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On a Shared Scooter Service with Opportunistic Riding under Ring Shape: the S3 Traffic Model and its Equilibrium

Abstract

Shared Scooter Services (S3) have been deployed in many cities to provide public transport to individual users on a free-floating basis. The article brings about an analytical traffic model for such service under ring shape, i.e. postulating there is a ring road along which the scooters are required to park and expected to be driven, in order to avoid detours and increase vehicle productivity as well as local availability to potential users. Opportunistic Riding is also postulated: on their way, every user takes the first available vehicle which they encounter. Also postulated are the homogeneity in both space and time. With respect to the ring circumference, the daily volume of potential trips, the statistical distributions of trip lengths, walk speeds and ride speeds, as well as the fleet size, operation period and transaction times, analytical formulas are provided for (i) access lengths and times, together with usage probability, (ii) ride lengths and times, (iii) vehicle occupation at the fleet level, (iv) the average number of available vehicles. Traffic equilibrium is shown to exist and to be unique. Its characteristic equation is recast as a fleet sizing rule according to demand volume and target access length, which stand respectively as the quantitative and qualitative service objectives.

Keywords: shared mobility services; free-floating scooters; vehicle occupation; fleet sizing; stochastic model; traffic equilibrium.

1/ Introduction

Background. In recent years, e-scooters have emerged as a convenient mode of travel for many individuals. As a vehicle, an e-scooter is agile and its motor enables for speeds up to 20 km/h and beyond: this is quite efficient in the urban setting. Furthermore, its small size and light weight make it easy to carry out e.g. in a car trunk or in transit vehicles (buses, trains).

The parallel development of connectedness both for vehicles and for people owing to smartphones has given rise to many shared mobility services. Among them, e-scooters sharing systems have been deployed in many big cities worldwide.

To what extent may such e-scooter sharing services contribute to public transport in urban areas? This question carries obvious stakes for mobility planners in territories. Shared e-scooters look environmental friendly compared to private cars, not only in terms of matters utilized and energy consumed but also to the cityscape: their parking requires little space and their traffic is by far less intrusive and dangerous to pedestrians.

Objective. A number of traffic models have been developed to perform traffic studies with much detail in space and time. This paper brings about a simple traffic model for an e-scooter shared mobility service under ring shape. The principle is to channel the users' trips in the territorial space, so as to increase vehicle availability along the ring, to avoid detours, to ease the service logistics (from vehicle maintenance to energy refills, passing by fleet

repositioning). By simultaneously reducing the operational costs and increasing vehicle productivity, a ring-shaped service is expected to reach commercial profitability under wider conditions of demand and territory, thereby enabling the planners to expand the spatial outreach of public transport.

In the model, the territorial conditions are represented as the circumference of the ring and the riding speed, a period volume of trips and the statistical distribution of their lengths. The fleet size is given and the service is operated on a free-floating basis. The potential service users take the first available scooter which they encounter on their way: there is no prior booking, in contrast to a related former model (Leurent, 2021). From these assumptions, the model determines the access conditions in terms of walk lengths and walk times, the ride lengths and riding times depending on users' ride speeds, the occupation of the vehicle fleet and its availability, as well as the probability of using the service among the potential demand.

Methodology. Homogeneity is postulated regarding both space and time. Spatial homogeneity is postulated for trip origin and destination endpoints, hence for the positions of available vehicles. Combined to the ring shape, this spatial homogeneity gives rise to circular symmetry. Temporal homogeneity is postulated along a duty period within the day, for instance a 4h period around the evening peak.

Traffic is modeled in disaggregate and stochastic way. Each service request is identified with its own service user. Trip lengths are modeled as a statistical distribution. Users are modeled in terms of walk speeds and ride speeds, with joint statistical distribution.

Each scooter is identified as an individual resource, of which the dynamic state is either Available (i.e. vacant) or Busy. At any instant, the number of available scooters is a random variable with specific distribution of which the mean is linked to the fleet workload. The service traffic involves two causalities: first, the more there are available scooters, the higher the rate of usage and the longer the ride lengths on the demand side, inducing higher traffic load on the supply side; second, a higher traffic load comes with more busy scooters and less available scooters at any instant. The two causalities balance each other, leading to traffic equilibrium. We demonstrate the existence and uniqueness of that equilibrium and derive some consequences, among which a fleet sizing rule according to potential demand volume and target access length.

Article structure. The rest of the article is in eight parts. Section 2 describes the system: territorial conditions, service supply and travel demand. Section 3 introduces the model architecture. After modeling successively Service access in Section 4, User rides in Section 5 and Fleet occupation in Section 6, the model is integrated in Section 7, which addresses traffic equilibrium. Section 8 discusses some properties and consequences of the equilibrium state. Lastly, Section 9 concludes and points to theoretical developments.

2/ System description and mathematical notation

Inside an urbanized territory, assume that a roadway circuit is designated as pathway for the shared e-scooters. By assumption, it can be used in either directions and its overall shape is a circular or quasi-circular ring of radius R and circumference C (which is about $2\pi R$ for a quasi-circular ring).

On the demand side, we consider that, during a time period of H , there are Q trips that would like to use the shared mobility service, half of which in each traffic direction. The trip origin points are distributed evenly along the ring. From their origin points the trips have spatial lengths L_T with statistical distribution that is identical from all origins. From the assumptions

about the trips' origins and lengths, it follows that the destination points are also evenly distributed along the ring.

Among the population of customers, each service user has his own walk speed w and ride speed v : assumedly, these are cruising speeds over the walk and ride legs respectively. Walk speeds are statistically distributed with CDF W and ride speeds with CDF S . Statistical independence is assumed between them and also with the generation of trips.

By assumption, the service is operated under free-floating mode all along the ring.

Coming to service usage, each potential user begins his trip of length L_T by walking along the ring towards his destination. On his way, if he finds an available scooter then he takes it to ride up to his destination: the walk length is denoted L_A , A for Access, and the ride length is denoted L_R . The condition of service utilization is that $\{L_A < L_T\}$: if negative then $L_R = 0$ and the service is not used, whereas if positive then $L_R := L_T - L_A$ is strictly positive and the associated service time $t_{RT} := t_R + t_T$ involves the ride time, $t_R = L_R/v$ according to ride length L_R and ride speed v , and also a transaction time denoted t_T for vehicle un-parking at the point of availability and re-parking at the trip destination point. We shall denote p_u the probability of utilizing the service among the potential customers.

On the supply side, there are N scooters constituting the service fleet, supposedly all in operational condition. At any instant, each of them has a probability p_0 of being occupied, which means Busy for riding as well as during the transaction operations. Thus, the trip time t_{RT} is also the resource occupation time for the scooter used for riding.

At any instant, the average number of vacant scooters, say \bar{k} , satisfies that

$$\bar{k} = N \cdot (1 - p_0). \quad (1)$$

From the postulates of spatial homogeneity for demand trips, we deduce that each available scooter has its position distributed evenly along the ring, and that the respective positions of the available scooters are statistically independent.

3/ Model architecture

Among the model components, the following parts are exogenous:

- on the demand side, trip volume Q , trip length distribution $F_T := \text{CDF}(L_T)$ with mean \bar{L}_T , walk speed CDF W and ride speed CDF S ,
- on the supply side, fleet size N , operations period H , transaction time t_T ,
- ring circumference C pertains to the territory as it is involved both with the demand and the supply of the shared mobility service,
- a stochastic model of vehicle availability, deriving the statistical distribution of the number of available scooters, k , and its mean value \bar{k} , from fleet size N and an “availability factor” denoted θ .

From these exogenous conditions, the model enables us to derive the following endogenous components:

- on the demand side, the probability of service utilization $p_{u|\ell}$ according to trip length ℓ , the statistical distribution of trip lengths L_T^u conditional on using the service, those of access lengths L_A and times t_A , those of ride lengths L_R and times t_R , that of trip times t_J , J for Journey, whether using the service or generally.

- on the supply side, the statistical distributions of ride lengths L_R^u and times t_R^u , the service times t_{RT} , the statistical distribution of the number of available scooters k and the fleet occupation probability p_0 .

The model architecture involves four sub-models:

- (i) the Access model first relates L_A to k hence to N and θ , then compares L_A to L_T to obtain $p_{u|\ell}$, L_T^u and L_A^u , and next derives t_A from L_A and W ;
- (ii) the Ride model establishes L_R^u with respect to L_T^u and L_A^u , then derives t_R^u from L_R^u and S ;
- (iii) the Occupation model determines t_{RT} and relates p_0 to \bar{t}_{RT} , H , N and $Q \cdot p_u$;
- (iv) the Traffic equilibrium model relates the average number of available vehicles \bar{k} to the occupation probability p_0 , and sets the availability factor θ so as to make the values of \bar{k} and p_0 consistent throughout the chain of influences along the previous three models.

Figure 1 exhibits the model architecture as a graph system of components and dependences. Each model component is represented by a box, i.e. a graph node. Each straightforward influence of one model component onto another is indicated by a link between their respective nodes, with orientation according to causality. The components and links are organized by sub-model: in other words, each sub-model consists in a sub-system of dependences. The figure shows that the traffic equilibrium model acts as a feedback to the other three sub-models.

In the following Sections, we will introduce in turn the Access model, the Ride model and the Occupation model, before turning to the Traffic equilibrium model. Thus, the order of presentation matches the logical thread in Figure 1.

