

Numerical Analysis in Optimal Control Problem for Aircraft Trajectories

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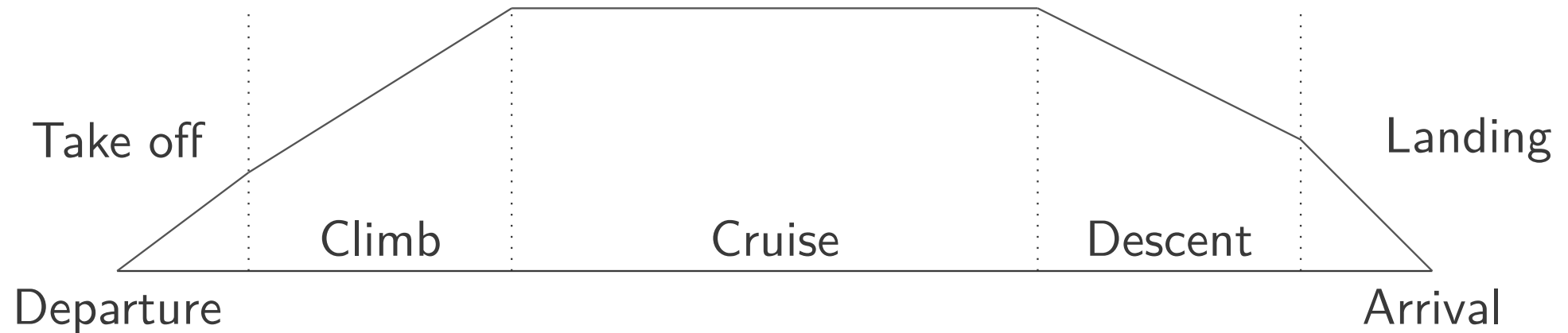
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Summary

- 1 Context of the study
- 2 Dynamic
- 3 Pontryagin's Maximum Principle
- 4 Numerical Results
- 5 Future work

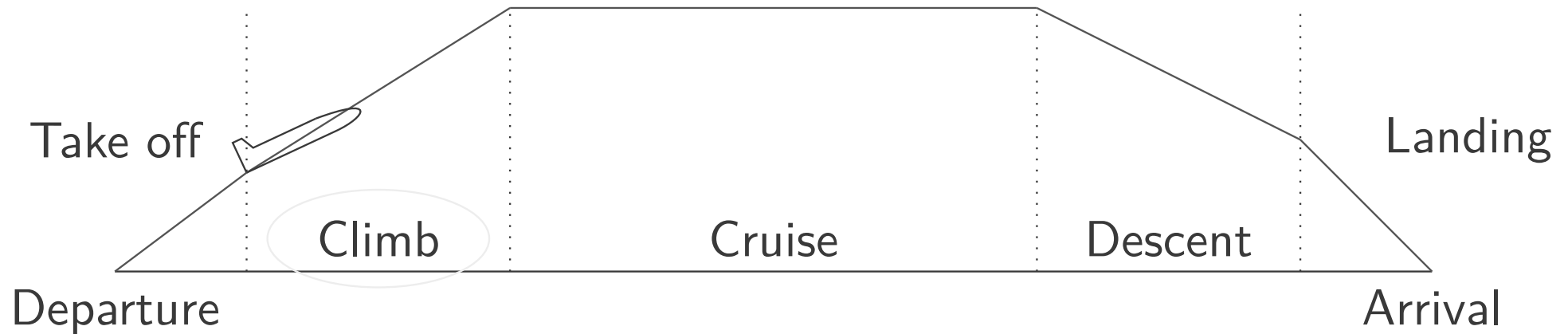
Aircraft trajectories include several flight phases with a lot of constraints



Each phase gets some constraints from :

