Existence analysis and numerical approximation for a second order model of traffic with orderliness marker

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Abstract

We propose a toy model for self-organized road traffic taking into account the state of orderliness in drivers' behavior. The model is reminiscent of the wide family of generalized second-order models (GSOM) of road traffic. It can also be seen as a phase-transition model. The orderliness marker is evolved along vehicles' trajectories and it influences the fundamental diagram of the traffic flow. The coupling we have in mind is non-local, leading to a kind of "weak decoupling" of the resulting 2×2 system; this makes the mathematical analysis similar to the analysis of the classical Keyfitz-Kranzer system. Taking advantage of the theory of weak and renormalized solutions of one-dimensional transport equations [Panov, 2008], which we further develop on this occasion in the first chapter, we prove the existence of admissible solutions defined via a mixture of the Kruzhkov and the Panov approaches; note that this approach to admissibility does not rely upon the classical hyperbolic structure for 2×2 systems. First, approximate solutions are obtained *via* a splitting strategy; compactification effects proper to the notion of solution we rely upon are carefully exploited, under general assumptions on the data. Second, we also address fully discrete approximation of the system, constructing a **BV**-stable Finite Volume numerical scheme and proving its convergence under the no-vacuum assumption and for data of bounded variation. As a byproduct of our approach, an original treatment of local GSOM-like models in the **BV** setting is briefly discussed, in relation to discontinuous-flux LWR models.

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Contents

1 Introduction

This paper is devoted to mathematical and numerical analysis of a 2×2 system of balance laws with non-local coupling. Our motivations come from macroscopic modeling of road traffic, and more specifically, from taking into account the distinction between ordered or disordered behaviors of drivers within the paradigm of the so-called Generalized Second-Order Models (GSOM).

1.1 Generalities on macroscopic PDE traffic models

Let us start by providing a brief account on advantages and drawbacks (in terms of modeling, but also in terms of completeness and flexibility of their mathematical and numerical analysis) of first-order and second-order hyperbolic models for road traffic, including phase transition models that combine both of the above. More information can be found, *e.g.*, in the surveys and monographs [13, 45, 47]. In Section 1.2, we will insert our work within this general picture and highlight the analytical purpose of our work that goes beyond its modeling purpose.

1.1.1 The fundamental flow equation

Although traffic description in terms of individual agents and their interactions is relevant, typically it leads to large ODE systems which mathematical analysis is cumbersome; moreover, they may encrypt the relevant traffic information (such as presence of shock waves) in a non-obvious way. The influence of fluid mechanics and the well developed mathematical machinery of hyperbolic PDEs and their approximation made macroscopic models very popular, starting from the pioneering Lighthill-Whitham and Richards model. All these models are based on the fundamental flow equation

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{1.1}$$

with ρ representing the density of the flow, bounded by some maximum value, and v representing the velocity. Different models are built upon this equation by adding functional and/or differential relations linking the two state variables ρ and v (or ρ and ρv).

1.1.2 First-order models

These models use an explicit closure relation linking v to ρ by a functional dependence, such as $v(\rho) = V_{\max}(1 - \dfrac{\rho}{\rho_{\max}})$. The classical Lighthill-Whitham and Richards model [40, 46] (LWR, in the sequel) is the prototype of the whole class. We refer to [12] for a survey of first-order models. The major advantage of such models is the possibility of their complete mathematical analysis, rigorous assessment of several approximation strategies, proved relation to certain microscopic many-particle models. Their theory is firmly attached to the classical theory of Kruzhkov entropy solutions to scalar conservation laws [37]. The robustness of the theory facilitates the introduction, into the first-order models, of additional features such as delays, non-locality, point constraints, variation of the number of lines, *etc*; see [12], see also [16, 18, 3, 14] for a few more recent examples. The clear drawback of the first-order models is their inadequacy to experimental data which exhibit a functional dependence of ρv on ρ only for low enough densities, see, *e.g.*, the experimental fundamental diagram in [27, Fig.1].

1.1.3 Second-order models

In the context of traffic flows, the name "second-order" is given to models describing the joint evolution of the state variables (ρ, v) (or $(\rho, \rho v)$) by means of a 2×2 system of PDEs. After the controversy of [23], the second-order model of Aw-Rascle and Zhang [11, 50] (ARZ, in the sequel) became popular. In [39, 38] a wide family of generalized second-order models (GSOM, in the sequel) was described. The mathematical structure of these models is a 2×2 system of conservation or balance laws, strictly hyperbolic away from the vacuum $\rho = 0$, with one genuinely nonlinear and one linearly degenerate characteristic fields. Selection criteria in terms of Riemann solver can be reformulated under the form of entropy conditions (see, in particular, [4] for a Kruzhkov-like choice of entropies). Variants of ARZ with additional features, as for the variants of the LWR model mentioned here above, were proposed. Existence analysis with, sometimes, numerical analysis could be extended to some of these variants, see, *e.g.*, [4]. However, the mathematical analysis of GSOM is not complete at the present stage, except for the case of the Riemann problems [38]. The additional complexity of ARZ and more generally, of GSOM is compensated by a better description of some of the features of traffic, yet for low densities and especially for vacuum the LWR model may represent a simpler and more reliable model.

1.1.4 Phase transition models

Phase transitions between a "free" and a "congested" states of flow were identified in the engineering literature, see *e.g.*, [32, 34], as the crucial property of real traffic flows responsible for the self-organization patterns such as the stop-and-go waves. The two phases are associated with two different regions of the experimental fundamental diagrams, like [27, Fig.1]. Several two-phase mathematical models with phase transitions were proposed. In particular, the model of [17] is close to the GSOM family, see [39]. In principle, these models offer a better description of traffic, combining the advantages of the first-order and the second-order models (*e.g.*, [28, 19]) and the insight from the engineering literature. This comes at the price of a much heavier mathematical treatment. Indeed, typically the phase-transition models are posed in terms of the Riemann solver (which describes, among other, the phase transition behavior) and the wave-front tracking algorithm with delicate control of variation is used for the existence analysis. Even slight modifications of such models may result in heavy modifications of the analysis of front interactions. We refer to [5] for one recent example of phase transition model enriched with point constraints and for a brief survey of mathematical literature on phase transition models.

1.2 Analytical and modeling purposes of the present work

Our purpose is two-fold. Our primary goal is to contribute to mathematical analysis for some GSOM models based upon the robust theory of scalar conservation laws like for the first-order case and on the theory of renormalization for the kind of transport equations encountered in typical GSOM. This line is an alternative to the classical line based on the general theory of hyperbolic systems of conservation laws, and it may allow for more flexibility when variants of the model are considered. Our secondary goal is to enrich the GSOM family of models with a variant built on taking into account the state of orderliness in drivers' behavior and its evolution along vehicles' trajectories. Our mathematical analysis is developed having in mind the key features of this non-local variant of GSOM, though it may have wider applications.

1.2.1 Contributions into analysis and approximation of GSOM-kind models and systems of the Keyfitz-Kranzer kind

We develop adequate analysis and approximation tools for an exemplary GSOM model featuring non-local coupling between the equation for the density ρ and the equation for the auxiliary marker $\mathbf{w}$. The non-locality has a regularizing effect that makes the system under study reminiscent, in terms of the analytical approach, of the classical Keyfitz-Kranzer system [35]. In this situation, the central role is played by the renormalization property for the component $\mathbf{w}$ of the solution which evolves along the trajectories of the flow. This surprising - in view of the involved and celebrated theory [25] of renormalized solutions - structural property was established in [42] for general weak solutions $\mathbf{w}$ of the transport equation $\partial_t \mathbf{w} + \partial_x(\mathbf{w}v) = 0$ with the velocity v involved in the continuity equation $\partial_t \rho + \partial_x(\rho v) = 0$ for the density ρ, having in mind application to the Keyfitz-Kranzer system. We further develop the tool of the weak/renormalized solution adding nonlinear source terms in the Panov setting [42] and uncovering a "propagation of compactness" mechanism proper to this linear equation.

Indeed, the renormalization structure yields compactness - either through the total variation control, or through the analysis in terms of Young measures. This structure also guides us in developing an original numerical strategy which enters, in a non-obvious way, the standard framework of finite volume approximations. It turns out that this numerical strategy can be seen as a generalization of the specific discretization strategy developed for the Keyfitz-Kranzer system [36]. Note that the renormalization property was already identified in [4] as a key ingredient in the study of the Aw-Rascle and Zhang system (ARZ, the best known example of GSOM) with point constraints at bottlenecks, and it can be instrumental as well for studying boundary-value problems for ARZ.

Applicable to a wider class of GSOM with non-local coupling, our analysis does not rely on the standard hyperbolic structure of the system. Instead, it relies upon a sort of decoupling due to the non-local dependence on $\mathbf{w}$ of the fundamental diagram $\rho \mapsto v(x,t,\rho)$. Moreover, we briefly discuss the possibility of pursuing this line of analysis for more standard local GSOM models, linking the question to the need for a deeper understanding of discontinuous-flux scalar conservation laws with moderately or wildly discontinuous in space flux function. Rigorous application of this approach to local GSOM is postponed to future work. Note that also the discretization strategy we pursue is applicable to the local GSOM.

1.2.2 Contribution to traffic modeling with GSOM

We propose a prototype model able to take into account the state of orderliness of drivers' behavior. Roughly speaking, we represent the state of the traffic by a family of fundamental diagrams $\rho \mapsto \rho v$ that depend on the additional orderliness parameter ω and interpolate between fundamental diagrams $\rho \mapsto \rho V_{\min}(\rho)$ (corresponding to $\omega = 0$, fully disordered traffic) and $\rho \mapsto \rho V_{\max}(\rho)$ (corresponding to $\omega = 1$, fully ordered traffic).

This idea was put forward by the authors in [10] with the goal to model self-organization (and disorganization) of traffic at bottlenecks, in the frame of the basic LWR model adapted to the presence of bottlenecks [3, 1, 2]. In [10], ω is a time-dependent parameter attached to the bottleneck; the passing capacity of the bottleneck is a function of the orderliness parameter ω. The dynamics of ω is governed by an ODE of the logistic type. This ODE is driven by averaged values of the density in the upstream neighbourhood of the bottleneck: this offers a mechanism of progressive ordering of the traffic (self-organization) in stable traffic conditions, and of quick disordering in the situations with abruptly growing averaged density upstream the bottleneck.

In view of the extensive evidence of self-organization of traffic beyond bottlenecks [33], we transpose this idea towards taking into account the influence of orderliness in drivers' behavior on the fundamental diagram of the flow in the bulk (so we do not focus on bottlenecks any more, unlike in [10]).

Many attempts have been made to model the self-organization in traffic and its salient features like the stop-and-go waves. One important paradigm for these models is phase transitions, resulting in formulation of two-phase models [34, 20]. Some of two-phase models are close, in their structure,

to the GSOM models [17, 39]. In the present paper, we propose a toy model which can be situated at the crossroads of the above mentioned ideas. It has the structure of GSOM with the Lagrangian marker interpreted as the orderliness parameter. It can be seen as a two-phase model, due to the fact that we take $V_{\min} \equiv V_{\max}$ for low densities. And it borrows from [10] the mechanism for the evolution of the orderliness marker $\mathbf{w}$ attached to individual vehicles. We define the orderliness parameter $\omega = \omega(x,t)$ of the fundamental diagram as the weighted average, over a small vicinity of every point (x,t), of the individual orderliness marker $\mathbf{w}$. The corresponding local model (with $\omega = \mathbf{w}$) makes sense and it is briefly discussed.
Any attempt to link the model we work with, or the values of the parameters of this model, to road traffic data is far beyond the scope of this paper. As a matter of fact, we have in mind the whole class of systems of non-local GSOM kind of which our exemplary model is a particular instance. Indeed, the mathematical analysis we carry out is suitable for a wide family of non-local GSOM models, including source terms for evolution of the Lagrangian marker.

2 The GSOM-kind model with orderliness

Once and for all, fix a time horizon $T > 0$ and denote $\Omega = \mathbb{R} \times (0,T)$. We consider that the maximal density $\rho_{\max}$ on the road equals 1. In our new model, the first equation on $[0,1]$-valued density ρ,

$$\partial_t \rho + \partial_x \left(\rho \mathbf{v}(x,t,\rho)\right) = 0, \quad (x,t) \in \Omega, \tag{2.1}$$

expresses the conservation of mass and it is driven by a time and space dependent velocity $\mathbf{v}$. This dependency reads:

$$\mathbf{v}(x,t,\rho) = (1-\omega(x,t))V_{\min}(\rho) + \omega(x,t)V_{\max}(\rho). \tag{2.2}$$

In (2.2), $V_{\min}, V_{\max}$ are the two levels of traffic velocity; the one for the ordered regime of traffic and the other for the disordered regime. As usual, we require both of them to be nonincreasing and nonnegative Lipschitz continuous functions defined for $\rho \in [0,1]$: naturally, $V_{\max} \geq V_{\min}$. The actual velocity $\mathbf{v}$ in (2.2) is a convex combination of the two regimes' velocities with $\omega(x,t) \in [0,1]$ representing the state of orderliness of the traffic at time t and position x. We further consider the orderliness parameter $\mathbf{w}$ associated to individual vehicles, which is evolved according to the transport equation

$$\partial_t \left(\rho \mathbf{w}\right) + \partial_x \left(\rho \mathbf{w} \mathbf{v}(x,t,\rho)\right) = \rho s(x,t,\mathbf{w}). \tag{2.3}$$

For a regular velocity field, equation (2.3) corresponds to the evolution of $\mathbf{w}$ according to the ODE $\dot{\mathbf{w}}(X(t),t) = s\big(X(t),t,\mathbf{w}(X(t),t)\big)$ along the integral curves $x = X(t)$ of the velocity field $\mathbf{v}$. In absence of regularity of $\mathbf{v}$, the rigorous meaning to such evolution is provided by the weak formulation (2.3) which, moreover, automatically implies the renormalization property (see Appendix A). The coupling of (2.1), (2.2) with (2.3) is provided by relations linking ω, s to $\mathbf{w}$, ρ.
First, we concentrate on the choice of the source term s in (2.3): it is directly inspired by our previous work [10] where self-organization at bottlenecks, governed by an analogous orderliness parameter ω, is considered. Let us take $s(x,t,\mathbf{w}) = \mathbf{K}\mathbf{w}(1-\mathbf{w})$ where $\mathbf{K}$, depending on ρ and $\partial_t \rho$ in a non-local way, reflects a mechanism of ordering/disordering subject to the traffic conditions in a vicinity of each point (x,t). To this end, we introduce the subjective density

$$\xi(x,t) = \int_{\mathbb{R}} \rho(y,t)\mu(x-y)\,\mathrm{d}y\,, \tag{2.4}$$

where $\mu \geq 0$, $\displaystyle\int_{\mathbb{R}} \mu(x)\,\mathrm{d}x = 1$, is a smooth weight function used to average ρ, similarly to non-local models of [3, 14, 10]. Further, we make $\mathbf{K}$ depend on ρ through the subjective density ξ and its time variations $\partial_t \xi$. For future use, let us precise that classical PDE computations using the weak formulation of (2.1) ensure that ξ admits a time derivative in the sense of the distributions and that for a.e. $(x,t) \in \Omega$,

$$\partial_t \xi(x,t) = -\int_{\mathbb{R}} \rho(y,t)\mathbf{v}(y,t,\rho)\mu'(x-y)\,\mathrm{d}y\,.$$

This comes from using $\varphi(y,t) = \mu(x-y)\psi(t)$ $(x \in \mathbb{R})$ as a test function in the weak formulation. To sum up, we take

$$s(x,t,\mathbf{w}) = \mathbf{K}(\xi, \partial_t \xi)\mathbf{w}(1-\mathbf{w}) \tag{2.5}$$

for some $\mathbf{K} : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$. To fix the ideas, in the simulations we will take, following [10],

$$\mathbf{K}(\xi,\chi) = C\Big(\frac{\xi}{\xi_c} - 1\Big)^+ \Big(1 - \frac{\chi^+}{D_+} - \frac{\chi^-}{D_-}\Big)$$

with some threshold $\xi_c \in (0,1)$ and constants $C > 0$, $D_+ \gg D_- > 0$ (see Figure 1). Mathematically speaking, we only suppose that $\mathbf{K} \in \mathbf{Lip_{loc}}([0,1] \times \mathbb{R})$. The idea behind the above choice of $\mathbf{K}$ is to allow for progressive ordering of the traffic with time when the traffic conditions are stable, and for a quick disordering when sudden and strong variations (especially in the case of densification) of the traffic occur. Note that random fluctuations of $\mathbf{w}$ could be considered, as a further step of modeling, but this is beyond the scope of our work.

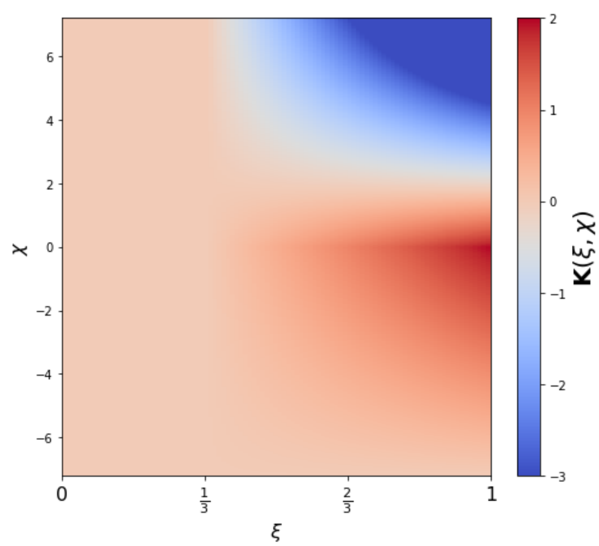

Figure 1: Typical behavior of the orderliness-driving function $\mathbf{K}$.

The key features of the dynamics of $\mathbf{w}$ encoded in (2.3)–(2.5) with the above choice of $\mathbf{K}$ are as follows:

- conservation of the "momentum" quantity $\rho\mathbf{w}$ in the region of low densities, because $\mathbf{K}$ is zero for low densities;
- rapid decrease of $\rho\mathbf{w}$ for moderate and particularly for high densities, under strong density variations (disordering);
- progressive increase of $\rho\mathbf{w}$ in dense and very dense traffic with small density variations (ordering).

Finally, let us write the link between ω in (2.2) and the individual ordering markers $\mathbf{w}$ as

$$\omega = \mathcal{M}[\mathbf{w}] \tag{2.6}$$

where $\mathcal{M}$ is an operator on $\mathbf{L}^\infty(\Omega;[0,1])$. We have in mind the following three choices. For the simplest one, $\mathcal{M} = \mathrm{Id}$, *i.e.*, $\omega = \mathbf{w}$, (2.1) can be seen as an LWR equation with space-time discontinuous flux. Its mathematical study still requires deeper analysis, despite much progress made in this direction. We briefly discuss the issue in Section 5.2. Because traffic involves only a limited number of agents in a neighbourhood of each point, in this paper we focus on non-local impact of the individual vehicle markers $\mathbf{w}$ on the global traffic orderliness ω. Two variants will be considered. In Section 4, the existence will be obtained with

$$\mathcal{M}[\mathbf{w}](x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,. \tag{2.7}$$

In (2.7), the function η is a weight function of the form $\eta(x,t) = \eta_1(x)\eta_2(t)$ with $\eta_1 \in \mathbf{C}^1_{\mathbf{c}}(\mathbb{R})$ and $\eta_2 \in \mathbf{BV}(\mathbb{R})$ and supported in a compact subset of $[0,T)$. Note also that to make sense of (2.7), we will extend $\mathbf{w}$ by the initial data $\mathbf{w}_0$ for negative times. Note that the space averaging means that the perception, by the drivers, of the traffic conditions relies on their observations of their immediate neighbourhood (typically, several dozens of meters downstream the flow) and the time averaging means that the drivers' perception of the situation is not instantaneous. Remark that the non-locality in time only looks in the past. In Section 5.1 and throughout Section 6, we assume a stronger reactivity of the drivers to instantaneous traffic conditions in their immediate neighbourhood, and take the mere space averaging

$$\mathcal{M}[\mathbf{w}](x,t) = \int_{\mathbb{R}} \mathbf{w}(y,t)\eta(x-y)\,\mathrm{d}y\,. \tag{2.8}$$

with $\eta \equiv \eta_1$. For the mathematical analysis of the resulting system, the difference between (2.7) and (2.8) is that that the latter one requires the $\mathbf{BV}$ framework for existence analysis, while the first choice is regularizing enough to deal with mere $\mathbf{L}^\infty$ solutions and data.

Finally, we stress that we have in mind the situation where

$$\exists \rho_f \in (0,1),\ \forall \rho \in [0,\rho_f], \quad V_{\min}(\rho) = V_{\max}(\rho) \tag{2.9}$$

so that (2.1)–(2.6) exhibits a two-phase behavior with $\rho \in [0,\rho_f]$ corresponding to the free traffic flow phase while $\rho > \rho_f$ correspond to the congested traffic.