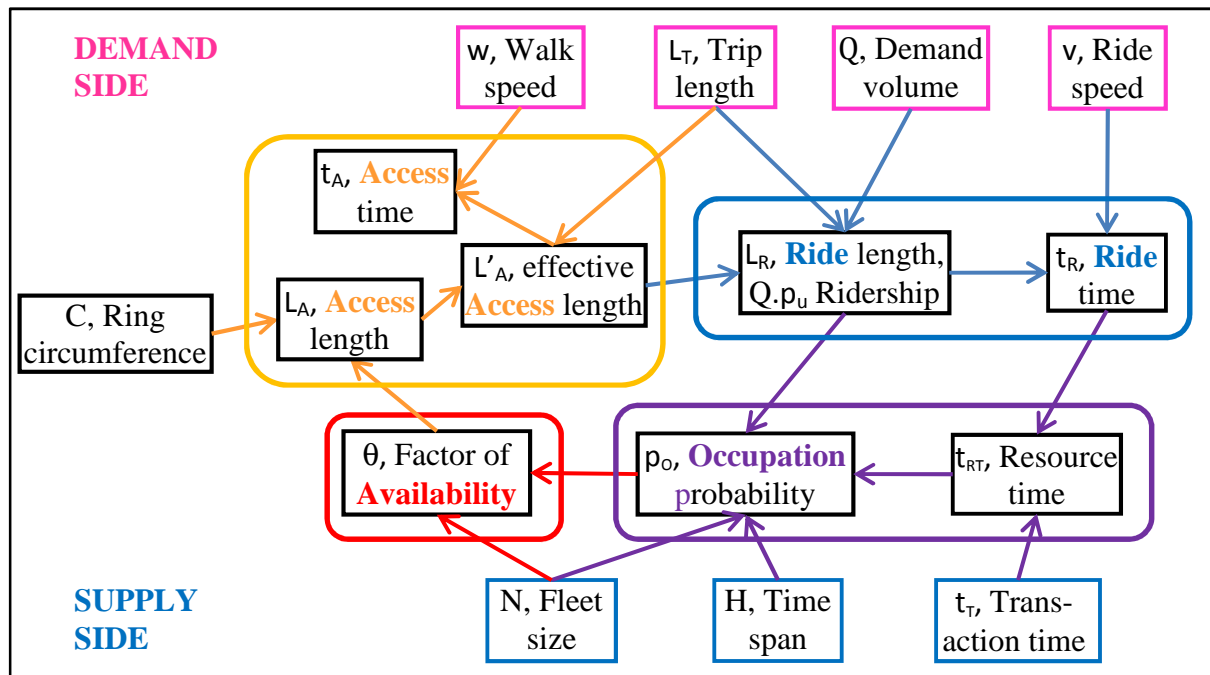


Fig. 1: S3 Traffic Model Architecture.

4/ The Access model

As potential rider, we consider an individual user that has subscribed to the service and is ready to use it, but will do so only if some riding opportunity occurs along his way from his origin point to his destination point.

4.1 Access length conditionally to the number of available vehicles

Each available scooter is positioned at a point distributed evenly along the ring: thus its distance from the trip origin point is distributed uniform between 0 and ring circumference C . Let us denote this access length to one available scooter as L_1 , its CDF as $F_1(x) := \min\{x/C, 1\}$ for $x \geq 0$ and $\bar{F}_1(x) := 1 - F_1(x)$ the distribution function cumulated from above.

When k scooters are available, statistical independence can safely be assumed and the access length is the minimum of k i.i.d. random variables $U[0, C]$. Its CDF cumulated from above then satisfies:

$$\bar{F}_{A|k}(x) := 1 - F_{A|k}(x) = (\bar{F}_1(x))^k. \quad (2)$$

Proof. For any available scooter $i \in \{1, \dots, k\}$, the length L_i from the customer origin point has CDF F_1 . The minimum $L_{A|k}$ of k independent lengths so distributed satisfies that

$$\Pr\{L_{A|k} > x\} = \Pr\{L_i > x : \forall i\} = \prod_{i=1}^k 1 - F_1(x) = (1 - F_1(x))^k.$$

4.2 The overall distribution of Access lengths

Now, the number k is itself a random variable: let us denote $\pi_k^{(N, \theta)} := \Pr\{k | N, \theta\}$ the probability values that stem from fleet size N and availability factor θ in a way that will be specified in the Traffic Equilibrium Model. The overall distribution of the access length comes from the conditional distributions weighted by the $\Pr\{k\}$. Thus, by integration over the values of k (and neglecting to denote the conditionality of L_A to N and θ),

$$\bar{F}_A(x) = \sum_{k=0}^N \pi_k^{(N, \theta)} \bar{F}_{A|k}(x). \quad (3)$$

Proof. Conditioning according to k , $\bar{F}_A(x) = \Pr\{L_A > x\} = \sum_{k=0}^N \Pr\{L_A > x \cap k\}$, hence $\bar{F}_A(x) = \sum_{k=0}^N \Pr\{L_{A|k} > x\} \cdot \Pr\{k\}$. When $k \geq 1$ we already know that $\bar{F}_{A|k}(x) = (\bar{F}_1(x))^k$ and in fact this formula is also suitable at $k = 0$ for which it says that the probability of not finding an available vehicle within length x is equal to 1.

4.3 The probability of service utilization

The service is utilized if and only if $\{L_A < L_T\}$. Given $L_T = \ell$, the probability of service utilization is

$$p_{u|\ell} = \Pr\{L_A < L_T | L_T = \ell\} = F_A(\ell). \quad (4)$$

Over the original distribution of trip lengths, the probability of service utilization is then

$$p_u = \int p_{u|\ell} dF_T(\ell) = 1 - \int \bar{F}_A(\ell) dF_T(\ell). \quad (5)$$

4.4 Trip lengths depending on service utilization

In turn, the condition of service utilization determines the statistical distribution of trip lengths among the population of service rides: let us denote L_T^u the conditional random variable. Its probability density function f_T^u is related to f_T and $p_{u|\ell}$ in the following way:

$$f_T^u(\ell) = p_u^{-1} \cdot p_{u|\ell} \cdot f_T(\ell). \quad (6)$$

Proof. As every trip has probability $p_{u|\ell}$ of utilizing the service depending on its length ℓ , in the population of rides that are service utilizations it holds that

$$f_T^u(\ell) \propto p_{u|\ell} \cdot f_T(\ell).$$

Denoting λ the factor of proportionality, by integration of both sides we get that $1 = \lambda \int p_{u|\ell} \cdot f_T(\ell) d\ell$, which implies that $\lambda = p_u^{-1}$.

Let us also provide a more rigorous demonstration:

$$\begin{aligned} \bar{F}_T^u(y) &:= \Pr\{L_T > y \mid L_T > L_A\} = \Pr\{L_T > y \cap L_T > L_A\} / \Pr\{L_T > L_A\} \\ &= p_u^{-1} \Pr\{L_T > \max\{y, L_A\}\} = p_u^{-1} \int \bar{F}_T(\max\{y, \ell\}) dF_A(\ell). \end{aligned}$$

Then, by differentiation,

$$\begin{aligned} f_T^u(y) &= -\frac{\partial}{\partial y} \bar{F}_T^u(y) = p_u^{-1} \int f_T(\max\{y, \ell\}) 1_{\{y \geq \ell\}} dF_A(\ell) \\ &= p_u^{-1} \int^y f_T(y) dF_A(\ell) = p_u^{-1} f_T(y) F_A(y). \end{aligned}$$

As $F_A(y) = p_{u|y}$, this completes the proof.

4.5 Effective access lengths

For one rider with trip length ℓ , the effective access length denoted L_A^u takes every nonnegative value x with probability density of

$$f_{A|\ell}^u(x) = f_A(x) 1_{\{x \leq \ell\}} / F_A(\ell). \quad (7)$$

Proof. $F_{A|\ell}^u(x) := \Pr\{L_A \leq x \mid L_T > L_A, L_T = \ell\}$

$$= \frac{\Pr\{L_A \leq \min\{x, L_T\} \cap L_T > L_A \mid L_T = \ell\}}{\Pr\{L_T > L_A \mid L_T = \ell\}} = \frac{\Pr\{L_A \leq \min\{x, \ell\}\}}{p_{u|\ell}} = \frac{F_A(\min\{x, \ell\})}{F_A(\ell)}$$

By differentiation with respect to x , we obtain that $f_{A|\ell}^u(x) = f_A(\min\{x, \ell\}) \cdot 1_{\{x \leq \ell\}} / F_A(\ell)$, hence the claim.

It is then easy to obtain the conditional average access length:

$$\bar{L}_{A|\ell}^u = \int x f_{A|\ell}^u(x) dx = \int_0^\ell x \cdot f_A(x) dx / F_A(\ell).$$

Integrating by parts, the numerator amounts to $[x F_A(x)]_0^\ell - \int_0^\ell F_A(x) dx$, so that

$$\bar{L}_{A|\ell}^u = \int_0^\ell (F_A(\ell) - F_A(x)) dx / F_A(\ell). \quad (8)$$

Over the population of rides that stems from that of trips and the access conditions, the statistical distribution of effective access lengths has CDF as follows:

$$\begin{aligned} F_A^u(x) &:= \Pr\{L_A^u \leq x\} = \int \Pr\{L_A^u \leq x \mid \ell\} dF_T^u(\ell) = p_u^{-1} \int F_A(\min\{x, \ell\}) dF_T(\ell) \\ &= p_u^{-1} [F_A(x) \bar{F}_T(x) + \int_0^x F_A(\ell) dF_T(\ell)] \end{aligned}$$

The associated PDF is

$$\begin{aligned} f_A^u(x) &= \int f_{A|\ell}^u(x) dF_T^u(\ell) = \int \frac{f_A(x)}{F_A(\ell)} 1_{\{x < \ell\}} dF_T^u(\ell) = p_u^{-1} f_A(x) \int_x^\infty dF_T(\ell), \text{ so that} \\ f_A^u(x) &= p_u^{-1} \cdot f_A(x) \cdot \bar{F}_T(x). \end{aligned} \quad (9)$$

Furthermore, the average effective access length over the population of service rides satisfies that $\bar{L}_A^u = \int \bar{L}_{A|\ell}^u dF_T^u(\ell)$, so that

$$\bar{L}_A^u = p_u^{-1} \int \left(\int_0^\ell (F_A(\ell) - F_A(x)) dx \right) dF_T(\ell). \quad (10)$$

4.6 Access times

Coming to the access time t_A , from the definition of cruising walk speeds w it holds that

$$t_A := L_A/w.$$

It can safely be assumed that L_A and w are independently distributed. Then, the average access time conditionally to $L_T = \ell$ satisfies that

$$\bar{t}_{A|\ell} = \bar{L}_{A|\ell} \cdot E[1/w]. \quad (11a)$$

The overall average access time is then:

$$\bar{t}_A := E[t_A] = \bar{L}_A \cdot E[1/w]. \quad (11b)$$

Similarly to access lengths, conditionally to service utilization the average access times conditional on $L_T = \ell$ or overall are respectively:

$$\bar{t}_{A|\ell}^u = \bar{L}_{A|\ell}^u \cdot E[1/w], \quad (11c)$$

$$\bar{t}_A^u = \bar{L}_A^u \cdot E[1/w]. \quad (11d)$$

Instance. Taking $w \approx \text{LN}(\mu_w, \sigma_w)$, then $1/w \approx \text{LN}(-\mu_w, \sigma_w)$ and $E[1/w] = \exp(-\mu_w + \sigma_w^2/2) = \exp(\sigma_w^2)/E[w]$. From the properties of LN laws, $\exp(\sigma_w^2) = 1 + \gamma_w^2$ with respect to the relative dispersion $\gamma_w := \text{SD}[w]/E[w]$. The relation $E[1/w] \approx (1 + \gamma_w^2)/E[w]$ holds as a good approximation for any positive random variable with not so large relative dispersion.