- Air traffic control
- Flight envelope
- Historical practices

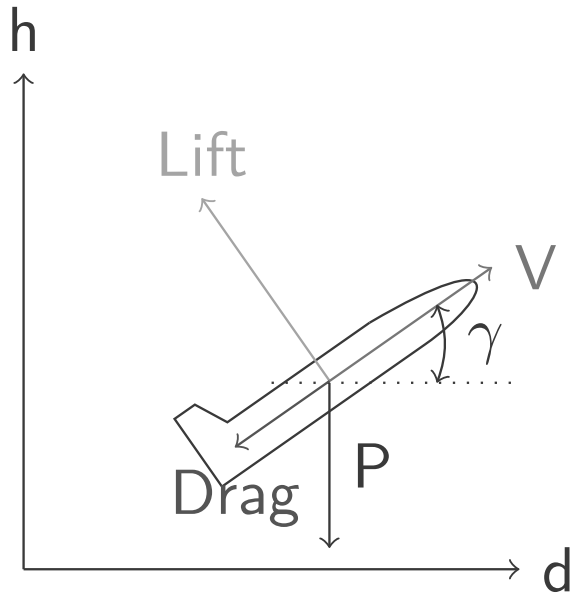
Climb phase for a middle-haul aircraft will be considered in this study



Constraints to be considered for climbing phase :

- maximum speed, or limited climb rate
- climb procedure to follow

Aircraft dynamics :



$$\frac{dh}{dt} = V \sin(\gamma)$$

$$\frac{dd}{dt} = V \cos(\gamma)$$

$$m \frac{dV}{dt} = \varepsilon T_{max}(h) - \underbrace{\frac{1}{2} \rho(h) S V^2 C_D(C_L)}_{\text{Drag}} - mg \sin(\gamma)$$

$$\frac{dm}{dt} = -\varepsilon C_s(V) T_{max}(h)$$

$$mV \frac{d\gamma}{dt} = \underbrace{\frac{1}{2} \rho(h) S V^2 C_L}_{\text{Lift}} - mg \cos(\gamma)$$

h : altitude

d : longitudinal distance

V : air speed

m : weight

γ : air slope

C_s : fuel flow

T_{max} : maximal Thrust

ρ : air density

C_L, C_D : lift, drag coefficients

ε : thrust ratio

S : wing area

g : gravitational constant

Atmospheric model :

ISA (*International Standard Atmosphere*) model

$$\begin{cases} T = T_0 - \beta h \\ P = P_0 \left(\frac{T}{T_0} \right)^{\frac{g}{\beta R}} \end{cases} \rightarrow \rho = \frac{P}{RT}, \text{ } R \text{ is the specific constant of air.}$$

Aircraft data model :

$$\text{BADA model} \left\{ \begin{array}{l} T_{\max} = C_{T_1} \left(1 - \frac{h}{C_{T_2}} + C_{T_3} h^2 \right) \text{ is the maximal thrust} \\ C_s = C_{s_1} \left(1 + \frac{V}{C_{s_2}} \right) \text{ is the fuel flow} \\ C_D = C_{D_1} + C_{D_2} \cdot C_L^2 \text{ is the drag coefficient} \end{array} \right.$$

- state : $x = (h, d, v, m, \gamma)$
- control : $u = (\varepsilon, C_L)$
- parameters : $\omega = (S, g, C_{T_1}, C_{T_2}, C_{T_3}, C_{D_1}, C_{D_2}, C_{S_1}, C_{S_2}, R, T_0, \beta, P_0)$
- auxiliary functions : $\theta(x, \omega) = (\theta_1(x, \omega), \theta_2(x, \omega), \theta_3(x, \omega), \theta_4(x, \omega))$

h is the altitude

d is the longitudinal distance

V is the air speed

γ is the air slope

m is the weight

ε is the thrust ratio

C_L is the lift coefficient

Then we write the dynamics :

$$\frac{dx}{dt} = f(x, u, \omega, \theta) = f_0(x, \omega, \theta) + u_1 f_1(x, \omega, \theta) + u_2 f_2(x, \omega, \theta) + u_2^2 f_3(x, \omega, \theta)$$

