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# Toward a Wong-Zakai approximation for big order generators

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## 1 Introduction

Let us consider a compact Riemannian manifold  $M$  of dimension  $d$  endowed with its normalized Riemannian measure  $dx$  ( $x \in M$ ).

Let us consider  $m$  smooth vector fields  $X_i$  (We will suppose later that they are without divergence). We consider the second order differential operator:

$$L = 1/2 \sum_{i=1}^m X_i^2 \quad (1)$$

It generates a Markovian semi-group  $P_t$  which acts on continuous function  $f$  on  $M$

$$\frac{\partial}{\partial t} P_t f = L P_t f ; P_0 f = f \quad (2)$$

$P_t f \geq 0$  if  $f \geq 0$ . It is represented by a stochastic differential equation in Stratonovitch sense ([3])

$$P_t f(x) = E[f(x_t(x))] \quad (3)$$

where

$$dx_t(x) = \sum_{i=1}^m X_i(x_t(x)) dw_t^i ; x_0(x) = x \quad (4)$$

where  $t \rightarrow w_t^i$  is a flat Brownian motion on  $\mathbb{R}^m$ . Classically, the Stratonovitch diffusion  $x_t(x)$  can be approximated by its Wong-Zakai approximation.

Let  $w_t^{n,i}$  be the polygonal approximation of the Brownian path  $t \rightarrow w_t^n$  for a subdivision of  $[0, 1]$  of length  $n$ .

We introduce the random ordinary differential equation

$$dx_t^n(x) = \sum_{i=1}^m X_i(x_t^n(x)) dw_t^{n,i} ; x_0^n(x) = x \quad (5)$$

Wong-Zakai theorem ([3]) states that if  $f$  is continuous

$$E[f(x_t^n(x))] \rightarrow E[f(x_t(x))] \quad (6)$$

We are motivated in this paper by an extension of (6) to bigger order generators.

Let us consider the generator  $L^k = (-1)^k \sum_I X_i^{2k}$ . We suppose that the vector fields  $X_i$  span the tangent space of  $M$  in all point of  $M$  and that they are divergent free.  $L^k$  is an elliptic positive essentially self-adjoint operator [1] which generates a contraction semi-group  $P_t^k$  on  $L^2(dx)$

Let  $L^{f,k}$  be the generator on  $\mathbb{R}^m$  ( $(w_i) \in \mathbb{R}^m$ ) By [1], it generates a semi-group  $P_t^{f,k}$  on  $C(\mathbb{R}^m)$ , the space of continuous functions on the flat space endowed with the uniform topology, which is represented by an heat-kernel:

$$P_t^{f,k}[f](w_0) = \int_{\mathbb{R}^m} f(w + w_0) p_t^{f,k}(w) \otimes dw_i \quad (7)$$

( $w = (w_i)$ ) In [7], we noticed that heuristically  $P_t^{f,k}$  is represented by a formal path space measure  $Q^{f,k}$  such that

$$\int_E f(w_t^k + w_0) dQ^{f,k}(w) = P_t^{k,f}(f)(w_0) \quad (8)$$

If we were able to construct a differential equation in the Stratonovitch sense

$$dx_t^k(x) = \sum_{i=1}^m X_i(x_t^k(x)) dw_{t,i}^k ; x_0^k(x) = x \quad (9)$$

$$P_t^{f,k}(x) = \int f(x_t^k(x)) dQ^{f,k} \quad (10)$$

These are formal considerations because in such a case the path measures are not defined. We will give an approach to (11) by showing that some convenient Wong-Zakai approximation converge to the semi-group. We introduce, according to [4] and [5] the Wong-Zakai operator

$$Q_t^k[f](x) = \int_{\mathbb{R}^m} f(x_i(w)(x)) p_t^{f,k}(w) \otimes dw_i = \int_{\mathbb{R}^m} f(x(t^{1/2k}w)(x)) p_1^{f,k}(w) dw \quad (11)$$

where

$$dx_1(w)(x) = \sum_{i=1}^m X_i(x_s(w)) w_i ds ; x_0(w)(x) = x \quad (12)$$

As a first theorem, we state:

