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1 Micro to macro traffic model: the case of a convergent 2 junction

3 N. El Khatib¹, N. Forcadel², M. Zaydan¹

4 April 13, 2021

5 Abstract

6 In this paper, we establish a rigorous connection between a microscopic and a macroscopic
7 traffic model on a convergent junction. At the microscopic scale, we assume that each driver
8 satisfies the "follow the leader" model before the junction point and, a priori, knows its turn
9 to enter the junction. At the macroscopic scale, we obtain a Hamilton-Jacobi equation with
10 a flux limiting condition at the junction point as in [27]. The novelty of this model is that we
11 are able to treat the case of a convergent junction, which wasn't covered in [18]. We explain
12 in the Appendix how to treat the general case with several incoming and outgoing roads.

13 **AMS Classification:** 35D40, 90B20, 35B27, 35F20, 45K05.

14 **Keywords:** specified homogenization, Hamilton-Jacobi equations, traffic flow, non-local opera-
15 tors, Slepčev formulation, viscosity solutions, microscopic model on junction.

16 1 Introduction

17 Several papers proposed partial differential equations to model the traffic on networks, see [30, 8,
18 22, 24]. Macroscopic traffic model are effective for describing the traffic at large scales but from
19 modeling point of view, since the dynamics of each driver can't be described individually, they
20 are hardly justifiable and incapable to predict all traffic situations (for example free or congested
21 flow). At the opposite, microscopic model are strong from modeling point of view but complicated
22 to implement in order to describe the traffic at large scales. Among works modeling traffic at the
23 micro scale, we refer to [4, 25, 33, 9], the last one appeared recently and considered the network
24 case.

25 A rigorous way to justify these macroscopic models is to establish a connection between them
26 and microscopic ones. Micro to macro connection on a simple road can be found in several paper,
27 we refer for example to [3, 14, 23, 13, 7, 10]. The case of network is more recent and very few
28 results are available. Let us cite for example [20, 19] in which the authors consider a very simple
29 junction with one incoming and one outgoing road (with a local perturbation at the origin). We
30 also like to mention [18] where the case of a divergent junction (one incoming road and several
31 outgoing roads) is considered.

32 In this paper, we will establish a micro-macro connection on a junction. Since this is the
33 novelty of the paper (see the paragraph below), we will present our results in details for the
34 case of a convergent junction. More precisely, we will consider a microscopic traffic model on a
35 merge, and will prove that the cumulative distribution function on each branch will converge to
36 the solution of a Hamilton-Jacobi equation studied by Imbert and Monneau in [27] and in [6, 32].

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1 At the end of this paper, we will show how to extend the results in the case of a general junction.
 2 We obtain our convergence result by using the theory of viscosity solution (see [12, 11]). The
 3 first homogenization result in this framework was obtained in [31] and since then, several works
 4 considered this problem, like [17, 16, 21, 1].

5 **Why the merge case?** We will present our results in the case of a merge for two reasons:
 6 firstly to simplify the presentation of our work and lessen the article. In fact, we will see that
 7 in the case of a junction, we need more "cumulative distribution functions" and so more partial
 8 differential equations. Secondly, the mathematical results of this work in the case of a merge
 9 can be easily extended to the case of a junction. In fact, the novelty of our work comes from
 10 considering convergent branches since the case of divergent ones is treated in [18].

11 **Comparison with the papers [18, 19, 20].** In [18, 19, 20], the authors proved the micro-
 12 macro limit in the case of a simple junction (one road with a local perturbation) and of a divergent
 13 junction. Although that the global strategy to get the convergence result is the same, our result
 14 requires new ideas. The difference with [19, 20] is a consequence of the geometry of the domain,
 15 the need of many non-local operators and hence to prove bounds for the solution components.
 16 Compared to [18], at the microscopic scale, we need here additional functions near the junction
 17 point to manage the order of entering and avoid cars collision. The control of oscillations of the
 18 solution is an important result that allows us to determine bounds on the gradient of the limit
 19 solution. In this paper, we provide a shorter proof (compared to [18]) of this result using the
 20 idea introduced in [29] and the definition of the security distance. In addition, we obtain better
 21 gradient estimates far from zero. A similar (but global) result is also needed when constructing
 22 the correctors. In this case, we should compare the components of the solution of an approximated
 23 problem on two different branches. We will use here the definition of a new distance d_0 which can
 24 be seen as a security distance while crossing the junction point.

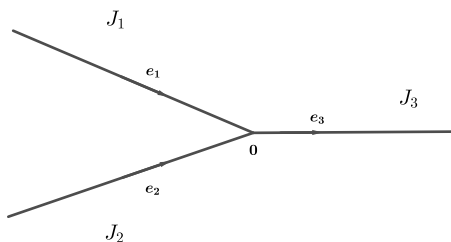
25 2 Main result

26 2.1 The microscopic model

We consider a microscopic traffic model on a merge with two incoming roads and one outgoing road. The junction is defined by $J = J_1 \cup J_2 \cup J_3$ where J_1 and J_2 are the incoming roads and isometric to \mathbb{R}^- and J_3 is the outgoing road and isometric to \mathbb{R}^+ . The branches of the junction are glued at the point 0. The point 0 is called the junction point and for all $k = 1, 2, 3$, we define $J_k^* = J_k \setminus \{0\}$. To be more precise, the definitions of the branches are given by

$$J_k = (-\infty, 0] \cdot e_k \quad \text{for } k = 1, 2, \quad J_3 = [0, +\infty) \cdot e_3,$$

27 with e_1, e_2 and e_3 three different unit vectors in \mathbb{R}^2 .



Let $x, y \in J$ with $x = x_k \cdot e_k, y = y_l \cdot e_l$: the distance $d(x, y)$ in J is defined by

$$d(x, y) = \begin{cases} |x_k - y_k| & \text{if } k = l, \\ |x_k| + |y_l| & \text{if } k \neq l. \end{cases}$$

1 Before writing the ordinary differential equations, we will first give the ideas of our model.

2 1) Vehicles distribution: We will assume a periodic distribution: two vehicles enter from the
3 branch J_1 and then one vehicle from J_2 .

4 2) Vehicles labels: Cars that are initially located on J_3 (resp. on $J_1^* \cup J_2^*$) have positive labels
5 (resp. strictly negative labels). On each branch, vehicle's positions is strictly increasing with
6 respect to their labels. In particular, if $i, j \in \mathbb{Z}$ with $i < j < 0$, then car j will pass by the
7 junction point before car i .

8 3) Velocity dependence: the velocity of a car far before the junction point depends on its initial
9 leader on the same branch. Approaching the junction point, the velocity of car $i \in \mathbb{Z} \setminus \mathbb{N}$ will
10 depend on the distance to "the last car assigned to pass by zero before it", i.e car with label
11 $i + 1$. To write our proposed model, we will need to introduce two continuous functions: one
12 modeling the transition of the micro-model before and after the point zero, and the other
13 one modeling the order of junction point entering.

14 4) Notations: The position of car i at time $t > 0$ is denoted by $W_i(t)$ and its velocity is $W_i'(t)$.
15 We define

$$\begin{cases} \mathcal{L}_1 = \{-1\} \cup \{-2 - 3k, k \in \mathbb{N}\}, \\ \mathcal{L}_2 = \{-4 - 3k, k \in \mathbb{N}\}, \\ \mathcal{L}_3 = \{-3k, k \in \mathbb{N}^*\}. \end{cases} \quad (2.1)$$

16 If $i \in \mathcal{L}_1 \cup \mathcal{L}_2$ (resp \mathcal{L}_3), then $W_i(0) \in J_1^*$ (resp. $W_i(0) \in J_2^*$). We will write a model that
17 satisfies:

- 18 – If $i \in \mathbb{N}$, then W_i will circulate on J_3 . Its leader is always W_{i+1} .
- 19 – If $i \in \mathcal{L}_1$, then W_i will go from J_1 to J_3 . Its leader is always W_{i+1} .
- 20 – If $i \in \mathcal{L}_2$, then W_i will go from J_1 to J_3 . Its leader on J_1 is W_{i+2} and is W_{i+1} on J_3 .
- 21 – If $i \in \mathcal{L}_3$, then W_i will go from J_2 to J_3 . Its leader on J_2 is W_{i+3} and is W_{i+1} on J_3 .

