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APPROXIMATELY CONTROLLABLE FINITE-DIMENSIONAL BILINEAR SYSTEMS ARE CONTROLLABLE

DANIELE CANNARSA AND MARIO SIGALOTTI

ABSTRACT. We show that a bilinear control system is approximately controllable if and only if it is controllable in $\mathbb{R}^n \setminus \{0\}$. We approach this property by looking at the foliation made by the orbits of the system, and by showing that there does not exist a codimension one foliation in $\mathbb{R}^n \setminus \{0\}$ with dense leaves that are everywhere transversal to the radial direction. The proposed geometric approach allows to extend the results to homogeneous systems that are angularly controllable.

Keywords: bilinear control systems, controllability, foliations

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this note we study control systems of the form

$$(\Sigma) \quad \dot{x} = M(t)x,$$

where the state x is in $\mathbb{R}^n \setminus \{0\}$, $n \geq 1$, and the admissible controls $M : [0, +\infty) \rightarrow \mathcal{M}$ take values in a subset \mathcal{M} of the space $M_n(\mathbb{R})$ of $n \times n$ matrices with real coefficients. For notational simplicity we shall take as admissible controls piecewise constant functions with values in \mathcal{M} . By a slight abuse of notation, we refer to control systems such as (Σ) as *bilinear control systems*, although the latter term usually denotes systems for which $M(t) = A + u^1(t)B_1 + \dots + u^m(t)B_m$ for some fixed $A, B_1, \dots, B_m \in M_n(\mathbb{R})$, with control $t \mapsto (u^1(t), \dots, u^m(t))$ taking values in some subset U of \mathbb{R}^m . For an introduction to bilinear control system we refer to [CK00] and [Eli09]. Notice that bilinear hybrid systems of the form $\dot{x} = (A_{q(t)} + u^1(t)B_{1,q(t)} + \dots + u^m(t)B_{m,q(t)})x(t)$, where $q(t)$ is a control parameter with discrete values (cf. [PvS10]) are also included in our setting.

Given $t > 0$, an initial state $x_0 \in \mathbb{R}^n$, and an admissible control $M : [0, +\infty) \rightarrow \mathcal{M}$, let us denote by $x(t, x_0, M)$ the value at time t of the solution of (Σ) with initial state x_0 and control M . The attainable set from an x_0 in \mathbb{R}^n is

$$\mathcal{A}_{x_0} = \{x(t, x_0, M) \mid t \geq 0, M : [0, +\infty) \rightarrow \mathcal{M} \text{ piecewise constant}\}.$$

System (Σ) is said to be *controllable* if, for all state x , one has $\mathcal{A}_x = \mathbb{R}^n \setminus \{0\}$, while it is said to be *approximately controllable* if, for all state x , one has $\text{cl}(\mathcal{A}_x) = \mathbb{R}^n$, where cl denotes the closure of a set in \mathbb{R}^n . The main result of this note is the following.

Theorem 1. (Σ) is approximately controllable if and only if (Σ) is controllable.

This result shows that the a priori weaker notion of approximate controllability implies controllability with no additional assumption, other than the finite-dimensional system being bilinear. This is useful when topological arguments lead directly to approximate controllability results, as it is the case for constructions based on control sets, whose definition involve the closure of attainable sets [CK00]. For example, in [CRS21] the authors use Theorem 1 to conclude that their sufficient condition for approximate controllability, expressed in terms of the Floquet spectrum of the bilinear system, actually yields controllability (see [CRS21, Corollary 3.19]).

Another possible way of applying Theorem 1 is the following: if for a bilinear system one is able to identify vector fields compatible with (Σ) in the sense of [AS02, Definition 8.4] that,

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when added to the admissible ones, lead to approximate controllability, then controllability of (Σ) follows, without the need of checking the Lie algebra rank condition. Such an extension argument by compatible vector fields is, e.g., at the core of the results in [Che05], which can therefore be improved by our result. In particular, Theorem 1 implies that the hypotheses ii) and iii) on the existence of stable and antistable equilibria can be dropped from [Che05, Theorem 4.3]. Similarly, Propositions 3.3 and 3.6 from the same paper can be strengthened by replacing in their conclusions approximate and practical controllability by controllability.

