptic curve. Notice that, under the diffeomorphism $\mathbf{T}(I_2(m)) \simeq \mathbb{R}^2/\mathbb{Z}^2$ induced by quotienting the 3-space $\widehat{V}$ by the radical of the Tits form $\widehat{B}_m$ of $\widehat{I_2(m)} = I_2(m)_{\mathrm{a}}$, the point $[v_0]$ corresponds to the origin.

We can easily identify the elliptic curves $\mathbf{T}(I_2(m))$ (for $m = 3, 4, 6$) in the moduli space $\mathcal{M}_{1,1} \simeq \mathbb{H}^2/PSL_2(\mathbb{Z})$ of complex elliptic curves, where $\mathbb{H}^2 = \{z \in \mathbb{C} \ ; \ \Im(z) > 0\}$ is the Poincaré half plane (see [Hai14, §2]). Recall that to $\tau \in \mathbb{H}^2$ we can associated a $j(\tau) \in \mathbb{C}$ and we have isomorphisms

$$\begin{array}{ccccc} \mathbb{C} & \xleftarrow{\sim} & \mathbb{H}^2/PSL_2(\mathbb{Z}) & \xrightarrow{\sim} & \mathcal{M}_{1,1} \\ j(\tau) & \longleftarrow & \tau & \longmapsto & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \end{array}$$

Recall also from [Ser70, Chapitre VII, §1.2] that $D := \{z \in \mathbb{H}^2 \ ; \ |\Re(z)| \leq 1/2, \ |z| \geq 1\}$ is a fundamental domain for $PSL_2(\mathbb{Z})$ acting on $\mathbb{H}^2$. We just have to determine a corresponding element $\tau \in D$ for each case. We have the following proposition:

Proposition 4.5.3. *Let $m \in \{3, 4, 6\}$ and let $\{\alpha^\vee, \beta^\vee\}$ be the simple coroots of the root system of type $I_2(m)$ and $V^* := \mathbb{R}\langle \alpha^\vee, \beta^\vee \rangle$. We normalize the roots in such a way that the short simple roots have norm 2. We denote by $\phi : V^* \to \mathbb{C}$ the unique isometry sending $\alpha^\vee$ to 1 and $\beta^\vee$ to an element of the upper-half plane $\mathbb{H}^2$. Then we have*

$$\phi(\beta^\vee) = \begin{cases} \exp\left(\frac{2i\pi}{3}\right) & \text{if} \quad m = 3, \\ \sqrt{2}\exp\left(\frac{3i\pi}{4}\right) & \text{if} \quad m = 4, \\ \sqrt{3}\exp\left(\frac{5i\pi}{6}\right) & \text{if} \quad m = 6. \end{cases}$$

In particular, for A_2 and G_2, the corresponding lattice is in the $PSL_2(\mathbb{Z})$-orbit of $\mathbb{Z} \oplus \tau\mathbb{Z}$ where $\tau = e^{\frac{2i\pi}{3}} \in D$, so $j(\tau) = 0$ and for B_2, we find $\tau = i$ and $j(\tau) = 1728$. Hence, the curves $\mathbf{T}(A_2)$, $\mathbf{T}(B_2)$ and $\mathbf{T}(G_2)$ are defined over $\mathbb{Q}$ and correspond to the orbifold points *of D, that is, the points in D having a non-trivial stabilizer in $PSL_2(\mathbb{R})$.*

Proof. For $I_2(3) = A_2$, we have $|\beta| = |\alpha| = 2$ and $\langle \alpha^\vee, \beta\rangle = \langle \beta^\vee, \alpha\rangle = -1$. Therefore, since ϕ is an isometry we should have

$$-\frac{1}{2} = \frac{1}{2}\langle \alpha^\vee, \beta\rangle = \langle \alpha^\vee, \beta^\vee\rangle = \langle \phi(\alpha^\vee), \phi(\beta^\vee)\rangle = \langle 1, \phi(\beta^\vee)\rangle \overset{\text{df}}{=} \Re(\overline{\phi(\beta^\vee)}) = \Re(\phi(\beta^\vee)).$$

On the other hand, we have $1 = \langle \beta^\vee, \beta^\vee\rangle = |\phi(\beta^\vee)|^2$ and this implies $\phi(\beta^\vee) \in \{-1/2 \pm i\sqrt{3}/2\}$ and if we impose that $\phi(\beta^\vee) \in \mathbf{H}^2$, then it should have a positive imaginary part and the only possibility is $\phi(\beta^\vee) = -1/2 + i\sqrt{3}/2 = \exp\left(\frac{2i\pi}{3}\right)$. The other cases are similar. □

Remark 4.5.4. *In Weierstrass forms, an equation for $\mathbf{T}(A_2)$ and $\mathbf{T}(G_2)$ is $y^2 = x^3 - 1$ and for $\mathbf{T}(B_2)$, we can take $y^2 = x^3 - x$. This is an unusual point of view on 2-dimensional tori. Indeed, they are first defined as Lie groups, hence as differentiable manifolds diffeomorphic to $\mathbb{S}^2$ and it turns out that they carry a natural rational elliptic curve structure. Moreover, they can be distinguished among complex elliptic curves by the fact that they correspond to the orbifold points of the Dirichlet domain.*

We now focus on the hyperbolic case where $g > 1$. We first notice the following coincidence between the Riemann surface $\mathbf{T}(I_2(m))$.

Proposition 4.5.5. *If $g > 1$, then we have an isometry (in particular, an isomorphism of Riemann surfaces)*

$$\mathbf{T}(I_2(4g+2)) \simeq \mathbf{T}(I_2(2g+1))$$

and these two are not isometric to the surface $\mathbf{T}(I_2(4g))$.

Proof. Using [Rat06, Theorem 8.1.5], it suffices to show that the groups Q_{2g+1} and Q_{4g+2} are conjugate in the positive Lorentz group $PO(1,2) \simeq \mathrm{Isom}(\mathbf{H}^2) \simeq PSL_2(\mathbb{R})$ and are not conjugate to Q_{4g}.