22 For $i \in \mathbb{Z}$, we have

$$W_i(t) \in J \iff W_i(t) = U_i(t) \cdot e_k \quad (2.2)$$

23 with

$$\begin{cases} U_i(t) \geq 0 \text{ and } k = 3 & \text{if } i \in \mathbb{N}, \\ (U_i(t) < 0 \text{ if } k = 1) \text{ and } (U_i(t) \geq 0 \text{ if } k = 3) & \text{if } i \in \mathcal{L}_1 \cup \mathcal{L}_2, \\ (U_i(t) < 0 \text{ if } k = 2) \text{ and } (U_i(t) \geq 0 \text{ if } k = 3) & \text{if } i \in \mathcal{L}_3. \end{cases}$$

24 Using the above equivalence (2.2), we will consider in the rest of the paper the variables U_i .
25 Our model is described by the following differential equations,

$$\begin{cases} U'_i(t) = \bar{\phi}(U_i(t), V_1(U_{i+1}(t) - U_i(t)), V_3(U_{i+1}(t) - U_i(t))) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}, \\ U'_i(t) = F(U_i(t), U_{i+1}(t), V_1(U_{i+2}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_2, \\ U'_i(t) = F(U_i(t), U_{i+1}(t), V_2(U_{i+3}(t) - U_i(t)), V_3(|U_{i+1}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_3, \end{cases} \quad (2.3)$$

with

$$F(x, y, a, b) = \bar{\phi}(x, a, b) \cdot \omega(x, y, |x| - |y|),$$

where $\bar{\phi}$ and ω are continuous functions modeling respectively the transition of velocities and the junction point entering order. Their definitions are given below in (2.4) and (2.5).

Remark 2.1. *The role of the function $\bar{\phi}$ is to model the change of optimal velocities on each branch of the junction. Far before the junction point, the optimal velocity is V_1 (resp. V_2) on J_1 (resp. on J_2) and after this point, the optimal velocity is V_3 . The role of the function ω is to avoid cars collision at the junction point. The function ω ensures that if car U_i is on the branch J_1 (resp. J_2), then the vehicle U_{i+1} located on J_2 (resp. J_1) will pass by the junction point before U_i . Let us remark that in the first equation of (2.3) we do not integrate ω since the leader's label does not change and no collision can occur.*

Remark 2.2. *We could consider a more general vehicles distribution (like 2 vehicles entering from J_1 and then two from J_2 , etc..). Since the simplest case is to consider one vehicle from J_1 and then one from J_2 , we decided to consider the above distribution to avoid the simplest case and the complex cases.*

We will now give some assumptions concerning the model and the dynamics.

Assumptions (A)

Assumptions on the velocities

(A1) For $i = 1, 2, 3$, the function $V_i : \mathbb{R} \rightarrow \mathbb{R}^+$ is Lipschitz continuous, non-negative and non-decreasing on \mathbb{R} .

(A2) There exists $h_0 \in (0, +\infty)$ such that for all $h \leq h_0$, $V_i(h) = 0$.

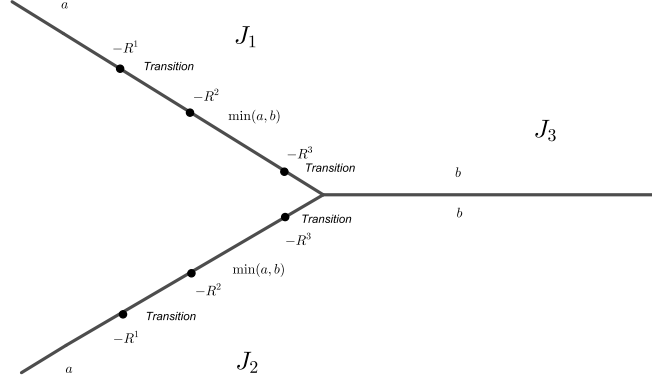
(A3) There exists $h_{max} > h_0$ such that for all $h \geq h_{max}$, $V_i(h_{max}) = V_{max}$.

(A4) There exists a real p_3 (resp. p_1, p_2) such that the function $p \mapsto pV_3(-1/p)$ (resp. $p \mapsto pV_1(-3/2p)$, $p \mapsto pV_2(-3/p)$) is decreasing on $[-1/h_0, p_3)$ (resp. $[-3/2h_0, p_1)$, $[-3/h_0, p_2)$) and increasing on $[p_3, 0)$ (resp. $[p_1, 0)$, $[p_2, 0)$).

Assumptions on the transition function

(A5) The function $\bar{\phi} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is defined as follow, for $R^1 > R^2 > R^3$,

$$\bar{\phi}(x, a, b) = \begin{cases} a & \text{if } x \leq -R^1, \\ \left(\frac{a - \min(a, b)}{R^2 - R^1} \right) (x - R^1) + a & \text{if } -R^1 < x \leq -R^2, \\ \min(a, b) & \text{if } -R^2 < x \leq -R^3, \\ \left(\frac{b - \min(a, b)}{R^3} \right) x + b & \text{if } -R^3 < x \leq 0, \\ b & \text{if } x > 0. \end{cases} \quad (2.4)$$


 Figure 1: Schematic representation of $\bar{\phi}$

1 **Assumptions on the functions ω**

2 (A6) The function $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by

$$\omega(x, y, p) = \alpha(x) + (1 - \alpha(x))\beta(y, p) \quad (2.5)$$

3 with α a lipshitz function defining the zone in which the car $U_i \in J_1^*$ (resp. $U_i \in J_2^*$) starts
 4 spotting the position of U_{i+1} on the other branch J_2 (resp. J_1). It's defined in the following
 5 way,

$$\alpha(x) = \begin{cases} 1 & \text{if } x < -R^3 - r \text{ or } x > r, \\ 0 & \text{if } -R^3 < x < 0. \end{cases} \quad (2.6)$$

6 The function β is a lipshitz function describing the fact that if car U_{i+1} is after the junction
 7 point, its influence on U_i disappears since collision is not possible in this case. It's defined
 8 by

$$\beta(y, p) = \begin{cases} 1 & \text{if } y > r, \\ \zeta(p) & \text{if } y \leq 0, \end{cases} \quad (2.7)$$

where ζ is a lipshitz function modeling the fact that U_{i+1} will enter the junction point before
 U_i ,

$$\zeta(p) = \begin{cases} 1 & \text{if } p > d_0 + r, \\ 0 & \text{if } p \leq d_0, \end{cases} \quad (2.8)$$

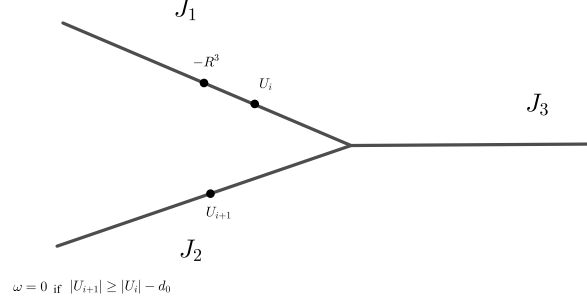
9 with

$$r, d_0 > 0 \quad \text{and} \quad r < R^1 - R^3. \quad (2.9)$$

Remark 2.3 (Comments on ω). *The role of the function ω is to manage the order of entering the junction point. Let $i \in \mathcal{L}_2$. If $U_i(t)$ is too close to the junction, i.e. $-R^3 < U_i(t) < 0$, then it will stop ($\omega = 0$) if $U_{i+1} \in J_2^*$ and if*

$$|U_i(t)| - |U_{i+1}(t)| < d_0.$$

- 1 The distance d_0 can be interpreted as a security distance when crossing the junction point.
 2 Remark also that when U_{i+1} enters the branch $J_3(U_{i+1}(t) > r)$, the influence of ω disappears
 3 ($\omega = 1$). Finally, let us remark that the definition of the function α on $[-R^3, 0]$ means that the
 4 influence of ω begins directly when the driver's velocity tends to depend on the distance to U_{i+1} .