**Theorem 1** (*Wong-Zakai*) *Let us suppose that the vector fields  $X_i$  commute. Then  $(Q_{1/n}^k)^n(f)$  converge in  $L^2(dx)$  to  $P_1^k f$  if  $f$  is in  $L^2(dx)$*

To give another example, we suppose that  $M$  is a compact Lie group  $G$  endowed with its normalized Haar measure  $dg$  and that the vector fields  $X_i$  are elements of the Lie algebra of  $G$  considered as right invariant vector fields. This means that if we consider the right action on  $L^2(dg)$   $R_{g_0}$

$$f \rightarrow (g \rightarrow (f(gg_0))) \quad (13)$$

we have

$$R_{g_0}[X_i f](\cdot) = X_i[R_{g_0} f](\cdot) \quad (14)$$

We consider the rightinvariant elliptic differential operator

$$L^k = (-1)^k \sum_{i=1}^m X_i^{2k} \quad (15)$$

It is an elliptic positive essentially selfadjoint operator. By elliptic theory ([1]), it has a positive spectrum  $\lambda$  associated to eigenvectors  $f_\lambda$ .  $\lambda \geq 0$  if  $\lambda$  belongs to the spectrum.

**Theorem 2** (*Wong-Zakai*) *Let  $f = \sum a_\lambda f_\lambda$  such that  $\sum_\lambda |a_\lambda|^2 C^\lambda < \infty$  for all  $C > 0$ . Then  $(Q_{1/n}^k)^n(f)$  converges in  $L^2(dg)$  to  $P_1^k f$ .*

We refer to the reviews [5], [6] [7] for the study of stochastic analysis without probability.

## 2 Proof of theorem 1

$L^k$  is an elliptic positive operator. By elliptic theory [?], it has a discrete spectrum  $\lambda$  associated to normalized eigenfunctions  $f_\lambda$ . Since  $\int_{\mathbb{R}^m} |p_1^{f,k}(w)|^2 dw < \infty$ ,  $Q_{1/n}^k$  is a bounded operator on  $L^2(dx)$ . Moreover

$$Q_{1/n}^k f = \sum a_\lambda Q_{1/n}^k f_\lambda \quad (16)$$

if

$$f = \sum a_\lambda f_\lambda \quad (17)$$

The main remark is that we can compute explicitly  $Q_{1/n}^k f_\lambda$ . We put  $t = 1/n$ . Formally

$$f_\lambda(x(t^{1/2k}w))(x) = \sum_{n'} 1/n'! (\sum_m X_i w_i t^{1/2k})^{n'} (f_\lambda)(x) \quad (18)$$

Namely, by ellipticity and because the vector fields  $X_i$  commute with  $L^k$ , we can conclude that the  $L^2$ -norm of  $X_{i_1}^{\alpha_1} X_{i_2}^{\alpha_2} \dots X_{i_l}^{\alpha_l} f_\lambda$  has a bound in  $\lambda^{\sum \alpha_i / 2k} C^{\sum \alpha_i}$  in order to deduce that the series in (18) converges. It is enough to compute

$$1/n'! \int_{\mathbb{R}^m} (\sum X_i w_i t^{1/2k})^{n'} f_\lambda(x) p_1^{k,f}(w) dw = B_{n'} \quad (19)$$

The main remark is if one of the  $l_i$  is not a multiple of  $2k$ , we have

$$\int_{\mathbb{R}^m} w_1^{l_1} \dots w_m^{l_m} p_1^{k,f}(w) dw = 0 \quad (20)$$

On the other hand, by using the semi-group properties of  $P_t^{k,f}$ , we have

$$\int_{\mathbb{R}^m} w_1^{2kl_1} \dots w_m^{2kl_m} p_1^{k,f}(w) dw = \frac{(2kl_1)!}{l_1!} \dots \frac{(2kl_m)!}{l_m!} \quad (21)$$