Let $m := 2g+1$. We first prove that Q_m and Q_{2m} are conjugate in $PO(1,2)$. Denote $I_2(2m) = \langle s, t \mid s^2 = t^2 = (st)^{2m} = 1\rangle$ and $\widehat{I_2(2m)} = \langle s, t, \widehat{s}_0\rangle$ its hyperbolic extension. Let $s' := s$, $t' := tst = s^t$ and $\widehat{s}'_0 := \widehat{s}_0$. Then $\langle s', t', \widehat{s}'_0\rangle = \widehat{I_2(m)}$ and $\langle s', t'\rangle = I_2(m)$. Recall moreover that we have the reflection $r_{2m} = (st)^{2g}s = ((st)^2)^g s = (s't')^g s' = r_m$. Let α, β and γ denote the simple roots of $\widehat{I_2(2m)}$ and $V_{2m} := \mathrm{span}_{\mathbb{R}}(\alpha, \beta, \gamma)$. We have the representation

$$\widehat{I_2(2m)} \overset{\sigma_{2m}}{\hookrightarrow} O(V_{2m}, B_{2m}),$$

where

$$B_{2m} = \begin{pmatrix} 1 & -\cos(\pi/2m) & -\cos(\pi/m) \\ -\cos(\pi/2m) & 1 & 0 \\ -\cos(\pi/m) & 0 & 1 \end{pmatrix}.$$

In the same way, denote $V_m := \mathrm{span}_{\mathbb{R}}(\alpha', \beta', \gamma')$ and $\sigma_m : \widehat{I_2(m)} \hookrightarrow O(V_m, B_m)$, where

$$B_m = \begin{pmatrix} 1 & -\cos(\pi/m) & -\cos(\pi/m) \\ -\cos(\pi/m) & 1 & -\cos(\pi/m) \\ -\cos(\pi/m) & \cos(\pi/m) & 1 \end{pmatrix}.$$

Consider the linear map $P : V_{2m} \to V_m$ with matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2\cos(\pi/2m) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have $B_m = {}^tPB_{2m}P$, so P induces an isomorphism $O(V_{2m}, B_{2m}) \to O(V_m, B_m)$ fitting in a commutative diagram

$$\begin{array}{ccccc} \widehat{I_2(m)} & \overset{\sigma_m}{\hookrightarrow} & O(V_m, B_m) & \overset{\sim}{\longrightarrow} & PO(1,2) \\ \downarrow & & \uparrow \sim & & \uparrow \sim \\ \widehat{I_2(2m)} & \underset{\sigma_{2m}}{\hookrightarrow} & O(V_{2m}, B_{2m}) & \overset{\sim}{\longrightarrow} & PO(1,2) \end{array}$$

and thus the group $\sigma_m(\widehat{I_2(m)})$ is conjugate in $PO(1,2)$ to a subgroup of $\sigma_{2m}(\widehat{I_2(2m)})$. Therefore, identifying $\widehat{I_2(m)}$ with its image in $\widehat{I_2(2m)}$, it suffices to prove that $Q_{2m} = Q_m$.

Recall that $q_{2m} \overset{\text{df}}{=} \widehat{s}_0 r_{2m} = \widehat{s}_0 r_m = q_m$, so $q_m^{\widehat{I_2(m)}} \subset q_{2m}^{\widehat{I_2(2m)}}$ and thus $Q_m \leq Q_{2m}$. Since we have

$$\begin{aligned} 2m[\widehat{I_2(2m)} : \widehat{I_2(m)}] &= [\widehat{I_2(2m)} : \widehat{I_2(m)}][\widehat{I_2(m)} : Q_m] = [\widehat{I_2(2m)} : Q_m] \\ &= [\widehat{I_2(2m)} : Q_{2m}][Q_{2m} : Q_m] = 4m[Q_{2m} : Q_m], \end{aligned}$$

we are left to show that $[\widehat{I_2(2m)} : \widehat{I_2(m)}] = 2$. Let $w \in \widehat{I_2(2m)}$. By induction on $\ell(w)$ and because t and $\widehat{s}_0$ commute, we immediately see that $w \in \widehat{I_2(m)}$ if and only if the number of occurrences of t in any reduced expression of w is even. Hence we have $[\widehat{I_2(2m)} : \widehat{I_2(m)}] \leq 2$ and since st and $\widehat{s}_0 t$ have even order, the map $\widehat{I_2(2m)} \to \mathbb{Z}/2\mathbb{Z}$ sending s and $\widehat{s}_0$ to 0 and t to 1 is a homomorphism whose kernel contains $\widehat{I_2(m)}$, hence the result.

We now prove that Q_{2g+1} and Q_{4g} are not conjugate in $PO(1,2)$. It is enough to prove that the elements $\sigma_{2g+1}(q_{2g+1}) \in PO(1,2)$ and $\sigma_{4g}(q_{4g})$ have different traces. Write $\widehat{I_2(2g+1)} = \langle s, t, \widehat{s}_0 \rangle$ and $\widehat{I_2(4g)} = \langle s', t', \widehat{s}'_0 \rangle$. We have $q_{2g+1} = \widehat{s}_0 (st)^g s$ and $q_{4g} = \widehat{s}'_0 (st)^{2g} s'$ and we can write explicitly the matrices of the simple reflections in the geometric representation. We diagonalize $st = PdP^{-1}$ and compute $\operatorname{tr}(q_{2g+1}) = \operatorname{tr}(d^g P^{-1} s \widehat{s}_0)$. After calculations, we find

$$\operatorname{tr}(q_{2g+1}) = 8\left(1 + \cos\left(\frac{\pi}{2g+1}\right)\right)\cot^2\left(\frac{\pi}{2g+1}\right) - 1.$$

Doing the same for g_{4g}, we find

$$\operatorname{tr}(q_{4g}) = 4\cot^2\left(\frac{\pi}{4g}\right) - 1.$$

And indeed, we get $\operatorname{tr}(q_{2g+1}) \neq \operatorname{tr}(q_{4g})$ for $g > 1$. □