The result in Theorem 1 is in sharp contrast with the case of bilinear systems in infinite-dimension: when the controlled operators B_i appearing in the representation $M(t) = A + u^1(t)B_1 + \dots + u^m(t)B_m$ are bounded, these systems cannot be controllable (see [BMS82, Theorem 3.6] and also [BCC20] for recent extensions), while there exist some criteria for approximate controllability (see, e.g., [Kha10, Chapters 4 and 9], [CFK17] and [BCS14]).

For general finite-dimensional systems (to which the notions of controllability and approximate controllability straightforwardly extend), while controllability clearly implies approximate controllability, the converse may fail to hold. A standard example can be provided using the irrational winding of a line in the torus \mathbb{T}^n , $n \geq 2$.

On the other hand, the equivalence stated in Theorem 1 is known to hold for some classes of control systems. This is the case for linear control systems in \mathbb{R}^n , i.e., systems of the form

$$(1) \quad \dot{x} = Ax + Bu(t), \quad u : [0, +\infty) \rightarrow \mathbb{R}^m, \quad x \in \mathbb{R}^n,$$

with $A \in M_n(\mathbb{R})$ and $B \in M_{n \times m}(\mathbb{R})$. Indeed, if system (1) is approximately controllable, then the attainable set from the origin \mathcal{A}_0 is dense in \mathbb{R}^n . Since \mathcal{A}_0 is a linear space (and in particular it is closed), it follows that $\mathcal{A}_0 = \mathbb{R}^n$, which is well known to be equivalent to the controllability of (1) due to the linear structure of the system.

Few other classes of control systems for which approximate controllability implies controllability are known: closed quantum systems on S^{n-1} [BGRS15, Theorem 17]; right-invariant control systems on simple Lie groups (as it follows from [JS72, Lemma 6.3] and [Smi42, Note at p. 312]); control systems obtained by projecting onto $\mathbb{R}\mathbb{P}^{n-1}$ systems of the form of (Σ) [BS20, Proposition 44].

By proving Theorem 1, we are able to identify bilinear control systems as a new class of systems for which approximate controllability and controllability are equivalent. The proposed proof of Theorem 1 (which can be found in Section 3), works first by deducing from [BS20] and [BV13] that, if the projection of (Σ) onto $\mathbb{R}\mathbb{P}^{n-1}$ is approximately controllable, then the orbits of (Σ) are transversal to the radial direction, and then by proving that there does not exist a codimension one foliation in $\mathbb{R}^n \setminus \{0\}$ with dense leaves which are transversal to the radial direction.

As a byproduct of this demonstration strategy, we can extend Theorem 1 to angularly controllable homogenous control systems (see Corollary 10).

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2. NOTATIONS AND RELATED RESULTS FROM THE LITERATURE

Given a matrix M in \mathcal{M} , let us denote by f_M the associated vector field

$$f_M : x \mapsto Mx, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Since the vector fields f_M are \mathbb{R} -homogenous, $\mathcal{F}^{\mathcal{M}} = \{f_M \mid M \in \mathcal{M}\}$ is a family of analytical, homogenous vector fields in $\mathbb{R}^n \setminus \{0\}$. System (Σ) is a special case of a general control system

$$(C) \quad \dot{x} = f_{u(t)}(x),$$

where the state x evolves in a n -dimensional manifold X following a fixed family \mathcal{F} of admissible vector fields indexed by a set U , i.e., $\mathcal{F} = \{f_u \mid u \in U\}$, with piecewise constant controls $u : [0, +\infty) \rightarrow U$. (For system (Σ) we have $X = \mathbb{R}^n \setminus \{0\}$ and $\mathcal{F} = \mathcal{F}^{\mathcal{M}}$.) For

notational simplicity, we assume that each vector field f in \mathcal{F} is complete. We denote by e^{tf} the flow at time t of the equation $\dot{x} = f(x)$. The *attainable set from a point x in X for (C)* is

$$\mathcal{A}_x = \{e^{t_1 f_1} \circ \dots \circ e^{t_k f_k}(x) \mid k \in \mathbb{N}, t_1, \dots, t_k \geq 0, f_1, \dots, f_k \in \mathcal{F}\};$$

similarly, the *orbit of x for (C)* is

$$\mathcal{O}_x = \{e^{t_1 f_1} \circ \dots \circ e^{t_k f_k}(x) \mid k \in \mathbb{N}, t_1, \dots, t_k \in \mathbb{R}, f_1, \dots, f_k \in \mathcal{F}\}.$$

The difference between orbits and attainable sets is that the orbits are constructed by using positive and negative time directions; thus, $\mathcal{A}_x \subset \mathcal{O}_x$ for all x in X . The orbits satisfy the following important property.