Therefore,  $B_{n'} = 0$  if  $n'$  is not a multiple of  $2k$  and is equal because the vector field commute, if  $n' = 2kl'$  to

$$\frac{1}{(2kl')!} \sum X_1^{2kl'_1} \dots X_m^{2kl'_m} \frac{(2kl_1)!}{l_1!} \dots \frac{(2kl_m)!}{l_m!} \frac{(2kl')!}{(2kl'_1)! \dots (2kl'_m)!} f_\lambda = 1/l'! (L^k)^{l'} f_\lambda \quad (22)$$

We deduce that

$$Q_{1/n}^k f_\lambda = \exp[-1/n\lambda] f_\lambda \quad (23)$$

and that

$$(Q_{1/n}^k)^n f_\lambda = \exp[-\lambda] f_\lambda \quad (24)$$

such that

$$(Q_{1/n}^k)^n f = \exp[-L] f \quad (25)$$

if  $f = \sum a_\lambda f_\lambda$ .

### 3 Proof of Theorem 2

Let  $E_\lambda$  be the space of eigenfunctions associated to the eigenvalue  $\lambda$  of  $L^k$ . Since  $L^k$  commute with the right action of  $G$ ,  $E_\lambda$  is a representation for the right action of  $G$  ([2]). Therefore rightinvariant vector fields acts on  $E_\lambda$ . If  $Z$  is a rightinvariant vector field, we can consider the  $L^2$  norm of  $Zf_\lambda$  for  $f_\lambda$  belonging to  $E_\lambda$ . We remark that  $(L^k + C)^{1/2k}$  is an elliptic pseudodifferential operator of order 1 ( $C$  is strictly ppositive). By Garding inequality [1],

$$\|Zf_\lambda\|_{L^2(G)} \leq C\|f_\lambda\|_{L^2(G)} + \|(L^k + C)^{1/2k} f_\lambda\|_{L^2(G)} \quad (26)$$

$f_\lambda$  is an eigenfunction associated to  $(L^k + C)^{1/2k}$  and the eigenvalue  $(\lambda + C)^{1/2k}$ .

Let us consider a polynomial  $X_{i_1}^{\alpha_1} \dots X_{i_l}^{\alpha_l} = Z_l$ . It acts on  $E_\lambda$  and is norm is bounded by  $((\lambda + C)^{1/2k} + C)^{\sum \alpha_i}$  for the  $L^2$  norm.

From that we deduce that if  $f_\lambda$  is an eigenfunction associated to  $\lambda$  of  $L^k$  that the series

$$\sum_l \frac{(X_i t^{1/2k} w_i)^l}{l!} f_\lambda \quad (27)$$

converges and is equal to  $f_\lambda(x(t^{1/2k}w)(x))$  By distinguishing if  $w$  is big or not and using (20), we see that if  $l \neq 2kl'$

$$\int_{\mathbb{R}^m} \left( \sum_{i=1}^m X_i t^{1/2k} w_i \right)^{l'} f_\lambda p_1^{f,k}(x) dw = 0 \quad (28)$$

Moreover, by (20) and (21)

$$\begin{aligned} \frac{1}{(2kl')!} \int_{\mathbb{R}^m} \left( \sum_{i=1}^m X_i t^{1/2k} w_i \right)^{2kl'} f_\lambda p_1^{f,k}(x) dw = \\ \frac{t^{l'}}{(2kl')!} \int_{\mathbb{R}^m} \sum X_{\alpha_1} \dots X_{\alpha_{2kl'}} f_\lambda w_1^{2kl'_1} \dots w_m^{2kl'_m} p_1^{f,k}(w) dw \end{aligned} \quad (29)$$

where  $2kl'_j$  is the number of  $\alpha_i$  equal to  $j$ . By using (20) and (21), we recognize in (29)

$$\frac{1t^{l'}}{(2kl')!} \sum_{\alpha_i} X_{\alpha_1} \dots X_{\alpha_{2kl'}} f_\lambda \frac{(2kl'_1)!}{l'_1!} \dots \frac{(2kl'_m)!}{l'_m!} \quad (30)$$