We are now in a position of presenting the outline of the paper. In Section 3 we fix the mathematical framework of our work. The equation (2.1) is understood in the sense of Kruzhkov entropy solutions [37] of LWR models. The equation prescribing the evolution of the orderliness marker (2.3) is understood in the weak and renormalized sense of Panov [42] for one-dimensional transport equations driven by zero-divergence coefficients, with necessary adaptations. Indeed, an important ingredient of our analysis is the refinement of the theory of weak (and renormalized) solutions of transport PDEs of the kind (2.3) under the key assumptions that the coefficients form a zero-divergence field in Ω, and for a wide class of source fields with separation on (x,t) and $\mathbf{w}$ dependence. We gather original results on this problem in Appendix A. Further, Section 4 is devoted to the proof of the existence of solutions of Problem (2.1) – (2.6) with the averaging choice (2.7). In Section 5 we discuss the extension of the existence analysis to other choices of $\mathcal{M}$ in (2.6). In Section 6 we build a numerical scheme adapted to the specific structure of the system at hand (LWR equation for ρ and a transport equation for $\mathbf{w}$). We make the simpler averaging choice (2.8) and prove that the scheme is **BV**-stable and convergent. We point out structural similarities between our scheme and the scheme of the authors of [36] developed for the classical Keyfitz-Kranzer system. Finally, Section 7 is devoted to performing numerical simulations to illustrate our model.

3 Notion of solution

We denote by f the time and space dependent flux $f(x,t,\rho) = \rho\mathbf{v}(x,t,\rho)$ and Φ its Kruzhkov entropy flux (see [37]):

$$\forall \rho, \kappa \in [0,1],\ \forall (x,t) \in \Omega,\ \Phi(x,t,\rho,\kappa) = \operatorname{sgn}(\rho-\kappa)\left(f(x,t,\rho) - f(x,t,\kappa)\right).$$

Relying upon [37] for the PDE describing the evolution of ρ and upon [42] (see also our Appendix A) for the PDE describing the evolution of $\mathbf{w}$, we give the following definition of solution to Problem (2.1) – (2.6).

Definition 3.1. A couple $(\rho, \mathbf{w}) \in \mathbf{L}^\infty(\Omega)^2$ is a solution to (2.1) – (2.6) with initial data $(\rho_0, \mathbf{w}_0) \in \mathbf{L}^\infty(\mathbb{R})^2$ if
(i) $\rho \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};[0,1]))$ and $\rho\mathbf{w} \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R};[0,1]))$, where $w^*\text{–}\mathbf{L}^\infty$ means the space $\mathbf{L}^\infty$ endowed with its topology of weak-$*$ convergence;
(ii) ρ is an entropy solution to (2.1) with initial data ρ_0 in the following sense: $\rho(\cdot,0) = \rho_0$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$; and for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+), \varphi \geq 0$, for all $\kappa \in [0,1]$ and for all $\tau, s \in [0,T]$ $(s<\tau)$,

$$\begin{aligned} &\int_s^\tau \int_{\mathbb{R}} \Big(|\rho-\kappa|\partial_t\varphi + \Phi(x,t,\rho,\kappa)\partial_x\varphi - \operatorname{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\mathbb{R}} |\rho(x,s)-\kappa|\varphi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau)-\kappa|\varphi(x,\tau)\,\mathrm{d}x \geq 0; \end{aligned} \tag{3.1}$$

(iii) $\mathbf{w}$ is a weak solution to (2.3) with initial data $\mathbf{w}_0$ in the following sense: $\rho(\cdot,0)\mathbf{w}(\cdot,0) = \rho_0\mathbf{w}_0$ in $\mathbf{L}^\infty(\mathbb{R})$-weakly*; and for all test functions $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+)$ and for all $\tau, s \in [0,T]$ $(s<\tau)$,

$$\begin{aligned} &\int_s^\tau \int_{\mathbb{R}} \Big((\rho\mathbf{w})\partial_t\phi + (\rho\mathbf{v}\mathbf{w})\partial_x\phi + \rho\mathbf{K}\left(\xi, \partial_t\xi\right)\mathbf{w}(1-\mathbf{w})\phi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\mathbb{R}} \rho(x,s)\mathbf{w}(x,s)\phi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} \rho(x,\tau)\mathbf{w}(x,\tau)\phi(x,\tau)\,\mathrm{d}x = 0, \end{aligned} \tag{3.2}$$

where ξ is linked to ρ by (2.4);

(iv) $\mathbf{v}$ and ω are linked by (2.2) and ω and $\mathbf{w}$ are linked by (2.6).

Remark 3.1. According to the result of Corollary A.8 based upon the theory of [42], given $\rho, \mathbf{v}$ and setting $g = \mathbf{K}\left(\xi, \partial_t\xi\right)$ with ξ given by (2.4), the solution $\mathbf{w}$ in the sense (3.2) automatically verifies the renormalization property, *cf.* Definition A.2. We will say, for short, that the weak solution in the sense (3.2) is also a renormalized solution, meaning that it fulfills this renormalization property. This aspect is essential for the compactness properties, and it also means that, in a sense, the solution is evolving as if characteristics could be defined (though the latter cannot be defined due to the possible irregularity of $\rho, \mathbf{v}$). The latter observation is the key to the construction of the the numerical scheme and it also ensures the propagation of the **BV** regularity, for **BV** initial data.

Remark 3.2 (On the time-continuity). It is more usual to formulate (3.1)-(3.2) with $s = 0$, $\tau = T$ and $\varphi, \phi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R} \times [0, T))$. Our present formulations are instrumental for the splitting argument we employ in our construction, see Section 4.1. The equivalence between the two formulations is due to the time-continuity of entropy solutions of LWR equation and of weak solutions of the transport equations at hand, see Definition 3.1(i).

In Section 4, we prove the following existence result.

Theorem 3.2. *Fix $\rho_0, \mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R}; [0, 1])$. Assume that $V_{\min} \leq V_{\max} \in \mathbf{C}^1([0, R])$ are nonnegative and that $V'_{\min}$ and $V'_{\max}$ do not vanish on any interval of $[0, 1]$. Then Problem (2.1) – (2.6),(2.7) admits at least one solution.*

In Section 6, we obtain the following results of numerical approximation and existence for the time-local variant (2.8) of our model; note that (2.7) can also be considered in our numerical framework.

Theorem 3.3. *Suppose that $\mathbf{TV}(\rho_0) < +\infty$ and that $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R}; [0, 1]), \mathbf{TV}(\mathbf{w}_0) < +\infty$. Moreover suppose that ρ_0 is separated from the vacuum in the sense that*

$$\exists \varepsilon \in (0, 1), \quad \varepsilon \leq \rho_0 \leq 1 \quad \textit{and} \quad V_{min}(\varepsilon) = V_{max}(\varepsilon). \tag{3.3}$$

Then up to a subsequence, the sequence of discrete solutions produced by the scheme of Section 6 converges to a solution of (2.1) – (2.6), (2.8).

Note that the second requirement in (3.3) follows from the assumption (2.9), while the first requirement in (3.3) is essential in order to define the CFL condition of the numerical scheme we develop.

Theorem 3.4. *Suppose that $\mathbf{TV}(\rho_0) < +\infty$ and ρ_0 satisfies (3.3), and that $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R}; [0, 1])$, $\mathbf{TV}(\mathbf{w}_0) < +\infty$. Then Problem (2.1) – (2.6), (2.8) admits at least one solution.*

Let us precise that the assumption (3.3) is only useful to construct and prove the convergence of the scheme developed in Section 6. The last existence result can be obtained without it, see the discussion in Section 5.1, by using the splitting construction borrowed the proof of Theorem 3.2 along with a **BV** stability argument ensuring compactness.

4 Existence of solutions *via* splitting

4.1 Time-splitting procedure and approximate solution

To prove existence of solutions to (2.1) – (2.6),(2.7), we use a time-splitting technique. This way, we split the model combining the notion of Kruzhkov entropy solution to LWR models with the notion of weak-and-renormalized solutions to transport equations under the specific form of Panov [42], extended in Appendix A in order to include the nonlinear source term.

Fix $\rho_0, \mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R}; [0, 1])$. Let $\nu > 0$ be a time step, denote for all $n \in \mathbb{Z}$, $t^n = n\nu$ and let $N \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1})$.
Initialization. For all $t \in \mathbb{R}$,

$$\rho^0(\cdot, t) = \rho_0 \quad \text{and} \quad \forall n \in \mathbb{Z}^-,\ \mathbf{w}^n(\cdot, t) = \mathbf{w}_0.$$

Induction. Fix $n \in \{1, \dots, N+1\}$.
(1) First define the orderliness parameter: $\forall t \in [t^{n-1}, t^n)$, $\forall x \in \mathbb{R}$,

$$\omega^n(x,t) = \int_{t^{n-2}}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}^{n-1}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s + \sum_{k \le n-2} \int_{t^{k-1}}^{t^k} \int_{\mathbb{R}} \mathbf{w}^k(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.$$

Remark that the values of ω^n only depend on the values of ρ and $\mathbf{w}$ before time t^{n-1}, which is the key to the splitting.
(2) We use ω^n to define the car velocity

$$\forall t \in [t^{n-1}, t^n),\ \forall x \in \mathbb{R},\ \mathbf{v}^n(x,t,\cdot) = (1-\omega^n(x,t))V_{\min}(\cdot) + \omega^n(x,t)V_{\max}(\cdot)$$

and the flux $f^n(x,t,\rho) = \rho\mathbf{v}^n(x,t,\rho)$.
(3) The flux function is smooth in x, Lipschitz in ρ and $\mathbf{BV}$ in t. Since $\rho^{n-1}(\cdot, t^{n-1})$ is bounded, we can define $\rho^n \in \mathbf{C}([t^{n-1}, t^n]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; [0,1]))$ as the unique entropy solution, in the sense of Definition 3.1 (i)-(ii), see [37, Theorem 1] and [21, Theorem 2.3], to

$$\begin{cases} \partial_t \rho^n + \partial_x \left(f^n(x,t,\rho^n)\right) = 0 \\ \rho^n(\cdot, t^{n-1}) = \rho^{n-1}(\cdot, t^{n-1}). \end{cases}$$

(4) Setting

$$\forall t \in [t^{n-1}, t^n),\ \forall x \in \mathbb{R},\ \xi^n(x,t) = \int_{\mathbb{R}} \rho^n(y,t)\mu(x-y)\,\mathrm{d}y\,,$$

and following Corollary A.8, we can define $\mathbf{w}^n \in \mathbf{L}^\infty(\mathbb{R} \times (t^{n-1}, t^n))$ as the unique weak solution to

$$\begin{cases} \partial_t \left(\rho^n \mathbf{w}^n\right) + \partial_x \left(f^n(x,t,\rho^n)\mathbf{w}^n\right) = \rho^n \mathbf{K}\left(\xi^n, \partial_t \xi^n\right) \mathbf{w}^n(1-\mathbf{w}^n) \\ \mathbf{w}^n(\cdot, t^{n-1}) = \mathbf{w}^{n-1}(\cdot, t^{n-1}). \end{cases}$$

Corollary A.8 ensures that $\mathbf{w}^n$ verifies the renormalization property, see Definition A.2; and Remark A.1 based upon [42, Lemma 1] provides the required regularity in time: $\mathbf{w}^n \in \mathbf{C}([t^{n-1}, t^n]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Note that by construction, $\mathbf{w}$ takes values in $[0,1]$.
Conclusion. Define the following functions: for a.e. $(x,t) \in \Omega$,

$$(\rho_\nu(\cdot,t), \mathbf{w}_\nu(\cdot,t)) = (\rho_0, \mathbf{w}_0)\,\mathbb{1}_{\mathbb{R}^-}(t) + \sum_{n=1}^{N+1} \left(\rho^n(\cdot,t), \mathbf{w}^n(\cdot,t)\right) \mathbb{1}_{(t^{n-1}, t^n]}(t);$$

$$(\mathbf{v}_\nu(x,t,\cdot), \omega_\nu(x,t), \xi_\nu(x,t)) = \sum_{n=1}^{N+1} \left(\mathbf{v}^n(x,t,\cdot), \omega^n(x,t), \xi^n(x,t)\right) \mathbb{1}_{[t^{n-1}, t^n)}(t)$$

$$f_\nu(x,t,\cdot) = \sum_{n=1}^{N+1} f^n(x,t,\cdot)\mathbb{1}_{[t^{n-1}, t^n)}(t).$$

Proposition 4.1. *The couple $(\rho_\nu, \mathbf{w}_\nu)$ constructed above is a solution in Ω to the following system:*

$$\begin{cases} \partial_t \rho_\nu + \partial_x \left(f_\nu(x,t,\rho_\nu)\right) = 0 \\ \mathbf{v}_\nu(x,t,\rho) = (1-\omega_\nu(x,t))V_{\min}(\rho) + \omega_\nu(x,t)V_{\max}(\rho) \\ \partial_t \left(\rho_\nu \mathbf{w}_\nu\right) + \partial_x \left(f_\nu(x,t,\rho_\nu)\mathbf{w}_\nu\right) = \rho_\nu \mathbf{K}\left(\xi_\nu, \partial_t \xi_\nu\right) \mathbf{w}_\nu(1-\mathbf{w}_\nu) \\ \omega_\nu(x,t) = \int_{-\infty}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,. \end{cases} \tag{4.1}$$

Proof. By construction, for all $n \in \{1, \ldots, N+1\}$, $\rho^n \in \mathbf{C}([t^{n-1}, t^n]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Combining this with the *stop-and-restart* conditions $\rho^n(\cdot, t^{n-1}) = \rho^{n-1}(\cdot, t^{n-1})$, we ensure that $\rho_\nu \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Using a similar reasoning, we obtain $\rho_\nu \mathbf{w}_\nu \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Fix now $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times \mathbb{R}^+)$, $\varphi \geq 0$ and $\kappa \in [0,1]$. Let us denote by Φ_ν the Kruzhkov entropy flux associated with f_ν. By construction, for every $n \in \{1, \ldots, N+1\}$, we have

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \left|\rho_\nu - \kappa\right| \partial_t \varphi + \Phi_\nu(x,t,\rho_\nu,\kappa) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \left|\rho^n - \kappa\right| \partial_t \varphi + \operatorname{sgn}(\rho^n - \kappa) \Big(f^n(x,t,\rho^n) - f^n(x,t,\kappa) \Big) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&\geq \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \operatorname{sgn}(\rho^n - \kappa) \partial_x f^n(x,t,\kappa) \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\mathbb{R}} \left|\rho^n(x,t^{n-1}) - \kappa\right| \varphi(x,t^{n-1}) \, \mathrm{d}x + \int_{\mathbb{R}} \left|\rho^n(x,t^n) - \kappa\right| \varphi(x,t^n) \, \mathrm{d}x \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \operatorname{sgn}(\rho^n - \kappa) \partial_x f_\nu(x,t,\kappa) \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\mathbb{R}} \left|\rho_\nu(x,t^{n-1}) - \kappa\right| \varphi(x,t^{n-1}) \, \mathrm{d}x + \int_{\mathbb{R}} \left|\rho_\nu(x,t^n) - \kappa\right| \varphi(x,t^n) \, \mathrm{d}x \, .
\end{aligned}
$$

From this inequality, it is straightforward to prove that for all $s, \tau \in [0,T]$ $(s < \tau)$, we have

$$
\begin{aligned}
&\int_s^\tau \int_{\mathbb{R}} \Big(|\rho_\nu - \kappa| \partial_t \varphi + \Phi_\nu(x,t,\rho_\nu,\kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\nu - \kappa) \partial_x f_\nu(x,t,\kappa) \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_{\mathbb{R}} |\rho_\nu(x,s) - \kappa| \varphi(x,s) \, \mathrm{d}x - \int_{\mathbb{R}} |\rho_\nu(x,\tau) - \kappa| \varphi(x,\tau) \, \mathrm{d}x \geq 0,
\end{aligned}
\tag{4.2}
$$

see [49] for an analogous calculation. Let us precise here the link between ρ_ν and ξ_ν. For all $t \in [0,T]$, if $t \in [t^{n-1}, t^n)$ for some $n \in \{1, \ldots, N+1\}$, then for all $x \in \mathbb{R}$,

$$
\xi_\nu(x,t) = \xi^n(x,t) = \int_{\mathbb{R}} \rho^n(y,t) \mu(x-y) \, \mathrm{d}y = \int_{\mathbb{R}} \rho_\nu(y,t) \mu(x-y) \, \mathrm{d}y \, .
$$

We now turn to the obtaining of an approximate weak formulation similar to (3.2). Let $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times \mathbb{R}^+)$. For every $n \in \{1, \ldots, N+1\}$, we have

$$
\begin{aligned}
&\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho_\nu \mathbf{w}_\nu \partial_t \phi + f_\nu(x,t,\rho_\nu) \mathbf{w}_\nu \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho^n \mathbf{w}^n \partial_t \phi + f^n(x,t,\rho^n) \mathbf{w}^n \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t \\
&= - \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho^n \mathbf{K} \left(\xi^n, \partial_t \xi^n \right) \mathbf{w}^n (1 - \mathbf{w}^n) \phi \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\mathbb{R}} \rho^n(x,t^{n-1}) \mathbf{w}^n(x,t^{n-1}) \phi(x,t^{n-1}) \, \mathrm{d}x + \int_{\mathbb{R}} \rho^n(x,t^n) \mathbf{w}^n(x,t^n) \phi(x,t^n) \, \mathrm{d}x \\
&= - \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho_\nu \mathbf{K} \left(\xi_\nu, \partial_t \xi_\nu \right) \mathbf{w}_\nu (1 - \mathbf{w}_\nu) \phi \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\mathbb{R}} \rho_\nu(x,t^{n-1}) \mathbf{w}_\nu(x,t^{n-1}) \phi(x,t^{n-1}) \, \mathrm{d}x + \int_{\mathbb{R}} \rho_\nu(x,t^n) \mathbf{w}_\nu(x,t^n) \phi(x,t^n) \, \mathrm{d}x \, ,
\end{aligned}
$$

and from this, once again, it is easy to prove that for all $s, \tau \in [0,T]$ $(s < \tau)$, we have

$$\begin{aligned}\int_s^\tau \int_{\mathbb{R}} \Big((\rho_\nu \mathbf{w}_\nu)\partial_t \phi + (\rho_\nu \mathbf{v}_\nu \mathbf{w}_\nu)\partial_x \phi + \rho_\nu \mathbf{K}\left(\xi_\nu, \partial_t \xi_\nu\right) \mathbf{w}_\nu (1-\mathbf{w}_\nu)\phi \Big) \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} \rho_\nu(x,s)\mathbf{w}_\nu(x,s)\phi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\nu(x,\tau)\mathbf{w}_\nu(x,\tau)\phi(x,\tau)\,\mathrm{d}x = 0.\end{aligned} \tag{4.3}$$

By construction, $\mathbf{v}_\nu$ and ω_ν are linked by the second equality in (4.1). Finally, if $t \in [t^{n-1}, t^n)$ for some $n \in \{1, \ldots, N+1\}$, then we have for all $x \in \mathbb{R}$,

$$\begin{aligned}\omega^n(x,t) &= \int_{t^{n-2}}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}^{n-1}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s + \sum_{k \le n-2} \int_{t^{k-1}}^{t^k} \int_{\mathbb{R}} \mathbf{w}^k(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{-\infty}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,,\end{aligned}$$

i.e. ω_ν and $\mathbf{w}_\nu$ are linked by the last equality in (4.1). □

4.2 Compactness and convergence

We now want to pass to the limit in (4.2)-(4.3), and for that we need sufficient compactness of the sequences involved. The difficulty lies in the obtaining of strong compactness for the sequence $(\mathbf{w}_\nu)_\nu$. For this sake, we developed the "compactness from renormalization" argument for one-dimensional transport equations addressed in [42], see Theorem A.6. To apply it, we need:

- uniform $\mathbf{L}^\infty$ bounds for the sequences $(\rho_\nu)_\nu$, $(\mathbf{v}_\nu)_\nu$, $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu))_\nu$ and $(\mathbf{w}_\nu)_\nu$;
- strong compactness for the sequences $(\rho_\nu)_\nu$, $(f_\nu(\cdot,\cdot,\rho_\nu))_\nu$, $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu))_\nu$;
- to prove that $(\mathbf{w}_\nu)_\nu$ is a sequence of weak solutions to the second PDE of (4.1), which implies that they verify the renormalization property, by virtue of Corollary A.8.

Note that we proved the last point in the proof of Proposition 4.1. We now focus on the two other requirements. Let us start with the $\mathbf{L}^\infty$ bounds.