5/ The Ride model

5.1 Ride lengths

The service is used depending on the access conditions and especially the access length in comparison to the trip length. The residual length $L_R := L_T - L'_A$ is the ride length only if strictly positive: among the population of rides we denote the ride lengths as $L_R^u := L_T^u - L'_A$. Conditionally to $L_T = \ell$, ride length L_R^u takes any value $y \in]0, \ell]$ with probability density as follows:

$$f_{R|\ell}^u(y) = f_A(\ell - y) 1_{\{0 \leq y \leq \ell\}} / F_A(\ell). \quad (12)$$

Proof. $\bar{F}_{R|\ell}^u(y) := \Pr\{L_R > y \mid L_T > L_A, L_T = \ell\} = \Pr\{L_A \leq \min\{L_T - y, L_T\} \mid L_T > L_A, L_T = \ell\}$
 $= \frac{\Pr\{L_A \leq \min\{L_T - y, L_T\} \cap L_T = \ell\}}{\Pr\{L_T > L_A \mid L_T = \ell\}} = \frac{\Pr\{L_A \leq \min\{\ell - y, \ell\}\}}{p_{u|\ell}} = \frac{F_A(\min\{\ell - y, \ell\})}{F_A(\ell)}$

By differentiation with respect to y , we get that $f_{R|\ell}^u(y) = f_A(\min\{\ell - y, \ell\}) \cdot 1_{\{0 \leq y \leq \ell\}} / F_A(\ell)$, hence the claim.

Then the conditional average ride length, $\bar{L}_{R|\ell}^u = \int_0^\ell y f_{R|\ell}^u(y) dy$, satisfies that

$$\bar{L}_{R|\ell}^u = \int_0^\ell \frac{F_A(x)}{F_A(\ell)} dx. \quad (13)$$

Proof. The reason is that $\int_0^\ell y f_A(\ell - y) dy = \int_0^\ell (\ell - x) f_A(x) dx = \ell F_A(\ell) - \int_0^\ell x f_A(x) dx$. Integration by parts of the latter term yields $\ell \cdot F_A(\ell) - \int_0^\ell F_A(x) dx$, so that the difference reduces to $\int_0^\ell F_A(x) dx$.

Over the population of rides, the ride length is distributed with CDF that satisfies

$$F_R^u(x) = p_u^{-1} \int_x^\infty (F_A(\ell) - F_A(\ell - x)) dF_T(\ell). \quad (14)$$

Proof. This is because $F_R^u(x) = \int F_{R|\ell}^u(x) dF_T^u(\ell) = \int_{\ell > x} \frac{F_A(\ell) - F_A(\ell - x)}{F_A(\ell)} dF_T^u(\ell)$ wherein $dF_T^u(\ell) = p_u^{-1} F_A(\ell) dF_T(\ell)$ and the integrand is nonzero only if $\ell > x$.

The associated PDF also stems from those conditionally to trip lengths:

$f_R^u(x) = \int f_{R|\ell}^u(x) dF_T^u(\ell)$, so that

$$f_R^u(x) = p_u^{-1} \int_x^\infty f_A(\ell - x) dF_T(\ell). \quad (15)$$

Similarly, the average ride length among the riders stems from the conditional expectations according to trip lengths: $\bar{L}_R^u = \int \bar{L}_{R|\ell}^u dF_T^u(\ell)$, hence

$$\bar{L}_R^u = p_u^{-1} \int \left(\int_0^\ell F_A(x) dx \right) dF_T(\ell). \quad (16)$$

5.2 Ride times

Coming to the ride time, it stems from the ride length and the ride speed v as $t_R = L_R/v$. It can safely be assumed that the lengths and speeds are independent. Conditionally to $L_T = \ell$, the average ride time satisfies that:

$$\bar{t}_{R|\ell} = \bar{L}_{R|\ell} \cdot E[1/v]. \quad (17a)$$

Similarly, the overall average ride time satisfies that

$$\bar{t}_R = \bar{L}_R \cdot E[1/v]. \quad (17b)$$

The effective average ride times are more representative of the service usage:

$$\bar{t}_{R|\ell}^u = \bar{L}_{R|\ell}^u \cdot E[1/v]. \quad (17c)$$

$$\bar{t}_R^u = \bar{L}_R^u \cdot E[1/v]. \quad (17d)$$

6/ The Occupation model at the fleet level

6.1 Fleet occupation

For every ride, the scooter is assigned to the user during the ride time t_R^u plus the transaction time of t_T which we take as exogenous. Let us denote the total service time for that ride as:

$$t_{RT} := t_R^u + t_T.$$

Its overall average value amounts to

$$\bar{t}_{RT} = \bar{t}_R^u + t_T. \quad (18)$$

At the service level, during the period of operations, the total vehicle occupation time amounts to

$$\mathcal{R} = Q \cdot p_u \cdot \bar{t}_{RT} = Q \cdot (p_u \cdot t_T + p_u \bar{L}_R^u E[1/v]). \quad (19)$$

To make this workload feasible in the period time H , the fleet size N must satisfy that

$$N \cdot H \geq Q \cdot p_u \cdot \bar{t}_{RT}. \quad (20)$$

Under this requirement, the proportion of occupation in resource time amounts to

$$p_O := Q \cdot p_u \cdot \bar{t}_{RT} / N \cdot H. \quad (21)$$

6.2 Fleet availability

From our postulates of homogeneity in time and space, proportion p_O can be taken as the probability of occupation for any scooter at any time, in the absence of any other information (such as the particular assignment of the scooter to some user request). The complementary probability $1 - p_O$ is the probability of the scooter being available. Then, on average over the vehicle fleet, it must hold that

$$\bar{k} = N \cdot (1 - p_O). \quad (22)$$

7/ Model integration and traffic equilibrium

7.1 Equilibrium characterization

We have chained up the influences between the state variables, from an exogenous factor of availability θ in the Access model, to the Occupation probability p_O in the Occupation model, passing by the formation of access lengths L_A^u and ride times t_R^u . To close the model it remains to link the θ factor to the occupation probability, thereby making θ endogenous.

In (22) the mean number of available vehicles, \bar{k} , is in fact a function $\bar{k}_{N,\theta}$ of the underlying stochastic model of vehicle availability. Replacing $N \cdot p_O$ with its expression from (21), i.e. $p_u \cdot \bar{t}_{RT} \cdot Q / H$, we obtain the following equation that determines θ and characterizes a state of traffic equilibrium for that mobility service:

$$\bar{k}_{N,\theta} = N - p_u \cdot \bar{t}_{RT} \cdot Q / H. \quad (23)$$

Traffic equilibrium of the S3 system is endowed with properties of existence and uniqueness that hold under mild conditions, as will be shown later on.

7.2 Equilibrium properties

Let us define the ‘‘Fleet Resizing function’’:

$$\theta \mapsto \sigma_{N,Q}(\theta) := \bar{k}_{N,\theta} + p_u \cdot \bar{t}_{RT} \cdot \frac{Q}{H}.$$

Proposition: Traffic Equilibrium Properties. (i) **Existence:** if $\sigma_{N,Q}$ is continuous with respect to θ and ranges from 0 to values beyond N , then there exists a solution to the characteristic equation: in other words, there exists a state of traffic equilibrium. (ii) **Uniqueness:** if the fleet resizing function is strictly increasing with respect to θ , then there can exist at most one solution to the characteristic equation. (iii) **Constructive determination:** if the two assumptions hold jointly, then the fleet resizing function can be inverted on its domain and its inverse function $\sigma_{N,Q}^{(-1)}$ is also strictly increasing and continuous on its own domain that encompasses $[0, N]$: thus, the solution of the characteristic equation is simply $\sigma_{N,Q}^{(-1)}(N)$.

Proof. Point (ii) about Uniqueness is obvious because between two values θ_1 and $\theta_2 \geq \theta_1$, the strict increasingness implies that $\sigma_{N,Q}(\theta_2) > \sigma_{N,Q}(\theta_1)$ so that if value N is taken on one side

it cannot be reached on the other side. As for point (i) about Existence, under the assumptions of continuity and variation range from 0 up to values beyond N , then by the Bolzano-Weierstrass theorem applied to $\sigma_{N,Q}$ on a bounded interval whose image encompasses $[0, N]$, there is at least one point in that interval, say $\theta_{N,Q}^*$, such that $\sigma_{N,Q}(\theta_{N,Q}^*) = N$. Coming to Point (iii), by combining the properties of strict increasingness, variation range encompassing $[0, N]$ and continuity for function $\theta \mapsto \sigma_{N,Q}(\theta)$, then it is invertible on its domain, with inverse function $\sigma_{N,Q}^{(-1)}$ that is also strictly increasing and continuous on its own domain that encompasses $[0, N]$. By construction, $\sigma_{N,Q}^{(-1)}(N)$ satisfies that $\sigma_{N,Q}(\sigma_{N,Q}^{(-1)}(N)) = N$, hence it is a solution to the characteristic equation, and the unique one.