For  $l' = 1$ , we recognize  $tL$ . Let us compute the  $L^2$  norm of the previous element. It is bounded by

$$\frac{t^{l'}}{(2kl')!} \sum_{\alpha_i} (\lambda^{1/2k} + C) \dots (\lambda^{1/2k} + C) \frac{(2kl'_1)!}{l'_1!} \dots \frac{(2kl'_m)!}{l'_m!} \quad (31)$$

For  $l' = 1$ , we recognize  $tL$ .

We recognize in the previous sum

$$\frac{t^{l'}}{(2kl')!} \sum \frac{(2kl')!}{(2kl'_1)! \dots (2kl'_m)!} \frac{(2kl'_1)!}{l'_1!} \dots \frac{(2kl'_m)!}{l'_m!} (\lambda^{1/2k} + C)^{2kl'} \quad (32)$$

We deduce a bound of the operation given by (29) in  $\frac{t^{l'} C^{2kl'}}{(l')!} (\lambda + C)^{l'}$ .

By the same argument, we have a bound of  $\frac{t^{l'}}{l'!} (L^k)^{l'}$  on  $E_\lambda$  in  $\frac{t^{l'}}{l'!} C^{l'} (\lambda + C)^{l'}$ . In order to conclude, we see that on  $E_\lambda$

$$Q_t^k = \exp[-\lambda t] Id + \sum_{l' > 1} \frac{t^{l'}}{l'!} Q_\lambda^{l',t} \quad (33)$$

where  $Q_\lambda^{l',t}$  has a bound on  $E_\lambda$  in  $C^{l'} (\lambda + C)^{l'}$ . We deduce that  $Q_t^k$  acts on  $E_\lambda$  by

$$\exp[-\lambda t] Id + t^2 Q_t^\lambda = R_t^\lambda \quad (34)$$

where the norm on  $E_\lambda$  of  $Q_t^\lambda$  is smaller that  $C \exp[C\lambda t]$ .

But if  $f = \sum a_\lambda f_\lambda$

$$(Q_{1/n}^k)^n f = \sum a_\lambda (R_{1/n}^\lambda)^n f_\lambda \quad (35)$$

Moreover

$$\begin{aligned} \|(Q_t^k)f\|_{L^2(G)} &= \int_G \left| \int_{\mathbb{R}^m} f(x(t^{1/2k}w)(g)) p_1^{f,k}(w) dw \right|^2 dg \leq \\ &C \int_G dg \int_{\mathbb{R}^m} |f(x(t^{1/2k}w)(g))|^2 |p_1^{f,k}(w)|^2 dw \leq C \int_{\mathbb{R}^m} |p_1^{f,k}(w)|^2 dw \int_G |f(x(t^{1/2k}w)(g))|^2 dg \end{aligned} \quad (36)$$

But

$$\int_G |f(x(t^{1/2k}w)(g))|^2 dg = \|f\|_{L^2(G)}^2 \quad (37)$$

because the vector fields are without divergence. If  $\lambda/n < C$ , the sum

$$\sum_{\lambda < Cn} a_\lambda (R_{1/n}^\lambda)^n f_\lambda \quad (38)$$

converges to

$$\sum a_\lambda \exp[-\lambda] f_\lambda \quad (39)$$

Moreover  $\sum_{\lambda > Cn} a_\lambda (R_{1/n}^\lambda)^n f_\lambda$  has a  $L^2$  norm bounded by  $(\sum_{\lambda > Cn} |a_\lambda|^2 C^\lambda)^{1/2}$  which goes to 0 when  $n \rightarrow \infty$ .

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