With

$$\left\{ \begin{array}{l} f_0 = \left(x_3 \sin(x_5), x_3 \cos(x_5), -\omega_6 \theta_3 - \omega_2 \sin(x_5), 0, -\frac{\omega_2}{x_3} \cos(x_5) \right)^T \\ f_1 = \left(0, 0, \frac{\theta_1}{x_4}, -\theta_1 \theta_2, 0 \right)^T \\ f_2 = \left(0, 0, 0, 0, \frac{\theta_3}{x_3} \right)^T \\ f_3 = \left(0, 0, -\omega_7 \theta_3, 0, 0 \right)^T \end{array} \right.$$

These considerations lead us to solve an optimal control problem (OCP) written in a *Mayer's formulation* and defined by :

$$\begin{array}{l}
 \min_{(t_f, x, u)} t_f \\
 \frac{dx}{dt}(t) = f(x(t), u(t)), \quad t \in [0, t_f] \text{ a.e.} \\
 u(t) \in \mathcal{U} = \{u \in \mathbb{R}^2, u_{1min} \leq u_1 \leq u_{1max}, u_{2min} \leq u_2 \leq u_{2max}\} \\
 x(t) \in \mathcal{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R} \rightarrow \text{no state constraints} \\
 x(0) = x_0, \quad x_f \in \mathcal{X}_f = \{x(t_f) \in \mathcal{X}, b_f(x(t_f)) = 0\} \subset \mathcal{X}
 \end{array}$$

$$b_f = \begin{pmatrix} x_{1f} - x_1(t_f) \\ x_{2f} - x_2(t_f) \\ x_{3f} - x_3(t_f) \\ x_{5f} - x_5(t_f) \end{pmatrix}, \quad \text{final weight is free}$$

$$(OCP) : \begin{cases} \min t_f \\ \frac{dx}{dt}(t) = f(x(t), u(t)), x(0) - x_0 = 0, b_f(x(t_f)) = 0, u \in \mathcal{U}, t \in [0, t_f] \text{ a.e} \end{cases}$$

$$\text{Hamiltonian : } H(p(t), x(t), u(t)) = \langle p(t), f(x(t), u(t)) \rangle$$

- Necessary conditions (PMP) : If (t_f^*, x^*, u^*) is optimal then $\exists p^* \in AC([0, t_f], (\mathbb{R}^n)^*)$, $(p^*, p^0) \neq (0, 0)$ such as a.e :

$$\left| \begin{array}{l} \dot{x}(t) = \frac{\partial H}{\partial p}(p^*(t), x^*(t), u^*(t)) \\ \dot{p}(t) = -\frac{\partial H}{\partial x}(p^*(t), x^*(t), u^*(t)) \\ H(p^*(t), x^*(t), u^*(t)) = \max_{u \in \mathcal{U}} H(p^*(t), x^*(t), u) \end{array} \right.$$

Besides, it exists $\lambda = (\lambda_1, \dots, \lambda_4) \in \mathbb{R}^4$ such as :

$$\left| \begin{array}{l} p^*(t_f) = \sum_{k=1}^4 \lambda_k \frac{\partial b_{f_k}}{\partial x}(x^*(t_f)) \\ H(p^*(t_f), x^*(t_f), u^*(t_f)) = -p^0 \end{array} \right.$$

Hamiltonian : $H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3$, $H_i = \langle p, f_i \rangle$, for $i=0, \dots, 3$

• We define the sets :

$$\Sigma_1 = \{H_1 = 0, \text{ and } H_3 \neq 0\}$$

$$\Sigma_2 = \{H_1 = H_2 = H_3 = 0\}$$

$$\Sigma_3 = \{H_1 = 0, H_2 \neq 0, H_3 = 0\}$$

Extremals outside $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ are controlled by $u = (u_1, u_2)$ defined

$$\text{by : } \begin{cases} u_1(t) = \begin{cases} u_{1\max}, & \text{if } H_1 > 0 \\ u_{1\min}, & \text{if } H_1 < 0 \end{cases} \\ u_2(t) = -\frac{H_2}{2H_3}, \text{ if } H_3 < 0, \text{ else } \begin{cases} u_{2\max}, & \text{if } H_2 > 0 \\ u_{2\min}, & \text{if } H_2 < 0 \end{cases} \end{cases}$$

• Transversality conditions : $p_4^*(t_f) = 0$, $H(p^*(t_f), x^*(t_f), u^*(t_f)) = -p^0$