 Figure 2: The role of ω .

5 2.2 The macroscopic model.

- 6 Let $k_0 = 1/h_0$. For $k = 1, 2, 3$, we define $\overline{H}_k : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\overline{H}_1(p) = \begin{cases} -p - \frac{3k_0}{2} & \text{for } p < -\frac{3k_0}{2}, \\ -V_1 \left(\frac{-3}{2p} \right) |p| & \text{for } -\frac{3k_0}{2} \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases} \quad (2.10)$$

7

$$\overline{H}_2(p) = \begin{cases} -p - 3k_0 & \text{for } p < -3k_0, \\ -V_2 \left(\frac{-3}{p} \right) |p| & \text{for } -3k_0 \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases} \quad (2.11)$$

8 and

$$\overline{H}_3(p) = \begin{cases} -p - k_0 & \text{for } p < -k_0, \\ -V_3 \left(\frac{-1}{p} \right) |p| & \text{for } -k_0 \leq p \leq 0, \\ p & \text{for } p > 0. \end{cases} \quad (2.12)$$

- 9 Note that for all $k = 1, 2, 3$, the function \overline{H}_k is continuous, coercive and because of (A4), there
 10 exists a unique point $p_k \leq 0$ such that

$$\begin{cases} \overline{H}_k \text{ is decreasing on } (-\infty, p_k), \\ \overline{H}_k \text{ is increasing on } (p_k, +\infty). \end{cases} \quad (2.13)$$

11 We denote by

$$H_0 = \max_{k \in \{1,2,3\}} H_0^k \quad (2.14)$$

12 with

$$H_0^k = \min_{p \in \mathbb{R}} \overline{H}_k(p). \quad (2.15)$$

We introduce now the definition of the gradient of a function defined on J . If $x \in J$, then we define

$$u_x(x) = \begin{cases} \partial_i u(x) & \text{if } x \in J_i^*, \\ (\partial_1 u(0), \partial_2 u(0), \partial_3 u(0)) & \text{if } x = 0, \end{cases}$$

1 with $\partial_i u(x)$ the derivative of the function u with respect to $x \in J_i$. We denote by $J^* = J \setminus \{0\}$. The
 2 macroscopic model is the Hamilton-Jacobi equation with flux limiting condition at the junction
 3 point as considered by Imbert and Monneau in [27] and given by

$$\begin{cases} u_t^0 + \overline{H}_k(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times J_k^*, \\ u_t^0 + F_{\overline{A}}(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\}, \\ u^0(0, x) = \overline{u}_0(x), \end{cases} \quad (2.16)$$

4 where \overline{A} is the flux limiter and $F_{\overline{A}}$ is defined by

$$F_{\overline{A}}(p_1, p_2, p_3) = \max\left(\overline{A}, \overline{H}_1^+(p_1), \overline{H}_2^+(p_2), \overline{H}_3^-(p_3)\right), \quad (2.17)$$

5 with

$$\overline{H}_k^-(p) = \begin{cases} \overline{H}(p) & \text{if } p \leq p_k, \\ \overline{H}(p_k) & \text{if } p \geq p_k, \end{cases} \quad \text{and} \quad \overline{H}_k^+(p) = \begin{cases} \overline{H}(p_k) & \text{if } p \leq p_k, \\ \overline{H}(p) & \text{if } p \geq p_k. \end{cases} \quad (2.18)$$

6 **Remark 2.4.** In [28], authors showed that equation (2.16) is equivalent (deriving in space) to the
 7 traffic model introduced in [30].

8 2.3 Micro-macro connection.

9 To establish the micro-macro connection, we need to introduce three cumulative functions ρ, σ and
 10 τ . The need of three functions arises from the different type of leaders: far before zero, on branch
 11 J_1 , the leader can be U_{i+1} or U_{i+2} and on branch J_2 , the leader is U_{i+3} . We have $\rho(t, U_i(t)) = -i$
 12 (resp. $\sigma(t, U_i(t)) = -i, \tau(t, U_i(t)) = -i$) for $i \in \mathcal{L}_1 \cup \mathbb{N}$ (resp. $i \in \mathcal{L}_2 \cup \mathbb{N}, i \in \mathcal{L}_3 \cup \mathbb{N}$). The
 13 functions ρ, σ and τ are defined as follows,

$$\begin{aligned} \rho(t, y) = & - \left(\sum_{i \geq 0} H(y - U_i(t)) + (-1 + H(y - U_{-1}(t))) + (-1 + H(y - U_{-2}(t))) \right) \\ & - \sum_{i=-2-3k, k \in \mathbb{N}^*} 3(-1 + H(y - U_i(t))), \end{aligned} \quad (2.19)$$

$$\begin{aligned} \sigma(t, y) = & - \left(\sum_{i \geq 0} H(y - U_i(t)) + 4(-1 + H(y - U_{-4}(t))) \right) \\ & - \sum_{i=-4-3k, k \in \mathbb{N}^*} 3(-1 + H(y - U_i(t))), \end{aligned} \quad (2.20)$$

14 and

$$\tau(t, y) = - \left(\sum_{i \geq 0} H(y - U_i(t)) + \sum_{i=-3k, k \in \mathbb{N}^*} 3(-1 + H(y - U_i(t))) \right), \quad (2.21)$$

15 with

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

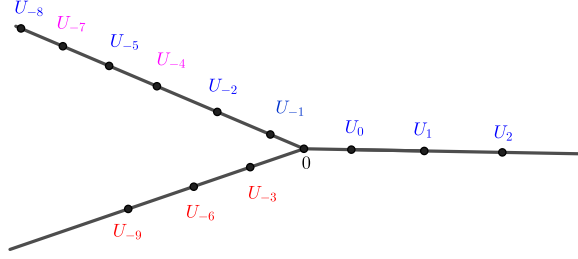


Figure 3: In blue (resp. pink and red): vehicles U_i whose leader's label at the initial time is $i + 1$ (resp. $i + 2$ and $i + 3$). The definition of the function ρ (resp. σ and τ) depends on vehicles colored in blue (resp. pink and red).

1 The main result of this paper is given by the following theorem.

Theorem 2.5 (Junction condition by homogenization: application to traffic flow). *Assume (A) and that at initial time, we have,*

$$\begin{cases} U_i(0) \leq U_{i+1}(0) - h_0 & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}, \\ U_i(0) \leq U_{i+2}(0) - h_0 & \text{if } i \in \mathcal{L}_2, \\ U_i(0) \leq U_{i+3}(0) - h_0 & \text{if } i \in \mathcal{L}_3. \end{cases} \quad (2.22)$$

We also assume that there exists $R > 0$ such that

$$\begin{cases} U_{i+1}(0) - U_i(0) = h_1 & \text{if } i \in \mathcal{L}_1 \text{ and } U_i(0) < -R, \\ U_{i+2}(0) - U_i(0) = h_1 & \text{if } i \in \mathcal{L}_2 \text{ and } U_i(0) < -R, \\ U_{i+3}(0) - U_i(0) = h_2 & \text{if } i \in \mathcal{L}_3 \text{ and } U_i(0) < -R, \\ U_{i+1}(0) - U_i(0) = h_3 & \text{if } i \in \mathbb{N} \text{ and } U_i(0) \geq R, \end{cases} \quad (2.23)$$