Lemma 4.2. *For all $\nu > 0$, we have the bounds:*

$$0 \le \rho_\nu, \mathbf{w}_\nu, \omega_\nu \le 1; \quad 0 \le \mathbf{v}_\nu \le V_{\max}; \quad |\mathbf{K}(\xi_\nu, \partial_t \xi_\nu)| \le \sup_{\substack{0 \le \xi \le 1 \\ |\chi| \le V_{\max} \|\mu'\|_{\mathbf{L}^1}}} |\mathbf{K}(\xi, \chi)|\,.$$

Proof. The bounds for $(\rho_\nu)_\nu$ and $(\mathbf{w}_\nu)_\nu$ are clear. Since η is a weight function, for all $\nu > 0$, we have

$$\forall (x,t) \in \Omega, \quad 0 \le \omega_\nu(x,t) \le \int_0^T \int_{\mathbb{R}} \eta(y,s)\,\mathrm{d}y\,\mathrm{d}s = 1,$$

which implies the desired bounds for $(\mathbf{v}_\nu)_\nu$ since it is a convex combination of $V_{\min}$ and $V_{\max}$. Now, once we recall that for a.e. $(x,t) \in \Omega$,

$$\partial_t \xi_\nu(x,t) = -\int_{\mathbb{R}} \rho_\nu(y,t)\mathbf{v}_\nu(y,t,\rho_\nu)\mu'(x-y)\,\mathrm{d}y\,,$$

we immediately get the bound for $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu))_\nu$. □

We now turn to the strong compactness for the sequences $(\rho_\nu)_\nu$, $(f(\cdot,\cdot,\rho_\nu))_\nu$, $(\mathbf{K}(\xi_\nu,\partial_t\xi_\nu))_\nu$. Let us start with $(f(\cdot,\cdot,\rho_\nu))_\nu$.

Lemma 4.3. *There exists $\omega \in \mathbf{C}(\Omega)$ such that up to the extraction of a subsequence, $(\omega_\nu)_\nu$ converges uniformly on compact sets to ω. Moreover, for all $(x,t) \in \Omega$, $\omega(x,t) \in [0,1]$.*

Proof. We now prove that the sequence $(\omega_\nu)_\nu$ is bounded in $\mathbf{W}^{1,\infty}(\Omega)$. We already proved in Lemma 4.2 that $(\omega_\nu)_\nu$ is bounded in $\mathbf{L}^\infty(\Omega)$. Fix now $(x,t),(\xi,\tau) \in \Omega$. On the one hand, we have

$$\begin{aligned}|\omega_\nu(x,t) - \omega_\nu(\xi,t)| &\leq \int_{-\infty}^{t-\nu}\int_{\mathbb{R}} |\eta(x-y,t-s) - \eta(\xi-y,t-s)|\,\mathrm{d}y\,\mathrm{d}s \\ &\leq |x-\xi| \int_{-\infty}^{t-\nu} \mathbf{TV}(\eta(\cdot,t-s))\,\mathrm{d}s \leq \|\eta\|_{\mathbf{L}^1((0,T);\mathbf{BV})}|x-\xi|.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\omega_\nu(x,t) - \omega_\nu(x,\tau)| \leq & \int_{-\infty}^{t-\nu}\int_{\mathbb{R}} |\eta(x-y,t-s) - \eta(x-y,\tau-s)|\,\mathrm{d}y\,\mathrm{d}s \\ &+ \left|\int_{t-\nu}^{\tau-\nu}\int_{\mathbb{R}} \eta(x-y,\tau-s)\,\mathrm{d}y\,\mathrm{d}s\right| \\ &\leq \left(\|\eta\|_{\mathbf{L}^1(\mathbb{R};\mathbf{BV})} + \|\eta\|_{\mathbf{L}^\infty((0,T);\mathbf{L}^1)}\right)|t-\tau|.\end{aligned}$$

The compactness result follows from the compact embedding $\mathbf{W}^{1,\infty}(\overset{\circ}{U}) \subset \mathbf{C}(U)$ when $U \subset \Omega$ is a compact subset. A standard diagonal process ensures then the existence of subsequence of $(\omega_\nu)_\nu$ that converges to some $\omega \in \mathbf{C}(\Omega)$ on every compact subset of Ω. $\square$

Corollary 4.4. *Define the velocity $\mathbf{v}(x,t,\rho) = (1-\omega(x,t))V_{\min}(\rho) + \omega(x,t)V_{\max}(\rho)$ and the flux $f(x,t,\rho) = \rho\mathbf{v}(x,t,\rho)$. Then, up to a subsequence, $(\mathbf{v}_\nu)_\nu$ and $(f_\nu)_\nu$ converge uniformly on compact subsets of $\Omega \times [0,1]$ to $\mathbf{v}$ and f, respectively.*

Proof. The claim is immediate because of the convergence of $(\omega_\nu)_\nu$. $\square$

We see here the effect of the non-locality of $(\omega_\nu)_\nu$. To obtain strong compactness of $(\rho_\nu)_\nu$, we impose a non-degeneracy assumption on the flux.

Lemma 4.5. *Suppose that $V'_{\min}$ and $V'_{\max}$ do not vanish on any interval of $[0,1]$. Then there exists a subsequence of $(\rho_\nu)_\nu$ which converges a.e. on Ω to some $\rho \in \mathbf{L}^\infty(\Omega)$. Moreover, for a.e. $(x,t) \in \Omega$, $\rho(x,t) \in [0,1]$.*

Proof. Fix U a bounded open subset of Ω, V a compact subset of Ω containing U and $\kappa \in [0,1]$. Using the formalism of [41, 42], we show that

$$\left(\mathrm{div}_{(t,x)}\begin{pmatrix}(\rho_\nu-\kappa)^+ \\ (\rho_\nu-\kappa)^+(f(x,t,\rho_\nu) - f(x,t,\kappa))\end{pmatrix}\right)_\nu \text{ is precompact in } \mathbf{H}^{-1}(U).$$

By construction, for all $\nu > 0$,

$$\begin{aligned}&2\partial_t(\rho_\nu-\kappa)^+ + 2(\rho_\nu-\kappa)^+(f(x,t,\rho_\nu) - f(x,t,\kappa)) \\ &= -\partial_x f(x,t,\kappa) + \partial_t|\rho_\nu-\kappa| + \partial_x\Phi(x,t,\rho_\nu,\kappa) \\ &+ \underbrace{\partial_x\left(f(x,t,\rho_\nu) - f_\nu(x,t,\rho_\nu)\right)}_{R_\nu(x,t)}\end{aligned} \tag{4.4}$$

For all $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(U)$, we have

$$\begin{aligned}\left|\iint_U R_\nu \varphi \,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\iint_U \left(f(x,t,\rho_\nu) - f_\nu(x,t,\rho_\nu)\right)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \|f - f_\nu\|_{\mathbf{L}^\infty(V)} \mathrm{mes}(U)^{1/3} \|\partial_x \varphi\|_{\mathbf{L}^{3/2}(U)} \\ &\leq \sup_{\nu>0}\left(\|f - f_\nu\|_{\mathbf{L}^\infty(V)}\right) \mathrm{mes}(U)^{1/3}\|\varphi\|_{\mathbf{W}^{1,3/2}(U)},\end{aligned}$$

which proves that the sequence $(R_\nu)_\nu$ is bounded in $\mathbf{W}^{-1,3}(U)$. Since $(R_\nu)_\nu$ is also clearly bounded in the space of finite signed Radon measures $\mathcal{M}_s(U)$, [26, Corollary 1.3.1] ensures that $(R_\nu)_\nu$ is precompact in $\mathbf{H}^{-1}(U)$. The same method applies to prove that the reminder of the right-hand side of (4.4) is precompact in $\mathbf{H}^{-1}(U)$. Hence,

$$\left(\mathrm{div}_{(t,x)}\begin{pmatrix}(\rho_\nu - \kappa)^+ \\ (\rho_\nu-\kappa)^+(f(x,t,\rho_\nu) - f(x,t,\kappa))\end{pmatrix}\right)_\nu \text{ is precompact in } \mathbf{H}^{-1}_{\mathbf{loc}}(\Omega).$$

Since $(\rho_\nu)_\nu \subset \mathbf{L}^\infty(\Omega)$ is bounded, for all $(x,t)\in\Omega$, the flux $f(x,t,\cdot)$ being non-degenerate in the sense required in [42] due to our assumption on $V_{\min}$, $V_{\max}$, [43, Corollary 2] yields a subsequence of $(\rho_\nu)_\nu$ that converges to some $\rho \in \mathbf{L}^\infty(\Omega)$ in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$. A further extraction yields the a.e. convergence on Ω. The fact that ρ takes values in $[0,1]$ comes from the $\mathbf{L}^\infty$ bound of Lemma 4.2. □

Corollary 4.6. *Define for all $(x,t)\in\Omega$,*

$$\xi(x,t) = \int_{\mathbb{R}} \rho(y,t)\mu(x-y)\,\mathrm{d}y\,; \quad \chi(x,t) = -\int_{\mathbb{R}}\rho(y,t)\mathbf{v}(y,t,\rho)\mu'(x-y)\,\mathrm{d}y\,.$$

Then, up to a subsequence, $(\xi_\nu)_\nu$, $(\partial_t\xi_\nu)_\nu$ and $(\mathbf{K}(\xi_\nu,\partial_t\xi_\nu))_\nu$ converge a.e. on Ω to ξ, χ and $\mathbf{K}(\xi,\chi)$, respectively.

Proof. The claim is immediate. □

We now assess the compactness of $(\mathbf{w}_\nu)_\nu$.

Corollary 4.7. *There exists $\mathbf{w}\in\mathbf{L}^\infty(\Omega;[0,1])$ such that $(\mathbf{w}_\nu)_\nu$ converges a.e. to $\mathbf{w}$ on Ω.*

Proof. Throughout this section, we ensured that all the hypotheses of Theorem A.6 are fulfilled, yielding the desired compactness. □

With the established compactness, we can prove the

Theorem 4.8. *The couple $(\rho,\mathbf{w})$ constructed in Lemma 4.5 and Corollary 4.7 is a solution to Problem (2.1) – (2.6),(2.7).*

Proof. For all $\nu>0$ and for all $(x,t)\in\Omega$, we have

$$\begin{aligned}\omega_\nu(x,t) &= \int_{-\infty}^{t-\nu}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= -\int_{t-\nu}^{t}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.\end{aligned}$$

The first term clearly vanishes as $\nu \to 0$, and since $\eta \in \mathbf{L}^1(\Omega)$, the second one converges to $\int_{-\infty}^{t} \int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s$ as $\nu \to 0$. Recall (*cf.* Lemma 4.3) that $(\omega_\nu)_\nu$ converges uniformly to ω on compact sets of Ω and we get:

$$\forall (x,t) \in \Omega,\ \omega(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.$$

It is clear from this formula that $\omega \in \mathbf{W}^{1,\infty}(\Omega)$. Apply now (4.2) with $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times [0,T)), \varphi \geq 0$, $\kappa \in [0,1]$, $s = 0$ and $\tau = T$ and let $\nu \to 0$. We get:

$$\int_0^T \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(x,t,\rho,\kappa)\partial_x \varphi - \mathrm{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x,0)\,\mathrm{d}x \geq 0.$$

This proves that ρ is an entropy solution to (2.1). Therefore, $\rho \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, see [21]. Moreover, it implies that ξ defined in Lemma 4.6 verifies for all $x \in \mathbb{R}$, $\xi(x,\cdot) \in \mathbf{W}^{1,\infty}((0,T))$ and that for a.e $t \in (0,T)$,

$$\partial_t \xi(x,t) = \chi(x,t),$$

where χ was defined in 4.6 as well. Now the convergences we have proved for $(\rho_\nu)_\nu$ and $(f_\nu)_\nu$ ensure that for a.e. $\tau, s \in [0,T]$ $(s < \tau)$,

$$\int_s^\tau \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(x,t,\rho,\kappa)\partial_x \varphi - \mathrm{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi\,\mathrm{d}x\,\mathrm{d}t \Big)$$
$$+ \int_{\mathbb{R}} |\rho(x,s) - \kappa| \varphi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau) - \kappa| \varphi(x,\tau)\,\mathrm{d}x \geq 0.$$

The expression in the left-hand side of the previous inequality is a continuous function of (s,τ) which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that ρ satisfies the entropy inequalities (3.1). To conclude the proof of the statement, we have to prove that $\mathbf{w}$ is a weak solution to (2.3). We apply (4.3) with $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times [0,T))$, $s = 0$ and $\tau = T$, and we let $\nu \to 0$. The strong convergence of $(\mathbf{w}_\nu)_\nu$ and $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu)_\nu)$ are crucial here. We obtain:

$$\int_0^T \int_{\mathbb{R}} \Big((\rho \mathbf{w})\partial_t \phi + (\rho \mathbf{v} \mathbf{w})\partial_x \phi + \rho \mathbf{K}\left(\xi, \partial_t \xi\right) \mathbf{w}(1-\mathbf{w})\phi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \rho_0(x)\mathbf{w}_0(x)\phi(x,0)\,\mathrm{d}x = 0,$$

implying in particular that $\rho\mathbf{w} \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Therefore, we can conclude the same way we did for ρ that $\mathbf{w}$ satisfies the weak formulation (3.2), concluding the proof. □

Proof of Theorem 3.2. The existence claim readily follows from Theorem 4.8. □

5 Variants of the model

In the previous section, we conducted the existence analysis of Problem (2.1) – (2.6) with (2.7). The averaging in both space and time of the orderliness marker (2.6),(2.7) allowed for a strong decoupling of the system (2.1)–(2.3) and thus led us to a proof of existence *via* a time-splitting technique with merely bounded initial datum. Notice however that, while optimal results on scalar

conservation laws feature merely $\mathbf{L}^\infty$ solutions ([37]), the assumption of bounded variation is typical in traffic modeling not only because of the numerous mathematical advantages it may offer and the consistency of the $\mathbf{BV}$-based theory, but also because it is natural in the context due to the relative smallness of the number of traffic agents.
In this section, we will adopt the setting of densities with bounded variation; within the $\mathbf{BV}$ framework, we will consider two variants of the model (2.1) – (2.6). In Section 5.1 we replace (2.7) with (2.8) with only space averaging of the orderliness marker. Note that this will be the framework of our Section 6 devoted to numerical analysis of the model. The essential property that allows for analysis and numerical analysis of this variant is the propagation of the initial $\mathbf{BV}$ regularity of the orderliness marker ω uniformly with respect to the dynamics of ρ, which is the specific feature of solutions to (2.3) intimately related to the renormalization property of [42]. Further, in Section 5.2 we will briefly discuss the local variant of the model without averaging of the orderliness marker,*i.e.* , the variant where ω is taken equal to $\mathbf{w}$. Up to the source term in (2.3) that keeps non-local character, such model boils down to a system of conservation laws, thus falling within the class of so-called GSOM (generalized second-order) models put forward in [39, 38]. The unconditional $\mathbf{BV}$ regularity for $\mathbf{w}$ (provided initial data are $\mathbf{BV}$) allows us to make a first step towards existence, however, we stress that mathematical tools for handling this situation are not ripe yet. Indeed, (2.1) becomes in this setting a conservation law with $\mathbf{BV}$ in space-time coefficients (see, *e.g.*, [43]) and one need to ensure that the candidate solutions fulfill selection criteria proper to the traffic context (see, *e.g.*, [9]) among infinitely many consistent selection criteria ([7]). The theory of (2.1),(2.2) is well understood for the case of isolated discontinuities in ω (*cf.* [31, 48]) but the case of interest, in the context of our model, requires much deeper investigation.

5.1 On the time-local model (2.1)–(2.6),(2.8)

Consider the variant of Problem (2.1) – (2.6) with averaging only in space of the orderliness marker (2.8). This simpler model keeps the non-local in space character reflecting the fact that, while the orderliness marker is attached to individual drivers, the impact (2.2),(2.6),(2.8) of the individual orderliness states on the fundamental diagram is taken in average.
The goal of this section is to sketch the existence theory, *via* convergence of the splitting approximations, based upon the propagation of the $\mathbf{BV}$ regularity of the initial datum $\mathbf{w}_0$. We do not expand this section, because the same problem is addressed in the setting of fully discrete numerical approximations in Section 6. We only point out the key arguments of the argumentation leading to convergence of the splitting approximations in this case.
To start with, we require $\rho_0, \mathbf{w}_0 \in \mathbf{BV}(\mathbb{R})$. The notion of solution is the one of Definition 3.1, with the necessary adjustment to replace (2.7) by (2.8); within the definition of solution, we can add the $\mathbf{BV}$ regularity of $\rho, \mathbf{w}$ since we achieve existence of such solutions. The splitting construction is unchanged. Our whole attention goes to the compactness issue, and at this point, we change the order of arguments and fully change the compactness analysis of $\mathbf{w}$. With $\mathbf{BV}$ datum ρ_0, compactness for $(\rho_\nu)_\nu$ is straightforward to obtain and it comes without the assumption on $V_{\min}, V_{\max}$ of Lemma 4.5. Indeed, due to the uniform space regularity of $(\omega_\nu)_\nu$ we can infer that $(\rho_\nu)_\nu$ is bounded in $\mathbf{L}^\infty([0,T];\mathbf{BV}(\mathbb{R}))$, see [21]. For $(\mathbf{w}_\nu)_\nu$, global $\mathbf{BV}$ bounds can be explained by the fact, highlighted in [42], that weak solutions to equations like (2.3) behave like if they were evolving along characteristics. In the basic sourceless case with piecewise constant data, this means that the solution at any time assumes the same states - and in the same order - as the initial datum, therefore its variation in space is controlled, for any time, by the variation of the initial data. For general $\mathbf{BV}$ datum and in presence of the source term, in order to infer this property one can rely upon the regularization approach of Appendix A and the renormalization property. We do not develop the

argument here, but we stress that the numerical counterpart of the $\mathbf{BV}$ bound for $(\mathbf{w}_\nu)_\nu$ is assessed in detail in Section 6. While in Section 6 we require the restriction $\rho_0 \geq \varepsilon > 0$ in the appropriate area, see (3.3), let us stress here that this restriction is needed only to define the scheme and to guarantee the appropriate CFL condition. As far as the splitting procedure is considered, there is no need to introduce this restriction, as one can see it from the arguments of Appendix A where the case of $\rho \geq 0$ can be handled *via* a regularization procedure.

5.2 On the local model (2.1)–(2.6)

In this subsection, we discuss the purely local variant of our model, taking $\mathcal{M} = \mathrm{Id}$ in (2.6); in other words, we consider the situation where the 2×2 system on $\rho, \mathbf{w}$ and ω is closed by identifying ω with $\mathbf{w}$. The resulting model is a variant of GSOM (generalized second-order) models proposed in [39, 38], inspired by the already classical Aw-Rascle and Zhang model (ARZ). However, due to the choice (2.2) of the velocity, in our case the model need not reduce to a hyperbolic system with one genuinely nonlinear and one linearly degenerate field. Let us sketch a non-standard approach to this kind of GSOM models. First, as in Section 5.1, the dynamics of $\mathbf{w}$ ensures the propagation of $\mathbf{BV}$ regularity if we assume $\mathbf{w}_0 \in \mathbf{BV}(\mathbb{R})$. For the sake of simplicity, consider first the case where $\mathbf{K} = 0$. Then it can be shown using the theory of [42] - due to the fact that the renormalization property is valid for general Borel functions - that piecewise constant $\mathbf{w}_0$ lead to piecewise constant $\mathbf{w}$ (*cf.* [38] for the analogous observation in the frame of GSOM). In this particular case equation (2.1) becomes a discontinuous-flux conservation law with separated interfaces. The theory (or, rather, multiple theories) of such equations were developed over more than 25 years, and we point out that it is possible to apply such theories in order to define the notion of solution to the model we are dealing with, and more generally, to GSOM models with or without the standard hyperbolicity structure. The key issue is to select the appropriate coupling conditions across discontinuities of $\omega \equiv \mathbf{w}$ (called interfaces), which is a clearly understood issue in the traffic context. According to phenomenological argumentation and to the numerical simulations involving the deterministic many-particle approximation (the so-called Follow-the-leader model), see [9], the coupling condition is the one maximizing the flux across interfaces. Either we do not pursue this line in the present paper, let us point out that - for piecewise constant initial datum $\mathbf{w}_0$ of the orderliness marker - it is possible to define solutions (admissible in the sense of maximizing the flow across interfaces) for the splitting scheme we used in Section 4, and pass to the limit in the scheme. The compactness of $(\rho_\nu)_\nu$ can be assessed relying on the non-degeneracy of the flux [43]. The general setting with piecewise $\mathbf{C}^1$ or merely $\mathbf{BV}$ component $\mathbf{w}$ of the solution is a challenging issue for which some elements of analysis are ready, and others are lacking. Let us pinpoint the two main issues we leave for future work:

- One needs a plausible (on heuristic grounds, such as the uniqueness for Riemann problems) characterization of admissible solutions suitable for general $\omega \equiv \mathbf{w} \in \mathbf{BV}$.We stress that the one of [43], obtained in a very general setting, does not lead to uniqueness for general flux configurations but may be sufficient in the setting we are considering. In particular, due to the fact that $V_{\max} \geq V_{\min}$ in our model, fundamental diagrams for different values of $\omega \equiv \mathbf{w}$ do not cross, so that the crossing condition of [30] is automatically fulfilled. In this situation, the optimal-flux entropy solutions we are interested in coincide with the so-called vanishing viscosity solutions studied in [30, 8](see also [6] and in [22]). Note that a subtler characterization of admissible vanishing viscosity solutions is provided in [8] and [22]; the particularity of [22] is that the analysis extends to the general $\mathbf{BV}$ structure of the flux, which is what we have in mind.