Definition : Considering F_1, F_2 two vector field and $z = (x, p) \in \mathcal{X} \times (\mathbb{R}^n)^*$

$$\text{Lie Bracket : } F_{12} = [F_1, F_2] = \frac{\partial F_1}{\partial x} F_2(x) - \frac{\partial F_2}{\partial x} F_1(x)$$

Poisson Bracket :

$$H_{12} = \{H_1, H_2\}(z) = dH_1(\vec{H}_2)(z) = H_{[F_1, F_2]}(z), \quad \vec{H}_2 = \begin{pmatrix} \nabla_p H_2 \\ -\nabla_q H_2 \end{pmatrix}$$

Lemma : The vector family $(f_0, f_1, f_2, f_3, f_{02})$ constitutes a basis of \mathbb{R}^5 for

$$x \in \bar{\mathcal{X}} = \{x \in \mathbb{R}^5, x_1 \neq \frac{\omega_{11}}{\omega_{12}}, x_1 \neq \frac{1 + \sqrt{1 - 4\omega_4^2 \omega_5}}{2\omega_4 \omega_5}, x_1 \neq \frac{1 - \sqrt{1 - 4\omega_4^2 \omega_5}}{2\omega_4 \omega_5}, x_3 \neq 0, x_3 \neq \frac{\omega_8}{\omega_9}, x_4 \neq 0\}.$$

Proof idea :

$$\det(f_0, f_1, f_2, f_3, f_{02}) = -\theta_1 \theta_2 \theta_3$$

Proposition : The singular set $\Sigma_2 = \{H_1 = H_2 = H_3 = 0\}$ cannot contain any extremals satisfying the transversality conditions.

Proof idea :

$$\begin{aligned} \text{-Extremals in } \Sigma_2 &\implies p \perp (f_0, f_1, f_2, f_3, f_{02}) \implies p = 0 \\ &\implies H = p^0 = 0 \implies \text{impossible} \end{aligned}$$

Hamiltonian : $H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3$, $H_i = \langle p, f_i \rangle$, for $i=0, \dots, 3$

- Considering the particular case $H_1 = 0, H_2 \neq 0, H_3 \neq 0$ inside the set $\Sigma_1 = \{H_1 = 0, \text{ and } H_3 \neq 0\}$:

Then

$$\frac{\partial H}{\partial u} = (0, H_2 + u_2 H_3) \implies \begin{cases} u_1 = ? \\ u_2 = -\frac{H_2}{2H_3}, \text{ if } H_3 < 0, \text{ else } \begin{cases} u_{2\max}, \text{ if } H_2 > 0 \\ u_{2\min}, \text{ if } H_2 < 0 \end{cases} \end{cases}$$

Determine u_1 :

$$\triangleright \dot{H}_1 = 0 = \{H_1, H\} = H_{10} + u_2 H_{12} + u_2^2 H_{13} \rightarrow u_1 \text{ does not appear}$$

$$\triangleright \ddot{H}_1 = 0 = u_1(H_{101} + u_2 H_{121} + u_2^2 H_{131}) + H_{100} + u_2(H_{102} + H_{120}) + u_2^2(H_{103} + H_{130} + H_{122}) + u_2^3(H_{123} + H_{132}) + u_2^4 H_{133}$$

where $H_{101} = \{\{H_1, H_0\}, H_1\}$

Bocop is a software using Direct Methods to solve Optimal Control Problem. This software transform an OCP into a Non linear Problem which is solved by the well known solver IPOPT.