Theorem 2 (Orbit theorem, [Sus73]). *For every $x \in X$, the orbit \mathcal{O}_x is a connected, immersed submanifold of X . Moreover, for all $y \in \mathcal{O}_x$,*

$$T_y \mathcal{O}_x = \text{span}_{\mathbb{R}} \{(e_*^{t_1 f_1} \circ \dots \circ e_*^{t_k f_k} g)(y) \mid k \in \mathbb{N}, t_1, \dots, t_k \in \mathbb{R}, f_1, \dots, f_k, g \in \mathcal{F}\},$$

where $e_*^{t_i f_i}$ denotes the operator of push-forward of vector fields along the flow $e^{t_i f_i}$.

It follows from the orbit theorem that the orbits define a *foliation*, i.e., a partition of X in immersed, connected, submanifolds (called *leaves*), possibly of varying dimension. In some texts the term foliation describes what we shall call here a *regular foliation*, i.e., a foliation whose leaves have all the same dimension d , and which admits locally around each point a coordinate trivialization of the form $\{\omega + p \mid p \in P\}$ for some connected submanifold $\omega \subset \mathbb{R}^n$ of dimension d and some connected submanifold P of dimension $n - d$ transversal to ω .

Another class of foliations of interest for our discussion is that of *integral* foliations, i.e., foliations such that for every x in X every vector $v \in T_x X$ tangent to the leaf passing through x can be extended to a smooth vector field everywhere tangent to the leaves of the foliation. The fact that foliations arising as orbits of control systems as in (C) are integral is a consequence of the orbit theorem. Conversely, any integral foliation is the orbit partition of a control system. For this, it suffices to take as family of admissible vector fields the collection of all vector fields that are everywhere tangent to the leaves.

For any family \mathcal{G} of vector fields, we denote by $D^{\mathcal{G}}$ the distribution defined by \mathcal{G} , that is,

$$D^{\mathcal{G}} = \{D_x^{\mathcal{G}} \mid x \in X\}, \quad D_x^{\mathcal{G}} = \text{span}\{f(x) \mid f \in \mathcal{G}\} \quad \forall x \in X.$$

The distribution $D^{\mathcal{G}}$ may not be a sub-bundle of TM . Now, let $\text{Lie } \mathcal{F}$ be the smallest Lie algebra of vector fields containing \mathcal{F} , i.e.,

$$\text{Lie } \mathcal{F} = \text{span}\{[f_1, \dots, [f_{k-1}, f_k] \dots] \mid k \in \mathbb{N}, f_1, \dots, f_k \in \mathcal{F}\}.$$

Since the vector $[f_i, f_j](x)$ is tangent to the trajectory $t \mapsto e^{-\sqrt{t} f_j} \circ e^{-\sqrt{t} f_i} \circ e^{\sqrt{t} f_j} \circ e^{\sqrt{t} f_i}(x)$, one has that

$$(2) \quad D_y^{\text{Lie } \mathcal{F}} \subset T_y \mathcal{O}_x, \quad \forall x \in X, y \in \mathcal{O}_x.$$

Under some rather general hypothesis, the inclusion in (2) turns out to be an equality, i.e.,

$$(LD) \quad D^{\text{Lie } \mathcal{F}} = T\mathcal{O}.$$

If this is the case, we say that (C) is *Lie-determined*. For example, any of the following classical hypotheses imply that system (C) is Lie-determined:

- (i) the family \mathcal{F} is analytic [Nag66];
- (ii) $\text{Lie } \mathcal{F}$ is locally finitely generated [Her62];
- (iii) $D^{\mathcal{F}}$ is \mathcal{F} -invariant [Sus73] [Ste74];
- (iv) $D^{\text{Lie } \mathcal{F}}$ is a constant rank distribution [Fro77]. In this case the foliation described by the orbits is regular.

For a short introduction to the subject we refer to [Lav18]. Notice that our bilinear system (Σ) is Lie-determined since the vector fields f_M are analytic. Our proof uses the following classical result about attainable sets. It says that, in the Lie-determined case, orbits cannot be of a greater dimension than attainable sets.