2 with $h_1, h_2, h_3 \geq h_0$. We define two functions u_0 and w_0 by

$$\begin{aligned} u_0(x) &= -\frac{3}{2h_1}x1_{\{x < 0\}} - \frac{1}{h_3}x1_{\{x \geq 0\}}, \\ w_0(x) &= -\frac{3}{h_2}x1_{\{x < 0\}} - \frac{1}{h_3}x1_{\{x \geq 0\}}. \end{aligned}$$

3 Let $\varepsilon > 0$ and $\chi^\varepsilon : \mathbb{R}^+ \times J \rightarrow \mathbb{R}$ be the function defined by

$$\chi^\varepsilon(t, x) = \begin{cases} \rho^\varepsilon(t, -d(0, x)) & \text{if } x \in J_1^*, \\ \tau^\varepsilon(t, -d(0, x)) & \text{if } x \in J_2^*, \\ \sigma^\varepsilon(t, d(0, x)) & \text{if } x \in J_3 \end{cases} \quad (2.24)$$

4 with

$$\rho^\varepsilon(t, x) = \varepsilon \rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad \sigma^\varepsilon(t, x) = \varepsilon \sigma\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad \tau^\varepsilon(t, x) = \varepsilon \tau\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

5 We define

$$\bar{u}_0(x) = \begin{cases} u_0(-d(0, x)) & \text{if } x \in J_1^*, \\ w_0(-d(0, x)) & \text{if } x \in J_2^*, \\ u_0(d(0, x)) & \text{if } x \in J_3. \end{cases}$$

1 Then there exists a unique $\bar{A} \in [H_0, 0]$ such that the function χ^ε converges towards the unique
2 solution u^0 of (2.16).

3 **Remark 2.6.** *The proof of this result will be divided into three steps:*

4 1) *Inject the cumulative distributions function into a non-local PDE.*

5 2) *Couple the PDE with suitable initial conditions and prove the convergence to the solution of*
6 *(2.16).*

7 3) *Deduce Theorem 2.5.*

8 3 The non-local equation

9 In this section, we will first define the non-local operators and then show that (ρ, σ, τ) is a viscosity
10 solutions of a non-local PDE. To do this, we will use the function $\phi(x, a, b) = -\bar{\phi}(x, -a, -b)$ defined
11 where $\bar{\phi}$ defined in (2.4). We remark that $\phi(x, a, \cdot)$ and $\phi(x, \cdot, b)$ are non-decreasing functions which
12 is a crucial property to obtain the comparison principle (see Proposition 4.1).

13 3.1 Definition of the non-local operators

14 In this subsection, we give the definition of the non-local operators.. To do this, we first introduce
15 the following functions. Let $a \in \mathbb{R}$, we define

$$E_a(z) = \begin{cases} 0 & \text{if } z \geq a \\ 1 & \text{if } z < a, \end{cases} \quad \tilde{E}_a(z) = \begin{cases} 0 & \text{if } z > a \\ 1 & \text{if } z \leq a, \end{cases} \quad (3.1)$$

16 and

$$F(z) = \begin{cases} 0 & \text{if } z > -1 \\ -1 & \text{if } z \leq -1, \end{cases} \quad \tilde{F}(z) = \begin{cases} 0 & \text{if } z \geq -1 \\ -1 & \text{if } z < -1. \end{cases} \quad (3.2)$$

17 Let $U, V, W : \mathbb{R} \rightarrow \mathbb{R}$. We define the following non-local operators: for $a = 1, 2, 3$,

$$M_{i,a}(U, [V])(x) = \int_0^{+\infty} V'_i(z) E_{-a}(V(x+z) - U(x)) dz - V_{max}, \quad i = 1, 2, 3, \quad (3.3)$$

$$N(U, [V])(x) = \int_{z \geq x} V'_3(|z| - x) E_{-1}(V(z) - U(x)) dz - V_{max}, \quad (3.4)$$

18 and

$$K(U, [V])(x) = \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F(V(z) - U(x)) dz + 1, \quad (3.5)$$

19 with ω defined in (2.5). Using the non-local operators defined above, we introduce

$$R_{1,1}(x, U, [V])(x) = \phi(x, M_{1,1}(U, [V])(x), M_{3,1}(U, [V])(x)), \quad (3.6)$$

$$R_{1,2}(x, V, [U], [W])(x) = \phi(x, M_{1,2}(V, [U])(x), N(V, [W])(x)) \cdot K(V, [W])(x), \quad (3.7)$$

$$R_{2,1}(x, W, [U])(x) = \phi(x, M_{2,3}(W, [W])(x), N(W, [U])(x)) \cdot K(W, [U])(x). \quad (3.8)$$

1 **Remark 3.1** (Comments on the non-local operators definition). *Using the non-local operators*
 2 *defined above, we can inject the ODE (2.3) into a non-local PDE (see Lemma 3.3). In particular,*
 3 *we have that*

$$\left\{ \begin{array}{ll} M_{1,1}(\rho(t, U_i(t)), [\sigma(t, \cdot)])(U_i(t)) = -V_1(U_{i+1}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ M_{3,1}(\rho(t, U_i(t)), [\sigma(t, \cdot)])(U_i(t)) = -V_3(U_{i+1}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ M_{1,2}(\sigma(t, U_i(t)), [\rho(t, \cdot)])(U_i(t)) = -V_1(U_{i+2}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_2 \cup \mathbb{N}, \\ M_{2,3}(\tau(t, U_i(t)), [\tau(t, \cdot)])(U_i(t)) = -V_2(U_{i+3}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_3, \\ N(\sigma(t, U_i(t)), [\tau(t, \cdot)])(U_i(t)) = -V_3(|U_{i+1}(t)| - U_i(t)) & \text{if } i \in \mathcal{L}_2 \cup \mathbb{N}, \\ N(\tau(t, U_i(t)), [\rho(t, \cdot)])(U_i(t)) = -V_3(|U_{i+1}(t)| - U_i(t)) & \text{if } i \in \mathcal{L}_3, \\ K(\sigma(t, U_i(t)), [\tau(t, \cdot)])(U_i(t)) = \omega(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|) & \text{if } i \in \mathcal{L}_2 \cup \mathbb{N}, \\ K(\tau(t, U_i(t)), [\rho(t, \cdot)])(U_i(t)) = \omega(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|) & \text{if } i \in \mathcal{L}_3. \end{array} \right.$$

4 *Remark that in the definition of $M_{i,a}$, the variable z is positive since it models the distance*
 5 *between U_i and its leader whose position is on the same branch or on J_3 (looking ahead). The*
 6 *different values of $a = 1, 2, 3$ represent the label of U_i 's leader which can be $i + 1, i + 2$ and $i + 3$.*
 7 *In the definition of N , the variable z models the leader on the other branch, U_{i+1} . We take $z > x$*
 8 *since for $z \leq x$, the velocity of U_i is zero due to the presence of the function ω_1 . In the definition*
 9 *of K the variable $z \in \mathbb{R}$ since it's modeling the car U_{i+1} . In the notation $R_{l,k}$, the labels l and k*
 10 *represent respectively the branches where the initial leader (resp. leader after the junction point)*
 11 *is located at $t = 0$.*

12 In the same way, we define $\tilde{M}_{i,a}, \tilde{N}$ and \tilde{K} replacing E_a and F respectively by \tilde{E}_a and \tilde{F} and
 13 then we get the definition of $\tilde{R}_{1,1}, \tilde{R}_{1,2}$ and $\tilde{R}_{2,1}$. Finally, we can easily remark that

$$-V_{max} \leq R_{1,1}(x, U, [V])(x), R_{1,2}(x, V, [U], [W])(x), R_{2,1}(x, W, [U])(x) \leq 0. \quad (3.9)$$