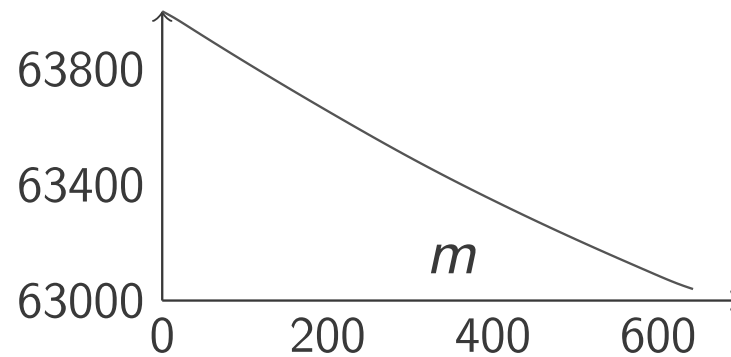
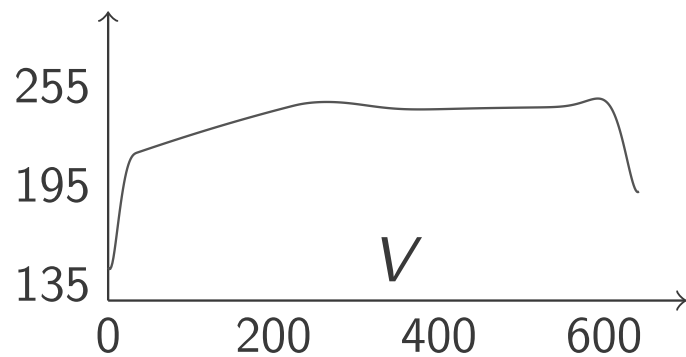
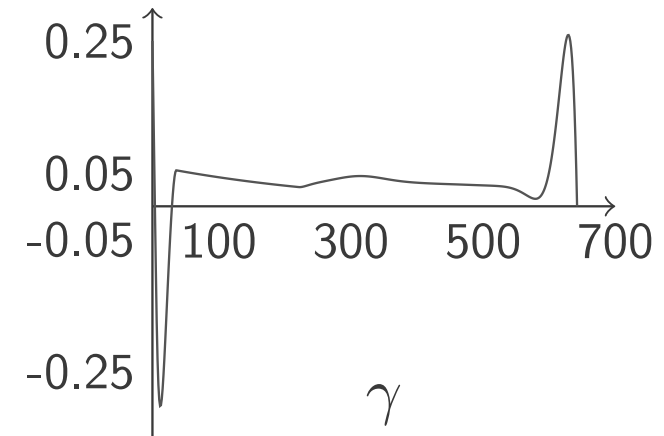
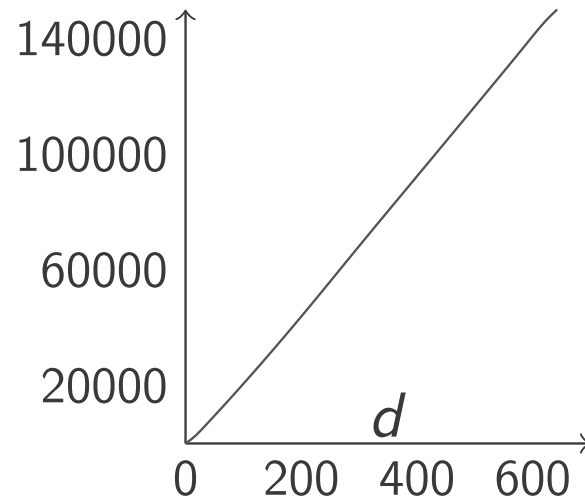
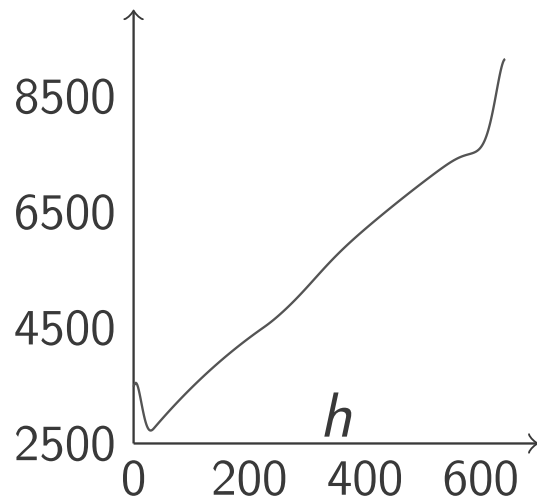
Data :

$$\left| \begin{array}{l} u_{1min} = 0.3, \quad u_{1max} = 1 \\ u_{2min} = 0, \quad u_{2max} = 1.6' \end{array} \right| \left| \begin{array}{l} x_0 = (3480, 0, 145, 64000, 0.25)^T \\ x_f = (9144, 150\ 000, 191, \text{free}, 0)^T \end{array} \right|$$