- Being understood that the uniqueness of solutions for the system is probably beyond the reach of full analysis, it would be interesting to assess uniqueness of ρ, given $\omega \equiv \mathbf{w} \in \mathbf{BV}(\mathbb{R})$. Towards this goal, delicate refinements of techniques of [30, 8, 22] need to be elaborated.

To sum up, the present investigation of the non-local problem (2.1)–(2.6) highlights a novel approach to the definition of admissibility of solutions of the local GSOM models, weakening at the same time the requirement on the hyperbolic structure of the system. Last but not least, the numerical strategy developed in Section 6 below for the spatially non-local problem of Section 5.1 is applicable also to the local problem of Section 5.2, provided consistent discretization of (2.1),(2.2) is used taking into account the possible sharp discontinuities in the expression of the flux function (*cf.* [48]).

6 Numerical approximation

In this section, we develop a finite volume numerical scheme for approximation of the model (2.1)–(2.6), with the averaging operator $\mathcal{M}$ in (2.6) given by (2.8). We analyze the **BV** stability and infer the convergence of the scheme. The approximation of the transport equation (2.3) is obtained exploiting the idea of propagation along characteristics; to state the idea clearly, we start with a simplified problem and expose the motivations behind the marching formula for the component $(\mathbf{w}^n_{j+1/2})_{j\in\mathbb{Z}}$ of the numerical solution. The scheme for the simplified problem turns out to be similar to the approximation of the Keyfitz-Kranzer [35] system put forward in [36], see Remark 6.1.

6.1 Motivation

We build a simple finite volume scheme and prove its convergence to a solution of (2.1) – (2.6) with (2.8) this time. Let us explain the ideas behind the construction of our scheme. For the sake of clarity, instead of (2.1) – (2.6), consider the problem

$$\left\{\begin{array}{rcl} \partial_t\rho + \partial_x\left(f(\rho)\right) &=& 0 \\ \partial_t\left(\rho\mathbf{w}\right) + \partial_x\left(f(\rho)\mathbf{w}\right) &=& \rho\mathbf{S}(x,t). \end{array}\right. \tag{6.1}$$

This system is a triangular one in the sense that we can solve the first equation and find ρ without $\mathbf{w}$, and then solve the second one. Numerically, this is what we do as well. The approximate density $\rho_\Delta = (\rho^n_{j+1/2})_{n,j}$ is constructed with a standard finite volume scheme:

$$(\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2})\Delta x + (f^n_{j+1} - f^n_j)\Delta t = 0,$$

where f^n_j is a suitable approximation of the flux $f(\rho)$, see (6.2). We then use these values to construct $\mathbf{w}_\Delta$. The starting point is that if all the involved functions are smooth and if $\rho > 0$, the second PDE in (6.1) can be solved with the method of characteristics. More precisely, if $x \in \mathbf{C}^1((0,T))$ and $u(t) = \mathbf{w}(x(t),t)$, assuming in addition that $\rho > 0$ in Ω, the second equation in (6.1) can be solved by solving the family of ODE systems

$$\begin{cases} x'(t) = \mathbf{v}(\rho(x(t),t)) = \dfrac{f(\rho(x(t),t))}{\rho(x(t),t)} \\ \\ u'(t) = \mathbf{S}(x(t),t). \end{cases}$$

On each time step $[t^n, t^{n+1})$, for all $j \in \mathbb{Z}$, we draw characteristics starting from x_j with slope $s^n_j := \dfrac{f^n_j}{\rho^{n+1}_{j+1/2}}$, which is our choice for the approximation of $\dfrac{f(\rho(x(t),t))}{\rho(x(t),t)}$. At this point we need to know that $\rho^{n+1}_{j+1/2} \geq \varepsilon > 0$, in order to guarantee the existence of a CFL condition ensuring that at time t^{n+1}, the characteristics which started at x_j ends up at point $X^{n+1}_j \in (x_j, x_{j+1})$, see Figure 2.

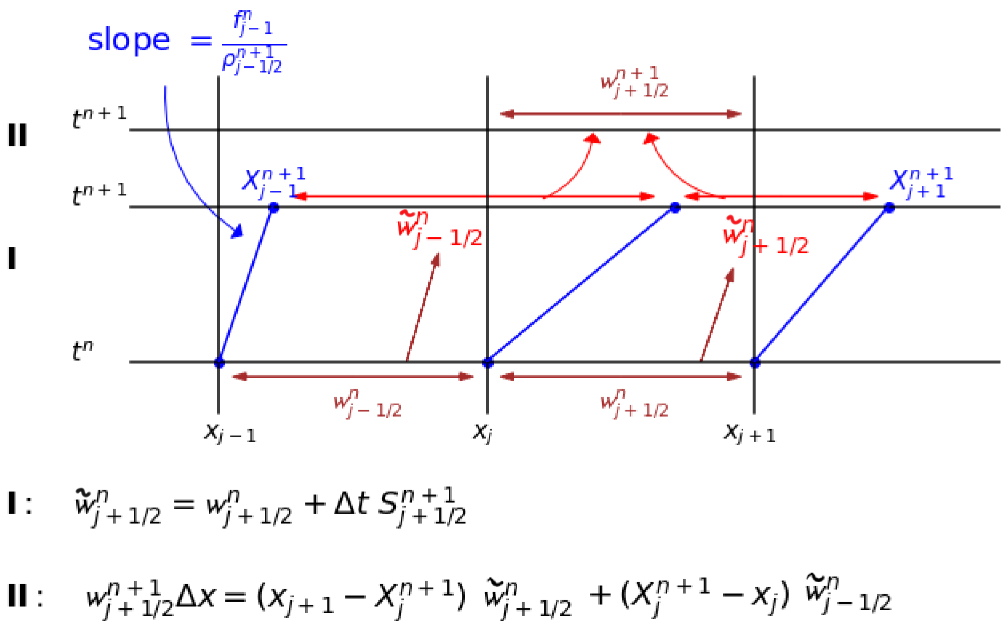

Figure 2: Illustration of the two steps of the construction of the scheme.

Now, the ODE solved by $u(t) = \mathbf{w}(x(t), t)$ tells us that

$$u(t^{n+1}) = u(t^n) + \int_{t^n}^{t^{n+1}} \mathbf{S}(x(t), t)\,\mathrm{d}t \;\; i.e.\;\; \mathbf{w}(X^{n+1}_j, t^{n+1}) \simeq \underbrace{\mathbf{w}(x_j, t^n) + \Delta t \mathbf{S}^{n+1}_{j+1/2}}_{\widetilde{\mathbf{w}}^n_{j+1/2}},$$

$\mathbf{S}^{n+1}_{j+1/2}$ being a suitable approximation of the source term on the cell $(x_j, x_{j+1}) \times (t^n, t^{n+1}]$. At the numerical level, we are led to assign the value $\widetilde{\mathbf{w}}^n_{j+1/2}$ on (X^{n+1}_j, x_{j+1}). At this point we choose to define $\mathbf{w}_\Delta(\cdot, t^{n+1})$ on (x_j, x_{j+1}) by averaging the values $\widetilde{\mathbf{w}}^n_{j-1/2}$ and $\widetilde{\mathbf{w}}^n_{j+1/2}$ on (x_j, x_{j+1}). This is expressed as:

$$\begin{aligned}
\mathbf{w}^{n+1}_{j+1/2}\Delta x &= (x_{j+1} - X^{n+1}_j)\widetilde{\mathbf{w}}^n_{j+1/2} + (X^{n+1}_j - x_j)\widetilde{\mathbf{w}}^n_{j-1/2} \\
&= \left(\left(1 - \frac{\Delta t}{\Delta x} s^n_j\right)\widetilde{\mathbf{w}}^n_{j+1/2} + \frac{\Delta t}{\Delta x} s^n_j \widetilde{\mathbf{w}}^n_{j-1/2}\right)\Delta x
\end{aligned}$$

The above choices lead to a conservative scheme for $\rho\mathbf{w}$. Looking at the simplest case $\mathbf{S} = 0$ ($\widetilde{\mathbf{w}}^n_{j+1/2} = \mathbf{w}^n_{j+1/2}$), by multiplying the last expression by $\rho^{n+1}_{j+1/2}$, we find that:

$$\begin{aligned}
&\left((\rho\mathbf{w})^{n+1}_{j+1/2} - (\rho\mathbf{w})^n_{j+1/2}\right)\Delta x \\
&= \rho^{n+1}_{j+1/2}\left(\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}\right)\Delta x + \left(\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}\right)\mathbf{w}^n_{j+1/2}\Delta x \\
&= -\rho^{n+1}_{j+1/2} s^n_j \left(\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right)\Delta t - (f^n_{j+1} - f^n_j)\mathbf{w}^n_{j+1/2}\Delta t \\
&= -f^n_j \left(\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right)\Delta t - (f^n_{j+1} - f^n_j)\mathbf{w}^n_{j+1/2}\Delta t \\
&= -\left(f^n_{j+1}\mathbf{w}^n_{j+1/2} - f^n_j \mathbf{w}^n_{j-1/2}\right)\Delta t,
\end{aligned}$$

so that the numerical flux for $\rho\mathbf{w}$ turns out to be $f_j^n \mathbf{w}_{j-1/2}^n$. This observation is a cornerstone of our convergence proof.

Remark 6.1. In the case $\mathbf{S} \equiv 0$, system (6.1) has the same structure as the classical Keyfitz-Kranzer system [35] up to the properties of the flux function f which is monotone in the Keyfitz-Kranzer case and which is bell-shaped in the case we are concerned with, see also [15]. Discretization of the Keyfitz-Kranzer system by finite difference schemes was addressed, in particular, in [36]. One of the schemes proposed in this reference (see [36, Section 5]) closely resembles our scheme. In the setting of [36] the flux has the form $f(\rho) = \rho\phi(\rho)$ but the assumptions on ϕ - different from our assumptions on $\mathbf{v}$ - ensure that f is increasing. Therefore the upwind choice is made for the numerical fluxes: $f_j^n = \rho_{j-1/2}^n \phi(\rho_{j-1/2}^n)$. The scheme of [36, Section 5] then reads:

$$\begin{aligned}
\rho_{j+1/2}^{n+1} &= \rho_{j+1/2}^n - \frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n)\\
\mathbf{w}_{j+1/2}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x}\bar{s}_j^n\right)\mathbf{w}_{j+1/2}^n + \frac{\Delta t}{\Delta x}\bar{s}_j^n \mathbf{w}_{j-1/2}^n.
\end{aligned}$$

with $\bar{s}_j^n = \frac{f_j^n}{\rho_{j-1/2}^n} = \phi(\rho_{j-1/2}^n)$ due to the upwind choice for f_j^n. This choice of $\bar{s}_j^n$ differs slightly from our choice of s_j^n. It does not require the lower bound on $\rho_{j-1/2}^n$, but this is due to the monotonicity of f and cannot be mimicked in the setting of bell-shaped f which is the ours.

The ideas to deal with Problem (2.1) – (2.6) are the same as the ones we just develop. The difference is the presence of the coupling between ρ and $\mathbf{w}$. The coupling is taken care of in Step 1 below. Section 6.2 details the construction of the scheme for (2.1) – (2.6), following the ideas developed above.

6.2 Definition of the scheme

In what concerns the initial density, we assume that $\mathbf{TV}(\rho_0) < +\infty$ and that ρ_0 is separated from the vacuum in the sense stated in assumption (3.3); for the initial orderliness, we assume that $\mathbf{TV}(\mathbf{w}_0) < +\infty$ and $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R};[0,1])$.
For a fixed spatial mesh size $\Delta x > 0$ and time mesh size $\Delta t > 0$, let $x_j = j\Delta x$ ($j \in \mathbb{Z}$), $t^n = n\Delta t$ ($n \in \mathbb{N}$) and $N \in \mathbb{N}^*$ such that $T \in (t^N, t^{N+1}]$. We define the cell grids:

$$\mathbb{R} \times (0,T] \subset \bigcup_{n=0}^{N} \bigcup_{j\in\mathbb{Z}} \mathcal{P}_{j+1/2}^{n+1}, \quad \mathcal{P}_{j+1/2}^{n+1} = (x_j, x_{j+1}) \times (t^n, t^{n+1}].$$

We aim at constructing an approximate solution $(\rho_\Delta, \mathbf{w}_\Delta)$ defined almost everywhere on Ω:

$$\begin{cases}
\rho_\Delta = \rho_0 \mathbb{1}_{\{t\le 0\}} + \displaystyle\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}} \rho_{j+1/2}^{n+1} \mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}}\\
\mathbf{w}_\Delta = \displaystyle\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}} \mathbf{w}_{j+1/2}^{n} \mathbb{1}_{(x_j,x_{j+1})\times[t^n,t^{n+1})}.
\end{cases}$$

First, we discretize the initial data ρ_0 (respect. $\mathbf{w}_0$) with $\left(\rho_{j+1/2}^0\right)_j$, (respect. with $\left(\mathbf{w}_{j+1/2}^0\right)_j$) where for all $j \in \mathbb{Z}$, $\rho_{j+1/2}^0$ (respect. $\mathbf{w}_{j+1/2}^0$) is its mean value on the cell (x_j, x_{j+1}). Fix $n \in \{0, \ldots, N\}$.

Step 1: Orderliness marker (mean value). For all $j \in \mathbb{Z}$, define

$$\omega_j^n = \int_{\mathbb{R}} \mathbf{w}_\Delta(x, t^n)\eta(x_j - y)\,\mathrm{d}y = \sum_{i\in\mathbb{Z}} \mathbf{w}_{i+1/2}^n \underbrace{\left(\int_{x_i}^{x_{i+1}} \eta(x_j - y)\,\mathrm{d}y\right)}_{\eta_{j-(i+1/2)}}.$$

Step 2: Finite volumes for the density. We use ω_j to define the velocity

$$\mathbf{v}_j^n(\rho) = (1 - \omega_j^n)V_{\min}(\rho) + \omega_j^n V_{\max}(\rho)$$

and the flux $f_j^n(\rho) = \rho\mathbf{v}_j^n(\rho)$. Introduce the notations:

$$f_{\min,\max}(\rho) = \rho V_{\min,\max}(\rho);\ \delta f = f_{\max} - f_{\min}.$$

Let $\mathbf{F}_j^n = \mathbf{F}_j^n(u, v)$ be a monotone, Lipschitz and consistent numerical flux associated with f_j^n. For the sake of simplicity, we use the Rusanov flux, that is for all $u, v \in [0, 1]$,

$$\mathbf{F}_j^n(u, v) = \frac{1}{2}\left(f_j^n(u) + f_j^n(v) + \mathbf{L}(u - v)\right),\ \mathbf{L} = \max\{\|f'_{\min}\|_{\mathbf{L}^\infty}, \|f'_{\max}\|_{\mathbf{L}^\infty}\}.$$

The conservation of ρ written in a cell $\mathcal{P}_{j+1/2}^{n+1}$ ($j \in \mathbb{Z}$) leads to the following marching formula:

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1}^n(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n)\right). \tag{6.2}$$

Eventually, it will be convenient to write the scheme under the form:

$$\rho_{j+1/2}^{n+1} = \mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right),$$

where $\mathbf{H}_j^n = \mathbf{H}_j^n(a, b, c)$ is given by the right-hand side of (6.2) with $\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n$ replaced by $a, b, c \in [0, 1]$.

Step 3: Source term. For all $j \in \mathbb{Z}$, we set

$$\begin{cases} \xi_{j+1/2}^{n+1} = \displaystyle\sum_{i\in\mathbb{Z}} \rho_{i+1/2}^{n+1} \underbrace{\left(\int_{x_i}^{x_{i+1}} \mu(x_{j+1/2} - y)\,\mathrm{d}y\right)}_{\mu_{j+1/2-(i+1/2)}} \\ \chi_{j+1/2}^{n+1} = -\displaystyle\sum_{i\in\mathbb{Z}} \mathbf{F}_i^n(\rho_{i-1/2}^{n+1}, \rho_{i+1/2}^{n+1}) \underbrace{\left(\int_{x_i}^{x_{i+1}} \mu'(x_{j+1/2} - y)\,\mathrm{d}y\right)}_{\mathrm{d}\mu_{j+1/2-(i+1/2)}}. \end{cases}$$

Note that hereabove, we discretize the expression for $\chi = \partial_t\xi$ that is obtained combining the definition of ξ and the weak formulation of the mass conservation equation.

Then we define the source term by

$$\forall j \in \mathbb{Z}, \quad \mathbf{S}_{j+1/2}^{n+1} = \mathbf{K}\left(\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1}\right)\mathbf{w}_{j+1/2}^n(1 - \mathbf{w}_{j+1/2}^n).$$

Step 4**:** Orderliness marker. Fix $j \in \mathbb{Z}$. Set

$$X_j^{n+1} = x_j + \Delta t \underbrace{\left(\frac{\mathbf{F}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n)}{\rho_{j+1/2}^{n+1}} \right)}_{s_j^n}.$$

We will prove that under (3.3) and a suitable CFL condition, see (6.4), the sequence $(X_j^{n+1})_j$ is well defined. Following the approach outlined in Section 6.1, we compute the updated orderliness marker as follows:

$$\begin{cases} \widetilde{\mathbf{w}}_{j+1/2}^n = \mathbf{w}_{j+1/2}^n + \Delta t \mathbf{S}_{j+1/2}^{n+1} \\ \mathbf{w}_{j+1/2}^{n+1} = \left(1 - \dfrac{\Delta t}{\Delta x} s_j^n\right) \widetilde{\mathbf{w}}_{j+1/2}^n + \dfrac{\Delta t}{\Delta x} s_j^n \widetilde{\mathbf{w}}_{j-1/2}^n. \end{cases} \tag{6.3}$$

We also define

$$(\xi_\Delta, \chi_\Delta, \mathbf{S}_\Delta) = \sum_{n=0}^{N} \sum_{j \in \mathbb{Z}} (\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1}, \mathbf{S}_{j+1/2}^{n+1}) \mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}}$$

and

$$\omega_\Delta = \sum_{n=0}^{N} \sum_{j \in \mathbb{Z}} \omega_j^n \mathbb{1}_{(x_j, x_{j+1}) \times [t^n, t^{n+1})}.$$

For later use, introduce the notations:

$$\|\mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{\varepsilon \le \xi \le 1 \\ |\chi| \le \mathbf{L} \times \mathbf{TV}(\mu)}} |\mathbf{K}(\xi, \chi)|; \quad \|\nabla \mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{\varepsilon \le \xi_1, \xi_2 \le 1 \\ |\chi_1|, |\chi_2| \le \mathbf{L} \times \mathbf{TV}(\mu)}} |\mathbf{K}(\xi_1, \chi_1) - \mathbf{K}(\xi_2, \chi_2)|$$

and

$$\|\delta f\|_{\mathbf{L}^\infty} = \sup_{0 \le \rho \le 1} \delta f(\rho); \quad \|\delta f'\|_{\mathbf{L}^\infty} = \sup_{0 \le \rho \le 1} |\delta f'(\rho)|.$$

6.3 $\mathbf{L}^\infty$ stability *via* monotonicity

Proposition 6.1. *Under the conditions*

$$\lambda \max\left\{2, \frac{1}{\varepsilon}\right\} \mathbf{L} \le 1; \quad \lambda = \frac{\Delta t}{\Delta x} \tag{6.4}$$

and

$$\Delta t \|\mathbf{K}\|_{\mathbf{L}^\infty} \le 1, \tag{6.5}$$

the scheme (6.2)-(6.3) *is monotone and* $\mathbf{L}^\infty$ *stable. More precisely, for all* $n \in \{0, \dots, N+1\}$ *and* $j \in \mathbb{Z}$*, we have*

$$\varepsilon \le \rho_{j+1/2}^n \le 1 \quad \textit{and} \quad 0 \le \mathbf{w}_{j+1/2}^n \le 1. \tag{6.6}$$