14 For $\varepsilon > 0$, we define the following non-local operators: for $a = 1, 2, 3$,

$$M_{i,a}^\varepsilon(U, [V])(x) = \int_0^{+\infty} V_i'(z) E_{-a}(V(x + \varepsilon z) - U(x)) dz - V_{max} \quad i = 1, 2, 3, \quad (3.10)$$

$$N^\varepsilon(U, [V])(x) = \int_{z \geq \frac{x}{\varepsilon}} V_3' \left(|z| - \frac{x}{\varepsilon} \right) E_{-1}(V(\varepsilon z) - U(x)) dz - V_{max}, \quad (3.11)$$

15 and

$$K^\varepsilon(U, [V])(x) = \int_{\mathbb{R}} \omega_z \left(\frac{x}{\varepsilon}, z, \left| \frac{x}{\varepsilon} \right| - |z| \right) F(V(\varepsilon z) - U(x)) dz + 1. \quad (3.12)$$

16 We then define

$$R_{1,1}^\varepsilon \left(\frac{x}{\varepsilon}, U, [V] \right) (x) = \phi \left(\frac{x}{\varepsilon}, M_{1,1}^\varepsilon(U, [V])(x), M_{3,1}^\varepsilon(U, [V])(x) \right), \quad (3.13)$$

$$R_{1,2}^\varepsilon \left(\frac{x}{\varepsilon}, V, [U], [W] \right) (x) = \phi \left(\frac{x}{\varepsilon}, M_{1,2}^\varepsilon(V, [U])(x), N^\varepsilon(V, [W])(x) \right) \cdot K^\varepsilon(V, [W])(x), \quad (3.14)$$

$$R_{2,1}^\varepsilon \left(\frac{x}{\varepsilon}, W, [U] \right) (x) = \phi \left(\frac{x}{\varepsilon}, M_{2,3}^\varepsilon(W, [W])(x), N^\varepsilon(W, [U])(x) \right) \cdot K^\varepsilon(W, [U])(x). \quad (3.15)$$

1 In the same way, we define $\tilde{M}_{i,a}^\varepsilon$, \tilde{N}^ε and \tilde{K}^ε replacing E_a and F respectively by \tilde{E}_a and \tilde{F} and
 2 then we get the definition of $\tilde{R}_{1,1}^\varepsilon$, $\tilde{R}_{1,2}^\varepsilon$ and $\tilde{R}_{2,1}^\varepsilon$.

3 3.2 The non-local PDE

4 In this subsection, we will prove that (ρ, σ, τ) is a discontinuous viscosity solutions of the following
 5 non-local PDE,

$$\begin{cases} \rho_t(t, x) + R_{1,1}(x, \rho(t, x), [\sigma(t, \cdot)])(x) |\rho_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \sigma_t(t, x) + R_{1,2}(x, \sigma(t, x), [\rho(t, \cdot)], [\tau(t, \cdot)])(x) |\sigma_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \tau_t(t, x) + R_{2,1}(x, \tau(t, x), [\rho(t, \cdot)])(x) |\tau_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \end{cases} \quad (3.16)$$

6 with $R_{1,1}$, $R_{1,2}$ and $R_{2,1}$ defined in (3.6), (3.7) and (3.8). We use the definition of viscosity
 7 solutions introduced in [34]. This definition allows to have a stability result for the non-local
 8 term. We refer to [16, Proposition 4.2] for the corresponding stability result. We give now the
 9 definition of viscosity solutions of (3.16).

10 **Definition 3.2** (Viscosity solutions for (3.16)). *Let $u, v, w : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ be upper semi-
 11 continuous (resp. lower semi-continuous) functions. We say that (u, v, w) is a viscosity sub-
 12 solution (resp. super-solution) of (3.16) on $(0, +\infty) \times \mathbb{R}$, if we have the following:*

1) *if $\varphi \in C^1([0, +\infty) \times \mathbb{R})$ such that $u - \varphi$ reaches a maximum (resp. a minimum) at the point
 (t, x) , we have*

$$\begin{aligned} & \varphi_t(t, x) + R_{1,1}(x, u(t, x), [w(t, \cdot)]) |\varphi_x(t, x)| \leq 0, \\ & \text{(resp. } \varphi_t(t, x) + \tilde{R}_{1,1}(x, u(t, x), [w(t, \cdot)]) |\varphi_x(t, x)| \geq 0 \text{)}. \end{aligned}$$

2) *If $\varphi \in C^1([0, +\infty) \times \mathbb{R})$ such that $v - \varphi$ reaches a maximum (resp. a minimum) at the point
 (t, x) , we have*

$$\begin{aligned} & \varphi_t(t, x) + R_{1,2}(x, v(t, x), [u(t, \cdot)], [w(t, \cdot)]) |\varphi_x(t, x)| \leq 0, \\ & \text{(resp. } \varphi_t(t, x) + \tilde{R}_{1,2}(x, v(t, x), [u(t, \cdot)], [w(t, \cdot)]) |\varphi_x(t, x)| \geq 0 \text{)}. \end{aligned}$$

3) *If $\varphi \in C^1([0, +\infty) \times \mathbb{R})$ such that $w - \varphi$ reaches a maximum (resp. a minimum) at the point
 (t, x) , we have*

$$\begin{aligned} & \varphi_t(t, x) + R_{2,1}(x, w(t, x), [u(t, \cdot)]) |\varphi_x(t, x)| \leq 0, \\ & \text{(resp. } \varphi_t(t, x) + \tilde{R}_{2,1}(x, w(t, x), [u(t, \cdot)]) |\varphi_x(t, x)| \geq 0 \text{)}. \end{aligned}$$

13 We say that (u, v, w) is a viscosity solution of (3.16) if (u^*, v^*, w^*) and (u_*, v_*, w_*) are respectively
 14 a sub-solution and a super-solution of (3.16).

15 We have the following theorem.

16 **Theorem 3.3.** *The function (ρ, σ, τ) is a discontinuous viscosity solution of (3.16). Conversely,
 17 if (u, v, w) are bounded and continuous viscosity solution of (3.16) satisfying for some time $T > 0$,
 18 and for all $t \in (0, T)$*

$$u(t, \cdot), v(t, \cdot), w(t, \cdot) \text{ are decreasing,}$$

then the points $U_i(t)$, defined by

$$\begin{cases} u(t, U_i(t)) = -i & \text{for } i \in \mathcal{L}_1 \cup \mathbb{N}, \\ v(t, U_i(t)) = -i & \text{for } i \in \mathcal{L}_2 \cup \mathbb{N}, \\ w(t, U_i(t)) = -i & \text{for } i \in \mathcal{L}_3 \cup \mathbb{N}, \end{cases}$$

19 are solutions of (2.3) on $(0, T)$.

1 The proof of Theorem 3.3 is a consequence of the following lemma.

Lemma 3.4. *Let $(U_i)_{i \in \mathbb{Z}}$ be the solution of (2.3). Then we have*

$$\begin{cases} U_i'(t) = -R_{1,1}(U_i(t), \rho(t, U_i(t)), [\sigma(t, \cdot)])(U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ U_i'(t) = -R_{1,2}(U_i(t), \sigma(t, U_i(t)), [\rho(t, \cdot)], [\tau(t, \cdot)])(U_i(t)) & \text{if } i \in \mathcal{L}_2 \cup \mathbb{N}, \\ U_i'(t) = -R_{2,1}(U_i(t), \tau(t, U_i(t)), [\rho(t, \cdot)])(U_i(t)) & \text{if } i \in \mathcal{L}_3. \end{cases} \quad (3.17)$$

2 *Proof.* We will only prove the second equality in (3.17) and the two others are derived in the same
3 way (and even simpler). In the rest of the proof, we will drop the time dependence of u, v and w
4 to simplify the presentation.