Numerical Test :

- Obtained with tolerance of 10^{-5}
- Use of Gauss Implicit scheme (second order), with 500 discretised points

Results from the unconstrained problem :

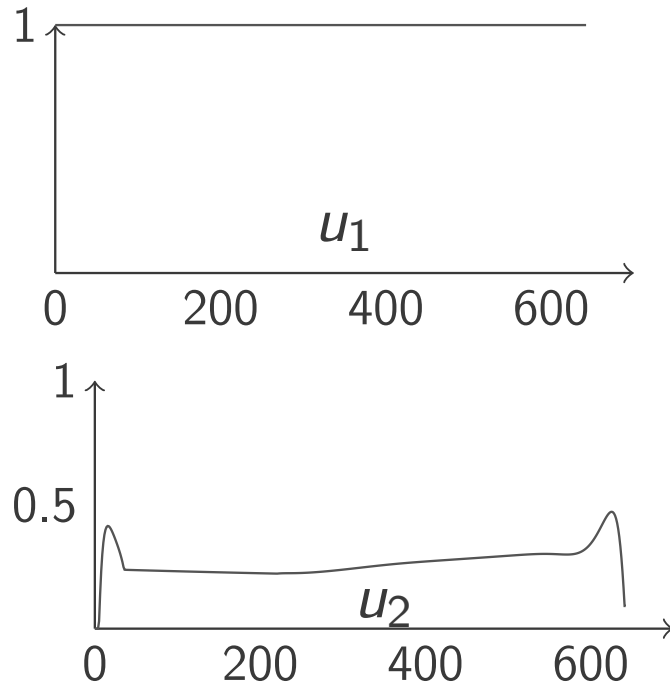


Solution is :

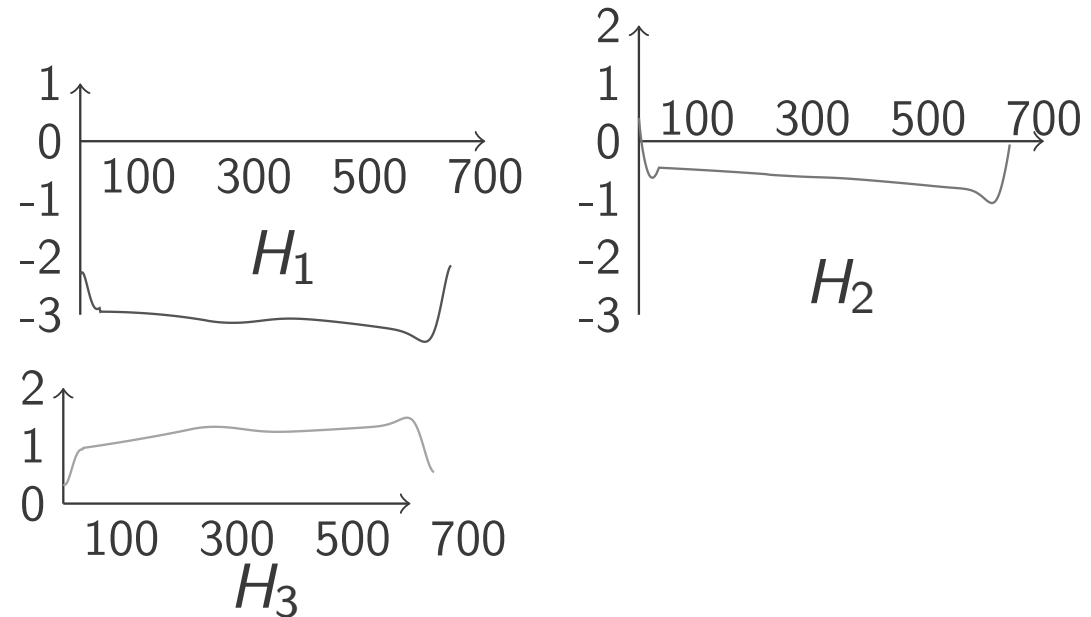
$$t_f = 643s$$

$$x_{4_f} = 63039kg$$

Controls from Bocop are :