Proof. We prove the result by induction on n.
The result is clearly true for $n = 0$ by definition of $\left(\rho_{j+1/2}^0\right)_j$ and $\left(\mathbf{w}_{j+1/2}^0\right)_j$. Suppose now that for some $n \in \{0, \dots, N\}$, (6.6) holds. Fix $j \in \mathbb{Z}$.
(i) Since $0 \le \mathbf{w}_\Delta(\cdot, t^n) \le 1$, we have

$$\omega_j^n = \int_{\mathbb{R}} \mathbf{w}_\Delta(y, t^n) \eta(x_j - y) \, \mathrm{d}y \in [0, 1],$$

from which we deduce that f_j^n is a convex combination of $f_{\min}$ and $f_{\max}$. Note also that

$$\begin{aligned}|\omega_{j+1}^n - \omega_j^n| &\leq \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} |\mathbf{w}_{j+1/2}^n| \cdot |\eta(x_{j+1} - y) - \eta(x_j - y)| \,\mathrm{d}y \\ &\leq \int_{\mathbb{R}} |\eta(y - \Delta x) - \eta(y)| \,\mathrm{d}y \leq \mathbf{TV}(\eta)\Delta x.\end{aligned}$$

(ii) Using the CFL condition, we can prove that the scheme (6.2) is monotone. More precisely, for a.e. $a, b, c \in [0,1]$, we have:

$$\frac{\partial \mathbf{H}_j^n}{\partial a}(a,b,c) = \lambda \frac{\partial \mathbf{F}_j^n}{\partial u}(a,b) \geq 0; \quad \frac{\partial \mathbf{H}_j^n}{\partial c}(a,b,c) = -\lambda \frac{\partial \mathbf{F}_{j+1}^n}{\partial v}(b,c) \geq 0$$

and

$$\frac{\partial \mathbf{H}_j^n}{\partial b}(a,b,c) = 1 - \lambda \left(\frac{\partial \mathbf{F}_{j+1}^n}{\partial u}(b,c) - \frac{\partial \mathbf{F}_j^n}{\partial u}(a,b) \right) \geq 1 - 2\lambda \mathbf{L} \geq 0.$$

Using the monotonicity of the scheme and the induction property, we deduce that

$$\rho_{j+1/2}^{n+1} = \mathbf{H}_j^n \left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n \right) \leq \mathbf{H}_j^n(1,1,1) = 1$$

and, since $\delta f(\varepsilon) = 0$ due to assumption (3.3),

$$\rho_{j+1/2}^{n+1} \geq \mathbf{H}_j^n(\varepsilon,\varepsilon,\varepsilon) = \varepsilon - \lambda(\omega_{j+1}^n - \omega_j^n)\delta f(\varepsilon) = \varepsilon.$$

(iii) Since $\varepsilon \leq \rho_\Delta(\cdot, t^{n+1}) \leq 1$, we have

$$\xi_{j+1/2}^{n+1} = \int_{\mathbb{R}} \rho_\Delta(y, t^{n+1}) \mu(x_{j+1/2} - y) \,\mathrm{d}y \in [\varepsilon, 1],$$

and clearly,

$$|\chi_{j+1/2}^{n+1}| \leq \mathbf{L} \times \mathbf{TV}(\mu).$$

(iv) Let us prove that $\widetilde{\mathbf{w}}_{j+1/2}^n \in [0,1]$. Introduce the function

$$g : \mathbf{w} \mapsto \mathbf{w} + \Delta t \mathbf{K} \left(\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1} \right) \mathbf{w}(1 - \mathbf{w}).$$

Using (6.5), we obtain that for all $\mathbf{w} \in [0,1]$,

$$g'(\mathbf{w}) = 1 + \Delta t \mathbf{K} \left(\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1} \right) (1 - 2\mathbf{w}) \geq 1 - \Delta t \left| \mathbf{K} \left(\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1} \right) \right| \geq 0.$$

Since $g(0) = 0$ and $g(1) = 1$, the monotonicity of g implies that $\widetilde{\mathbf{w}}_{j+1/2}^n = g(\mathbf{w}_{j+1/2}^n) \in [0,1]$. Due to the CFL condition, $\mathbf{w}_{j+1/2}^{n+1}$ is a convex combination of $\widetilde{\mathbf{w}}_{j+1/2}^n$ and $\widetilde{\mathbf{w}}_{j-1/2}^n$. This implies that $\mathbf{w}_{j+1/2}^{n+1} \in [0,1]$, which completes the induction argument. □

Remark 6.2. The stability estimates (6.6) immediately imply:

$$\varepsilon \leq \rho_\Delta, \xi_\Delta \leq 1; \quad 0 \leq \mathbf{w}_\Delta, \omega_\Delta \leq 1; \quad |\chi_\Delta| \leq \mathbf{L} \times \mathbf{TV}(\mu); \quad |\mathbf{S}_\Delta| \leq \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}.$$

For all $a, b \in [0,1]$, set

$$a \wedge b = \min\{a,b\} \quad a \vee b = \max\{a,b\}.$$

Corollary 6.2 (Discrete entropy inequalities)**.** *The numerical scheme (6.2) fulfills the following discrete entropy inequalities for all $n \in \{0, \dots, N\}$, $j \in \mathbb{Z}$ and $\kappa \in [0,1]$:*

$$\begin{aligned} &\left(\left|\rho^{n+1}_{j+1/2} - \kappa\right| - \left|\rho^n_{j+1/2} - \kappa\right|\right)\Delta x + (\Phi^n_{j+1} - \Phi^n_j)\Delta t \\ &\leq -\operatorname{sgn}\left(\rho^{n+1}_{j+1/2} - \kappa\right) \times (f^n_{j+1}(\kappa) - f^n_j(\kappa))\Delta t, \end{aligned} \tag{6.7}$$

where Φ^n_j denotes the numerical entropy flux:

$$\Phi^n_j = \mathbf{F}^n_j\left(\rho^n_{j-1/2} \vee \kappa, \rho^n_{j+1/2} \vee \kappa\right) - \mathbf{F}^n_j\left(\rho^n_{j-1/2} \wedge \kappa, \rho^n_{j+1/2} \wedge \kappa\right).$$

Proof. This is mostly a consequence of the scheme monotonicity. Remark that

$$\forall j \in \mathbb{Z}, \quad \mathbf{H}^n_j(\kappa, \kappa, \kappa) = \kappa - \lambda(f^n_{j+1}(\kappa) - f^n_j(\kappa)).$$

We combine this with the convexity of the function $|\cdot - \kappa|$ to obtain:

$$\begin{aligned} &\left|\rho^{n+1}_{j+1/2} - \kappa\right| \\ &= \left|\mathbf{H}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}, \rho^n_{j+3/2}\right) - \kappa\right| \\ &\leq \left|\mathbf{H}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}, \rho^n_{j+3/2}\right) - \mathbf{H}^n_j(\kappa,\kappa,\kappa)\right| + \operatorname{sgn}\left(\rho^{n+1}_{j+1/2} - \kappa\right) \times \left(\mathbf{H}^n_j(\kappa,\kappa,\kappa) - \kappa\right) \\ &\leq \mathbf{H}^n_j\left(\rho^n_{j-1/2} \vee \kappa, \rho^n_{j+1/2} \vee \kappa, \rho^n_{j+3/2} \vee \kappa\right) - \mathbf{H}^n_j\left(\rho^n_{j-1/2} \wedge \kappa, \rho^n_{j+1/2} \wedge \kappa, \rho^n_{j+3/2} \wedge \kappa\right) \\ &- \lambda \operatorname{sgn}\left(\rho^{n+1}_{j+1/2} - \kappa\right) \times (f^n_{j+1}(\kappa) - f^n_j(\kappa)) \\ &= \left|\rho^n_{j+1/2} - \kappa\right| - \lambda(\Phi^n_{j+1} - \Phi^n_j) - \lambda \operatorname{sgn}\left(\rho^{n+1}_{j+1/2} - \kappa\right) \times (f^n_{j+1}(\kappa) - f^n_j(\kappa)). \end{aligned}$$

□

6.4 Compactness *via* BV stability

The key to obtain compactness is to derive global **BV** bounds for $(\rho_\Delta, \mathbf{w}_\Delta)_\Delta$.

Theorem 6.3. *There exists a constant $c > 0$ such that for all $n \in \{1, \dots, N\}$:*

$$\mathbf{TV}(\rho_\Delta(\cdot, t^n)) + \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) \leq (\mathbf{TV}(\rho_0) + \mathbf{TV}(\mathbf{w}_0))e^{(2c + c^2\Delta t)t^n}. \tag{6.8}$$

Proof. Fix $n \in \{0, \dots, N\}$ and $j \in \mathbb{Z}$. For the sake of clarity, set

$$\mathcal{F}^n_j = \mathbf{F}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}\right).$$

We start by writing the scheme (6.2) under the form:

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \underbrace{\lambda\left(\frac{\mathcal{F}_{j+1}^n - \mathbf{F}_{j+1}^n\left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right)}{\rho_{j+3/2}^n - \rho_{j+1/2}^n}\right)}_{-B_{j+1}}\left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right)$$

$$- \underbrace{\lambda\left(\frac{\mathbf{F}_j^n\left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right) - \mathcal{F}_j^n}{\rho_{j+1/2}^n - \rho_{j-1/2}^n}\right)}_{A_j}\left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right) - \lambda(\omega_{j+1}^n - \omega_j^n)\delta f(\rho_{j+1/2}^n).$$

The monotonicity of $\mathbf{F}_{j+1}^n$ and $\mathbf{F}_j^n$ ensures that $A_j, B_{j+1} \geq 0$. We deduce that

$$\begin{aligned}
\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1} &= (1 - A_j - B_j)\left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right) \\
&\quad + A_{j-1}\left(\rho_{j-1/2}^n - \rho_{j-3/2}^n\right) + B_{j+1}\left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right) \\
&\quad - \lambda(\omega_{j+1}^n - \omega_j^n)\delta f(\rho_{j+1/2}^n) + \lambda(\omega_j^n - \omega_{j-1}^n)\delta f(\rho_{j-1/2}^n).
\end{aligned}$$

Making use of the CFL condition (6.4), we have

$$|A_j| + |B_j| \leq 2\lambda \mathbf{L} \leq 1,$$

hence:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\left|\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}\right| &\leq \sum_{j\in\mathbb{Z}}(1 - A_j - B_j)\left|\rho_{j+1/2}^n - \rho_{j-1/2}^n\right| \\
&\quad + \sum_{j\in\mathbb{Z}} A_{j-1}\left|\rho_{j-1/2}^n - \rho_{j-3/2}^n\right| + \sum_{j\in\mathbb{Z}} B_{j+1}\left|\rho_{j+3/2}^n - \rho_{j+1/2}^n\right| \\
&\quad + \lambda\sum_{j\in\mathbb{Z}}\left|(\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n)\delta f(\rho_{j+1/2}^n)\right| \\
&\quad + \lambda\sum_{j\in\mathbb{Z}}\left|(\omega_j^n - \omega_{j-1}^n)\left(\delta f(\rho_{j+1/2}^n) - \delta f(\rho_{j-1/2}^n)\right)\right| \\
&\leq \left(1 + \Delta t \mathbf{TV}(\eta)\|\delta f'\|_{\mathbf{L}^\infty}\right)\sum_{j\in\mathbb{Z}}\left|\rho_{j+1/2}^n - \rho_{j-1/2}^n\right| \\
&\quad + \lambda\|\delta f\|_{\mathbf{L}^\infty}\sum_{j\in\mathbb{Z}}\left|\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n\right|.
\end{aligned}$$

We now rewrite the last term of the inequality using the Abel procedure. For all $j \in \mathbb{Z}$, we have

$$\begin{aligned}
\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n &= \sum_{i\in\mathbb{Z}} \mathbf{w}_{i+1/2}^n\Big(\left(\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}\right) - \left(\eta_{j-(i+1/2)} - \eta_{j-(i+3/2)}\right)\Big) \\
&= \sum_{i\in\mathbb{Z}}\left(\mathbf{w}_{i+1/2}^n - \mathbf{w}_{i-1/2}^n\right)\left(\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}\right),
\end{aligned}$$

from which we deduce:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}|\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n| &\leq \sum_{i\in\mathbb{Z}}|\mathbf{w}_{i+1/2}^n - \mathbf{w}_{i-1/2}^n|\left(\sum_{j\in\mathbb{Z}}|\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}|\right) \\
&\leq \mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta x.
\end{aligned}$$

We now derive a similar estimate for $(\mathbf{w}_\Delta)_\Delta$. We have

$$\begin{aligned}\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^{n+1}_{j-1/2} &= \left(1-\lambda s^n_j\right)\left(\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right) + \lambda s^n_{j-1}\left(\mathbf{w}^n_{j-1/2} - \mathbf{w}^n_{j-3/2}\right)\\ &\quad + \Delta t\left\{\left(1-\lambda s^n_j\right)\left(\mathbf{S}^{n+1}_{j+1/2} - \mathbf{S}^{n+1}_{j-1/2}\right) + \lambda s^n_{j-1}\left(\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j-3/2}\right)\right\}.\end{aligned}$$

Since $0 \le \lambda s^n_j \le 1$ due to the CFL condition, we obtain

$$\sum_{j\in\mathbb{Z}}\left|\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^{n+1}_{j-1/2}\right| \le \sum_{j\in\mathbb{Z}}\left|\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right| + \Delta t\sum_{j\in\mathbb{Z}}\left|\mathbf{S}^{n+1}_{j+1/2} - \mathbf{S}^{n+1}_{j-1/2}\right|.$$

But

$$\begin{aligned}\left|\mathbf{S}^{n+1}_{j+1/2} - \mathbf{S}^{n+1}_{j-1/2}\right| &\le \|\mathbf{K}\|_{\mathbf{L}^\infty}\left|\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right|\\ &\quad + \frac{\|\nabla\mathbf{K}\|_{\mathbf{L}^\infty}}{4}\left(\left|\xi^{n+1}_{j+1/2} - \xi^{n+1}_{j-1/2}\right| + \left|\chi^{n+1}_{j+1/2} - \chi^{n+1}_{j-1/2}\right|\right),\end{aligned}$$

so that from

$$\begin{aligned}\xi^{n+1}_{j+1/2} - \xi^{n+1}_{j-1/2} &= \sum_{i\in\mathbb{Z}}\rho^{n+1}_{i+1/2}(\mu_{j+1/2-(i+1/2)} - \mu_{j-1/2-(i+1/2)})\\ &= \sum_{i\in\mathbb{Z}}\rho^{n+1}_{i+1/2}(\mu_{j+1/2-(i+1/2)} - \mu_{j+1/2-(i-1/2)})\\ &= \sum_{i\in\mathbb{Z}}(\rho^{n+1}_{i+1/2} - \rho^{n+1}_{i+3/2})\mu_{j+1/2-(i+1/2)},\end{aligned}$$

we deduce (remember that μ is a weight function):

$$\sum_{j\in\mathbb{Z}}\left|\xi^{n+1}_{j+1/2} - \xi^{n+1}_{j-1/2}\right| \le \mathbf{TV}(\rho_\Delta(\cdot,t^{n+1})).$$

We prove in the same way that

$$\sum_{j\in\mathbb{Z}}\left|\chi^{n+1}_{j+1/2} - \chi^{n+1}_{j-1/2}\right| \le 2\mathbf{L}\times\mathbf{TV}(\mu)\mathbf{TV}(\rho_\Delta(\cdot,t^{n+1})).$$

Finally, we proved that

$$\begin{cases}\mathbf{TV}(\rho_\Delta(\cdot,t^{n+1})) \le \left(1+\Delta t\mathbf{TV}(\eta)\|\delta f'\|_{\mathbf{L}^\infty}\right)\mathbf{TV}(\rho_\Delta(\cdot,t^n))\\ \qquad\qquad + \|\delta f\|_{\mathbf{L}^\infty}\mathbf{TV}(\eta)\Delta t\mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))\\ \mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^{n+1})) \le \left(1+\Delta t\|\mathbf{K}\|_{\mathbf{L}^\infty}\right)\mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))\\ \qquad\qquad + \Delta t\dfrac{\|\nabla\mathbf{K}\|_{\mathbf{L}^\infty}(1+2\mathbf{L}\times\mathbf{TV}(\mu))}{4}\mathbf{TV}(\rho_\Delta(\cdot,t^{n+1})),\end{cases}\tag{6.9}$$

i.e. by setting $u_n = \mathbf{TV}(\rho_\Delta(\cdot,t^n))$ and $v_n = \mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))$,

$$\begin{cases}u_{n+1} \le (1+c_1\Delta t)u_n + c_2\Delta t v_n\\ v_{n+1} \le (1+c_3\Delta t + c_2c_4\Delta t^2)v_n + (1+c_1\Delta t)c_4\Delta t u_n.\end{cases}$$

Putting the above inequalities into a matrix form, with standard linear algebra computations we are led to (6.8) with $c = \max_{1\le i\le 4} c_i$. □

Remark 6.3 ($\mathbf{L}^1$ stability)**.** Under the additional assumption that $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R})$, the scheme (6.3) is $\mathbf{L}^1$ stable. Indeed, for all $n \in \{0, \dots, N-1\}$,

$$
\begin{aligned}
\|\mathbf{w}_\Delta(\cdot, t^{n+1})\|_{\mathbf{L}^1} &= \sum_{j\in\mathbb{Z}} \mathbf{w}^{n+1}_{j+1/2}\Delta x \\
&= \sum_{j\in\mathbb{Z}} \mathbf{w}^{n}_{j+1/2}\Delta x + \sum_{j\in\mathbb{Z}} \mathbf{S}^{n+1}_{j+1/2}\Delta x\Delta t \\
&+ \sum_{j\in\mathbb{Z}} \lambda s^n_j(\mathbf{w}^n_{j-1/2} - \mathbf{w}^n_{j+1/2}) + \sum_{j\in\mathbb{Z}} \lambda s^n_j(\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j+1/2})\Delta t \\
&\leq (1 + \|\mathbf{K}\|_{\mathbf{L}^\infty}\Delta t)\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1} + \frac{\mathbf{L}}{\varepsilon} \times \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t \\
&+ \frac{\mathbf{L}}{\varepsilon} \times \frac{\|\nabla\mathbf{K}\|_{\mathbf{L}^\infty}(1 + 2\mathbf{L} \times \mathbf{TV}(\mu))}{4}\mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t \\
&\leq (1 + c\Delta t)\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1} + \frac{\mathbf{L}}{\varepsilon} \times \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t \\
&+ \frac{\mathbf{L}}{\varepsilon} \times c\mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t.
\end{aligned}
$$

Gronwall lemma yields $\sup_\Delta \|\mathbf{w}_\Delta\|_{\mathbf{L}^\infty((0,T);\mathbf{L}^1(\mathbb{R}))} < +\infty$.

Corollary 6.4. *We have:*

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}|\Delta x &\leq \Big(2\mathbf{L} \times \mathbf{TV}(\rho_\Delta(\cdot, t^n)) + \|\delta f\|_{\mathbf{L}^\infty}\mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Big)\Delta t \\
\sum_{j\in\mathbb{Z}} |\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}|\Delta x &\leq \left(\frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) + c\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1}))\right) + c\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1}\right)\Delta t
\end{aligned}
\tag{6.10}
$$

Consequently, there exist $\rho, \mathbf{w} \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, such that along a subsequence, $(\rho_\Delta, \mathbf{w}_\Delta)_\Delta \to (\rho, \mathbf{w})$ a.e. on Ω.