7.3 Availability factor and increasing fleet resizing function

Definition: A random variable Y is smaller than Z in a weak sense iff $\forall x, F_Y(x) \geq F_Z(x)$.

Lemma: Monotonicity of utilization probability and average ride length. Given two access random variables $L_A^{(1)}$ and $L_A^{(2)}$ such that $L_A^{(2)}$ is smaller than $L_A^{(1)}$ in the weak sense, then $p_u^{(2)} \geq p_u^{(1)}$ and $p_u^{(2)} \bar{L}_R^{u(2)} \geq p_u^{(1)} \bar{L}_R^{u(1)}$.

Proof: as $L_A^{(2)}$ is weakly smaller than $L_A^{(1)}$, we have that $F_A^{(2)}(x) \geq F_A^{(1)}(x)$ everywhere in x . From (4) it follows that $p_{u|\ell}^{(2)} \geq p_{u|\ell}^{(1)}$ everywhere in ℓ . Then, integration over the original distribution of trip lengths yields that $p_u^{(2)} \geq p_u^{(1)}$. Coming to the ride length averaged over all trips, $\bar{L}_R = p_u \bar{L}_R^u$, from (16) we have that $p_u \bar{L}_R^u = \int \left(\int_0^\ell F_A(x) dx \right) dF_T(\ell)$. As $F_A^{(2)} \geq F_A^{(1)}$, $\int_0^\ell F_A^{(2)} \geq \int_0^\ell F_A^{(1)}$ everywhere in ℓ and in turn $p_u^{(2)} \bar{L}_R^{u(2)} \geq p_u^{(1)} \bar{L}_R^{u(1)}$.

Consequently, the period service workload $p_u \cdot \bar{t}_{RT} \cdot Q = Q(p_u \cdot t_T + p_u \bar{L}_R^u E[1/v])$ is greater under smaller access lengths $L_A^{(2)}$ than under greater ones $L_A^{(1)}$.

The ‘‘Monotonicity’’ turns into ‘‘Increasingness’’ with respect to θ as Availability factor which decreases the access lengths

We are now ready to define an Accessibility factor in a more formal way.

Definition: Availability factor: A parameter θ of an access law F_A is an Availability factor if it makes the access lengths decrease in the weak sense: i.e. between two values θ_1 and $\theta_2 \geq \theta_1$, the associated access laws satisfy that $L_A^{(2)}$ is weakly smaller than $L_A^{(1)}$.

Thus, when θ increases the CDF $F_{A|\theta}$ increases everywhere. In turn, the service workload is an increasing function of the availability factor. If furthermore the average number of available vehicles, $\bar{k}_{N,\theta}$, increases with θ , then so does the fleet resizing function. It is strictly increasing if one of its components is so, for instance the average number of available vehicles.

7.4 Stochastic models of Fleet Availability

Let us put forward three alternative sets of assumptions that constitute three models of availability: by order of increasing complexity, the Concentrated model, the Poisson model and the Truncated Poisson model.

In the Concentrated model, it is assumed that the number of available vehicles, k , is about constant hence approximately equal to its average value \bar{k} . Then $\theta = \bar{k}$ is rightly an availability factor, since $F_{A|\theta}(x) = 1 - (1 - x/C)^\theta$ is an increasing function of θ everywhere

in $x \in [0, C]$. Furthermore, $\bar{k}_{N,\theta} = \theta$ increases strictly with θ from 0 to infinity, and so does the fleet resizing function since p_u tends to 0 when θ does so (it is obvious for every $p_{u|\ell}$, then by aggregation). Complementarily, $F_{A|\theta}(x)$ is a continuous function of θ for every x , and in turn p_u and \bar{t}_{RT} are continuous with respect to θ , making the fleet resizing function continuous. Thus the requirements for the existence and uniqueness of traffic equilibrium are satisfied.

In the Poisson model, a Poisson law with intensity θ is postulated for k : thus $\Pr\{k\} = e^{-\theta} \theta^k / k!$, which is a continuous function of θ . This postulate is obviously an approximation since it assigns positive probabilities to values of k beyond fleet size N ... A specific appendix is dedicated to the Poisson model, which is relatively simple. Its advantages are (i) that $\bar{k}_{N,\theta} = \theta$, (ii) $\bar{F}_{A|\theta}(x) = e^{-\theta x/C}$, a continuous function of θ that decreases with it for every $x > 0$, thereby making θ a correct availability factor, (iii) functions p_u and \bar{t}_{RT} are continuous with respect to θ . Thus the Poisson model also satisfies the requirements for the existence and uniqueness of traffic equilibrium. A salient feature of this model is that the Fleet Resizing Function does not depend on N , which makes it a convenient Fleet Sizing Function depending on a target access length C/θ and a period demand volume Q .

The third model is a truncated Poisson model, still with parameter θ and such that the number of available vehicles is limited to the $\{0, 1, \dots, N\}$ range. Then, denoting $X_N(x) := \sum_{k=0}^N x^k / k!$, we have that

$$\pi_k^{(N,\theta)} = \theta^k / [k! X_N(\theta)]. \quad (24)$$

It is shown in Appendix C that $\bar{k}_{N,\theta} = \Psi_N(\theta)$, wherein $\Psi_N(\theta) := \frac{\theta \cdot X_{N-1}(\theta)}{X_N(\theta)}$ is continuous and increases from 0 to N^- as θ varies from 0 to ∞ , Cf. Leurent (2019b). Furthermore, the access lengths are distributed according to the following CDF from above:

$$\bar{F}_{A|\theta}(x) = \frac{X_N(\theta \bar{F}_{A|1}(x))}{X_N(\theta)}. \quad (25)$$

The property that θ is a correct availability factor is shown in Appendix C. Thus, the truncated Poisson model also induces the existence and uniqueness of traffic equilibrium.

8/ Discussion and numerical study

The model outcomes consist in (i) the distributions of trip access lengths and times, together with the probability of service usage according to trip length, (ii) the distributions of ride lengths and times, (iii) service fleet occupation measured as a period workload for the vehicle fleet, (iv) a characteristic equation of traffic equilibrium, with respect to availability factor θ which is involved in the stochastic model of fleet availability. The characteristic equation can be turned into a fleet sizing rule according to target demand volume and average access length.

As the traffic equilibrium is endowed with existence and uniqueness properties, all of the model outcomes constitute functions of the AF. And as the outcomes are characterized by explicit analytical formulas, this makes the model fully analytical.

The analytical nature of the model makes it especially suitable to sensitivity analysis. We have shown that both the usage probability and the actual average ride length are increasing functions of the AF. In a similar way, the influence of all model parameters onto all model outcomes can be traced analytically to demonstrate their respective directions of variations,

either increasing or decreasing. This is the topic of a subsequent article. These influences depend on one another. For instance, given the fleet size, the demand volume exerts a decreasing influence on the AF: in turn, this makes the access length increase and the ride length decrease, as well as the usage probability.

If S3 services under ring shape were implemented, then the analytical formulas would make them observable. For instance, knowing the fleet size, the service volume could be estimated from measuring in the field the average access length.

8.1 Numerical study

Numerical instance. Let us consider a ring road with radius $R = 1\text{km}$ and circumference $C = 6.3\text{km}$. By assumption, trip lengths are distributed uniformly on $[0, C/2]$ yielding average trip length \bar{L}_T of $C/4 = 1,6\text{km}$. Transaction time t_T is set to 2' (for scooter un-parking then re-parking, putting one's helmet etc.) and daily service period $H = 14\text{h}$, riding speeds such that $1/E[1/v] = 15\text{km/h}$, daily trip volume $Q = 1000$. Based on the Poisson model (cf. Appendix B), Figure 2 exhibits the fleet size according to θ (part a) and to target access length C/θ (part b).

To serve a given volume of demand, halving the target access length from 400m to 200m requires doubling the fleet size. Of course, shorter access lengths require a larger number of available vehicles, at the cost of a larger share of idle time for each vehicle in the fleet.

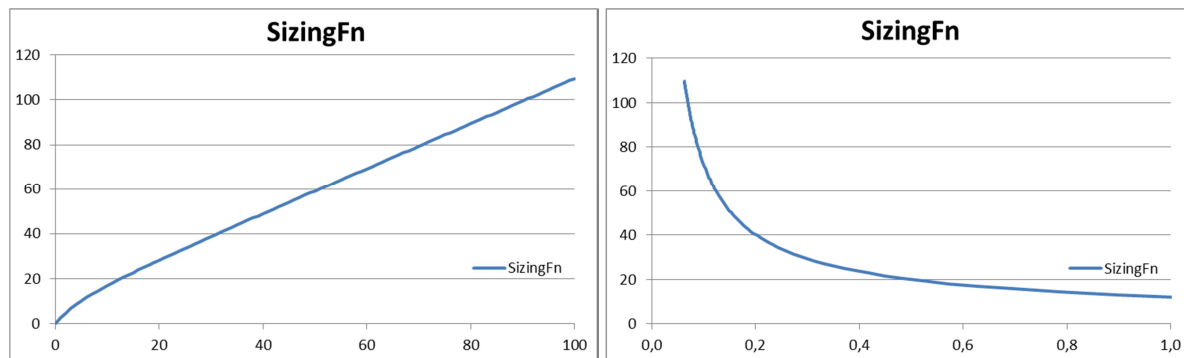


Fig. 2. Fleet size according to (a) Theta, (b) Target access length.

9/ Conclusion

We have provided an analytical model for a shared scooter service (S3) under ring form, free-floating and opportunistic usage. Owing to postulates of homogeneity in space and time, we established simple formulas for the average access length and time, ride time, probability of occupation. Given the fleet size and the potential demand volume, the number of available vehicles is involved in a balance equation that characterizes a state of traffic equilibrium for that system. This equation can be solved with respect to a variable called “availability factor” as it determines the statistical distribution of the number of available vehicles. We established some mild conditions under which a traffic equilibrium does exist and is unique. Moreover, the characteristic equation constitutes a fleet sizing rule for a service operator targeting both the demand volume (a quantitative objective of service) and the access length (a qualitative objective).