Theorem 3 (Krener's theorem [Kre74]). *Assume that (C) is Lie-determined. Then, for every x in X , the attainable set \mathcal{A}_x has a nonempty interior in the orbit \mathcal{O}_x .*

The following lemma provides some informations about the orbits of an approximately controllable system.

Lemma 4. *Assume that system (C) is approximately controllable. Then, the orbits of (C) form a regular foliation of X with dense leaves.*

Proof. As noticed above, the orbits of system (C) form a partition of X in immersed submanifolds. Since the attainable sets are contained in the orbits, the approximate controllability implies that the orbits are dense. Finally, due to the expression of the tangent space of the orbits in Theorem 2, the dimension of the orbits is lower semi-continuous, i.e., for all x in X , there exists a neighbourhood $V(x)$ of x such that

$$\dim \mathcal{O}_x \leq \dim \mathcal{O}_y, \quad \forall y \in V(x).$$

Now let \mathcal{O}_x be an orbit of maximal dimension; since \mathcal{O}_x is dense, all other orbits have the same dimension as \mathcal{O}_x . Finally, an integral foliation of constant rank is a regular foliation. \square

The family \mathcal{F} is said to satisfy the *Lie algebra rank condition* at $x \in X$ if the evaluation at x of the Lie algebra generated by \mathcal{F} has maximal dimension, i.e., $D_x^{\text{Lie } \mathcal{F}} = T_x X$.

Corollary 5. *Assume that system (C) is Lie-determined and approximately controllable. Then, exactly one of the following holds:*

- (a) \mathcal{F} satisfies the Lie algebra rank condition at all points in X ; thus, system (C) is controllable.
- (b) There exists an integer k with $0 < k < \dim X$ such that the orbits of (C) form a regular k -dimensional foliation of X with dense leaves.

Proof. If there exists a point x in X at which \mathcal{F} satisfies the Lie algebra rank condition, then (C) has a single orbit. Indeed, Krener's theorem implies that the interior of \mathcal{A}_x is nonempty. Hence, due to the approximate controllability assumption, the attainable set from any other point intersects \mathcal{A}_x , and therefore is contained in \mathcal{O}_x . Thus, (C) has a single orbit and, due to the Lie-determinedness property (LD), every point in X satisfies the Lie algebra rank condition. The controllability follows as a corollary of Krener's theorem ([AS02, Corollary 8.3]).

Otherwise, assume that \mathcal{F} does not satisfy the Lie algebra rank condition at any point. Then, Lemma 4 shows that the orbits of (C) form a regular foliation, whose leaves are dense since they contain the attainable sets. Finally, the dimension of the orbits is less than the dimension of X , otherwise \mathcal{F} would satisfy the Lie algebra rank condition due to (LD). \square

Remark 6. Corollary 5 is useful if one can exclude the existence of a foliation with the properties described in (b); this might be possible thanks to the particular form of system (C) or some topological properties of X . Most of the results in this direction are for codimension-one regular foliations: for example, it is known that even dimensional spheres do not admit codimension-one regular foliations [Dur72]. We recall also that a compact manifold with finite fundamental group has no analytic regular foliations of codimension one [Hae57]; some additional results can be found in [Law74].

If we drop the hypothesis that the system is Lie-determined, it is possible to construct an approximately controllable and not controllable system having only one orbit. This is shown in the following example.

Example 7. Let $X = \mathbb{R}^2$ and consider the family $\mathcal{F} = \{f_1, f_2, f_3^+, f_3^-\}$ with

$$f_1 = \frac{\partial}{\partial y}, \quad f_2 = -\phi(x, y)f_1, \quad f_3^\pm = \pm\phi(x, y)\frac{\partial}{\partial x},$$

where $\phi : \mathbb{R}^2 \rightarrow [0, +\infty)$ is a smooth function such that, for all $(x, y) \in \mathbb{R}^2$, $\phi(x, y) = 0$ if and only if $x = 0$ and $y \leq 0$. This four vector fields are illustrated in Figure 1. It is not hard to check that this system has a single orbit, is approximately controllable, and still it is not controllable.