Proof. Estimates (6.10) come from a combination of estimates (6.8) and the scheme (6.2)-(6.3). More precisely,

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}|\Delta x &\leq \sum_{j\in\mathbb{Z}} \left|\mathbf{F}^n_{j+1}\left(\rho^n_{j+1/2}, \rho^n_{j+3/2}\right) - \mathbf{F}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}\right)\right|\Delta t \\
&\leq 2\mathbf{L}\sum_{j\in\mathbb{Z}} |\rho^n_{j+1/2} - \rho^n_{j-1/2}|\Delta t + \sum_{j\in\mathbb{Z}} |f^n_{j+1}(\rho^n_{j+1/2}) - f^n_j(\rho^n_{j+1/2})|\Delta t \\
&\leq 2\mathbf{L} \times \mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t + \|\delta f\|_{\mathbf{L}^\infty}\mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t.
\end{aligned}
$$

Regarding $(\mathbf{w}_\Delta)_\Delta$, we write

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} \left|\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}\right|\Delta x &\leq \frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) + \sum_{j\in\mathbb{Z}}\left|\mathbf{S}^{n+1}_{j+1/2} - \mathbf{S}^{n+1}_{j-1/2}\right|\right)\Delta t + \sum_{j\in\mathbb{Z}}\left|\mathbf{S}^{n+1}_{j+1/2}\right|\Delta x\Delta t \\
&\leq \frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) + c\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1}))\right)\Delta t + c\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1}\Delta t.
\end{aligned}
$$

The compactness comes from [29, Appendix A] since we have the bounds (6.6)-(6.8)-(6.10). $\square$

6.5 Approximate entropy inequalities and weak formulation

We derive approximate entropy inequalities verified by ρ_Δ and an approximate version of the weak formulation (3.2) satisfied by $\mathbf{w}_\Delta$. We start with ρ_Δ. With Φ_j^n defined in Corollary 6.2, we define the approximate entropy flux and the $\mathbf{w}_\Delta$-related contribution:

$$\Phi_\Delta(\rho_\Delta, \kappa) = \sum_{n=0}^{N} \sum_{j\in\mathbb{Z}} \Phi_j^n \mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}}; \quad \partial_\Delta f(x,t,\kappa) = \left(\int_{\mathbb{R}} \mathbf{w}_\Delta(y,t) \eta'(x-y)\,\mathrm{d}y \right) \delta f(\kappa). \tag{6.11}$$

Theorem 6.5 (Approximate entropy inequalities)**.** *Fix* $\varphi \in \mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R}\times\mathbb{R}^+), \varphi \geq 0$, $\kappa \in [0,1]$ *and* $n \in \{0,\ldots,N\}$. *Then as* $\Delta \to 0$, *we have:*

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi - \operatorname{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathbb{R}} |\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x,t^{n+1})-\kappa|\varphi(x,t^{n+1})\,\mathrm{d}x \\ &\geq O(\Delta x \Delta t) + O\big(\Delta t^2\big)\,. \end{aligned} \tag{6.12}$$

Proof. Fix $n \in \{0,\ldots,N\}$, $j \in \mathbb{Z}$, $\varphi \in \mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R}\times\mathbb{R}^+), \varphi \geq 0$, $\kappa \in [0,1]$ and set

$$\varphi_{j+1/2}^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \varphi(x,t^n)\,\mathrm{d}x\,.$$

Multiply the discrete entropy inequalities (6.7) by $\varphi_{j+1/2}^n$ and take the sum over $j \in \mathbb{Z}$. Proceeding to the Abel summation, we obtain:

$$\begin{aligned} &\underbrace{\sum_{j\in\mathbb{Z}} |\rho_{j+1/2}^{n+1} - \kappa|\varphi_{j+1/2}^{n+1}\Delta x - \sum_{j\in\mathbb{Z}} |\rho_{j+1/2}^{n} - \kappa|\varphi_{j+1/2}^{n}\Delta x}_{A} \\ &\underbrace{- \sum_{j\in\mathbb{Z}} |\rho_{j+1/2}^{n+1} - \kappa| \left(\varphi_{j+1/2}^{n+1} - \varphi_{j+1/2}^{n}\right)\Delta x}_{B} - \underbrace{\sum_{j\in\mathbb{Z}} \Phi_{j+1/2}^n \left(\varphi_{j+1/2}^{n} - \varphi_{j-1/2}^{n}\right)\Delta t}_{C} \\ &\leq \underbrace{- \sum_{j\in\mathbb{Z}} \operatorname{sgn}(\rho_{j+1/2}^{n+1} - \kappa)(f_{j+1}^n(\kappa) - f_j^n(\kappa))\varphi_{j+1/2}^n \Delta x \Delta t}_{D}\,. \end{aligned}$$

Remark that

$$A - B = \int_{\mathbb{R}} |\rho_\Delta(x,t^{n+1}) - \kappa|\varphi(x,t^{n+1})\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x - \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}} |\rho_\Delta - \kappa|\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

We now compare the other members of the inequality to their continuous counterparts.
Estimating C**.** We write:

$$\begin{aligned} C &= \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}} \Phi_\Delta(x,\rho_\Delta,\kappa)\partial_x\varphi(x,t^n)\,\mathrm{d}x\,\mathrm{d}t + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}}\int_{x-\Delta x}^{x}\int_x^y \Phi_{j+1/2}^n \partial_x\varphi(z,t^n)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}_{C_1} \\ &= \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}} \Phi_\Delta(x,\rho_\Delta,\kappa)\partial_x\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + C_1 + \underbrace{\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\int_t^{t^n} \Phi_\Delta(x,\rho_\Delta,\kappa)\partial_{tx}^2\varphi(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t}_{C_2}, \end{aligned}$$

and we have the estimations:

$$|C_1| \leq 4\mathbf{L} \sup_{t\geq 0} \|\partial^2_{xx}\varphi(\cdot,t)\|_{\mathbf{L}^1} \Delta x \Delta t; \quad |C_2| \leq \mathbf{L} \sup_{t\geq 0} \|\partial^2_{tx}\varphi(\cdot,t)\|_{\mathbf{L}^1} \Delta t^2.$$

Estimating D**.** With the notation (6.11), we have

$$\begin{aligned}
D &= \sum_{j\in\mathbb{Z}} \Delta t \int_{x_j}^{x_{j+1}} \operatorname{sgn}(\rho^{n+1}_{j+1/2} - \kappa)\partial_\Delta f(x,t^n,\kappa)\varphi(x,t^n)\,\mathrm{d}x \\
&+ \underbrace{\sum_{j\in\mathbb{Z}} \lambda \int_{x_j}^{x_{j+1}} \int_{\mathbb{R}} \int_{x_j}^{x_{j+1}} \int_x^z \operatorname{sgn}(\rho^{n+1}_{j+1/2} - \kappa)\mathbf{w}_\Delta(y,t^n)\eta''(u-y)\delta f(\kappa)\varphi(x,t^n)\,\mathrm{d}u\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}_{D_1} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \operatorname{sgn}(\rho_\Delta - \kappa)\partial_\Delta f(x,t,\kappa)\varphi(x,t)\,\mathrm{d}x + D_1 \\
&+ \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \operatorname{sgn}(\rho_\Delta - \kappa)\partial_\Delta f(x,t^n,\kappa)(\varphi(x,t^n) - \varphi(x,t))\,\mathrm{d}x}_{D_2},
\end{aligned}$$

which we combine with the bounds:

$$\begin{aligned}
|D_1| &\leq \|\eta''\|_{\mathbf{L}^1} \|\delta f\|_{\mathbf{L}^\infty} \sup_{t\geq 0} \|\varphi(\cdot,t)\|_{\mathbf{L}^1} \Delta x \Delta t \\
|D_2| &\leq \|\eta'\|_{\mathbf{L}^1} \|\delta f\|_{\mathbf{L}^\infty} \sup_{t\geq 0} \|\partial_t\varphi(\cdot,t)\|_{\mathbf{L}^1} \Delta t^2.
\end{aligned}$$

□

We now turn to $\mathbf{w}_\Delta$. Let us define the approximate flux function:

$$f_\Delta(x,t,\rho) = (1-\omega_\Delta(x,t))f_{\min}(\rho) + \omega_\Delta(x,t)f_{\max}(\rho).$$

Theorem 6.6 (Approximate weak formulation)**.** *Fix* $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+)$ *and* $n \in \{0,\ldots,N\}$*. Then as* $\Delta \to 0$*, we have:*

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Big((\rho_\Delta \mathbf{w}_\Delta)\partial_t\phi + (f_\Delta(x,t,\rho_\Delta)\mathbf{w}_\Delta)\partial_x\phi - \rho_\Delta \mathbf{S}_\Delta \phi \Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^n)\phi(x,t^n)\,\mathrm{d}x - \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^{n+1})\phi(x,t^{n+1})\,\mathrm{d}x \\
&= O(\Delta x \Delta t) + O\big(\Delta t^2\big)\,.
\end{aligned} \tag{6.13}$$

Proof. This proof follows the same steps as the one of Theorem 6.5.
Fix $n \in \{0,\ldots,N\}$ and $j \in \mathbb{Z}$. Let us multiply (6.3) by $\rho^{n+1}_{j+1/2}$ and combine the result with (6.2). More precisely, we write:

$$\begin{aligned}
&\left((\rho\mathbf{w})^{n+1}_{j+1/2} - (\rho\mathbf{w})^n_{j+1/2}\right)\Delta x \\
&= \rho^{n+1}_{j+1/2}\left(\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}\right)\Delta x + \left(\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}\right)\mathbf{w}^n_{j+1/2}\Delta x \\
&= \mathcal{F}^n_j\left(\mathbf{w}^n_{j-1/2} - \mathbf{w}^n_{j+1/2}\right)\Delta t + \rho^{n+1}_{j+1/2}\mathbf{S}^{n+1}_{j+1/2}\Delta x\Delta t + \mathcal{F}^n_j \times (\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j+1/2})\Delta t^2 \\
&- \left(\mathcal{F}^n_{j+1} - \mathcal{F}^n_j\right)\mathbf{w}^n_{j+1/2}\Delta t \\
&= -\left(\mathcal{F}^n_{j+1}\mathbf{w}^n_{j+1/2} - \mathcal{F}^n_j\mathbf{w}^n_{j-1/2}\right)\Delta t + \rho^{n+1}_{j+1/2}\mathbf{S}^{n+1}_{j+1/2}\Delta x\Delta t + \mathcal{F}^n_j \times (\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j+1/2})\Delta t^2.
\end{aligned}$$

These computations are the analogous of the ones we did in Section 6.1. This last equality expresses the consistency of our scheme.
Fix now $\phi \in \mathbf{C_c^\infty}(\mathbb{R} \times \mathbb{R}^+)$ and set

$$\phi^n_{j+1/2} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \phi(x, t^n)\, \mathrm{d}x\,.$$

Multiply the previous equality by $\phi^{n+1}_{j+1/2}$ and take the sum over $j \in \mathbb{Z}$. Proceeding to the Abel summation, we obtain:

$$\begin{aligned}
&\underbrace{\sum_{j\in\mathbb{Z}} (\rho\mathbf{w})^{n+1}_{j+1/2} \phi^{n+1}_{j+1/2} \Delta x - \sum_{j\in\mathbb{Z}} (\rho\mathbf{w})^{n}_{j+1/2} \phi^{n}_{j+1/2} \Delta x}_{A} - \underbrace{\sum_{j\in\mathbb{Z}} (\rho\mathbf{w})^{n}_{j+1/2} \left(\phi^{n+1}_{j+1/2} - \phi^{n}_{j+1/2}\right) \Delta x}_{B} \\
&- \underbrace{\sum_{j\in\mathbb{Z}} \mathcal{F}^n_{j+1} \mathbf{w}^n_{j+1/2} \left(\phi^{n+1}_{j+3/2} - \phi^{n+1}_{j+1/2}\right) \Delta t}_{C} - \underbrace{\sum_{j\in\mathbb{Z}} \rho^{n+1}_{j+1/2} \mathbf{S}^{n+1}_{j+1/2} \phi^{n+1}_{j+1/2} \Delta x \Delta t}_{D} \\
&- \underbrace{\sum_{j\in\mathbb{Z}} \mathcal{F}^n_j \times (\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j+1/2}) \phi^{n+1}_{j+1/2} \Delta t^2}_{E} = 0.
\end{aligned}$$

The remaining part of the proof consists in estimating each member of this last equality, having in mind the previously established estimates such as (6.8). Like in the previous proof, we immediately see that:

$$A = \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x, t^{n+1}) \phi(x, t^{n+1})\, \mathrm{d}x - \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x, t^n) \phi(x, t^n)\, \mathrm{d}x\,.$$

Moreover,

$$B = \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta) \partial_t \phi\, \mathrm{d}x\, \mathrm{d}t + \underbrace{\sum_{j\in\mathbb{Z}} (\rho^n_{j+1/2} - \rho^{n+1}_{j+1/2}) \mathbf{w}^n_{j+1/2} \left(\phi^{n+1}_{j+1/2} - \phi^n_{j+1/2}\right) \Delta x}_{B_1}$$

and, using Theorem 6.3 and Corollary 6.4, we have

$$|B_1| \leq \left(\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}| \Delta x \right) \|\partial_t \phi\|_{\mathbf{L}^\infty} \Delta t = O\big(\Delta t^2\big)\,.$$

Estimating C:

$$
\begin{aligned}
C &= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} \mathcal{F}^n_{j+1} \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} f^n_{j+1}(\rho^n_{j+1/2}) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x \\
&\quad \underbrace{+ \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} (\mathcal{F}^n_{j+1} - f^n_{j+1}(\rho^n_{j+1/2})) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_1} \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} f^n_{j}(\rho^n_{j+1/2}) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x + C_1 \\
&\quad \underbrace{+ \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} (f^n_{j+1}(\rho^n_{j+1/2}) - f^n_{j}(\rho^n_{j+1/2})) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_2} \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} f^n_{j}(\rho^{n+1}_{j+1/2}) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x + C_1 + C_2 \\
&\quad \underbrace{+ \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} (f^n_{j}(\rho^n_{j+1/2}) - f^n_{j}(\rho^{n+1}_{j+1/2})) \mathbf{w}^n_{j+1/2} \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_3} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x,t,\rho_\Delta) \mathbf{w}_\Delta) \partial_x \phi(x, t^{n+1}) \,\mathrm{d}x \,\mathrm{d}t + C_1 + C_2 + C_3 \\
&\quad \underbrace{+ \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_x^{x+\Delta x} f^n_{j}(\rho^{n+1}_{j+1/2}) \mathbf{w}^n_{j+1/2} (\partial_x \phi(y, t^{n+1}) - \partial_x \phi(x, t^{n+1})) \,\mathrm{d}y \,\mathrm{d}x}_{C_4} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x,t,\rho_\Delta) \mathbf{w}_\Delta) \partial_x \phi(x, t) \,\mathrm{d}x \,\mathrm{d}t + C_1 + C_2 + C_3 + C_4 \\
&\quad \underbrace{+ \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x,t,\rho_\Delta) \mathbf{w}_\Delta) (\partial_x \phi(x, t^{n+1}) - \partial_x \phi(x,t)) \,\mathrm{d}x \,\mathrm{d}t,}_{C_5}
\end{aligned}
$$

and we have the estimations:

$$
|C_1| \le 2\mathbf{L} \|\partial_x \phi\|_{\mathbf{L}^\infty} \mathbf{TV}(\rho_\Delta(\cdot, t^n)) \Delta x \Delta t; \quad |C_2| \le 2\|\delta f\|_{\mathbf{L}^\infty} \mathbf{TV}(\eta) \sup_{t\ge 0} \|\partial_x \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta x \Delta t;
$$

$$
|C_3| \le \mathbf{L} \left(\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}| \Delta x \right) \|\partial_x \phi\|_{\mathbf{L}^\infty} \Delta t = O\big(\Delta t^2\big) \quad \text{due to Corollary 6.4};
$$

$$
|C_4| \le 4\mathbf{L} \sup_{t\ge 0} \|\partial^2_{xx} \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta x \Delta t; \quad |C_5| \le \mathbf{L} \sup_{t\ge 0} \|\partial^2_{tx} \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta t^2.
$$

Estimating D**.** We write

$$\begin{aligned} D &= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t)\mathbf{S}_\Delta(x,t)\phi(x,t^{n+1})\,\mathrm{d}x \\ &= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t)\mathbf{S}_\Delta(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t)\mathbf{S}_\Delta(x,t)(\phi(x,t^{n+1} - \phi(x,t))\,\mathrm{d}x\,\mathrm{d}t}_{D_1}, \end{aligned}$$

and we have the bound:

$$|D_1| \leq \|\mathbf{S}_\Delta\|_{\mathbf{L}^\infty} \sup_{t\geq 0} \|\partial_t\phi(\cdot,t)\|_{\mathbf{L}^1}\,\Delta t^2.$$

To **estimate** E, we directly write:

$$|E| \leq c\mathbf{L}\|\phi\|_{\mathbf{L}^\infty}\left(\mathbf{TV}(\rho_\Delta(\cdot,t^n)) + \mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))\right)\Delta t^2,$$

concluding the proof. □

6.6 Convergence and existence statement

Before proving the convergence result, remark that the strong convergence of $(\rho_\Delta)_\Delta$ and $(\mathbf{w}_\Delta)_\Delta$ implies the strong convergence of $(\xi_\Delta)_\Delta$, $(\chi_\Delta)_\Delta$, $(\omega_\Delta)_\Delta$, $(f_\Delta(\cdot,\cdot,\rho_\Delta))_\Delta$ and $(\mathbf{S}_\Delta)_\Delta$. More precisely, fix $(x,t)\in\Omega$. Given Δ, let $n\in\{0,\ldots,N\}$, $j\in\mathbb{Z}$ be such that $(x,t)\in\mathcal{P}_{j+1/2}^{n+1}$. We have:

$$\xi_\Delta(x,t) = \xi_{j+1/2}^{n+1} = \int_{\mathbb{R}} \rho_\Delta(y,t)\mu(x_{j+1/2}-y)\,\mathrm{d}y \underset{\Delta\to 0}{\longrightarrow} \int_{\mathbb{R}} \rho(y,t)\mu(x-y)\,\mathrm{d}y := \xi(x,t).$$

Moreover,

$$\omega_\Delta(x,t) = \omega_j^n = \int_{\mathbb{R}} \mathbf{w}_\Delta(y,t)\eta(x_j-y)\,\mathrm{d}y \underset{\Delta\to 0}{\longrightarrow} \int_{\mathbb{R}} \mathbf{w}(y,t)\eta(x-y)\,\mathrm{d}y := \omega(x,t).$$

Consequently,

$$f_\Delta(x,t,\rho_\Delta(x,t)) \underset{\Delta\to 0}{\longrightarrow} (1-\omega(x,t))f_{\min}(\rho(x,t)) + \omega(x,t)f_{\max}(\rho(x,t)) := f(x,t,\rho(x,t)),$$

from which we deduce:

$$\begin{aligned} \chi_\Delta(x,t) = \chi_{j+1/2}^{n+1} &= -\int_{\mathbb{R}} f_\Delta(y,t,\rho_\Delta)\mu'(x_{j+1/2}-y)\,\mathrm{d}y \\ &\quad \underbrace{-\sum_{i\in\mathbb{Z}}(\mathbf{F}_i^n(\rho_{i-1/2}^{n+1},\rho_{i+1/2}^{n+1}) - f_i^n(\rho_{i+1/2}^{n+1}))\int_{x_i}^{x_{i+1}} \mu'(x_{j+1/2}-y)\,\mathrm{d}y}_{=O(\Delta x)} \\ &\underset{\Delta\to 0}{\longrightarrow} -\int_{\mathbb{R}} f(y,t,\rho)\mu'(x-y)\,\mathrm{d}y := \chi(x,t). \end{aligned}$$

Also, by continuity of $\mathbf{K}$,

$$\begin{aligned} \mathbf{S}_\Delta(x,t) &= \mathbf{K}(\xi_\Delta(x,t),\chi_\Delta(x,t))\mathbf{w}_\Delta(x,t)(1-\mathbf{w}_\Delta(x,t)) \\ &\underset{\Delta\to 0}{\longrightarrow} \mathbf{K}(\xi(x,t),\chi(x,t))\mathbf{w}(x,t)(1-\mathbf{w}(x,t)) := \mathbf{S}(x,t). \end{aligned}$$

We now turn to the

Proof of Theorem 3.3. We verify that $(\rho, \mathbf{w})$ satisfies all the points of Definition 3.1.
(i) Fix $\varphi \in \mathbf{C}^{\infty}_{\mathbf{c}}(\mathbb{R} \times [0,T)), \varphi \geq 0$, $\kappa \in [0,1]$ and $\tau, s \in [0,T]$ $(\tau < s)$. Being given $\Delta > 0$, let $n, m \in \{0, \dots, N+1\}$ such that $\tau \in [t^n, t^{n+1})$ and $s \in [t^m, t^{m+1})$ By summing (6.12) over $k \in \{n, \dots, m-1\}$, we obtain:

$$\begin{aligned}
&\int_s^\tau \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\Delta - \kappa) \partial_\Delta f(x,t,\kappa) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&= - \int_{t^n}^s \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\Delta - \kappa) \partial_\Delta f(x,t,\kappa) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \sum_{k=n}^{m-1} \int_{t^k}^{t^{k+1}} \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\Delta - \kappa) \partial_\Delta f(x,t,\kappa) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_{t^m}^\tau \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\Delta - \kappa) \partial_\Delta f(x,t,\kappa) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \,.
\end{aligned} \tag{6.14}$$

Using the uniform $\mathbf{L}^\infty$ bounds, we see that the first and last term of the right-hand side of this equality can be written as $O(\Delta t)$. By (6.12),

$$\begin{aligned}
&\sum_{k=n}^{m-1} \int_{t^k}^{t^{k+1}} \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi - \operatorname{sgn}(\rho_\Delta - \kappa) \partial_\Delta f(x,t,\kappa) \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\geq \int_{\mathbb{R}} |\rho_\Delta(x,t^m) - \kappa| \varphi(x,t^m) \,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x,t^n) - \kappa| \varphi(x,t^n) \,\mathrm{d}x + \sum_{k=n}^{m-1} \big(O(\Delta x \Delta t) + O(\Delta t^2) \big) \\
&\geq \int_{\mathbb{R}} |\rho_\Delta(x,s) - \kappa| \varphi(x,s) \,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x,\tau) - \kappa| \varphi(x,\tau) \,\mathrm{d}x - T(\Delta x + \Delta t) \\
&+ \int_{\mathbb{R}} \left(|\rho_\Delta(x,t^m) - \kappa| \varphi(x,t^m) - |\rho_\Delta(x,s) - \kappa| \varphi(x,s) \right) \mathrm{d}x \\
&- \int_{\mathbb{R}} \left(|\rho_\Delta(x,t^n) - \kappa| \varphi(x,t^n) - |\rho_\Delta(x,\tau) - \kappa| \varphi(x,\tau) \right) \mathrm{d}x \,.
\end{aligned}$$