The S3 traffic model is a physical and technical model. It stands as the scooter counterpart of the ring-shaped taxi service model (Leurent, 2019a) and the ring-shaped Shuttle service

model C3 (Leurent, 2019b). It may represent a ring-shaped bike-sharing system as well, if operated in free-floating mode.

The economic theory of the S3 model can be developed along the same path as that for the taxi and C3 models, all the more so as the characteristic equation is analogous to that of the taxi service. Yet service production for a free-floating S3 is specific as it involves charging operations and vehicle maintenance, which we did not address in detail for the other modes (since vehicle driving, be it human or robotic, enables for offline maintenance and energy refill).

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Appendix A: The Concentrated Model of Availability

The postulate of a concentrated number of available vehicles involves a continuous approximation: the discrete value of k is put in equivalence with the continuous value \bar{k} , i.e. θ . We have that:

$$\Pr\{k\} = 1_{\{k=\theta\}}. \quad (\text{A.1})$$

A.1 On access lengths

The access length has a simple CDF from above:

$$\bar{F}_{A|\theta}(x) = (1 - \min\{1, x/C\})^\theta. \quad (\text{A.2})$$

Given $x \in]0, C[$, then $1 - x/C \in]0, 1[$: when θ increases, $\bar{F}_{A|\theta}(x)$ decreases: thus $F_{A|\theta}(x)$ increases with θ , which is thus a suitable availability factor.

The associated PDF is simply:

$$f_{A|\theta}(x) = \frac{\theta}{C} (1 - x/C)^{\theta-1} 1_{\{0 \leq x < C\}}. \quad (\text{A.3})$$

A.2 Probability of utilization

The probability of utilization conditional to trip length is generic whatever the initial distribution:

$$p_{u|\ell} = F_A(\ell) = 1 - (1 - \min\{1, \ell/C\})^\theta. \quad (\text{A.4})$$

To go further, let us from now on consider a uniform distribution of trip lengths in the domain $[0, C/2]$. Thus, $f_T(\ell) = \frac{2}{C} 1_{\{0 \leq \ell < C/2\}}$ and

$$F_T(\ell) = \min\{1, 2\ell/C\}.$$

Over the initial distribution of trip lengths, the probability of service utilization satisfies

$$\begin{aligned}
p_u &= 1 - \int \bar{F}_{A|\theta}(\ell) dF_T(\ell) = 1 - \frac{2}{C} \int_0^{C/2} (1 - \ell/C)^\theta d\ell \\
&= 1 - 2 \int_0^{.5} (1 - z)^\theta dz = 1 - 2 \left[-\frac{1}{\theta+1} (1 - z)^{\theta+1} \right]_0^{.5} = 1 - \frac{2}{\theta+1} (1 - .5^{\theta+1})
\end{aligned}$$

Thus,

$$p_u = \frac{\theta - 1 + .5^\theta}{\theta + 1}. \quad (\text{A.5})$$

It depends only on θ , not on C , nor N or Q .

A.3 Trip lengths among the rides

Coming to the trip lengths among the rides, from (6) we deduce that:

$$f_T^u(\ell) = p_u^{-1} \cdot (1 - (1 - \ell/C)^\theta) \cdot \frac{2}{C} 1_{\{0 \leq \ell < C/2\}}. \quad (\text{A.6})$$

It follows that

$$\begin{aligned}
p_u \cdot \int \ell f_T^u(\ell) d\ell &= \frac{2}{C} \int_0^{C/2} \ell [1 - (1 - \ell/C)^\theta] d\ell \\
&= \frac{2}{C} \frac{1}{2} \left(\frac{C}{2}\right)^2 - \frac{2}{C} \int_0^{C/2} \ell (1 - \ell/C)^\theta d\ell = \frac{C}{4} - 2C \int_0^{1/2} z(1 - z)^\theta dz \\
&= \frac{C}{4} + 2C \left[z \frac{1}{\theta+1} (1 - z)^{\theta+1} \right]_0^{1/2} - 2C \int_0^{1/2} \frac{1}{\theta+1} (1 - z)^{\theta+1} dz \\
&= \frac{C}{4} + \frac{2C}{\theta+1} (.5)^{\theta+2} + \frac{2C}{(\theta+1)(\theta+2)} [(1 - z)^{\theta+2}]_0^{1/2} = \frac{C}{4} + \frac{C}{\theta+1} (.5)^{\theta+1} - \frac{2C}{(\theta+1)(\theta+2)} (1 - (.5)^{\theta+2}) \\
&= \frac{C}{4(\theta+1)} (\theta + 1 + 2^{1-\theta} - \frac{1}{\theta+2} (8 - 2^{1-\theta})).
\end{aligned}$$

So that:

$$\bar{L}_T^u = \frac{C}{4} \cdot \frac{\theta + 1 + 2^{1-\theta} - \frac{1}{\theta+2} (8 - 2^{1-\theta})}{\theta - 1 + .5^\theta}. \quad (\text{A.7})$$

A.4 Access lengths among the rides

As for the effective access lengths, their PDF conditionally to trip length $\ell \leq C$ is

$$f_{A|\ell}^u(x) = \frac{\theta}{C} \frac{(1-x/C)^{\theta-1}}{1 - (1-\ell/C)^\theta} 1_{\{x \leq \ell\}}. \quad (\text{A.8})$$

The conditional average access length is such that:

$$\begin{aligned}
F_A(\ell) \cdot \bar{L}_{A|\ell}^u &= \int_0^\ell x \cdot f_A(x) dx = \frac{\theta}{C} \int_0^\ell x \cdot (1 - x/C)^{\theta-1} dx \\
&= \theta C \int_0^{\ell/C} z \cdot (1 - z)^{\theta-1} dz = C [-z(1 - z)^\theta]_0^{\ell/C} + C \int_0^{\ell/C} (1 - z)^\theta dz \\
&= -\ell \left(1 - \frac{\ell}{C}\right)^\theta + C \left[-\frac{1}{\theta+1} (1 - z)^{\theta+1} \right]_0^{\ell/C} = \frac{C}{\theta+1} (1 - (1 - \frac{\ell}{C})^{\theta+1}) - \ell \left(1 - \frac{\ell}{C}\right)^\theta \\
&= \frac{C}{\theta+1} \left(1 - (1 - \frac{\ell}{C})^\theta + \frac{\ell}{C} (1 - \frac{\ell}{C})^\theta\right) - \ell \left(1 - \frac{\ell}{C}\right)^\theta = \frac{C}{\theta+1} \left(1 - (1 - \frac{\ell}{C})^\theta - \theta \frac{\ell}{C} (1 - \frac{\ell}{C})^\theta\right)
\end{aligned}$$

Thus

$$\bar{L}_{A|\ell}^u = \frac{C}{\theta+1} (1 - \theta \frac{\ell}{C} / ((1 - \frac{\ell}{C})^{-\theta} - 1)). \quad (\text{A.9})$$

Over the population of all rides, from (9) we get that:

$$f_A^u(x) = p_u^{-1} \cdot \frac{\theta}{c} \left(1 - \frac{x}{c}\right)^{\theta-1} \cdot (1 - 2\frac{x}{c}) 1_{\{0 \leq x < c/2\}}. \quad (\text{A.10})$$

The average effective access length over the population of service rides comes from (10):

$$\begin{aligned} p_u \bar{L}_A^u &= \int \bar{L}_{A|\ell}^u p_u f_T^u(\ell) d\ell = \frac{2}{c} \int_0^{c/2} \frac{c}{\theta+1} \left(1 - \left(1 - \frac{\ell}{c}\right)^\theta - \theta \frac{\ell}{c} \left(1 - \frac{\ell}{c}\right)^\theta\right) d\ell \\ &= \frac{2}{\theta+1} C \int_0^{.5} (1 - (1-z)^\theta - \theta z (1-z)^\theta) dz \\ &= \frac{2}{\theta+1} C \left(\frac{1}{2} + \left[\frac{1}{\theta+1} (1-z)^{\theta+1}\right]_0^{.5} + \left[\frac{\theta z}{\theta+1} (1-z)^{\theta+1}\right]_0^{.5} - \frac{\theta}{\theta+1} \int_0^{.5} (1-z)^{\theta+1} dz\right) \\ &= \frac{2}{\theta+1} C \left(\frac{1}{2} + \frac{1}{\theta+1} (.5^{\theta+1} - 1) + \frac{\theta}{\theta+1} .5^{\theta+2} + \frac{\theta}{\theta+1} \left[\frac{1}{\theta+2} (1-z)^{\theta+2}\right]_0^{.5}\right) \\ &= \frac{2}{\theta+1} C \left(\frac{1}{2} + \frac{1}{\theta+1} (.5^{\theta+1} - 1) + \frac{\theta}{\theta+1} .5^{\theta+2} + \frac{\theta}{\theta+1} \frac{1}{\theta+2} (.5^{\theta+2} - 1)\right) \\ &= \frac{2}{\theta+1} C \left(\frac{1}{2} + \frac{2}{\theta+1} \left(.5^{\theta+2} - \frac{1}{2}\right) + \frac{\theta}{\theta+1} - \frac{\theta}{\theta+1} (1 - .5^{\theta+2}) + \frac{\theta}{\theta+1} \frac{1}{\theta+2} (.5^{\theta+2} - 1)\right) \\ &= \frac{2}{\theta+1} C \left(\frac{1}{2} + \frac{\theta}{\theta+1} + \frac{1}{\theta+1} + \frac{2}{\theta+1} (.5^{\theta+2} - 1) - (1 - .5^{\theta+2}) \frac{\theta}{\theta+1} \left(1 - \frac{1}{\theta+2}\right)\right) \\ &= \frac{2}{\theta+1} C \left(\frac{3}{2} - \frac{1}{\theta+1} \left(2 + \theta - \frac{\theta}{\theta+2}\right) (1 - .5^{\theta+2})\right) \end{aligned}$$

So that

$$\bar{L}_A^u = \frac{2}{\theta-1+.5^\theta} C \left(\frac{3}{2} - \frac{1}{\theta+1} \left(2 + \theta - \frac{\theta}{\theta+2}\right) (1 - .5^{\theta+2})\right). \quad (\text{A.11})$$