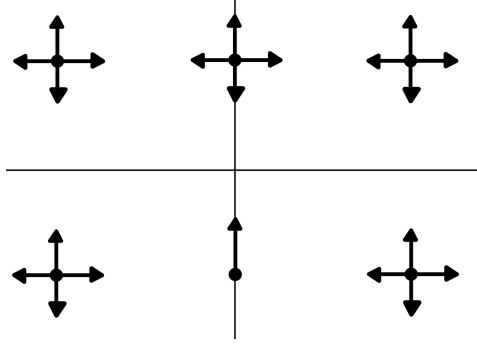


FIGURE 1. The admissible vector fields for the control system introduced in Example 7.

3. PROOF OF THEOREM 1

Assume that system (Σ) is approximately controllable and that $n \geq 2$, the case $n = 1$ being trivial. As already remarked, system (Σ) is Lie-determined due to the analyticity of each linear vector field. Thus, Corollary 5 applies: either system (Σ) is controllable, or the partition of $\mathbb{R}^n \setminus \{0\}$ into the orbits of (Σ) forms a regular foliation of positive codimension as described in (b). Let us assume the latter, and show that this leads to a contradiction.

Let us introduce two systems which can be naturally associated with (Σ) : the projections of system (Σ) onto S^{n-1} and $\mathbb{R}\mathbb{P}^{n-1}$. First, consider the projection $\pi : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ defined by $\pi(x) = x/|x|$. For every x in S^{n-1} and every $v \in T_x\mathbb{R}^n$, consider the pushforward $\pi_*(v) = v - \langle x, v \rangle x$, and let us introduce the control system

$$(\text{S}\Sigma) \quad \dot{x} = \pi_*(M(t)x), \quad M(t) \in \mathcal{M}, \quad x \in S^{n-1}.$$

Due to homogeneity, the trajectories of $(\text{S}\Sigma)$ are the image of the trajectories of (Σ) via π ; thus, the orbits \mathcal{O}^S of $(\text{S}\Sigma)$ are projections of the orbits of (Σ) , that is

$$(3) \quad \mathcal{O}_{\pi(y)}^S = \pi(\mathcal{O}_y), \quad \forall y \in \mathbb{R}^n \setminus \{0\}.$$

We say that (Σ) is *angularly controllable* if $(\text{S}\Sigma)$ is controllable. Similarly, consider the canonical projection $\omega : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}\mathbb{P}^{n-1}$ and the system

$$(\text{P}\Sigma) \quad \dot{q} = \omega_*(M(t)x), \quad M(t) \in \mathcal{M}, \quad q = \omega(x) \in \mathbb{R}\mathbb{P}^{n-1}.$$

This system is well-defined because $\omega_*(M(t)x)$ depends only on q and not on the choice of the specific $x \in \mathbb{R}^n \setminus \{0\}$ such that $q = \omega(x)$. As mentioned in the introduction, Theorem 1 holds if one replaces (Σ) by $(\text{P}\Sigma)$, as shown in [BS20, Proposition 44]. Using this result, let us deduce the following property.

Lemma 8. *If (Σ) is approximately controllable then (Σ) is angularly controllable.*

Proof. Since the projection of the trajectories of (Σ) are trajectories of $(\text{P}\Sigma)$, if the former is approximately controllable then the same holds for the latter. Due to [BS20, Proposition 44], if $(\text{P}\Sigma)$ is approximately controllable then it is controllable. In [BV13, Theorem 1] the authors show that system $(\text{P}\Sigma)$ is controllable if and only if system $(\text{S}\Sigma)$ is controllable. Therefore, system $(\text{S}\Sigma)$ is controllable, meaning that system (Σ) is angularly controllable. \square

Let us denote by $\mathcal{O} = \{\mathcal{O}_x \mid x \in \mathbb{R}^n \setminus \{0\}\}$ the orbit partition of system (Σ) . Due to (3) and Lemma 8, one has

$$\pi_*(T_y\mathcal{O}_x) = T_{\pi(y)}\mathcal{O}_{\pi(x)}^S = T_{\pi(y)}S^{n-1}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall y \in \mathcal{O}_x.$$

Therefore, we have that

$$(4) \quad T_y\mathcal{O}_x + \mathbb{R}y = \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall y \in \mathcal{O}_x,$$

which is to say that the orbits are transversal to the radial direction. Additionally, since we assumed to be in case (b) of Corollary 5, this implies that $\dim T_y\mathcal{O}_x = n - 1$. It follows that \mathcal{O} is a codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction

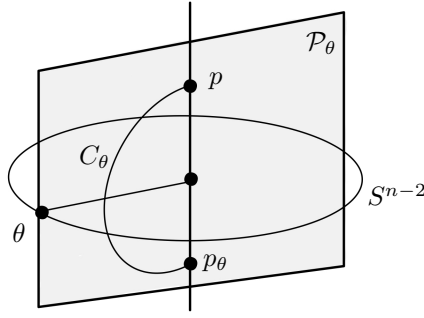


FIGURE 2

and with dense leaves. Moreover \mathcal{O} is *homogenous* in the sense that, for all $\lambda > 0$ and x in $\mathbb{R}^n \setminus \{0\}$, one has that $\lambda \mathcal{O}_x = \mathcal{O}_{\lambda x}$. In the following lemma we show that such a foliation cannot exist.

Lemma 9. *Assume that $n \geq 2$. Then, there does not exist a homogenous, codimension-one regular foliation of $\mathbb{R}^n \setminus \{0\}$ transversal to the radial direction and with dense leaves.*

The hypothesis of transversality between the foliation and the radial direction is necessary, as counterexamples can be constructed otherwise. For instance, in [Hec76] the author presents an example of codimension-one regular foliation of \mathbb{R}^3 with dense leaves; this construction is presented with additional details in [CN13, Chapter 4].

Proof of Lemma 9. By contradiction, suppose there exists a codimension-one regular foliation $\mathcal{L} = \{L_\alpha \mid \alpha \in A\}$ of $\mathbb{R}^n \setminus \{0\}$ with dense leaves transversal to the radial direction.

Let us first consider the case $n = 2$. Orienting the foliation using the clockwise direction and applying Whitney's theorem (see [ABZ96, Theorem 2.3 at p. 23]), the foliation can be identified with the set of trajectories of a vector field. Using the stereographic projection, the flow of such a vector field can be pushed to the sphere S^2 minus two points. However, a flow with dense trajectories on S^2 minus finitely many points does not exist: see, for instance, [ABZ96, Lemma 2.4 at p. 56].

Assume that $n \geq 3$. Let us fix the point $p = (0, \dots, 0, 1) \in \mathbb{R}^n$, and denote by S^{n-2} the embedded sphere $S^{n-2} \times \{0\} \subset \mathbb{R}^n$. For every θ in S^{n-2} , let \mathcal{P}_θ be the plane

$$\mathcal{P}_\theta = \text{span}\{p, \theta\},$$

as depicted in Figure 2. Because of the transversality between the leaves of \mathcal{L} and the radial direction, the linear subbundle $I_\theta = \mathcal{P}_\theta \cap T\mathcal{L}|_{\mathcal{P}_\theta \setminus \{0\}}$ is a 1-dimensional distribution on $\mathcal{P}_\theta \setminus \{0\}$ satisfying

$$(5) \quad I_\theta(x) \oplus \mathbb{R}x = T_x \mathcal{P}_\theta, \quad \forall x \in \mathcal{P}_\theta \setminus \{0\}.$$

By definition, the intersection $L_p \cap \mathcal{P}_\theta$ contains the integral curve to I_θ starting at p . The integral curves of I_θ are nothing but the leaves of the 1-dimensional foliation defined by the distribution I_θ in $\mathcal{P}_\theta \setminus \{0\}$. We claim that such a foliation is orientable. Indeed, for each x in $\mathcal{P}_\theta \setminus \{0\}$, we can say that a nonzero vector v in $I_\theta(x)$ has a positive orientation if (v, x) is an oriented frame of \mathcal{P}_θ . It follows from the already cited Whitney's theorem that the foliation defined by I_θ is the orbit partition of $\mathcal{P}_\theta \setminus \{0\}$ by the flow of a vector field g_θ everywhere transversal to the radial direction.

Because of the transversality condition and the homogeneity, the flow of g_θ makes a turn around the origin, in the sense that its angular component is monotone with respect to time and the radial component does not converge to zero nor diverge in finite time. Let us choose the vector field g_θ such that, starting from p , one intersects $\mathbb{R}_{>0}\theta$ before $-\mathbb{R}_{>0}\theta$. Define p_θ to be the point of first intersection between the integral curve of g_θ starting at p and the ray $-\mathbb{R}_{>0}p$, and C_θ to be the arc between p and p_θ (see Figure 2).