Using the time **BV** estimate (6.10), we deduce that the last two members of this inequality can be written as $O(\Delta t)$ as well. Putting everything together, when letting $\Delta \to 0$ in (6.14), we obtain that ρ is an entropy solution to

$$\partial_t \rho + \partial_x \left(f(x,t,\rho) \right) = 0.$$

(ii) From (6.13), and using the same ideas as in the previous reasoning, with in this case the second time **BV** estimate of Corollary 6.10, we easily obtain that for all $\phi \in \mathbf{C}^{\infty}_{\mathbf{c}}(\mathbb{R} \times [0,T))$ and $\tau, s \in [0,T]$ $(\tau < s)$, we have:

$$\begin{aligned}
&\int_s^\tau \int_{\mathbb{R}} \Big((\rho_\Delta \mathbf{w}_\Delta) \partial_t \phi + (f_\Delta(x,t,\rho_\Delta) \mathbf{w}_\Delta) \partial_x \phi - \rho_\Delta \mathbf{S}_\Delta \phi \Big) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_{\mathbb{R}} \rho_\Delta(x,s) \mathbf{w}_\Delta(x,s) \phi(x,s) \,\mathrm{d}x - \int_{\mathbb{R}} \rho_\Delta(x,\tau) \mathbf{w}_\Delta(x,\tau) \phi(x,\tau) \,\mathrm{d}x = O(\Delta x) + O(\Delta t) \,,
\end{aligned}$$

which by taking the limit as $\Delta \to 0$ implies that $\mathbf{w}$ is a weak solution of

$$\partial_t (\rho \mathbf{w}) + \partial_x (f(x,t,\rho)\mathbf{w}) = \rho \mathbf{K}(\xi, \chi)\, \mathbf{w}(1-\mathbf{w}).$$

Finally, since ρ is a weak solution to $\partial_t \rho + \partial_x (f(x,t,\rho)) = 0$, we deduce that ξ is differentiable with respect to t, with derivative χ.
(iii) We proved that $\mathbf{v}$ and ω are linked and that ω and $\mathbf{w}$ are linked by (2.6) at the beginning of the section. The proof is completed. □

Proof of Theorem 3.4. The existence claim readily follows from Theorem 3.3. □

7 Numerical simulation

In this section, we present a numerical test performed with the scheme analyzed in Section 6. For $f_{\min}$, we take the uniformly concave flux $f_{\min}(\rho) = \rho(1-\rho)$, and for $f_{\max}$, we take

$$f_{\max}(\rho) = \begin{cases} f_{\min}(\rho) & \text{if } 0 \le \rho \le \rho_c \\ P(\rho) & \text{if } \rho_c < \rho \le 1, \end{cases}$$

where ρ_c is some critical threshold and P is polynomial of degree 3 satisfying:

$$P \ge 0 \text{ on } [\rho_c, 1]; \quad P(\rho_c) = f_{\min}(\rho_c); \quad P'(\rho_c) = f'_{\min}(\rho_c); \quad P(1) = 0,$$

as depicted in Figure 3, left. For the sake of simplicity, we choose $\eta = \mu$, both equal to a suitable regularization of the triangle-shaped function $x \mapsto 2(1-2|x|)\mathbb{1}_{\{|x| \le \frac{1}{2}\}}$. We deal with a road parametrized by the interval $[-2, 5]$ and time horizon $T = 6.0$. We choose initial data satisfying the hypotheses of Theorem 3.3:

$$\rho_0(x) = \begin{cases} 0.4 & \text{if } \quad -1 < x < 0 \\ 0.8 & \text{if } \quad 1 < x < 2 \\ 0.10 & \text{otherwise;} \end{cases} \qquad \mathbf{w}_0(x) = \begin{cases} 0.5 & \text{if } \quad |x| \le 10 \\ 0 & \text{otherwise,} \end{cases}$$

as represented in Figure 3, right.
Let us comment on the profile of the numerical solutions represented in Figure 4. Quite expectedly, as we can see from Figure 4 at time T, the introduction of the orderliness marker has favored the global velocity of the density. Now let us look more precisely at the different profiles of the numerical solution. We see that at times $t = 1.64$ and $t = 3.01$, the highest peaks of density correspond to the areas where the orderliness is the lowest. In the meantime, notice how this peak of the density is followed by an increase of the orderliness value, suggesting the emergence of an organizing pattern upstream the bottleneck. Finally, as incorporated in the model, everywhere the density is lesser than the threshold ρ_c, the value of $\mathbf{w}$ does not vary.

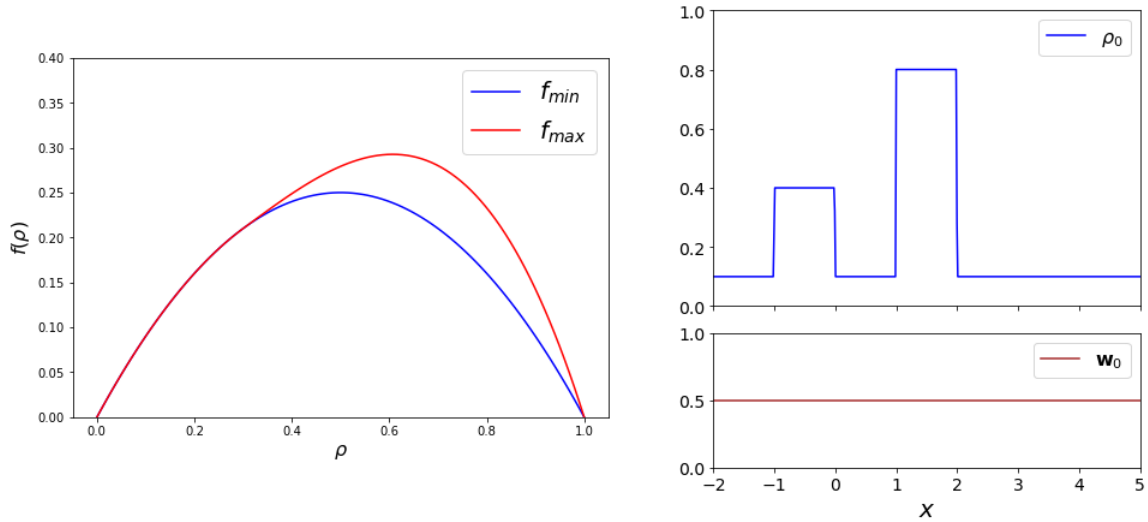

Figure 3: Typical choice of fundamental diagrams and initial data.

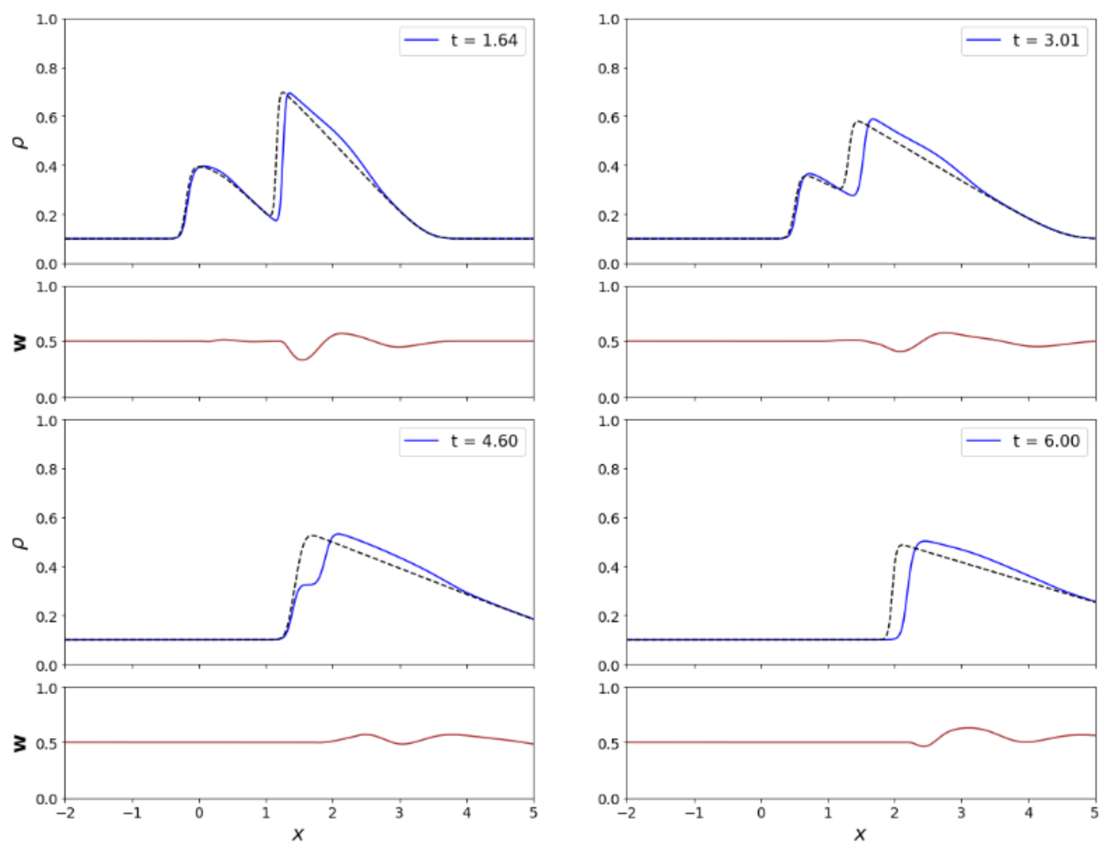

Figure 4: The numerically computed solutions $\rho_\Delta(\cdot,t), \mathbf{w}_\Delta(\cdot,t)$ at different fixed times t; dashed lines correspond to the reference solution in absence of orderliness marker, *i.e.* for $\omega \equiv 0$ in (2.2); for an animated evolution of the numerical solution, follow: https://utbox.univ-tours.fr/s/s9ecPQaq5qLCeLH.

A Well-posedness and compactification of renormalized solutions to a semilinear one-dimensional transport equation

In this appendix, we extend the results put forward by Panov in [42]. Recall that we write Ω for $\mathbb{R} \times (0,T)$. For the sake of completeness, let us recall the working framework. Fix $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ such that

$$\rho \geq 0; \quad \partial_t \rho + \partial_x (\rho \mathbf{v}) = 0 \text{ in } \mathcal{D}(\Omega). \tag{A.1}$$

Given a source term $\mathbf{S} \in \mathbf{L}^\infty(\Omega)$ and an initial datum $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$, introduce the transport equation formally written as $\partial_t \mathbf{w} + \mathbf{v}\partial_x \mathbf{w} = \mathbf{S}$, $\mathbf{w}(\cdot,0) = \mathbf{w}_0$ and reformulated as:

$$\begin{cases} \partial_t (\rho \mathbf{w}) + \partial_x (\rho \mathbf{v} \mathbf{w}) = \rho \mathbf{S} \\ \qquad \rho(\cdot,0)\mathbf{w}(\cdot,0) = \rho(\cdot,0)\mathbf{w}_0. \end{cases} \tag{A.2}$$

Following [42], we give the following notions of solution for Problem (A.2).

Definition A.1. A function $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$ is a weak solution to (A.2) with initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$ if for all test functions $\phi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R} \times [0,T))$, the following weak formulation is satisfied:

$$\int_0^T \int_{\mathbb{R}} \Big((\rho \mathbf{w})\partial_t \phi + (\rho \mathbf{v}\mathbf{w})\partial_x \phi + (\rho \mathbf{S})\phi \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \rho(x,0)\mathbf{w}_0(x)\phi(x,0)\,\mathrm{d}x = 0. \tag{A.3}$$

Remark A.1. Since ρ is a distributional solution to $\partial_t \rho + \partial_x(\rho \mathbf{v}) = 0$, we know (see [42, Lemma 1]) that $t \mapsto \rho(\cdot,t)$ is weakly* continuous in $\mathbf{L}^\infty(\mathbb{R})$, and the quantity $\rho(\cdot,0)$ has to be understood as the weak* limit of $\rho(\cdot,t)$ as $t \to 0^+$. Further, applying [42, Lemma 1] to the field $(\tilde{A}, B)$,

$$\tilde{A} : (x,t) \mapsto (\rho \mathbf{w})(x,t) - \int_0^t (\rho S \mathbf{w})(x,s)\,\mathrm{d}s\,, \quad B = \rho v \mathbf{w}$$

satisfying $\partial_t \tilde{A} + \partial_x B = 0$, we see that $\tilde{A} \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$ and since the integral term in the definition of $\tilde{A}$ is in $\mathbf{C}([0,T]; \mathbf{L}^\infty(\mathbb{R}))$, we also have $\rho\mathbf{w} \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. In particular, $\rho \mathbf{w}$ assumes the initial datum $\rho(\cdot,0)\mathbf{w}_0$ in the sense of the weak* limit in $\mathbf{L}^\infty(\mathbb{R})$.

Definition A.2. We say that a weak solution $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$ to (A.2) with initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$ verifies the renormalization property if for any function $p \in \mathbf{C}^1(\mathbb{R})$, $u = p(\mathbf{w})$ is a weak solution to

$$\begin{cases} \partial_t (\rho u) + \partial_x (\rho \mathbf{v} u) = \rho \mathbf{S} p'(\mathbf{w}) \\ \qquad \rho(\cdot,0)u(\cdot,0) = \rho(\cdot,0)\,(p \circ \mathbf{w}_0)(\cdot). \end{cases} \tag{A.4}$$

Let us recall the following results, put forward in [42].

Theorem A.3. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (A.1) and let $\mathbf{S} \in \mathbf{L}^\infty(\Omega)$.*
(i) For any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$, the transport equation (A.2) admits a unique weak solution. Moreover, this weak solution verifies the renormalization property.
(ii) If $\mathbf{w}^1$ and $\mathbf{w}^2$ are two weak solutions to (A.2) associated with data $(\mathbf{w}^1_0, \mathbf{S}^1)$ and $(\mathbf{w}^2_0, \mathbf{S}^2)$, respectively, then the following stability estimate holds: for a.e. $t \in (0,T)$,

$$\|\mathbf{w}^1(\cdot,t) - \mathbf{w}^2(\cdot,t)\|_{\mathbf{L}^\infty} \leq \|\mathbf{w}^1_0 - \mathbf{w}^2_0\|_{\mathbf{L}^\infty} + \int_0^t \|\mathbf{S}^1(\cdot,s) - \mathbf{S}^2(\cdot,s)\|_{\mathbf{L}^\infty}\,\mathrm{d}s\,. \tag{A.5}$$

Remark A.2. The author of [42] even extended these results with source terms:

$$\mathbf{S}(x,t) = g(x,t)\mathbf{w}(x,t) + h(x,t); \quad g, h \in \mathbf{L}^\infty(\Omega),$$

$\mathbf{w}$ being the unknown.

The contribution of this appendix is to prove an analogous to Theorem A.3 when the source term of (A.2) takes the form

$$\mathbf{S}(x,t) = g(x,t)\mathbf{F}(\mathbf{w}(x,t)); \quad g \in \mathbf{L}^\infty(\Omega). \tag{A.6}$$

Remark that when the function $\mathbf{F}$ is separated from zero in the sense described below, existence of a weak solution for a given initial datum follows from the renormalization property.

Lemma A.4. *Suppose that $\mathbf{F} \in \mathbf{C}(\mathbb{R})$ and that there exists $\delta > 0$ such that $\mathbf{F} \geq \delta$. Then for any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$, the transport equation (A.2) with source term $\mathbf{S}$ given by (A.6) admits a weak solution.*

Proof. Introduce the $\mathbf{C}^1$ function

$$\forall w \in \mathbb{R}, \quad p(w) = \int_0^w \frac{\mathrm{d}y}{\mathbf{F}(y)}.$$

Note that the assumption on $\mathbf{F}$ implies that p is a $\mathbf{C}^1$-diffeomorphism on its image. From Theorem A.3 *(i)*, we know that the transport equation

$$\begin{cases} \partial_t (\rho u) + \partial_x (\rho \mathbf{v} u) = \rho g \\ \qquad \rho(\cdot,0)u(\cdot,0) = \rho(\cdot,0)\ (p \circ \mathbf{w}_0)(\cdot). \end{cases}$$

admits a unique weak solution u. Since u verifies the renormalization property, by remarking that $(p^{-1})'(u) = \mathbf{F}(\mathbf{w})$, we deduce that $\mathbf{w} = p^{-1} \circ u$ is a weak solution to (A.2). □

Under the mere local assumption on $\mathbf{F}$, uniqueness for the transport equation with source terms of the form (A.6) follows.

Proposition A.5. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (A.1), $g \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{F} \in \mathbf{Lip_{loc}}(\mathbb{R})$. Then for any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$, the transport equation (A.2) with source term $\mathbf{S}$ given by (A.6) admits at most one weak solution.*

Proof. Let $\mathbf{w}_0^1, \mathbf{w}_0^2 \in \mathbf{L}^\infty(\mathbb{R})$. We denote by $\mathbf{w}^1$ (respect. $\mathbf{w}^2$) a weak solution to (A.2) associated with initial data $\mathbf{w}_0^1$ (respect. $\mathbf{w}_0^2$). Remark in the particular that $\mathbf{w}^1$ (respect. $\mathbf{w}^2$) is a weak solution to (A.2) with source term $\mathbf{S}^1 = g\mathbf{F}(\mathbf{w}^1)$ (respect. $\mathbf{S}^2 = g\mathbf{F}(\mathbf{w}^2)$). Using the stability estimate (A.5), we obtain that for a.e. $t \in (0,T)$,

$$\|\mathbf{w}^1(\cdot,t) - \mathbf{w}^2(\cdot,t)\|_{\mathbf{L}^\infty} \leq \|\mathbf{w}_0^1 - \mathbf{w}_0^2\|_{\mathbf{L}^\infty} + \|g\|_{\mathbf{L}^\infty}\|\mathbf{F}'\|_{\mathbf{L}^\infty} \int_0^t \|\mathbf{w}^1(\cdot,s) - \mathbf{w}^2(\cdot,s)\|_{\mathbf{L}^\infty}\,\mathrm{d}s\,.$$

Gronwall lemma yields a stability estimate and the uniqueness follows. □

We now prove the main result of compactness/stability regarding weak solutions verifying the renormalization property.