A.5 Ride lengths

Turning to ride lengths L_R^u , their PDF conditionally to trip length ℓ comes from (12):

$$f_{R|\ell}^u(y) = 1_{\{0 \leq y \leq \ell\}} \frac{\theta}{c} (1 - (\ell - y)/c)^{\theta-1} / [1 - (1 - \ell/c)^\theta]. \quad (\text{A.12})$$

The average ride length conditionally to trip lengths ℓ satisfies that:

$$\begin{aligned} F_A(\ell) \cdot \bar{L}_{R|\ell}^u &= \int_0^\ell F_A(x) dx = \int_0^\ell [1 - (1 - x/c)^\theta] dx \\ &= \ell + \left[\frac{c}{\theta+1} (1 - x/c)^{\theta+1}\right]_0^\ell = \ell - \frac{c}{\theta+1} (1 - (1 - \ell/c)^{\theta+1}) \\ &= \ell - \frac{c}{\theta+1} \left(1 - \left(1 - \frac{\ell}{c}\right)^\theta + \frac{\ell}{c} \left(1 - \frac{\ell}{c}\right)^\theta\right) = \ell \left(1 - \frac{1}{\theta+1} \left(1 - \frac{\ell}{c}\right)^\theta\right) - \frac{c}{\theta+1} \left(1 - \left(1 - \frac{\ell}{c}\right)^\theta\right) \\ &= \ell \frac{\theta}{\theta+1} - \frac{c-\ell}{\theta+1} \left(1 - \left(1 - \frac{\ell}{c}\right)^\theta\right) \end{aligned}$$

Thus,

$$\bar{L}_{R|\ell}^u = \ell \frac{\theta}{\theta+1} \left(1 - \left(1 - \frac{\ell}{c}\right)^\theta\right)^{-1} - \frac{c-\ell}{\theta+1}. \quad (\text{A.13})$$

Over the population of all rides, the PDF and expectation of the ride lengths stem from (15) and (16) respectively:

$$p_u f_R^u(y) = \frac{2}{c} \int_y^{c/2} p_{u|\ell} f_{R|\ell}^u(y) d\ell = \frac{2}{c} \int_y^{c/2} \frac{\theta}{c} (1 - (\ell - y)/c)^{\theta-1} d\ell$$

$$\begin{aligned}
&= \frac{2}{c} \left[-(1 - (\ell - y)/C)^\theta \right]_y^{C/2} = \frac{2}{c} \left(1 - \left(\frac{1}{2} + \frac{y}{c} \right)^\theta \right) \\
&f_R^u(y) = \frac{2}{c} \frac{\theta+1}{\theta-1+5^\theta} \left(1 - \left(\frac{1}{2} + \frac{y}{c} \right)^\theta \right). \tag{A.14}
\end{aligned}$$

$$\begin{aligned}
p_u \bar{L}_R^u &= \int_0^{C/2} \ell p_u f_R^u(\ell) d\ell = \frac{2}{c} \int_0^{C/2} \ell \left(1 - \left(\frac{1}{2} + \frac{\ell}{c} \right)^\theta \right) d\ell \\
&= \frac{2}{c} \frac{1}{2} \left(\frac{C}{2} \right)^2 - 2C \int_0^{.5} z \left(\frac{1}{2} + z \right)^\theta dz = \frac{C}{4} - 2C \left[\frac{z}{\theta+1} \left(\frac{1}{2} + z \right)^{\theta+1} \right]_0^{.5} + \frac{2C}{\theta+1} \int_0^{.5} \left(\frac{1}{2} + z \right)^{\theta+1} dz \\
&= \frac{C}{4} - \frac{C}{\theta+1} + \frac{2C}{\theta+1} \left[\frac{1}{\theta+2} \left(\frac{1}{2} + z \right)^{\theta+2} \right]_0^{.5} = C \left(\frac{1}{4} - \frac{1}{\theta+1} + \frac{2}{\theta+1} \frac{1}{\theta+2} (1 - .5^{\theta+2}) \right)
\end{aligned}$$

Thus:

$$\bar{L}_R^u = \frac{C}{4} \cdot \frac{\theta-3+\frac{2}{\theta+2}(4-.5^\theta)}{\theta-1+5^\theta}. \tag{A.15}$$

The respective influences of C and θ on the average ride length are under product form for a uniform distribution of initial trip lengths.

A.6 Fleet sizing function

Owing to the generic formulation of the S3 traffic model and as the access law specific to the concentrated model does not depend on N , it follows that in this specific model the fleet resizing function does not depend on N . It is then a straightforward ‘‘fleet sizing function’’. Its exact formula, denoting $\alpha := Q \cdot t_T/H$ and $\beta := Q \cdot E[1/v]/H$, is as follows:

$\sigma_Q(\theta) := \theta + \alpha \cdot p_u + \beta \cdot p_u \bar{L}_R^u$, so that

$$\sigma_Q(\theta) := \theta + \alpha \cdot \frac{\theta-1+2^{-\theta}}{\theta+1} + \beta \cdot \frac{C}{4} \cdot \frac{\theta-3+\frac{2}{\theta+2}(4-2^{-\theta})}{\theta+1}, \tag{A.16a}$$

For values of θ above 5, we can neglect $2^{-\theta}$ and obtain a first approximate sizing function:

$$\sigma_Q(\theta) \approx \theta + \alpha \cdot \frac{\theta-1}{\theta+1} + \beta \cdot \frac{C}{4} \cdot \frac{\theta-3+\frac{8}{\theta+2}}{\theta+1}, \tag{A.16b}$$

For larger θ e.g. beyond 20, we may neglect $\frac{8}{\theta+2}$ and approximate $\frac{\theta-1}{\theta+1} \approx \frac{\theta-3}{\theta+1} \approx 1$, yielding:

$$\sigma_Q(\theta) \approx \theta + \alpha + \beta \cdot \frac{C}{4}, \tag{A.16c}$$

Then, the solution to the characteristic equation is obtained straightforwardly:

$$\theta_{N,Q}^* \approx N - (t_T + \frac{C}{4} E[1/v])Q/H. \tag{A.17}$$

Complementarily, for large values of θ , we have $p_u \approx 1$ and $\bar{L}_R^u \approx \bar{L}_T = \frac{C}{4}$ for the uniform distribution of trip lengths.

Appendix B: The Poisson Model of Availability

B.0 Postulate and basic properties

Assuming a Poisson distribution with parameter θ , then

$$\Pr\{k\} = e^{-\theta} \theta^k / k!. \quad (\text{B.1})$$

The mean number of available vehicles is equal to the distribution parameter:

$$\bar{k}_{N,\theta} = \theta. \quad (\text{B.2})$$

It is a well-known property of Poisson laws. For completeness, let us recall the proof that is very simple:

$$\bar{k}_{N,\theta} = \sum_{k \geq 0} \Pr\{k\} \cdot k = e^{-\theta} \sum_{k \geq 0} k \frac{\theta^k}{k!} = e^{-\theta} \theta \sum_{k \geq 1} \frac{\theta^{k-1}}{(k-1)!} = e^{-\theta} \theta \sum_{k \geq 0} \frac{\theta^k}{k!} = e^{-\theta} \theta e^{\theta} = \theta.$$

Another well-known property of the Poisson law is that its variance is equal to θ , too. Thus the relative dispersion $\gamma_k := SD(k)/\bar{k}$ is equal to $1/\sqrt{\theta}$: the larger θ , the more concentrated is k in relative terms.

B.1 Access lengths

$$\bar{F}_A(x) = \exp -\theta F_1(x). \quad (\text{B.3})$$

Proof. $\bar{F}_A(x) = \sum_{k \geq 0} \bar{F}_{A|k}(x) \Pr\{k\} = e^{-\theta} \sum_{k \geq 0} \frac{(\bar{F}_1(x))^k \theta^k}{k!} = e^{-\theta} \sum_{k \geq 0} \frac{(\bar{F}_1(x)\theta)^k}{k!} = e^{-\theta} e^{\theta \bar{F}_1(x)}$, yielding the claimed result. This proof is recalled here for the sake of completeness – it is a well-known property of Poisson laws that the length until the first occurrence has an exponential distribution with parameter equal to the intensity of the Poisson law (e.g. Kleinrock, 1975).

B.2 Utilization probability

Given $L_T = \ell \leq C/2$, the probability of service utilization is

$$p_{u|\ell} = \Pr\{L_A < L_T \mid L_T = \ell\} = F_A(\ell) = 1 - e^{-\ell\theta/C}. \quad (\text{B.4})$$

Over the distribution of trip lengths, the probability of service usage is then

$$p_u = \int p_{u|\ell} dF_T(\ell) = 1 - \int e^{-\ell\theta/C} dF_T(\ell). \quad (\text{B.5a})$$

The probability of NOT using the service amounts to the Laplace transform f_T^* of PDF $f_T := \frac{d}{d\ell} F_T$ at point θ/C which is the spatial density of available scooters.

For trip lengths uniformly distributed from 0 to $C/2$, the average is $\bar{L}_T = C/4$ and we have that $\int e^{-\ell\theta/C} dF_T(\ell) = \frac{2}{C} \int_0^{C/2} e^{-\frac{\ell\theta}{C}} d\ell = \frac{2}{\theta} (1 - e^{-\frac{\theta}{2}})$, thus:

$$p_u = 1 - \frac{2}{\theta} (1 - e^{-\frac{\theta}{2}}). \quad (\text{B.5b})$$

It depends only on θ , neither on the ring circumference nor on the fleet size or the demand volume.