Now, let us define the map $\Phi : S^{n-2} \rightarrow -\mathbb{R}_{>0}p$ by $\Phi(\theta) = p_\theta$. The map Φ is well-defined, in the sense that p_θ does not depend on the vector field g_θ (once the latter is chosen with

the appropriate orientation). Moreover, the map Φ is continuous, as it follows from the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2} . In addition, the image of Φ is contained in the intersection $L_p \cap -\mathbb{R}_{>0}p$, which has empty interior because of the transversality between $T\mathcal{L}$ and the radial direction. Since S^{n-2} is connected ($n > 2$), it follows that Φ is constant. Let us define

$$S = \bigcup_{\theta \in S^{n-2}} C_\theta.$$

By the transversality between $I_\theta(-p)$ and $-\mathbb{R}_{>0}p$ for all θ in S^{n-2} it follows that we can parameterize C_θ as a continuous arc on $[0, 1]$ continuously with respect to θ . Hence, the topology of S as a subset of \mathbb{R}^n is that of $(S^{n-2} \times [0, 1]) / \sim$, where \sim is the equivalence relation which identifies the points in $S^{n-2} \times \{0\}$ to a single equivalence class, and analogously for the points in $S^{n-2} \times \{1\}$. That is, S is a topological manifold homeomorphic to the sphere S^{n-1} . In particular, $S \subset L_p$ is closed in \mathbb{R}^n , and therefore it is closed in L_p . Since S has the same topological dimension as L_p , we have that S is open in the topology of L_p . Since L_p is connected, we can conclude that $S = L_p$. This contradicts the assumptions that the leaves are dense. \square

Lemma 9 shows that the supposition that we are in case (b) of Corollary 5 leads to a contradiction. Therefore, we are in case (a), and (Σ) is controllable. This concludes the proof of Theorem 1. \square

The result in Lemma 9 implies that Theorem 1 generalises for control systems which are Lie-determined, homogeneous, and angularly controllable. We say that (C) is *homogeneous* if $X = \mathbb{R}^n \setminus \{0\}$ and for every $x \in X$, $u \in U$, and $\lambda > 0$, $\pi_*(f_u(\lambda x)) = \pi_*(f_u(x))$, where $\pi : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ still denotes the canonical projection. Just as in the bilinear case, the projection of such systems on the sphere S^{n-1} is

$$(SC) \quad \dot{x} = \pi_*(f_{u(t)}(x)), \quad f_{u(t)} \in \mathcal{F}, \quad x \in S^{n-1},$$

and we say that (C) is angularly controllable if (SC) is controllable.

Corollary 10. *Let $n \geq 2$. Assume that the control system (C) is Lie-determined, homogeneous, and angularly controllable. Then, (C) is approximately controllable if and only if it is controllable.*

Since for $n = 2$ system (C) projects to S^1 , in this case the hypothesis of angular controllability can be easily removed.

Remark 11. For all n odd, $n = 2k + 1$ with $k \geq 1$, the hypothesis of angular controllability in Corollary 10 is superfluous. Indeed, if system (C) is Lie-determined, homogeneous, and approximately controllable, then Corollary 5 implies that either (C) is angularly controllable, or the projection of its orbits forms a nontrivial regular foliation of the even-dimensional sphere S^{2k} . Since even dimensional spheres do not admit nontrivial regular foliations (indeed their tangent spaces do not admit any nontrivial subbundles - see, e.g., [MS74, Problem 9C]), the angular controllability follows.

However, it has not been possible to fully remove the hypothesis of angular controllability in Corollary 10. In this regard, let us discuss the case $n = 4$. Due to Corollary 5, if (C) is Lie-determined, homogeneous, approximately controllable, and (SC) is not controllable, then the orbits of (SC) form a regular foliation of S^3 of either dimension one or codimension one. The latter gives a contradiction, since the Novikov compact leaf theorem implies that any codimension-one regular foliation of the sphere S^3 has a compact leaf [Nov65]. The former implies that the orbits of (SC) are given by the flow of a minimal vector field, i.e., a vector field whose orbits are dense. The existence of such flows has been raised as an open question in [Got58] for compact metric spaces, and for the sphere S^3 has been mentioned by Smale in [Sma98] under the name Gottschalk conjecture; further details can be found in [FPW07]. If the Gottschalk conjecture were to be true, it would imply the existence of Lie-determined, homogeneous, approximately controllable, yet not controllable systems, showing that Corollary 5 fails to hold if we remove the angular controllability hypothesis.

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