Recall that a *Belyi function* on a Riemann surface X is a holomorphic map $\beta : X \to \widehat{\mathbb{C}}$ which is ramified only over three points of $\widehat{\mathbb{C}}$. Since $\widehat{I_2(m)}$ is a compact triangle group and $Q_m \trianglelefteq \widehat{I_2(m)}$ is torsion-free and of finite index. Thus, by [JW16, Theorem 3.10], the projection

$$\beta : \mathbf{T}(I_2(m)) = \mathbb{H}^2/Q_m \longrightarrow\!\!\!\!\!\rightarrow \mathbb{H}^2/\widehat{I_2(m)} \simeq \widehat{\mathbb{C}}$$

is a Belyi function on $\mathbf{T}(I_2(m))$ of degree $[\widehat{I_2(m)} : Q_m] = 2m$. Using [JW16, Theorem 1.3], this implies the following result:

Proposition 4.5.6. *For any $m \geq 3$, the Riemann surface $\mathbf{T}(I_2(m))$ may be defined over a number field (or equivalently, may be defined over $\overline{\mathbb{Q}}$). Moreover, if $m = 5$ or $m \geq 7$, then the 1-skeleton of the tessellation $\Delta(\widehat{I_2(m)})/Q_m$ defines a dessin d'enfant on $\mathbf{T}(I_2(m))$.*

Remark 4.5.7. *It is a reasonable to expect that $\mathbf{T}(I_2(m))$ is definable over $\mathbb{Q}(\cos(2\pi/m))$. This is coherent with the isomorphism $\mathbf{T}(2g+1) \simeq \mathbf{T}(4g+2)$ and with the vertices of the tessellation of $\mathbb{H}^2$, whose coordinates may be chosen in this field. However, we haven't found a proof of this yet.*

Example 4.5.8. *The triangulation $\Delta(\widehat{I_2(5)})$ is the classical tessellation $\{3, 10\}$ of the Poincaré disk. More precisely, the Tits form $\widehat{B}$ is given by*

$$\widehat{B} = \begin{pmatrix} 1 & -c & -c \\ -c & 1 & -c \\ -c & -c & 1 \end{pmatrix} \quad \text{with} \quad c = \cos(\pi/5)$$

and, if $v^* \in V^*$ is a normalized eigenvector for the unique negative eigenvalue of $\widehat{B}$, then we have an identification with the hyperbolic plane

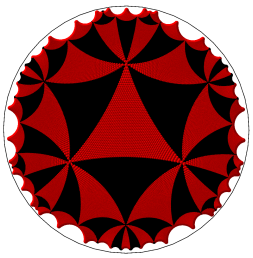
$$\mathcal{H} := \{\lambda \in V^* \ ; \ \widehat{B}(\lambda, \lambda) = -1, \ \widehat{B}(v^*, \lambda) < 0\} \simeq \mathbb{H}^2$$

and the stereographic projection on the hyperplane $\widehat{B}(v^*, -) = 0$ with pole λ_0 gives the Poincaré disk model for $\mathbb{H}^2$. Under this representation we represent the tessellation $\Delta(\widehat{I_2(5)}) =$

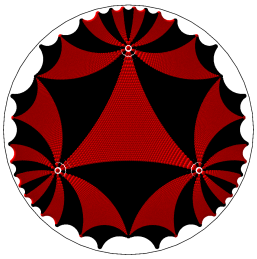
$\{3, 10\}$ *of* $\mathcal{H}$ *as in Figure 4a, where the black triangles are the images of the fundamental triangle* $\overline{C}/\mathbb{R}_+^* \simeq \overline{C} \cap \mathcal{H}$ *under elements of odd length. In this tessellation, we can identify the triangles that are in the* Q*-orbit of* $\overline{C} \cap \mathcal{H}$*. These are displayed in green in Figure 5. Collapsing these triangles in one gives the surface* $\mathbf{T}(I_2(5))$*.*

We remark that we can extract a fundamental domain for Q *on* $\mathbf{T}(I_2(5))$ *as the projection of the domain displayed in Figure 6a. Rearranging the figure we obtain the triangulation displayed in the Figure 6b, where the points with the same name (resp. the edges with the same color) are identified. We notice that the resulting space is indeed a closed surface of genus 2.*

The case of $I_2(m)$ *for* m *odd is pretty similar and we obtain the* $\{3, 2m\}$*-tessellation of the Poincaré disk. For instance, the Figure 4b shows the case of* $I_2(7)$*.*



(A) The $\{3, 10\}$-tessellation of $\Sigma(\widehat{I_2(5)}) \simeq \mathbb{H}^2$.



(B) The $\{3, 14\}$-tessellation of $\Sigma(\widehat{I_2(7)}) \simeq \mathbb{H}^2$.

FIGURE 4. Two regular tessellations of the Poincaré disk.

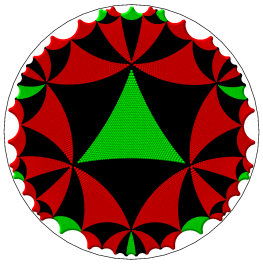
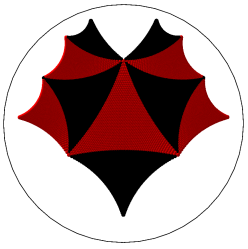
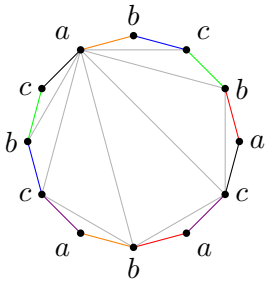


FIGURE 5. The green triangles form the Q-orbit of the fundamental triangle $\overline{C} \cap \mathcal{H}$ inside the Poincaré disk.



(A) Fundamental domain for Q in the Poincaré disk.



(B) Fundamental domain for $I_2(5)$ in $\mathbf{T}(I_2(5)) = \mathbf{T}_2$.

FIGURE 6. Fundamental domain for Q and its image in $\mathbf{T}(I_2(5))$.