B.3 Trip lengths over the population of rides

Coming to the trip lengths among the rides, from (6) we deduce that:

$$f_T^u(\ell) = p_u^{-1} \cdot (1 - e^{-\ell\theta/C}) \cdot f_T(\ell). \quad (\text{B.6a})$$

Starting from a uniform initial distribution, the resulting distribution is:

$$f_T^u(\ell) = p_u^{-1} \cdot (1 - e^{-\ell\theta/C}) \cdot \frac{2}{c} \mathbf{1}_{\{0 \leq \ell < C/2\}}. \quad (\text{B.6b})$$

The average trip length among the population of rides satisfies that:

$$p_u \cdot \int \ell f_T^u(\ell) d\ell = \int \ell \left(1 - e^{-\frac{\ell\theta}{c}}\right) f_T(\ell) d\ell, \text{ so that}$$

$$\bar{L}_T^u = p_u^{-1} \cdot (\bar{L}_T - \int \ell e^{-\frac{\ell\theta}{c}} f_T(\ell) d\ell). \quad (\text{B.7a})$$

From the uniform distribution,

$$\begin{aligned} \frac{2}{c} \int_0^{C/2} \ell e^{-\ell\theta/C} d\ell &= \frac{2}{c} \left[-\ell \frac{c}{\theta} e^{-\ell\theta/C} \right]_0^{C/2} + \frac{c}{\theta} \int_0^{C/2} e^{-\ell\theta/C} d\ell \\ &= -\frac{c}{\theta} e^{-\frac{\theta}{2}} + \frac{c}{\theta} (1 - p_u) = \frac{c}{\theta} \left(1 - p_u - e^{-\frac{\theta}{2}}\right) = \frac{c}{\theta} \left(\frac{\theta}{2} (1 - p_u) - p_u\right) = \frac{c}{2} (1 - p_u) - \frac{c}{\theta} p_u \end{aligned}$$

$$\text{So that: } \bar{L}_T^u = p_u^{-1} \cdot \left(\frac{c}{4} - \frac{c}{2} (1 - p_u) + \frac{c}{\theta} p_u\right) = \frac{c}{\theta} + \frac{c}{2} - \frac{c}{4} p_u^{-1}$$

$$\bar{L}_T^u = \frac{c}{4} \left(\frac{4}{\theta} + 2 - \frac{1}{p_u}\right). \quad (\text{B.7b})$$

B.4 Effective access lengths (over the population of rides)

Let us define the effective access length as $L'_A := \min\{L_A, L_T\}$. Conditionally to $L_T = \ell$, L'_A takes value ℓ with probability $\bar{F}_A(\ell)$ and on every value $x < \ell$ it has probability density function of $e^{-x\theta/C} \theta/C$. The conditional average effective access length is then

$$\bar{L}_A | \ell = \frac{c}{\theta} \left(1 - e^{-\frac{\ell\theta}{c}}\right).$$

Proof. Value ℓ is taken with probability $\bar{F}_A(\ell) = e^{-\frac{\ell\theta}{c}}$ yielding contribution of $\ell e^{-\frac{\ell\theta}{c}}$ to the average. The values $x < \ell$ contribute an amount of

$$\int_0^{\ell} \frac{\theta}{c} x e^{-x\theta/C} dx = \left[-x e^{-x\theta/C}\right]_0^{\ell} + \int_0^{\ell} e^{-x\theta/C} dx = -\ell e^{-\frac{\ell\theta}{c}} + \frac{c}{\theta} \left(1 - e^{-\frac{\ell\theta}{c}}\right).$$

The contribution of the first term to the average is canceled out by that of the specific value.

Overall, the average access length is

$$\bar{L}_A = \frac{c}{\theta} p_u.$$

Proof. $\bar{L}_A = \int \bar{L}_A | L_T = \ell dF_T(\ell) = \int \frac{c}{\theta} \left(1 - e^{-\frac{\ell\theta}{c}}\right) dF_T(\ell) = \frac{c}{\theta} \left(1 - \int e^{-\frac{\ell\theta}{c}} dF_T(\ell)\right)$ in which we recognize the formula of p_u .

For those effectively using the service, the effective access length $L_{A|\ell}^u$ conditionally to $L_T = \ell$ has PDF as follows:

$$f_{A|\ell}^u(x) = \frac{\exp(-\frac{\ell\theta}{c}) \frac{\theta}{c}}{1 - \exp(-\frac{\ell\theta}{c})} \mathbf{1}_{\{0 \leq x \leq \ell\}}. \quad (\text{B.8})$$

Its average value is thus:

$$\bar{L}_{A|\ell}^u = \frac{\int_0^{\ell} \frac{\theta}{c} x e^{-x\theta/C} dx}{F_A(\ell)} = \frac{c}{\theta} - \frac{\ell}{\exp(\frac{\ell\theta}{c}) - 1}. \quad (\text{B.9})$$

Over all effective users, the effective access length L_A^u has PDF as follows:

$$f_A^u(x) = p_u^{-1} \cdot f_A(x) \cdot \bar{F}_T(x) = p_u^{-1} \exp(-\frac{\ell\theta}{c}) \frac{\theta}{c} \bar{F}_T(x). \quad (\text{B.10})$$

To obtain the effective access length averaged we have to integrate the conditional average taking into account their respective weights that are proportional to $F_A(\ell)$. $dF_T(\ell)$:

$$\bar{L}_A^u = p_u^{-1} \int (-\ell \exp(-\frac{\ell\theta}{c}) + \frac{c}{\theta}(1 - \exp(-\frac{\ell\theta}{c}))) dF_T(\ell), \text{ so that}$$

$$\bar{L}_A^u = \frac{c}{\theta} - p_u^{-1} \int \ell \exp(-\frac{\ell\theta}{c}) dF_T(\ell). \quad (\text{B.11a})$$

Then, it is less than the target length $\frac{c}{\theta}$ among riders. Recalling the derivation of (B7.d) for a uniform distribution of trip lengths we know that $p_u^{-1} \int \ell e^{-\frac{\ell\theta}{c}} f_T(\ell) d\ell = \frac{c}{2}(p_u^{-1} - 1) - \frac{c}{\theta}$.

As $p_u^{-1} - 1 = \frac{1-p_u}{p_u} = \frac{2}{\theta} \left(\frac{1 - \exp(-\frac{\theta}{2})}{1 - \frac{2}{\theta}(1 - \exp(-\frac{\theta}{2}))} \right)$ so that

$$\bar{L}_A^u = \frac{c}{\theta} \left(2 - \frac{1 - \exp(-\frac{\theta}{2})}{1 - \frac{2}{\theta}(1 - \exp(-\frac{\theta}{2}))} \right). \quad (\text{B.11b})$$

When θ is large, $\exp(-\frac{\theta}{2})$ is negligible and $\bar{L}_A^u \approx \frac{c}{\theta} (2 - (1 + \frac{2}{\theta})) \approx \frac{c}{\theta} (1 - \frac{2}{\theta})$. Thus $\frac{c}{\theta}$ is a good indicator of the average access length not only overall but also among riders, who tend to obtain shorter lengths.

B.5 Ride lengths

The service is used depending on the access conditions and especially the access length in comparison to the trip length. The residual length $L_{T-A} := L_T - L'_A$ is the ride length only if strictly positive. Conditionally to $L_T = \ell$, it takes value 0 with probability $\bar{F}_A(\ell)$ and each value $y \in]0, \ell]$ with probability density $e^{(y-\ell)\theta/c} \theta / c$.

The conditional average residual trip length is thus:

$$\bar{L}_{T-A} |_{L_T=\ell} = \int_0^\ell \frac{\theta}{c} y e^{(y-\ell)\theta/c} dx = \ell - \frac{c}{\theta} (1 - e^{-\ell\theta/c}).$$

The overall average residual length satisfies that

$$\bar{L}_{T-A} = \bar{L}_T - \frac{c}{\theta} p_u.$$

Proof. The overall average residual length stems from the integration of the conditional average residual lengths over the distribution of trip lengths. Thus

$$\bar{L}_{T-A} = \int \bar{L}_{T-A} |_{L_T=\ell} dF_T(\ell) = \bar{L}_T - \frac{c}{\theta} (1 - \int e^{-\frac{\ell\theta}{c}} dF_T(\ell)) \text{ in which we recognize } p_u.$$

For those using the service, the ride length conditional on trip length has PDF

$$f_{R|\ell}^u(y) = f_A(\ell - y) 1_{\{0 \leq y \leq \ell\}} / F_A(\ell). \quad (\text{B.12})$$

In their population, the average ride length conditional on trip length is

$$\bar{L}_{R|\ell}^u = \frac{\bar{L}_{R|L_T=\ell}}{F_A(\ell)} = \frac{\ell}{1 - e^{-\ell\theta/c}} - \frac{c}{\theta}. \quad (\text{B.13})$$

Over all effective users, the ride length L_R^u has PDF as follows:

$$f_R^u(x) = p_u^{-1} \int_x^\infty f_A(\ell - x) dF_T(\ell). \quad (\text{B.14})$$

For those using the service, the effective average ride length is thus:

$$\bar{L}_R^u = \frac{1}{p_u} \bar{L}_T - \frac{C}{\theta}. \quad (\text{B.15})$$

It depends on f_T only through \bar{L}_T and $p_u = 1 - f_T^*(\theta/C)$, wherein θ/C is the spatial density of available vehicles.

For trip lengths uniformly distributed from 0 to $C/2$, their initial average is $\bar{L}_T = C/4$ and we have that $p_u = 1 - \frac{2}{\theta}(1 - e^{-\frac{\theta}{2}})$, thus:

$$\bar{L}_R^u = \frac{C}{4} \left[1 - \frac{2}{\theta}(1 - e^{-\frac{\theta}{2}}) \right]^{-1} - \frac{C}{\theta}. \quad (\text{B.16})$$

Under large θ , $\bar{L}_R^u \approx \frac{C}{4}(1 + \frac{2}{\theta}) - \frac{C}{\theta} \approx \frac{C}{4}(1 - \frac{2}{\theta})$

B.6 Fleet sizing function

Proposition: Fleet sizing function in the Poisson model. The function $\theta \mapsto \sigma_Q(\theta) := \theta + Q \cdot p_u \cdot \bar{t}_{RT}/H$ is continuous and it increases strictly from 0 to infinity as θ does so.

Proof. As for increasingness, the functions p_u and \bar{t}_{RT} are increasing with respect to θ because it is a suitable availability factor. As both functions are nonnegative, their product also is an increasing function of θ , and so is function $\theta \mapsto \sigma_Q(\theta) := \theta + p_u \bar{t}_{RT} Q/H$. Increasingness is strict because the part $p_u \bar{t}_{RT} Q/H$ is increasing and part θ is strictly increasing.