5 We distinguish two cases.

6 **Case 1:** $i \in \mathbb{N}$. Since $U_i(0) \geq 0$, we know that

$$U_i'(t) = V_3(U_{i+1}(t) - U_i(t)).$$

7 Using the fact that $U_i' \geq 0$, we deduce that for all $t > 0$, $U_i(t) \geq 0$ which implies that

$$R_{1,2}(U_i(t), v, [u], [w])(U_i(t)) = N(v, [w])(U_i(t)) \cdot K(v, [w])(U_i(t)).$$

8 Firstly, using the definition of ω (see (2.5)), we have that $\omega_z(U_i(t), z, |U_i(t)| - |z|) = 0$ which
9 implies that

$$K(v, [w])(U_i(t)) = 1.$$

10 Secondly, we have

$$\begin{aligned} N(v, [w])(U_i(t)) + V_{max} &= M_{3,1}(v, [w])(U_i(t)) + V_{max} \\ &= \int_0^{+\infty} V_3'(z) E_{-1}(w(U_i(t) + z) - v(U_i(t))) dz \\ &= \int_0^{U_{i+1}(t) - U_i(t)} V_3'(z) E_{-1}(w(U_i(t) + z) - v(U_i(t))) dz \\ &\quad + \int_{U_{i+1}(t) - U_i(t)}^{+\infty} V_3'(z) E_{-1}(w(U_i(t) + z) - v(U_i(t))) dz. \end{aligned}$$

11 If $z < U_{i+1}(t) - U_i(t)$, using the fact that w is decreasing, we have

$$w(U_i(t) + z) - v(U_i(t)) > w(U_{i+1}(t)) - v(U_i(t)) = -i - 1 + i = -1.$$

12 Similarly, if $z \geq U_{i+1}(t) - U_i(t)$, then

$$w(U_i(t) + z) - v(U_i(t)) \leq w(U_{i+1}(t)) - v(U_i(t)) = -1.$$

13 Recalling the definition of E_a (3.1), we obtain that

$$N(v, [w])(U_i(t)) + V_{max} = \int_{U_{i+1}(t) - U_i(t)}^{+\infty} V_3'(z) dz = V_{max} - V_3(U_{i+1}(t) - U_i(t))$$

14 where we used assumption (A3).

Case 2: $i \in \mathcal{L}_2$. In this case, we have that $i + 2 \in \mathcal{L}_1$ which implies

$$u(t, U_{i+2}(t)) = \rho(t, U_{i+2}(t)) = -(i + 2).$$

In addition, since $i + 1 \in \mathcal{L}_3$, we get

$$w(t, U_{i+1}(t)) = \tau(t, U_{i+1}(t)) = -(i + 1).$$

1 We claim that

$$\begin{cases} M_{1,2}(v, [u])(U_i(t)) &= -V_1(U_{i+2}(t) - U_i(t)), \\ K(v, [w])(U_i(t)) &= \omega(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|), \\ N(v, [w])(U_i(t)) &= -V_3(|U_{i+1}(t)| - U_i(t)). \end{cases} \quad (3.18)$$

2 We have

$$M_{1,2}(v, [u])(U_i(t)) + V_{max} = \int_0^{+\infty} V_1'(z) E_{-2}(u(U_i(t) + z) - v(U_i(t))) dz.$$

If $z < U_{i+2}(t) - U_i(t)$, then $u(U_i(t) + z) - v(U_i(t)) > u(U_{i+2}(t)) - v(U_i(t)) = -2$ where we have used the fact that u is decreasing. Similarly, for $z \geq U_{i+2}(t) - U_i(t)$, we have

$$u(U_i(t) + z) - v(U_i(t)) \leq u(U_{i+2}(t)) - v(U_i(t)) = -2.$$

3 Using the definition of E_{-2} (see (3.1)), we obtain

$$\begin{aligned} M_{1,2}(v, [u])(U_i(t)) + V_{max} &= \int_{U_{i+2}(t) - U_i(t)}^{+\infty} V_1'(z) dz \\ &= V_{max} - V_1(U_{i+2}(t) - U_i(t)). \end{aligned}$$

4 We turn now to the proof of the last two equality in (3.18). We have

$$\begin{aligned} K(v, [w])(U_i(t)) - 1 &= \int_{\mathbb{R}} \omega_z(U_i(t), z, |U_i(t)| - |z|) F(w(z) - v(U_i(t))) dz \\ &= \int_{z < U_{i+1}(t)} \omega_z(U_i(t), z, |U_i(t)| - |z|) F(w(z) - v(U_i(t))) dz \\ &\quad + \int_{z \geq U_{i+1}(t)} \omega_z(U_i(t), z, |U_i(t)| - |z|) F(w(z) - v(U_i(t))) dz. \end{aligned}$$

5 If $z < U_{i+1}(t)$, then $w(z) - v(U_i(t)) > w(U_{i+1}(t)) - v(U_i(t)) = -1$ where we have used the fact
6 that w is decreasing. Similarly, for $z \geq U_{i+1}(t)$, we have

$$w(z) - v(U_i(t)) \leq w(U_{i+1}(t)) - v(U_i(t)) = -1.$$

7 Using the definition of F (see (3.2)), we obtain

$$\begin{aligned} K(v, [w])(U_i(t)) - 1 &= - \int_{z \geq U_{i+1}(t)} \omega_z(U_i(t), z, |U_i(t)| - |z|) dz \\ &= -1 + \omega(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|) \end{aligned}$$

8 where we used the fact that $\omega(x, y, p) = 1$ for $y > r$ (see (2.7)). We have

$$N(v, [w])(U_i(t)) + V_{max} = \int_{z \geq U_i(t)} V_3'(|z| - U_i(t)) E_{-1}(w(z) - v(U_i(t))) dz.$$

1 Arguing as above, we get

$$N(v, [w])(U_i(t)) + V_{max} = \int_{z \geq U_{i+1}(t)} V_3'(|z| - U_i(t)) dz = V_{max} - V_3(|U_{i+1}(t)| - U_i(t)).$$

2 We obtain (3.18) and then the second equality in (3.17). To obtain the two remaining equality in
3 (3.17), we prove (in a similar way) that

$$\begin{cases} M_{1,1}(u(t, U_i(t)), [v(t, \cdot)])(U_i(t)) = -V_1(U_{i+1}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ M_{2,3}(w(t, U_i(t)), [w(t, \cdot)])(U_i(t)) = -V_2(U_{i+3}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_3, \\ M_{3,1}(u(t, U_i(t)), [v(t, \cdot)])(U_i(t)) = -V_3(U_{i+1}(t) - U_i(t)) & \text{if } i \in \mathcal{L}_1, \\ N(w(t, U_i(t)), [u(t, \cdot)])(U_i(t)) = -V_3(|U_{i+1}(t)| - U_i(t)) & \text{if } i \in \mathcal{L}_3, \\ K(w(t, U_i(t)), [u(t, \cdot)])(U_i(t)) = \omega(U_i(t), U_{i+1}(t), |U_i(t)| - |U_{i+1}(t)|) & \text{if } i \in \mathcal{L}_3. \end{cases}$$

4

□

5 *Proof of Theorem 3.3.* Using Lemma 3.4 and Definition 3.2, we can see that $(\rho_*, \sigma_*, \tau_*)$ is a discontinuous viscosity super-solution of (3.16). We obtain a similar result for $(\rho^*, \sigma^*, \tau^*)$ and therefore,
6 (ρ, σ, τ) is a discontinuous viscosity solution of (3.16).
7

8 We prove the converse. Using Proposition 4.8 from [16], we can conclude that

$(\rho_* = [u], \sigma_* = [v], \tau_* = [w])$ is a super-solution of

$$\begin{cases} \rho_t - \tilde{c}_1(t, x) \rho_x = 0, \\ \sigma_t - \tilde{c}_2(t, x) \sigma_x = 0, \\ \tau_t - \tilde{c}_3(t, x) \tau_x = 0, \end{cases}$$

with

$$\begin{aligned} \tilde{c}_1(t, x) &= \tilde{R}_{1,1}(x, u(t, x), [v(t, \cdot)])(x), \\ \tilde{c}_2(t, x) &= \tilde{R}_{1,2}(x, v(t, x), [u(t, \cdot)], [w(t, \cdot)])(x), \\ \tilde{c}_3(t, x) &= \tilde{R}_{2,1}(x, w(t, x), [u(t, \cdot)])(x). \end{aligned}$$

Using the fact that u, v and w are decreasing in space, we define

$$\begin{cases} U_i(t) = \inf\{x, u(t, x) \leq -i\} = (u(t, \cdot))^{-1}(-i) & \text{if } i \in \mathcal{L}_1 \cup \mathbb{N}, \\ U_i(t) = \inf\{x, v(t, x) \leq -i\} = (v(t, \cdot))^{-1}(-i) & \text{if } i \in \mathcal{L}_2, \\ U_i(t) = \inf\{x, w(t, x) \leq -i\} = (w(t, \cdot))^{-1}(-i) & \text{if } i \in \mathcal{L}_3, \end{cases}$$

9 and we consider the functions $t \mapsto U_i(t)$. They are continuous because u, v and w are decreasing
10 in x and are continuous in (t, x) .