5. Equivariant chain complex of $\mathbf{T}(W)$ and computation of homology

5.1. The W-dg-algebra of $\mathbf{T}(W)$.

The combinatorics of the complex $C^*_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ is fairly similar to the one of the complex $C^*_{\mathrm{cell}}(T, W; \mathbb{Z})$ we constructed in the first part and the proofs given above can be applied *verbatim* to this new situation. We obtain the following results:

Theorem 5.1.1. *The $\mathbb{Z}[W]$-dg-algebra $C^*_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z})$ associated to the W-triangulation $\Delta(\widehat{W}, \widehat{S})/Q$ of $\mathbf{T}(W)$ has homogeneous components*

$$C^k_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z}) = \bigoplus_{\substack{I \subset \widehat{S} \\ |I| = n-k}} \mathbb{Z}[\pi({}^I\widehat{W})] \simeq \bigoplus_{\substack{I \subset \widehat{S} \\ |I| = n-k}} \mathbb{Z}[\pi(\widehat{W}_I)\backslash \widehat{W}],$$

differentials given, for any $I \subset \widehat{S}$ and $w \in \widehat{W}$, by

$$d^k(\pi({}^I w)) = \sum_{\substack{0 \leq u \leq k+1 \\ j_{u-1} < j < j_u}} (-1)^u \pi\left(\epsilon_I^{I \setminus \{j\}} w\right), \ \epsilon_I^J = \sum_{x \in {}^J_I\widehat{W}} x,$$

where $\{j_0 < \cdots < j_k\} := \widehat{S} \setminus I$. Its product

$$C^p_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z}) \otimes_{\mathbb{Z}} C^q_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z}) \xrightarrow{\cup} C^{p+q}_{\mathrm{cell}}(\mathbf{T}(W), W; \mathbb{Z})$$

is induced by the deflation from $\widehat{W}$ to W of the unique map

$$\mathbb{Z}[[{}^I\widehat{W}]] \otimes_{\mathbb{Z}} \mathbb{Z}[[{}^J\widehat{W}]] \longrightarrow \mathbb{Z}[[{}^{I \cap J}\widehat{W}]]$$

satisfying the formula

$$\forall x, y \in \widehat{W}, \ {}^I x \cup {}^J y = \delta_{\max(I^\complement), \min(J^\complement)} \times \begin{cases} {}^{I \cap J}((xy^{-1})_J y) & \text{if } xy^{-1} \in \widehat{W}_I \widehat{W}_J \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.1.2. *As explained in* [BR04, §2.3]*, a quotient simplicial complex of the form $\Delta(\widehat{W}, \widehat{S})/H$ (with $H \leq \widehat{W}$) has a an interpretation in terms of double cosets. In our case, we have an isomorphism of posets*

$$\begin{array}{ccc} \left(\mathcal{P}(\Delta(\widehat{W}, \widehat{S})/Q), \subseteq\right) & \xrightarrow{\sim} & \left(\{(I, Qw\widehat{W}_I)\}_{I \subsetneq \widehat{S}, \ w \in \widehat{W}}, \preceq\right) \\ \pi(\widehat{W}^I) \ni \pi(w^I) & \longmapsto & (I, Qw\widehat{W}_I) \end{array}$$

where the order $\preceq$ on the second factor is defined by

$$(I, Qw\widehat{W}_I) \preceq (J, Qw'\widehat{W}_J) \overset{df}{\iff} \begin{cases} I \supseteq J \\ Qw\widehat{W}_I \supseteq Qw'\widehat{W}_J \end{cases}$$

and we may rephrase the above results using this poset.

5.2. The homology W-representation of $\mathbf{T}(W)$.

We can now determine the action of W on $H_*(\mathbf{T}(W), \mathbb{Z})$. Recall from [GP00, Theorem 5.3.8] that a splitting field for W is given by

$$\mathbb{Q}(W) = \mathbb{Q}(\cos(2\pi/m_{s,t}), \ s, t \in S) = \mathbb{Q}(\chi_\sigma(w), \ w \in W) \subset \mathbb{R},$$

where $\chi_\sigma = \mathrm{tr}(\sigma)$ is the character of the geometric representation $\sigma : W \to GL(V)$ of W. If W is a Weyl group, then $\mathbb{Q}(W) = \mathbb{Q}$ and we have

$$\mathbb{Q}(I_2(m)) = \mathbb{Q}(\cos(2\pi/m)) \ \text{ and } \ \mathbb{Q}(H_3) = \mathbb{Q}(H_4) = \mathbb{Q}(\sqrt{5}).$$

We suppose from now on that W is one of the groups H_3, H_4 or $I_2(m)$, with $m \geq 3$ and we keep the notation of the previous section. The first groups to be determined are the top and bottom homology of $\mathbf{T}(W)$. Recall that we have $n = \mathrm{rk}(W) = \dim \mathbf{T}(W)$.

Proposition 5.2.1. *Let $\mathbb{1}$ and ε be the trivial and signature modules over $\mathbb{Z}[W]$, respectively. We have isomorphisms of $\mathbb{Z}[W]$-modules*

$$\begin{cases} H_0(\mathbf{T}(W),\mathbb{Z}) \simeq \mathbb{1}, \\ H_n(\mathbf{T}(W),\mathbb{Z}) \simeq \varepsilon. \end{cases}$$

Proof. Since $\widehat{\Sigma}$ is path-connected, its quotient $\mathbf{T}(W)$ is path-connected too and is orientable by Theorem 4.3.3. Thus, we have an isomorphism of abelian groups

$$H_0(\mathbf{T}(W),\mathbb{Z}) \simeq \mathbb{Z} \simeq H_n(\mathbf{T}(W),\mathbb{Z}).$$