Coming to smoothness, function $\theta \mapsto p_u(\theta)$ is continuous and differentiable with respect to θ because so are the exponential functions $\theta \mapsto e^{-\ell\theta/C}$ and the integral of such functions over the distribution of trip lengths. When $\theta \rightarrow 0$ then $e^{-\ell\theta/C} \approx 1 - \ell\theta/C$ so that $1 - p_u \approx \bar{L}_T \theta/C$ – indeed a standard property of Laplace transforms. Thus, function $\theta \mapsto p_u(\theta)/\theta$ is continuous at point 0 as well as for any $\theta > 0$. In turn, function $\theta \mapsto p_u \cdot \bar{t}_{RT} = p_u t_T + E[1/v](\bar{L}_T - C p_u(\theta)/\theta)$ has the same smoothness properties as p_u/θ .

As for the range of variations, when $\theta \rightarrow 0$ so does $p_u \cdot \bar{t}_{RT}$, implying that $\sigma_Q \rightarrow 0$. And as $\sigma_Q \geq \theta$ it will tend to infinity when θ does so. Overall, function σ_Q is continuous and it increases strictly from zero to infinity with θ . QED.

In fact function σ_Q is the most essential outcome of the Poisson model: given Q and θ it yields the fleet size $N = \sigma_Q(\theta)$ that meets the demand volume Q and the quality objective θ . Another way to interpret that quality objective is to recall that C/θ is the spatial density of available vehicles: thus, it stands as the average length that a potential user has to walk in order to find an available vehicle.

For a uniform distribution of initial trip lengths, the fleet sizing function is specified as:

$$\sigma_Q(\theta) := \theta + (p_u \cdot t_T + p_u \bar{L}_R^u E\left[\frac{1}{v}\right]) \frac{Q}{H}, \text{ so that} \quad (\text{B.18a})$$

$$\sigma_Q(\theta) = \theta + (p_u \cdot t_T + C(\frac{1}{4} - \frac{1}{\theta} p_u) E\left[\frac{1}{v}\right]) \frac{Q}{H}.$$

For large θ , $p_u \approx 1 - 2/\theta$ can be approximated by 1 and p_u/θ may be neglected compared to $\frac{1}{4}$, so that

$$\sigma_Q(\theta) \approx \theta + (t_T + \frac{C}{4} E\left[\frac{1}{v}\right]) \frac{Q}{H}, \quad (\text{B.18b})$$

This is the same approximate function as in the concentrated model.

Depending on a target access length ℓ_A and also on Q , the fleet sizing function is thus:

$$\widehat{N}(\ell_A, Q) \approx \frac{c}{\ell_A} + (t_T + \frac{c}{4} E \left[\frac{1}{v} \right]) \frac{Q}{H}, \quad (\text{B.19})$$

The shares of idle (available) and busy times for the scooters are respectively:

$$p_A = \frac{c}{\ell_A \widehat{N}} \quad \text{and} \quad p_B = (t_T + \frac{c}{4} E \left[\frac{1}{v} \right]) \frac{Q}{H \widehat{N}}$$

For a scooter, being busy is a dynamic state with odds ratio of $\frac{p_B}{p_A} = \frac{c}{\ell_A} \cdot \frac{H}{Q} / (t_T + \frac{c}{4} E \left[\frac{1}{v} \right])$.

Put in words: both ℓ_A and Q influence vehicle productivity in a proportional way and the odds ratio in an inversely proportional way. The smaller the target access length, the larger the odds ratio. The larger the demand volume, the smaller the odds ratio.

Appendix C: The Truncated Poisson Model of Availability

The truncated Poisson model has parameter θ and is such that the number of available vehicles is limited to the $\{0, 1, \dots, N\}$ range.

Denoting $X_N(x) := \sum_{k=0}^N x^k / k!$, we have that $\pi_k^{(N, \theta)} = \theta^k / [k! X_N(\theta)]$.

It holds that $\bar{k}_{N, \theta} = \sum_{k=0}^N k \frac{\theta^k}{k! X_N(\theta)} = \frac{\theta}{X_N(\theta)} \sum_{k=1}^N \frac{\theta^{k-1}}{(k-1)!} = \Psi_N(\theta)$, wherein $\Psi_N(\theta) := \frac{\theta \cdot X_{N-1}(\theta)}{X_N(\theta)}$ is continuous and increases from 0 to N^- as θ varies from 0 to ∞ , Cf. Leurent (2019a).

It remains to demonstrate that θ is truly an availability factor for the access law in the truncated Poisson model.

From (25), $\bar{F}_{A|\theta}(x) = \sum_{k=0}^N (\bar{F}_{A|1}(x))^k \frac{\theta^k}{k! X_N(\theta)}$, so that

$$\bar{F}_{A|\theta}(x) = \frac{X_N(\theta \bar{F}_{A|1}(x))}{X_N(\theta)}. \quad (\text{C.1})$$

Thus $F_{A|\theta}(x) = 1 - \bar{F}_{A|\theta}(x) = \frac{X_N(\theta) - X_N(\theta \lambda)}{X_N(\theta)}$, wherein $\lambda := \bar{F}_{A|1}(x) \in [0, 1]$.

Let us show that function $\theta \mapsto f_\lambda(\theta) = 1 - X_N(\theta \lambda) / X_N(\theta)$ increases with respect to θ .

Its derivative is as follows:

$$\begin{aligned} \frac{\partial}{\partial \theta} f_\lambda(\theta) &= -\lambda \frac{\dot{X}_N(\theta \lambda)}{X_N(\theta \lambda)} + \frac{X_N(\theta \lambda) \cdot \dot{X}_N(\theta)}{X_N(\theta)^2} = \frac{X_N(\theta \lambda)}{X_N(\theta)} \left(\frac{\dot{X}_N(\theta)}{X_N(\theta)} - \lambda \frac{\dot{X}_N(\theta \lambda)}{X_N(\theta \lambda)} \right) \\ &= \frac{X_N(\theta \lambda)}{X_N(\theta)} \frac{1}{\theta} (\Psi_N(\theta) - \Psi_N(\theta \lambda)). \end{aligned}$$

As $\lambda \in [0, 1]$, $0 \leq \theta \lambda \leq \theta$ so that, by the increasingness of Ψ_N , we have that $\frac{\partial}{\partial \theta} f_\lambda \geq 0$. Therefore function $f_\lambda = F_{A|\theta}(x)$ increases with θ , which implies that θ is a rightful availability factor.

Appendix D: a two-state Markov chain model of vehicle occupation

A shared scooter may be modeled as a stochastic system taking one out of two states A/B in alternation: state A means no current user hence Available to any newcomer, whereas state B stands for Busy, i.e. Occupied by a user. The two states communicate with each other, i.e. there are transitions from each state to the other one. Figure 3 depicts the two states and the stochastic transitions between them, with respective transition rates of λ' and μ .

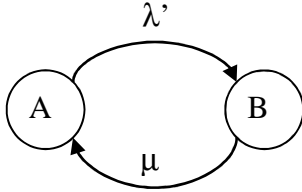


Fig. 3. States and Transitions.

Transition rate μ is the ending rate of an occupation under way. From the duration τ , it is the ratio between the probability $\Pr\{t_{RT} \leq \tau + \delta\tau \mid t_{RT} > \tau\}$ that such occupation will end during an elementary interval $[0, \delta\tau]$, and its incremental duration $\delta\tau$. Making the crude approximation that the random variable of vehicle occupation, t_{RT} , is distributed exponential, then the exponential parameter is $1/\bar{t}_{RT}$. Thus, the transition rate satisfies

$$\begin{aligned} \mu &:= \lim_{\delta\tau \rightarrow 0} \frac{\Pr\{t_{RT} \leq \tau + \delta\tau \mid t_{RT} > \tau\}}{\delta\tau} = \lim_{\delta\tau \rightarrow 0} \frac{\Pr\{t_{RT} \leq \tau + \delta\tau \cap t_{RT} > \tau\}}{\Pr\{t_{RT} > \tau\} \cdot \delta\tau} \\ &= \frac{\text{PDF}(t_{RT})}{1 - \text{CDF}(t_{RT})} = \frac{\exp(-\frac{\tau}{\bar{t}_{RT}})/\bar{t}_{RT}}{\exp(-\frac{\tau}{\bar{t}_{RT}})} = 1/\bar{t}_{RT}. \end{aligned}$$

$$\mu = 1/\bar{t}_{RT}. \quad (\text{D.1})$$

As for the transition rate from State A (Available) to State B (Busy), it comes from the overall rate of customers arrivals per time unit, $Q \cdot p_u / H$, divided by the average number of available vehicles, \bar{k} :

$$\lambda' = Q \cdot p_u / (H \cdot \bar{k}). \quad (\text{D.2})$$

The system is in stochastic equilibrium with stationary probability distribution $[p_A, p_B]$ such that $p_A, p_B \geq 0$ and $p_A + p_B = 1$, and the probability flow is conserved in each state:

$$p_A \cdot \lambda' = p_B \cdot \mu. \quad (\text{D.3})$$

Put in words, the probability flow going out of State A and to State B per time unit is equal to that coming in A from B. Substituting $1 - p_A$ for p_B , the balance condition becomes that

$$p_A \cdot \lambda' = (1 - p_A)\mu \text{ hence } p_A = \mu / (\mu + \lambda') = 1 / (1 + \lambda' / \mu) = 1 / (1 + Q \cdot p_u \cdot \bar{t}_{RT} / (H \cdot \bar{k}))$$

At every instant, it also holds that $p_A = \bar{k} / N$.

Substituting, the balance equation becomes an equation in \bar{k} : $\frac{\bar{k}}{N} \cdot \left(1 + Q \cdot p_u \cdot \frac{\bar{t}_{RT}}{H \cdot \bar{k}}\right) = 1$, i.e.

$$N = \bar{k} + p_u \bar{t}_{RT} Q / H. \quad (\text{D.4})$$

Thus, the balance condition of the Markov model is exactly the characteristic equation of the traffic model.