Theorem A.6. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (A.1), $g \in \mathbf{L}^\infty(\Omega)$, $\mathbf{F} \in \mathbf{Lip}(\mathbb{R})$ and $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$. Let $(\rho_\nu)_\nu$, $(\mathbf{v}_\nu)_\nu$, $(g_\nu)_\nu$, $(\mathbf{w}_{0,\nu})_\nu$ be sequences of uniformly bounded functions such that:*

$$\forall \nu > 0,\ \rho_\nu \geq 0; \quad (\rho_\nu)_\nu, (\rho_\nu \mathbf{v}_\nu)_\nu, (g_\nu)_\nu \underset{\nu \to 0}{\longrightarrow} \rho, \rho\mathbf{v}, g \text{ a.e. on } \Omega.$$

Moreover, suppose that there exist $a, b \in \mathbb{R}$ such that $\mathbf{F}_{|(a,b)} > 0$ and

$$\forall \nu > 0,\ a \leq \mathbf{w}_{0,\nu} \leq b; \quad \mathbf{w}_{0,\nu} \underset{\nu \to 0}{\longrightarrow} \mathbf{w}_0 \text{ a.e. on } \mathbb{R}.$$

Suppose that $(\mathbf{w}_\nu)_\nu \subset \mathbf{L}^\infty(\Omega)$ is a sequence of weak solutions to

$$\begin{cases} \partial_t\left(\rho_\nu \mathbf{w}_\nu\right) + \partial_x\left(\rho_\nu \mathbf{v}_\nu \mathbf{w}_\nu\right) = \rho_\nu g_\nu \mathbf{F}(\mathbf{w}_\nu) \\ \qquad\qquad \rho_\nu(\cdot, 0)\mathbf{w}_\nu(\cdot, 0) = \rho_\nu(\cdot, 0)\mathbf{w}_{0,\nu}, \end{cases} \tag{A.7}$$

verifying the renormalization property. Then:

1. *There exists $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$ such that $(\mathbf{w}_\nu)_\nu \to \mathbf{w}$ a.e. on Ω.*
2. *The function $\mathbf{w}$ is a weak solution to the transport equation (A.2) with source term given by (A.6), and it verifies the renormalization property.*

Proof. *1.* We split the study into two steps.
Step 1. The uniform $\mathbf{L}^\infty$ bound of $(\mathbf{w}_\nu)_\nu$ provides the existence, up to the extraction of a subsequence (not relabeled), for a.e. $(x,t) \in \Omega$ of a Borel probability measure $m_{(x,t)}$ on $\mathbb{R}$ such that for each $\varphi \in \mathbf{C}(\mathbb{R})$, $(\varphi(\mathbf{w}_\nu))_\nu$ converges $\mathbf{L}^\infty$-weakly* to $\overline{\varphi}$ where for a.e. $(x,t) \in \Omega$:

$$\overline{\varphi}(x,t) = \int_{\mathbb{R}} \varphi(y)\, \mathrm{d}m_{(x,t)}(y),$$

see for example [24, 44]. Suppose that there exists $\varepsilon > 0$ such that for all $\nu > 0$, $a + \varepsilon \leq \mathbf{w}_\nu \leq b - \varepsilon$. Introduce the $\mathbf{C}^1([a+\varepsilon, b-\varepsilon])$ function

$$p(w) = \int_{(a+b)/2}^{w} \frac{\mathrm{d}y}{\mathbf{F}(y)}.$$

By the renormalization property, for all $\nu > 0$, $u_\nu = p(\mathbf{w}_\nu) \in \mathbf{L}^\infty(\Omega)$ is a weak solution to

$$\begin{cases} \partial_t\left(\rho_\nu u_\nu\right) + \partial_x\left(\rho_\nu \mathbf{v}_\nu u_\nu\right) = \rho_\nu g_\nu \\ \qquad\qquad \rho_\nu(\cdot, 0)u_\nu(\cdot, 0) = \rho_\nu(\cdot, 0)\left(p \circ \mathbf{w}_{0,\nu}\right)(\cdot). \end{cases} \tag{A.8}$$

Note that the source term does not depend on u_ν; this is the reason behind the choice of p above. Moreover, Theorem A.3 ensures that u_ν verifies the renormalization property. By definition, for all test functions $\phi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R} \times [0,T))$, we have

$$\int_0^T \int_{\mathbb{R}} \Big((\rho_\nu u_\nu)\partial_t \phi + (\rho_\nu \mathbf{v}_\nu u_\nu)\partial_x \phi + (\rho_\nu g_\nu)\phi \Big)\, \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} \rho_\nu(x,0) p(\mathbf{w}_{0,\nu}(x))\phi(x,0)\, \mathrm{d}x = 0. \tag{A.9}$$

Now from this, we take two routes.

Route 1: limit first, renormalization second. We can safely pass to the limit in (A.9). This proves that $\overline{p}$ is a weak solution to

$$\begin{cases} \partial_t\left(\rho\overline{p}\right) + \partial_x\left(\rho\mathbf{v}\overline{p}\right) = \rho g \\ \qquad \rho_\nu(\cdot,0)\overline{p}(\cdot,0) = \rho_\nu(\cdot,0)\ (p\circ\mathbf{w}_0)(\cdot). \end{cases}$$

Since the source term of this last transport equation is of the form covered by Theorem A.3 (Remark A.2), we are assured that $\overline{p}$ verifies the renormalization property. Applying it with $p = \exp$, we obtain that $\overline{u} = \exp(\overline{p})$ is a weak solution to

$$\begin{cases} \partial_t\left(\rho\overline{u}\right) + \partial_x\left(\rho\mathbf{v}\overline{u}\right) = \rho g\overline{u} \\ \qquad \rho_\nu(\cdot,0)\overline{u}(\cdot,0) = \rho_\nu(\cdot,0)\exp(p\circ\mathbf{w}_0). \end{cases} \tag{A.10}$$

Route 2: renormalization first, limit second. From (A.8), we apply the renormalization property to u_ν ($\nu > 0$) with exp. This ensures that $U_\nu = \exp(u_\nu)$ is a weak solution to

$$\begin{cases} \partial_t\left(\rho_\nu U_\nu\right) + \partial_x\left(\rho_\nu\mathbf{v}_\nu U_\nu\right) = \rho_\nu g_\nu U_\nu \\ \qquad \rho_\nu(\cdot,0)U_\nu(\cdot,0) = \rho_\nu(\cdot,0)\exp(p\circ\mathbf{w}_{0,\nu}), \end{cases}$$

i.e. for all test functions $\phi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}\times[0,T))$, we have

$$\int_0^T\int_\mathbb{R}\Big((\rho_\nu U_\nu)\partial_t\phi + (\rho_\nu\mathbf{v}_\nu U_\nu)\partial_x\phi + (\rho_\nu g_\nu U_\nu)\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_\mathbb{R}\rho_\nu(x,0)\exp(p(\mathbf{w}_{0,\nu}(x)))\phi(x,0)\,\mathrm{d}x = 0.$$

We now let $\nu \to 0$ in this formulation to obtain that $\overline{\exp\circ p}$ is a weak solution to (A.10). By uniqueness (see Theorem A.3 and Remark A.2), $\overline{\exp\circ p} = \exp\circ\overline{p}$ a.e. on Ω. Consequently, for a.e. $(x,t)\in\Omega$,

$$\begin{aligned} \exp\left(\overline{p}(x,t)\right) &= \exp\left(\int_\mathbb{R} p(y)\,\mathrm{d}m_{(x,t)}\left(y\right)\right) \\ &\le \int_\mathbb{R}\exp\left(p(y)\right)\mathrm{d}m_{(x,t)}\left(y\right) = \overline{\exp(p)}(x,t) = \exp\left(\overline{p}(x,t)\right). \end{aligned}$$

Since exp is strictly convex, the function $y \mapsto p(y)$ is constant $m_{(x,t)}$-a.e. and consequently, for a.e. $(x,t)\in\Omega$, $m_{(x,t)} = m_{\alpha(x,t)}$ for some function $\alpha : \Omega \to \mathbb{R}$. Finally, for all $\nu > 0$, and for all bounded open subsets $U \subset \Omega$,

$$\begin{aligned} \|\mathbf{w}_\nu\|^2_{\mathbf{L}^2(U)} &= \int_0^T\int_\mathbb{R}\mathbf{w}_\nu^2\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t \\ &\underset{\nu\to0}{\longrightarrow}\int_0^T\int_\mathbb{R}\left(\int_\mathbb{R} y^2\,\mathrm{d}m_{(x,t)}\left(y\right)\right)\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\mathbb{R}\alpha(x,t)^2\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t = \|\mathbf{w}\|^2_{\mathbf{L}^2(U)}, \end{aligned}$$

which implies that $\mathbf{w}_\nu \to \mathbf{w}$ in $\mathbf{L}^2_{\mathbf{loc}}(\Omega)$. A standard diagonal process yields a subsequence of $(\mathbf{w}_\nu)_\nu$ that converges a.e. on Ω to $\mathbf{w}$.

Step 2. We now get back to the general case. Fixe $\varepsilon > 0$ and consider the cut-off functions

$$\mathbf{F}_\varepsilon(r) = \max\{F(r),\varepsilon\};\quad T_\varepsilon(w) = \min\{\min\{a+\varepsilon, w\}, b-\varepsilon\}.$$

Since $\mathbf{F}_\varepsilon \in \mathbf{C}(\mathbb{R})$ and $\mathbf{F}_\varepsilon \geq \varepsilon > 0$, Lemma A.4 ensures that the transport equation

$$\begin{cases} \partial_t (\rho_\nu \mathbf{w}) + \partial_x (\rho_\nu \mathbf{v}_\nu \mathbf{w}) = \rho_\nu g_\nu \mathbf{F}_\varepsilon(\mathbf{w}) \\ \qquad\qquad \rho_\nu(\cdot,0)\mathbf{w}(\cdot,0) = \rho_\nu(\cdot,0) T_\varepsilon(\mathbf{w}_{0,\nu}) \end{cases}$$

admits a weak solution $\mathbf{w}_{\nu,\varepsilon}$. Note that from Proposition A.5 for all $\nu, \varepsilon > 0$,

$$\begin{aligned} \|\mathbf{w}_{\nu,\varepsilon}(\cdot,t) - \mathbf{w}_\nu(\cdot,t)\|_{\mathbf{L}^\infty} &\leq \|\mathbf{w}_{\nu,\varepsilon}(\cdot,t) - \mathbf{w}_\nu(\cdot,t)\|_{\mathbf{L}^\infty} \\ &\quad + \sup_{\nu>0} \|g_\nu\|_{\mathbf{L}^\infty} \int_0^t \|\mathbf{F}(\mathbf{w}_\nu(\cdot,s)) - \mathbf{F}_\varepsilon(\mathbf{w}_{\nu,\varepsilon}(\cdot,s))\|_{\mathbf{L}^\infty} \, \mathrm{d}s \\ &\leq \varepsilon + \sup_{\nu>0} \|g_\nu\|_{\mathbf{L}^\infty} \left(\|\mathbf{F}'\|_{\mathbf{L}^\infty} \int_0^t \|\mathbf{w}_\nu(\cdot,s) - \mathbf{w}_{\nu,\varepsilon}(\cdot,s)\|_{\mathbf{L}^\infty} \, \mathrm{d}s + \varepsilon t \right), \end{aligned}$$

since $\|\mathbf{F} - \mathbf{F}_\varepsilon\|_{\mathbf{L}^\infty} \leq \varepsilon$. From this, we deduce with Gronwall lemma, that

$$\forall \nu, \varepsilon > 0, \quad \|\mathbf{w}_{\nu,\varepsilon} - \mathbf{w}_\nu\|_{\mathbf{L}^\infty(\Omega)} \leq \varepsilon \underbrace{(1 + \sup_{\nu>0} \|g_\nu\|_{\mathbf{L}^\infty} \|\mathbf{F}'\|_{\mathbf{L}^\infty} T) \exp(\sup_{\nu>0} \|g_\nu\|_{\mathbf{L}^\infty} \|\mathbf{F}'\|_{\mathbf{L}^\infty} T)}_{\mathbf{C}}. \tag{A.11}$$

Clearly, if $0 < \varepsilon \leq 1$, inequality (A.11) establishes a uniform $\mathbf{L}^\infty$ bound for the sequence $(\mathbf{w}_{\nu,\varepsilon})_\nu$ since $(\mathbf{w}_\nu)_\nu$ is bounded in $\mathbf{L}^\infty$ by assumption. Consequently, since $\mathbf{F}_\varepsilon \geq \varepsilon > 0$, Step 1 provides the existence of $\mathbf{w}_\varepsilon \in \mathbf{L}^\infty(\Omega)$ such that a subsequence of $(\mathbf{w}_{\nu,\varepsilon})_\nu$ converges a.e. on Ω to $\mathbf{w}_\varepsilon$. Now, by a standard topological argument we prove that (A.11) leads to strong compactness for the sequence $(\mathbf{w}_\nu)_\nu$. More precisely, we are to prove that $(\mathbf{w}_\nu)_\nu$ is relatively compact in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$. Fix $\mathbf{K} \subset \Omega$ a compact subset of Ω and fix $\delta > 0$. Since for all $\varepsilon > 0$, $(\mathbf{w}_{\nu,\varepsilon})_\nu$ converges a.e. on Ω and is uniformly bounded in $\mathbf{L}^\infty$, the sequence converges in $\mathbf{L}^1(\mathbf{K})$. Consequently, for all $\varepsilon > 0$, $(\mathbf{w}_{\nu,\varepsilon})_\nu$ is relatively compact in $\mathbf{L}^1_{\mathbf{loc}}(\mathbf{K})$. Fix $\varepsilon > 0$ such that, with $\mathbf{C}$ defined in (A.11),

$$\mathrm{mes}(\mathbf{K})\mathbf{C}\varepsilon \leq \frac{\delta}{2}.$$

Now use the precompactness of $(\mathbf{w}_{\nu,\varepsilon})_\nu$ to introduce a finite covering

$$\left\{ \mathbf{B}_{\mathbf{L}^1}\left(u_i, \frac{\delta}{2}\right) \right\}_{1 \leq i \leq J}; \quad u_i \in \mathbf{L}^1(\mathbf{K}), \ J \in \mathbb{N}^*.$$

By construction, $\{\mathbf{B}_{\mathbf{L}^1}(u_i, \delta)\}_{1 \leq i \leq J}$ is a covering of $(\mathbf{w}_\nu)_\nu$. We can conclude that a subsequence of $(\mathbf{w}_\nu)_\nu$ converges in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$ to some $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$. A further extraction establishes the a.e. convergence.

2. Passing to the limit in the weak formulation satisfied by $(\mathbf{w}_\nu)_\nu$, we obtain that $\mathbf{w}$ is a weak solution to (A.2) with source term given by (A.6). By uniqueness of such a weak solution, see Proposition A.5, the whole sequence $(\mathbf{w}_\nu)_\nu$ converges to $\mathbf{w}$. Finally, Theorem A.3 *(i)* applied with

$$\mathbf{S}(x,t) = g(x,t)\mathbf{F}(\mathbf{w}(x,t))$$

ensures that $\mathbf{w}$ satisfies the renormalization property, concluding the proof. □

We conclude this appendix by a well-posedness result for the transport equation (A.2) with source term (A.6) where we consider functions $\mathbf{F}$ which satisfy:

$$\exists a, b \in \mathbb{R}\ (a < b), \quad \mathbf{F} \in \mathbf{Lip}([a,b]), \ \ \mathbf{F}(a) = \mathbf{F}(b) = 0 \text{ and } \mathbf{F} > 0 \text{ on } (a,b). \tag{A.12}$$

Our study is motivated by the particular case $a, b = 0, 1$ and $\mathbf{F}(\mathbf{w}) = \mathbf{w}(1-\mathbf{w})$.

Theorem A.7. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (A.1), $g \in \mathbf{L}^\infty(\Omega)$, $\mathbf{F}$ satisfying (A.12) and $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R};[a,b])$. Then the transport equation (A.2) with source term given by (A.6) admits at least a weak solution. Moreover, this solution verifies the renormalization property.*

Proof. The idea is to construct sequences $(\rho_k)_k, (\mathbf{v}_k)_k, (g_k)_k$ satisfying the assumptions of Theorem A.6. For the sake of consistency, let us extend $\mathbf{F}$ on $\mathbb{R}\backslash[a,b]$ so that $\mathbf{F} \in \mathbf{Lip}(\mathbb{R})$ and verifies the assumption of Theorem A.6. Fix $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}), \varphi \geq 0$ a test function of mass 1 and supported in $[-1,0]$. For all $k \in \mathbb{N}^*$, consider the function

$$\theta_k(x,t) = \frac{\varphi(kx)\varphi(kt)}{k^2} \in \mathbf{C}^\infty_\mathbf{c}((\mathbb{R}^2;\mathbb{R}^+)).$$

We now introduce the smooth approximations of the coefficients:

$$\rho_k = \rho * \theta_k + \frac{1}{k}; \quad \mathbf{V}_k = (\rho\mathbf{v}) * \theta_k + \frac{1}{k}; \quad g_k = g * \theta_k.$$

The sequences $(\rho_k)_k, (\mathbf{V}_k)_k$ and $(g_k)_k$ are sequences of smooth functions that converge in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$ to $\rho, \rho\mathbf{v}$ and g, respectively, and even if it means taking subsequences, we can assume that the convergence is a.e. on Ω. Note also that since $\rho \geq 0$, then $\rho_k \geq \dfrac{1}{k} > 0$. Fix $\phi \in \mathbf{C}^\infty_\mathbf{c}(\Omega)$. It is readily checked that $\partial_t\rho_k + \partial_x\mathbf{V}_k = 0$ in $\mathcal{D}'(\Omega)$, and since ρ_k and $\mathbf{V}_k$ are smooth, the equality holds pointwise. Consider now $(\mathbf{w}_{0,k})_k \subset \mathbf{C}^1(\mathbb{R})$ such that

$$\forall k \in \mathbb{N}^*,\ a \leq \mathbf{w}_{0,k} \leq b \quad \text{and} \quad \mathbf{w}_{0,k} \underset{k\to+\infty}{\longrightarrow} \mathbf{w}_0 \text{ a.e. on } \mathbb{R}.$$

Since ρ_k does not vanish, the function $\mathbf{v}_k = \dfrac{\mathbf{V}_k}{\rho_k}$ is smooth, moreover, it verifies the uniform $\mathbf{L}^\infty$ bound:

$$\forall k \in \mathbb{N}^*, \quad |\mathbf{v}_k| = \frac{|\mathbf{V}_k|}{\rho_k} = \frac{|(\rho\mathbf{v}) * \theta_k + 1/k|}{\rho * \theta_k + 1/k} \leq \|\mathbf{v}\|_{\mathbf{L}^\infty} + 1.$$

We can define $\mathbf{w}_k \in \mathbf{Lip}(\Omega)$ as the classical solution to the following transport equation:

$$\begin{cases} \partial_t\mathbf{w}_k + \mathbf{v}_k\partial_x\mathbf{w}_k = g_k\mathbf{F}(\mathbf{w}_k) \\ \mathbf{w}_k(\cdot,0) = \mathbf{w}_{0,k}. \end{cases} \tag{A.13}$$

Indeed, we can solve this PDE using the method of characteristics. More precisely, fix $(x,t) \in \Omega$. First, we solve the following system of ODEs $(0 < s < t)$:

$$\begin{cases} \dot{\xi}_k(s) &= \mathbf{v}_k(\xi_k(s),s) \\ \xi_k(t) &= x \end{cases} \qquad \begin{cases} \dot{u}_k(s) &= g_k(\xi_k(s),s)\mathbf{F}(u_k(s)) \\ u_k(0) &= \mathbf{w}_{0,k}(\xi_k(0)). \end{cases}$$

The first ODE admits a unique global solution since $\mathbf{v}_k$ is smooth and bounded. Moreover, since $(s,u) \mapsto g_k(\xi_k(s),s)\mathbf{F}(u)$ is continuous and Lipschitz continuous with respect to the u variable, the second ODE admits a unique solution. This defines $\mathbf{w}_k$ everywhere in Ω. Note that since

$u_k(0) \in [a,b]$, Assumption (A.12) ensures that $u(s) \in [a,b]$ for all $s \in [0,t]$. Consequently, $\mathbf{w}_k(x,t) = \mathbf{w}_k(\xi(t),t) = u(t) \in [a,b]$. Hence:

$$\forall k \in \mathbb{N}^*,\ \forall (x,t) \in \Omega, \quad a \leq \mathbf{w}_k(x,t) \leq b. \tag{A.14}$$

It is classical that $\mathbf{w}_k$ defined that way is a classical solution to the PDE (A.13) and also to

$$\begin{cases} \partial_t(\rho_k \mathbf{w}_k) + \partial_x(\rho_k \mathbf{v}_k \mathbf{w}_k) = g_k \mathbf{F}(\mathbf{w}_k) \\ \qquad\qquad \rho_k(\cdot,0)\mathbf{w}_k(\cdot,0) = \rho_k(\cdot,0)\mathbf{w}_{0,k}. \end{cases} \tag{A.15}$$

since $\rho_k > 0$. Therefore $\mathbf{w}_k$ is also a weak solution to (A.15). Since we also have, for any $p \in \mathbf{C}^1(\mathbb{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p(u_k(s))\right) = g_k(\xi_k(s),s)\mathbf{F}(u_k(s))p'(u_k(s)),$$

we deduce the same way that $U_k = p(\mathbf{w}_k)$ is a weak solution to

$$\begin{cases} \partial_t(\rho_k U_k) + \partial_x(\rho_k \mathbf{v}_k U_k) = g_k \mathbf{F}(\mathbf{w}_k) p'(\mathbf{w}_k) \\ \qquad\qquad \rho_k(\cdot,0)U_k(\cdot,0) = \rho_k(\cdot,0)\ (p \circ \mathbf{w}_{0,k})(\cdot), \end{cases} \tag{A.16}$$

i.e. the sequence $(\mathbf{w}_k)_k$ is a sequence of weak solutions to (A.15) which satisfy the renormalization property.

All the hypotheses of Theorem A.6 are fulfilled. Consequently, there exists $\mathbf{w} \in \mathbf{L}^\infty(\Omega;[a,b])$ such that $(\mathbf{w}_k)_k$ converges a.e. to $\mathbf{w}$, $\mathbf{w}$ is a weak solution to (A.2) and it verifies the renormalization property. □

Putting together Proposition A.5 and Theorem A.7, we proved:

Corollary A.8. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (A.1), $g \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{F}$ satisfying (A.12). Then for any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R};[a,b])$, the transport equation*

$$\begin{cases} \partial_t\left(\rho \mathbf{w}\right) + \partial_x\left(\rho \mathbf{v} \mathbf{w}\right) = \rho g \mathbf{F}(\mathbf{w}) \\ \qquad\qquad \rho(\cdot,0)\mathbf{w}(\cdot,0) = \rho(\cdot,0)\mathbf{w}_0. \end{cases}$$

admits a unique weak solution $\mathbf{w} \in \mathbf{L}^\infty(\Omega;[a,b])$. Moreover, $\mathbf{w}$ verifies the renormalization property.

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