It is clear that $H_0(\mathbf{T}(W),\mathbb{Z})$ is the trivial module and, as $\mathbb{Z}[W]$-modules we have $H_n(\mathbf{T}(W),\mathbb{Z}) = \ker(\partial_n)$ with

$$\begin{array}{rccc} \partial_n & : & \mathbb{Z}[W] & \longrightarrow & \bigoplus_{i=0}^n \mathbb{Z}[W/\langle s_i\rangle] \\ & & w & \longmapsto & \sum_{i=0}^n (-1)^i w\langle s_i\rangle \end{array}$$

where $s_i = \pi(\widehat{s}_i)$ is a simple reflection of W for $i \geq 1$ and $s_0 := r_W = \pi(\widehat{s}_0)$. Define $e := \sum_w \varepsilon(w)w \in \mathbb{Z}[W]$ with $\varepsilon(w) = (-1)^{\ell(w)}$ and notice that $we = \varepsilon(w)e$ for $w \in W$ and $\partial_n(e) = 0$. Let $x = \sum_w x_w w \in \mathbb{Z}[W]$ such that $\partial_n(x) = 0$. Then, for all $0 \leq i \leq n$, we have $\sum_w x_w w\langle s_i\rangle = 0$. Fixing $1 \leq i \leq n$, we can choose a set $\{w_1,\dots,w_k\}$ of representatives of the left coset $W/\langle s_i\rangle$ (the minimal length representatives for instance). We have

$$0 = \sum_{w\in W} x_w w\langle s_i\rangle = \sum_{j=1}^k x_{w_j} w_j\langle s_i\rangle + \sum_{j=1}^k x_{w_j s_i} w_j s_i\langle s_i\rangle = \sum_j (x_{w_j} + x_{w_j s_i}) w_j\langle s_i\rangle,$$

hence $x_{w_j} + x_{w_j s_i} = 0$ for all $1 \leq j \leq k$. This implies $x_w + x_{ws_i} = 0$ for all $w \in W$ and doing this for every $i \geq 1$ gives $x_w + x_{ws} = 0$ for all $w \in W$ and $s \in S$, in other words, $x_w = \varepsilon(w)x_1$ for $w \in W$ and $x = x_1 e \in \mathbb{Z}e$. □

Proposition 5.2.2. *The homology $H_*(\mathbf{T}(W),\mathbb{Z})$ is torsion-free and the Poincaré duality on $\mathbf{T}(W)$ induces isomorphisms of $\mathbb{Z}[W]$-modules*

$$H_{n-i}(\mathbf{T}(W),\mathbb{Z})^\vee \simeq H_i(\mathbf{T}(W),\mathbb{Z})^\vee \otimes_{\mathbb{Z}} \varepsilon.$$

Proof. It suffices to prove that $H_*(\mathbf{T}(W),\mathbb{Z})$ is torsion-free, the Poincaré pairing $H^i(\mathbf{T}(W),\mathbb{Z}) \otimes H_{n-i}(\mathbf{T}(W),\mathbb{Z}) \to H_n(\mathbf{T}(W),\mathbb{Z}) = \varepsilon$ and the universal coefficient theorem implying the second one. But since the dimension is at most 4, we only have to prove that $H_1(\mathbf{T}(W),\mathbb{Z})$ is torsion-free and this holds by Corollaries 4.4.5 and 4.5.2. □

The above Lemma, combined with the *Hopf trace formula* (see [Spa81, Chap. 4, §7, Theorem 6] or [Lin, Lemma 2.4]) provides enough information to determine the homology representation of $\mathbf{T}(W)$. More precisely, letting G be a (discrete) group, $H \leq G$ is a subgroup and if M is an H-module, we denote by $M{\uparrow}_H^G$ the *induced module* of M; it is a G-module. Similarly, the *restricted module* of a G-module N is denoted $N{\downarrow}_H^G$. Observe that we have a canonical isomorphism of $\mathbb{Q}[G]$-modules $\mathbb{Q}[G/H] \simeq \mathbb{1}{\uparrow}_H^G$. Recall also that if $N \trianglelefteq G$ and if M is a G-module, then its *deflation* $\mathrm{Def}_{G/N}^G(M)$ is a G/N-module.

In our context, we have isomorphisms of $\mathbb{Q}[\widehat{W}]$-modules

$$C_k^{\mathrm{cell}}(\widehat{\Sigma},\widehat{W};\mathbb{Q}) = \bigoplus_{I\subset\widehat{S}\ ;\ |I|=n-k} \mathbb{1}{\uparrow}_{\widehat{W}_I}^{\widehat{W}}.$$

Thus

$$C_k^{\mathrm{cell}}(\mathbf{T}(W),W;\mathbb{Q}) = \mathrm{Def}_W^{\widehat{W}}(C_k^{\mathrm{cell}}(\widehat{\Sigma},\widehat{W};\mathbb{Q})) = \bigoplus_{I\subset\widehat{S}\ ;\ |I|=n-k} \mathbb{1}{\uparrow}_{\pi(\widehat{W}_I)}^{W},$$

and applying Hopf's formula yields the following result:

Lemma 5.2.3. *We have the following equality of virtual rational characters of W*

$$\sum_{I\subsetneq\widehat{S}} (-1)^{|I|}\mathrm{Def}_W^{\widehat{W}}\left(\mathbb{1}{\uparrow}_{\widehat{W}_I}^{\widehat{W}}\right) = \sum_{I\subsetneq\widehat{S}} (-1)^{|I|}\mathbb{1}{\uparrow}_{\pi(\widehat{W}_I)}^{W} = (-1)^n \sum_{i=0}^n (-1)^i H_i(\mathbf{T}(W),\mathbb{Q}).$$

For notation simplicity, we shall use the conventions of [GP00] to denote the irreducible characters of W. We start with $I_2(m)$.

Theorem 5.2.4. *Let $m \geq 3$. Following* [GP00, §5.3.4]*, for $1 \leq j \leq \lfloor (m-1)/2 \rfloor$, we consider the following representation of $I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$*

$$\widetilde{\rho_j} : I_2(m) \to GL_2(\mathbb{R}) \text{ defined by } \widetilde{\rho_j}(s) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \widetilde{\rho_j}(st) := \begin{pmatrix} \cos(j\theta_m) & -\sin(j\theta_m) \\ \sin(j\theta_m) & \cos(j\theta_m) \end{pmatrix},$$

where $\theta_m := 2\pi/m$ and we let ρ_j be a realization of $\widetilde{\rho_j}$ on the splitting field $\mathbb{Q}(\theta_m)$ of $I_2(m)$.