Computed Hamiltonian lifts are :



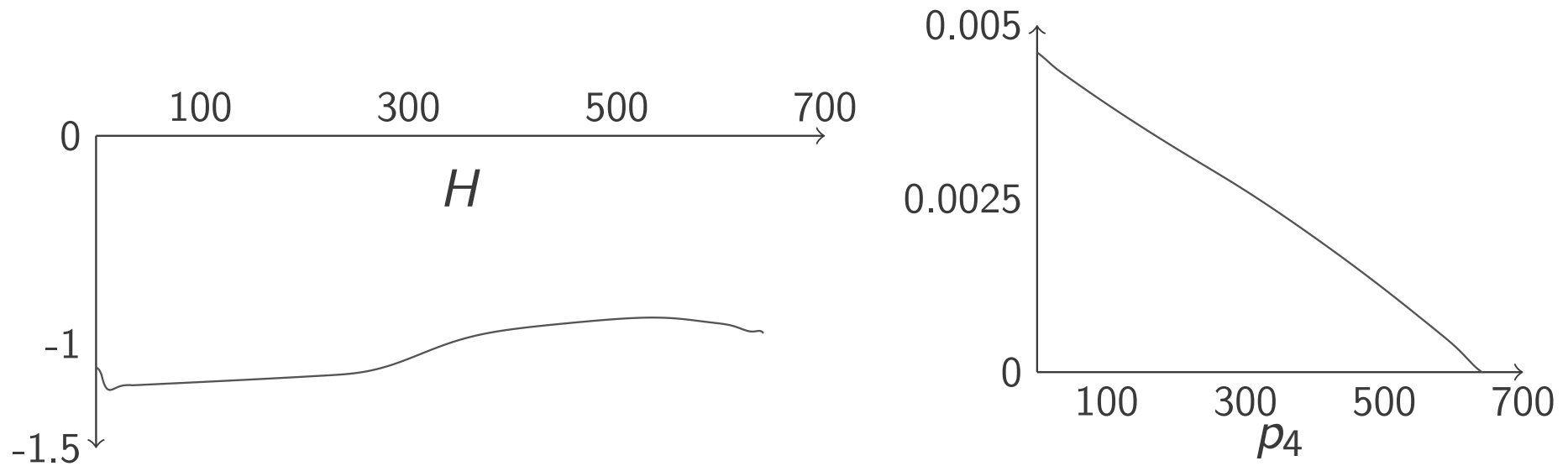
Average error on $|u_{2_{theo}} - u_{2_{num}}| = 1,52 \cdot 10^{-3}$ and $\|u_{2_{theo}} - u_{2_{num}}\|_{\infty} = 5,8 \cdot 10^{-2}$

This confirms that control follows the law defined by :

$$u_1(t) = \begin{cases} u_{1\max}, & \text{if } H_1 < 0 \\ u_{1\min}, & \text{if } H_1 > 0 \end{cases}, \quad u_2(t) = -\frac{H_2}{2H_3} \text{ if } H_3 > 0, \text{ else } \begin{cases} u_{2\max}, & \text{if } H_2 < 0 \\ u_{2\min}, & \text{if } H_2 > 0 \end{cases}$$

Transversality conditions give : $H(p^*(t_f), x^*(t_f), u^*(t_f)) = -1, p_4^*(t_f) = 0$
 As the system is autonomous, we have : $H(p^*(t), x^*(t), u^*(t)) = \text{Cst} = -1$

The Hamiltonian extracted from Bocop :



Numerical results confirm theoretical behavior of this climbing problem.

This work is the preliminary part of a deeper investigation which includes :

- Numerical solution of this problem using indirect method
- Study of minimum time problem with saturated state constraints, especially on x_5
- Study of a minimal fuel consumption trajectory : $g(t_f, x(t_f)) = -x_4(t_f)$