11 We now prove that the functions U_i are viscosity solutions of (2.3). We will do the proof for
12 $i \in \mathcal{L}_1 \cup \mathbb{N}$ since other cases can be done in a similar way. Let φ be a test function such that
13 $\varphi(t) \leq U_i(t)$ and $\varphi(t_0) = U_i(t_0)$. Let us now define $\hat{\varphi}(t, x) = -i + \varphi(t) - x$. It satisfies

$$\hat{\varphi}(t_0, U_i(t_0)) = \rho_*(t_0, U_i(t_0)),$$

1 and

$$\hat{\varphi}(t, x) \leq \rho_*(t, x) \quad \text{for } U_i(t) - 1 < x < U_{i+1}(t).$$

This implies that

$$\varphi_t(t_0) + \tilde{c}_1(t_0, U_i(t_0)) \geq 0 \Leftrightarrow \varphi_t(t_0) = U_i'(t_0) \geq -\tilde{c}_1(t_0, U_i(t_0))$$

with

$$\begin{aligned} -\tilde{c}_1(t_0, U_i(t_0)) &= -\phi(U_i(t_0), -V_1(U_{i+1}(t_0) - U_i(t_0)), -V_3(U_{i+1}(t_0) - U_i(t_0))) \\ &= \bar{\phi}(U_i(t_0), V_1(U_{i+1}(t_0) - U_i(t_0)), V_3(U_{i+1}(t_0) - U_i(t_0))). \end{aligned}$$

2 This proves that for $i \in \mathcal{L}_1 \cup \mathbb{N}$, the functions U_i are viscosity super-solutions of the first
3 equation in (2.3). In a similar way we can prove that for $i \in \mathcal{L}_2$ (resp. $i \in \mathcal{L}_3$), the functions U_i
4 are viscosity super-solutions of the second (resp. third) equation in (2.3).

5 The proof for sub-solutions is similar and we skip it. Moreover, since $\tilde{c}_i(\cdot, U_i(\cdot))$ is continuous
6 for $i = 1, 2, 3$, we deduce that $U_i \in C^1$ and it is therefore a classical solution of (2.3).

7

□

8 4 Results for the non-local PDE with initial conditions

9 In this section, we will consider the following PDE:

$$\begin{cases} u_t(t, x) + R_{1,1}(x, u(t, x), [v(t, \cdot)](x))|u_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v_t(t, x) + R_{1,2}(x, v(t, x), [u(t, \cdot)], [w(t, \cdot)](x))|v_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ w_t(t, x) + R_{2,1}(x, w(t, x), [u(t, \cdot)](x))|w_x(t, x)| = 0 & \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}, \\ v(0, x) = v_0(x) & \text{for } x \in \mathbb{R}, \\ w(0, x) = w_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad (4.1)$$

10 We will give first some classical results like the comparison principle and existence via Perron's
11 method, then we will prove a gradient estimate result far from the junction point. The initial
12 conditions u_0, v_0 and w_0 are lipschitz continuous functions. In addition, to control the gradient of
13 the limit solution of (2.16), we need the following assumption. Let $k_0 = -1/h_0$.

(A0) The functions u_0, v_0 and w_0 are non-increasing. Let R^1 the positive parameter appearing
in the definition (2.4) of ϕ . We assume that for $y < x < -R^1$,

$$\begin{cases} \frac{-3k_0}{2}(x - y) - 1 \leq v_0(x) - u_0(y), \\ -3k_0(x - y) - 3 \leq w_0(x) - w_0(y), \end{cases}$$

14 and for $x > y > 0$,

$$-k_0(x - y) - 1 \leq v_0(x) - u_0(y).$$

15 4.1 Existence and uniqueness results for (4.1)

The definition of viscosity solutions for (4.1) is the same as in Definition 3.2 for $t > 0$ and for
 $t = 0$, we will add the following inequality for the sub-solution (resp. super-solution),

$$\begin{aligned} u(0, x) &\leq u_0(x), \quad v(0, x) \leq v_0(x) \quad \text{and} \quad w(0, x) \leq w_0(x), \\ \text{(resp. } u(0, x) &\geq u_0(x), \quad v(0, x) \geq v_0(x) \quad \text{and} \quad w(0, x) \geq w_0(x)). \end{aligned}$$

16 We begin first with the comparison principle.

1 **Proposition 4.1** (Comparison principle for (4.1)). *Assume (A). Let (u, v, w) be a viscosity sub-*
 2 *solution of (4.1) and $(\hat{u}, \hat{v}, \hat{w})$ be a viscosity super-solution of (4.1) in the sense introduced of*
 3 *Definition 3.2. Let us also assume that there exists a constant $K > 0$ such that for all $t > 0$,*

$$\begin{aligned} u(t, x) &\leq u_0(x) + Kt \quad \text{and} \quad -\hat{u}(t, x) \leq -u_0(x) + Kt \quad \text{for } x \in \mathbb{R}, \\ v(t, x) &\leq v_0(x) + Kt \quad \text{and} \quad -\hat{v}(t, x) \leq -v_0(x) + Kt \quad \text{for } x \in \mathbb{R}, \\ w(t, x) &\leq w_0(x) + Kt \quad \text{and} \quad -\hat{w}(t, x) \leq -w_0(x) + Kt \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

4 *Then we have for all $t > 0$,*

$$\begin{aligned} u(t, x) &\leq \hat{u}(t, x) \quad \text{for } x \in \mathbb{R}, \\ v(t, x) &\leq \hat{v}(t, x) \quad \text{for } x \in \mathbb{R}, \\ w(t, x) &\leq \hat{w}(t, x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

5

6 *Proof.* The proof is very similar to the one in [19] and uses the monotony of E, F, ϕ_k and w_k , so
 7 we skip it.

8

□

In the proof of convergence (see Section 6), we need a comparison principle on bounded sets. For a given point $(t_0, x_0) \in (0, T) \times \mathbb{R}$ and for $\bar{r}, \bar{R} > 0$, we define the set

$$\mathcal{P}_{\bar{r}, \bar{R}}(t_0, x_0) = (t_0 - \bar{r}, t_0 + \bar{r}) \times (x_0 - \bar{R}, x_0 + \bar{R}). \quad (4.2)$$

Theorem 4.2 (Comparison principle on bounded sets for (4.1)). *Assume (A). If (u, v, w) and $(\hat{u}, \hat{v}, \hat{w})$ are respectively sub-solution and super-solution of (4.1) on the open set $\mathcal{P}_{\bar{r}, \bar{R}}$ such that*

$$u \leq \hat{u} \quad v \leq \hat{v} \quad w \leq \hat{w} \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{R}},$$

then

$$u \leq \hat{u} \quad v \leq \hat{v} \quad w \leq \hat{w} \quad \text{on } \mathcal{P}_{\bar{r}, \bar{R}}.$$

9

10 *Proof.* The proof of this theorem is similar to the one of Proposition 4.1, so we skip it.