Then, the first homology representation of $\mathbf{T}(I_2(m))$ is given by

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \begin{cases} \displaystyle\bigoplus_{1 \leq j \leq (m-1)/2} \rho_j & \text{if } m \text{ is odd,} \\ \displaystyle\bigoplus_{\substack{1 \leq j \leq m/2-1 \\ j \text{ odd}}} \rho_j & \text{if } m \text{ is even.} \end{cases}$$

Recall also that $H_0(\mathbf{T}(I_2(m)), \mathbb{Q}) = \mathbb{1}$ and $H_2(\mathbf{T}(I_2(m)), \mathbb{Q}) = \varepsilon$.

Proof. We already have obtained the last statement above in Proposition 5.2.1. For the first homology, we let $\chi_j := \mathrm{tr}(\rho_j)$ be the character of ρ_j and, denoting by $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{Q}(\theta_m)[I_2(m)]$ the regular module, lemma 5.2.3 yields the following equality of virtual characters of $I_2(m)$

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} - \sum_{\emptyset \neq I \subsetneq \{\widehat{s}_0, s, t\}} (-1)^{|I|} \mathbb{1}\uparrow_{\pi(\widehat{I_2(m)}_I)}^{I_2(m)}.$$

We deal with each case separately. Recall the computations of the images in $I_2(m)$ of the parabolic subgroups of $\widehat{I_2(m)}$ from the proof of the Corollary 4.5.1.

- $m = 2k+1$ is odd. We have $r := r_W = (st)^k s$. Hence $s^r = t$ and $t^r = s$ so $I_2(m) = \langle s, r \rangle = \langle t, r \rangle$. Furthermore, in this case we have (cf [GP00, §5.3.4]) $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{1} + \varepsilon + \sum_j 2\chi_j$ and the above formula reduces to

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = 3 \cdot \mathbb{1}\uparrow_{\langle s \rangle}^{I_2(m)} - 3 \cdot \mathbb{1} - \sum_j 2\chi_j.$$

  Now, by [GP00, §6.3.5] we have $\mathbb{1}\uparrow_{\langle s \rangle}^{I_2(m)} = \mathbb{1} + \sum_j \chi_j$ and thus

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \sum_j \chi_j.$$

- $m = 4k$. In this case we have $r = (st)^{2k-1}s$ and let $a := st$. The conjugacy classes of $I_2(m)$ are given as follows

Representative	1	a	a^2	$\cdots$	a^{2k}	a^{2k+1}	s	t
Cardinality	1	2	2	$\cdots$	2	1	$2k+1$	$2k+1$

  First, we determine the characters $\mathbb{1}\uparrow_{\langle x, r \rangle}^{I_2(m)}$ for $x = s, t$. In the proof of 4.5.1 we have seen that $t = (sr)^{2k-1}s$, so $\langle s, r \rangle = I_2(m)$. Next, as detailed in [GP00, §5.3.4], the character χ_j is given by

$$\chi_j(a^i) = 2\cos\left(\frac{2\pi i j}{m}\right) \quad \text{and} \quad \chi_j(sa^i) = 0.$$

  We have $\langle t, r \rangle = \{1, t, r, a^{2k}\} \simeq C_2 \times C_2$ and by Frobenius reciprocity

$$\forall j, \ \left(\mathbb{1}\uparrow_{\langle t,r \rangle}^{I_2(m)}, \chi_j\right)_W = \left(\mathbb{1}, \chi_j\downarrow_{\langle t,r \rangle}^{I_2(m)}\right)_{\langle t,r \rangle} = \frac{\chi_j(1) + \chi_j(t) + \chi_j(r) + \chi_j(a^{2k})}{4}$$

$$= \frac{\chi_j(1) + 2\chi_j(t) + \chi_j(a^{2k})}{4} = \frac{1+\cos(\pi j)}{2} = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

The 1-dimensional irreducible representations of $I_2(m)$ other that $\mathbb{1}$ and ε are given by $\varepsilon_s(s) = \varepsilon_t(t) = 1$ and $\varepsilon_s(t) = \varepsilon_t(s) = -1$. Therefore, $\mathrm{Reg}_{\mathbb{Q}(\theta_m)} = \mathbb{1} + \varepsilon + \varepsilon_s + \varepsilon_t + \sum_j 2\chi_j$. We directly compute using Frobenius reciprocity

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon\right)_{I_2(m)} = 0$$

and

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_t\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)} = 1$$

and hence

$$\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_t + \sum_{j \text{ even}} \chi_j.$$

On the other hand, by [GP00, §6.3.5], we have $\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_s + \sum_j \chi_j$ and $\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_t + \sum_j \chi_j$. Putting everything together and remembering that t and r are conjugate yields

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} - \sum_{\emptyset \neq I \subsetneq \{\widehat{s}_0, s, t\}} (-1)^{|I|} \mathbb{1}\uparrow_{\pi(\widehat{I_2(m)}_I)}^{I_2(m)}$$

$$= \mathbb{1} + \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} + \mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} + 2\cdot\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} - 2\cdot\mathbb{1} = \sum_{j \text{ odd}} \chi_j.$$

- $m = 4k+2$. We proceed in the same way, noticing that $r = (st)^{2k}s = a^{2k}s$. The characters $\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)}$ and $\mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)}$ are determined as above. We compute

$$\left(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}, \varepsilon_s\downarrow_{\langle s,r\rangle}^{I_2(m)}\right)_{\langle s,r\rangle} = 1 = \left(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)}$$

but since $\deg(\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)}) = [I_2(m) : \langle s,r\rangle] = [I_2(m) : \langle s, a^{2k}\rangle] = 2$ this implies $\mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)} = \mathbb{1} + \varepsilon_s$. Now, we have $\langle t,r\rangle = \{1, t, r, a^{2k+1}\} \simeq C_2 \times C_2$ and using again the Frobenius reciprocity we obtain

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \chi_j\right)_{I_2(m)} = \left(\mathbb{1}, \chi_j\downarrow_{\langle t,r\rangle}^{I_2(m)}\right)_{\langle t,r\rangle} = \frac{\chi_j(1) + \chi_j(t) + \chi_j(r) + \chi_j(a^{2k+1})}{4}$$

$$= \frac{\chi_j(1) + \chi_j(a^{2k+1})}{4} = \frac{1+\cos(\pi j)}{2}.$$

Since $\mathbb{1}\downarrow_{\langle t,r\rangle}^{I_2(m)} \neq \mathbb{1} \neq \mathbb{1}\downarrow_{\langle t,r\rangle}^{I_2(m)}$ we also get

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_s\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon_t\right)_{I_2(m)} = \left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \varepsilon\right)_{I_2(m)} = 0$$

and

$$\left(\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)}, \mathbb{1}\right)_{I_2(m)} = 1.$$

Finally,

$$\mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \mathbb{1} + \sum_{j \text{ even}} \chi_j.$$

As above, we conclude that

$$H_1(\mathbf{T}(I_2(m)), \mathbb{Q}(\theta_m)) = \varepsilon - \mathrm{Reg}_{\mathbb{Q}(\theta_m)} + 2\cdot\mathbb{1}\uparrow_{\langle s\rangle}^{I_2(m)} + \mathbb{1}\uparrow_{\langle t\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle s,r\rangle}^{I_2(m)} - \mathbb{1}\uparrow_{\langle t,r\rangle}^{I_2(m)} = \sum_{j \text{ odd}} \chi_j,$$

as claimed.

□

Theorem 5.2.5. *With the notation of* [GP00, Appendix C, Table C.1]*, we have*

$$\forall 0 \leq i \leq 3,\ H_i(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = \begin{cases} \mathbb{1} & \text{if } i = 0, \\ 3'_s \oplus \overline{3'_s} \oplus 5_r & \text{if } i = 1, \\ 3_s \oplus \overline{3_s} \oplus 5'_r & \text{if } i = 2, \\ \varepsilon & \text{if } i = 3. \end{cases}$$

Proof. Consider the virtual character $\chi_H := \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \mathbb{1}\uparrow^{H_3}_{\pi(\widehat{H_{3I}})}$. For $\chi \in \mathrm{Irr}(H_3)$, we may compute

$$(\chi, \chi)_{H_3} = \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \left(\mathbb{1}\uparrow^{H_3}_{\pi(\widehat{H_{3I}})}, \chi_H \right)_{H_3}$$

$$= \sum_{I \subsetneq \widehat{S}} (-1)^{|I|+1} \left(\mathbb{1}, \chi\downarrow^{H_3}_{\pi(\widehat{H_{3I}})} \right)_{\pi(\widehat{H_{3I}})} = \sum_{\substack{I \subsetneq \widehat{S} \\ w \in \pi(\widehat{H_{3I}})}} (-1)^{|I|+1} \chi(w).$$

We obtain

$$\chi_H = \varepsilon - \mathbb{1} - 3_s - \overline{3_s} + 3'_s + \overline{3'_s} + 5_r - 5'_r$$

and therefore, using lemma 5.2.3

$$H_2(\mathbf{T}(H_3)) - H_1(\mathbf{T}(H_3)) = -3_s - \overline{3_s} + 3'_s + \overline{3'_s} + 5_r - 5'_r.$$

But from Lemma 4.4.5, we have $\dim(H_1(\mathbf{T}(H_3))) = \dim(H_2(\mathbf{T}(H_3))) = 11 = \dim(3_s + \overline{3_s} + 5'_r)$, so

$$H_1(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = 3'_s + \overline{3'_s} + 5_r \text{ and } H_2(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) = 3_s + \overline{3_s} + 5'_r = H_1(\mathbf{T}(H_3), \mathbb{Q}(\sqrt{5})) \otimes \varepsilon.$$

□

Finally we treat the case of H_4.

Theorem 5.2.6. *With the notation of* [GP00, Appendix C, Table C.2]*, we have*

$$\forall 0 \leq i \leq 4,\ H_i(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5})) = \begin{cases} \mathbb{1} & \text{if } i = 0, \\ 4_t \oplus \overline{4_t} \oplus 16'_r & \text{if } i = 1, \\ 6_s \oplus \overline{6_s} \oplus 30_s \oplus \overline{30_s} & \text{if } i = 2, \\ 4'_t \oplus \overline{4'_t} \oplus 16_r & \text{if } i = 3, \\ \varepsilon & \text{if } i = 4. \end{cases}$$

Proof. As for the previous proof, we let $\chi_H := \sum_{I \subsetneq \widehat{S}} (-1)^{|I|} \mathbb{1}\uparrow^{H_4}_{\pi(\widehat{H_{4I}})}$ and

$$\forall \chi \in \mathrm{Irr}(H_4),\ (\chi, \chi_H)_{H_4} = \sum_{I \subsetneq \widehat{S},\ w \in \pi(\widehat{H_{4I}})} (-1)^{|I|} \chi(w).$$

This leads to

$$\chi_H = \mathbb{1} + \varepsilon - 4_t - \overline{4_t} - 4'_t - \overline{4'_t} + 6_s + \overline{6_s} - 16_r - 16'_r + 30_s + \overline{30_s}.$$

Since $\dim(H_1(\mathbf{T}(H_4))) = \dim(H_3(\mathbf{T}(H_4))) = 24$ we obtain

$$H_2(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5})) = 30_s + \overline{30_s} + 6_s + \overline{6_s}$$

and

$$H_1(\mathbf{T}(H_4)) + H_3(\mathbf{T}(H_4)) = 4_t + 4'_t + \overline{4_t} + \overline{4'_t} + 16_r + 16'_r.$$

But since the representations $H_1(\mathbf{T}(H_4))$ and $H_3(\mathbf{T}(H_4))$ must be realizable over $\mathbb{Q}$ and because of the Poincaré duality pairing between the two, we are left with the following four possibilities:

$H_1(\mathbf{T}(H_4))$	$4_t + \overline{4_t} + 16_r$	$4'_t + \overline{4'_t} + 16_r$	$4_t + \overline{4_t} + 16'_r$	$4'_t + \overline{4'_t} + 16'_r$
$H_3(\mathbf{T}(H_4))$	$4'_t + \overline{4'_t} + 16'_r$	$4_t + \overline{4_t} + 16'_r$	$4'_t + \overline{4'_t} + 16_r$	$4_t + \overline{4_t} + 16_r$

However, the $\mathbb{Q}[H_4]$-module $H_1(\mathbf{T}(H_4), \mathbb{Q})$ is a sub-quotient of the module

$$C_1^{\text{cell}}(\mathbf{T}(H_4), H_4; \mathbb{Q}) = \sum_{\substack{I \subsetneq \widehat{S} \\ |I|=3}} \mathbb{1}\uparrow_{\pi(\widehat{H_4}_I)}^{H_4}$$

and we compute

$$\left(C_1^{\text{cell}}(\mathbf{T}(H_4)), 16_r\right)_{H_4} = 0 \implies (H_1(\mathbf{T}(H_4)), 16_r)_{H_4} = 0.$$

Hence, only $16'_r$ can be a direct factor of $H_1(\mathbf{T}(H_4), \mathbb{Q}(\sqrt{5}))$. In the same fashion we compute

$$\left(C_1^{\text{cell}}(\mathbf{T}(H_4)), 4'_t\right)_{H_4} = 0 \implies \left(H_1(\mathbf{T}(H_4)), 4'_t\right)_{H_4} = 0$$

and thus only the third column of the table above is possible. □

Remark 5.2.7. *In* [RT01, §3] *and* [Mar15, §2.2]*, the homology of* $\mathbf{T}(H_4)$ *is also described, but only as a* $\mathbb{Z}$*-module.*

Finally, we exhibit another algebraic meaning of the Euler characteristic of $\mathbf{T}(W)$. The Poincaré series of $\widehat{I_2(m)}$, $\widehat{H_3}$ and $\widehat{H_4}$ can be found in [CLS10, §3.1, Table 7.4 and Table 7.5]. Using these expressions, we immediately obtain the following corollary:

Corollary 5.2.8. *Let* W *be a finite irreducible Coxeter group. If* $W(q)$ *(resp.* $\widehat{W}(q)$*) denotes the Poincaré series of* W *(resp. of its extension* $\widehat{W}$*), then the Euler characteristic of* $\mathbf{T}(W)$ *is given by*

$$\chi(\mathbf{T}(W)) = \left.\frac{W(q)}{\widehat{W}(q)}\right|_{q=1}$$

Moreover, the geometric representation σ *of* W *is always a direct summand of* $H_1(\mathbf{T}(W), \mathbb{Q}(W))$ *for every* W *and the two are equal if and only if* W *is crystallographic. In particular*

$$\bigoplus_{\alpha \in \mathrm{Gal}(\mathbb{Q}(W)/\mathbb{Q})} \sigma^\alpha \text{ is a direct summand of } H_1(\mathbf{T}(W), \mathbb{Q}).$$

Remark 5.2.9. *With* [CLS10] *it can be seen that the quotient* $W(q)/\widehat{W}(q)$ *is a polynomial in* q*, but we cannot hope for a generalization of the Bott factorization theorem as in the affine case, i.e. a formula of the form*

$$\frac{W(q)}{W_{\mathrm{a}}(q)} = \prod_{i=1}^{n} 1 - q^{d_i - 1},$$

with $\{d_i\}$ *the degrees of* W*. Indeed, the polynomial* $H_4(q)/\widehat{H_4}(q)$ *is irreducible of degree* 60*.*

Appendix A. Hyperbolic extensions of finite irreducible Weyl groups

As the irreducible hyperbolic Coxeter groups have rank ≤ 10 and are all classified (see [Che69]), we can check each reflection of each irreducible finite Weyl group to see which one of them give hyperbolic extensions. There may are other possible reflections and extensions, but the resulting Coxeter diagram must appear in the following table. The computations were made using [GAP21]. Of course, for the case of $G_2 = I_2(6)$, we find in particular the diagram corresponding to $\widehat{I_2(6)}$ defined above.

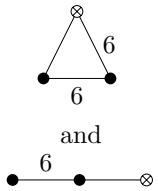
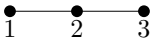
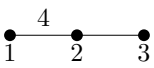
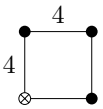
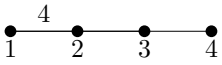
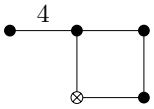
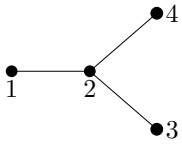
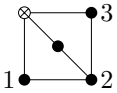
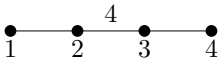
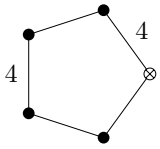
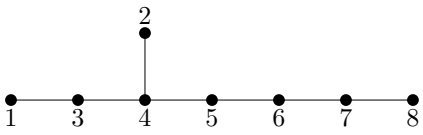
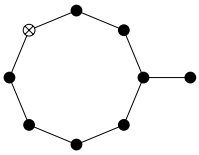
Type	Dynkin diagram	Hyperbolic diagram	Reflection	Compact?
G_2			$s_1^{s_2}$ and $s_1^{s_2s_1}$	both
A_3			$s_3^{s_2}$	no
C_3			$s_2^{s_1}$	no
C_4			$s_3^{s_1^{s_2}} = s_3^{s_2s_1s_2}$	no
D_4			$s_4^{s_2s_3}$	no
F_4			$s_3^{s_2s_1}$	no
E_8			$s_7^{(s_3s_2)^{s_4s_5s_6}(s_5s_2)^{s_4s_3s_1}}$	no

TABLE 5. Hyperbolic extensions of finite irreducible Weyl groups.

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