11

□

Lemma 4.3 (Existence of barriers for (4.1)). *Assume (A0) and (A). Let $K_1 = \frac{3}{2}V_{max}k_0$ and $K_2 = 3V_{max}k_0$. We define*

$$\begin{aligned} (u^+(t, x), v^+(t, x), w^+(t, x)) &= (K_1 t + u_0(x), K_1 t + v_0(x), K_2 t + w_0(x)), \\ (u^-(t, x), v^-(t, x), w^-(t, x)) &= (u_0(x), v_0(x), w_0(x)). \end{aligned}$$

12 *Then (u^+, v^+, w^+) and (u^-, v^-, w^-) are respectively super and sub solution of (4.1).*

13 *Proof.* The proof is very simple. We just use the bounds of $R_{1,1}, R_{1,2}$ and $R_{2,1}$ (see (3.9)) and the
 14 gradient bounds (A0). □

15 Applying Perron's method (see [2] or [26] to see how to apply Perron's method for problems
 16 with non-local terms), joint to the comparison principle, we obtain the following result.

17 **Theorem 4.4** (Existence and uniqueness of viscosity solutions for (4.1)). *Assume (A0) and (A).
 18 Then, there exists a unique continuous solution of (4.1) which satisfies*

$$|u(t, x) - u_0(x)| \leq K_1 t, \quad |v(t, x) - v_0(x)| \leq K_1 t \quad \text{and} \quad |w(t, x) - w_0(x)| \leq K_2 t. \quad (4.3)$$

19

4.2 Gradient estimates

Theorem 4.5 (Control of the oscillations). *Assume (A0)-(A). Let $T > 0$ and (u, v, w) be the solution of (4.1) on $(0, T)$ given in Theorem 4.4. For all $t > 0$, the functions $u(t, \cdot)$, $v(t, \cdot)$ and $w(t, \cdot)$ are non-increasing. Let $x < -R^1$ and $h > 0$ small enough such that $x + h < -R^1$ and $h < h_0$. For all $t \in (0, T)$, we have*

$$\frac{-3k_0}{2}h - 1 \leq v(t, x + h) - u(t, x), \quad (4.4)$$

$$-3k_0h - 3 \leq w(t, x + h) - w(t, x). \quad (4.5)$$

Let $x > 0$ and $0 < h < h_0$. For all $t \in (0, T)$, we have

$$-k_0h - 1 \leq v(t, x + h) - u(t, x). \quad (4.6)$$

Proof. The proof of monotony is similar to the one in [20, Theorem 4.10]. We will only do the proof of (4.4) since the other inequalities can be proved in the same way. We will use the technique introduced in [29] for the proof of local gradient estimates. Let $x_0 < -R^1$ and $\delta > 0$ small enough such that

$$\begin{cases} h + 2\delta < h_0, \\ x_0 + h + \delta < -R^1. \end{cases} \quad (4.7)$$

We will prove for all $t \in (0, T)$, $y \in B_\delta(x_0)$ and $x \in B_\delta(x_0 + h)$, that

$$\frac{-3k_0}{2}(x - y) - 1 - L_\delta(y - x_0)^2 - L_\delta(x - x_0 - h)^2 \leq v(t, x) - u(t, y) \quad (4.8)$$

with $L_\delta = \frac{K_1 T}{\delta^2}$ and K_1 defined in (4.3). In particular, taking $y = x_0$ and $x = x_0 + h$, we obtain (4.4). To prove (4.8), we introduce

$$\Delta = [0, T] \times B_\delta(x_0 + h) \times B_\delta(x_0)$$

and consider the following supremum

$$M = \sup_{(t, x, y) \in \Delta} \left\{ u(t, y) - v(t, x) - \frac{3k_0}{2}(x - y) - 1 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2 \right\}.$$

We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

Step 1: the test function. For $\eta > 0$ small, we define

$$\varphi(t, x, y) = u(t, y) - v(t, x) - \frac{3k_0}{2}(x - y) - 1 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2 - \frac{\eta}{T - t}.$$

Since φ is continuous, it reaches a maximum on Δ at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically we have for η small enough,

$$0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}).$$

Step 2: $\bar{t} > 0$. By contradiction, assume first that $\bar{t} = 0$. Then we have

$$\frac{\eta}{T} < u_0(\bar{y}) - v_0(\bar{x}) - \frac{3k_0}{2}(\bar{x} - \bar{y}) - 1 \leq 0,$$

where we used (A0).

- 1 **Step 3:** $|\bar{x} - x_0 - h| \neq \delta$ and $|\bar{y} - x_0| \neq \delta$. By contradiction, assume that $|\bar{x} - x_0 - h| = \delta$.
 2 Using the barriers and (A0), we get that

$$\begin{aligned} 0 < \frac{\eta}{T} &\leq u_0(\bar{y}) - v_0(\bar{x}) - \frac{3k_0}{2}(\bar{x} - \bar{y}) - 1 - K_1T - L_\delta\delta^2 \\ &\leq K_1T - L_\delta\delta^2 = 0 \end{aligned}$$

- 3 where we used the definition of L_δ . In the same way, we have $|\bar{y} - x_0| \neq \delta$.
 4 **Step 4: utilization of the equation.** By doing a duplication of the time variable and passing
 5 to the limit in this duplication parameter, we get that

$$\begin{aligned} \frac{\eta}{(T - \bar{t})^2} &\leq \tilde{M}_{1,2}(v(\bar{t}, \bar{x}), [u(\bar{t}, \cdot)]) (\bar{x}) \cdot \left| \frac{-3k_0}{2} - 2L_\delta(\bar{x} - x_0 - h) \right| \\ &\quad - M_{1,1}(u(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)]) (\bar{y}) \cdot \left| \frac{-3k_0}{2} + 2L_\delta(\bar{y} - x_0) \right| \end{aligned} \quad (4.9)$$

- 6 where we used the fact that $\bar{\phi}(x, a, b) = a$ and $\omega(x, y, p) = 1$ for $x < -R^1$. We claim that

$$M_{1,1}(u(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)]) (\bar{y}) = 0. \quad (4.10)$$

- 7 In fact if (4.10) is true we will obtain a contradiction in (4.9) since $\tilde{M}_{1,2} \leq 0$. The definition of
 8 $M_{1,1}$ can be written for $z \geq h_0$ since for $z < h_0$, we have $V'_1 = 0$. We have

$$M_{1,1}(u(\bar{t}, \bar{y}), [v(\bar{t}, \cdot)]) (\bar{y}) = \int_{h_0}^{+\infty} V'_1(z) E_{-1}(v(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) dz - V_{max}.$$

Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we have

$$v(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -1.$$

If $z > h_0$, using (4.7), we have

$$\bar{y} + z > x_0 - \delta + h_0 > x_0 + \delta + h > \bar{x}.$$

- 9 Since $v(t, \cdot)$ is non-increasing, we get for $z > h_0$,

$$v(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y}) \leq v(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) < -1$$

- 10 which implies $E_{-1}(v(\bar{t}, \bar{y} + z) - u(\bar{t}, \bar{y})) = 1$ and then (4.10). This ends the proof.
 11 □

12 5 Construction of correctors

- 13 In this section, we construct the corrector far and near the junction point, which allow us to use
 14 the perturbed test method introduced by Evans in [15].

5.1 Corrector far from the junction point.

Proposition 5.1 (Homogenization far from the junction point). *Assume (A). For $p \leq 0$, we define the following non-local operator:*

$$M_{i,a}^p(U, [V])(x) = \int_0^{+\infty} V'_i(z) E_{-a}(V(x+z) - U(x) + pz) dz - V_{max}, \quad (5.1)$$

with E_a defined in (3.1).

1) For every $p \leq 0$, there exists a unique $\lambda_1 \in \mathbb{R}$ such that there exists a viscosity solution (u, v) of

$$\begin{cases} M_{1,1}^p(u(x), [v])(x) |p + u_x| = \lambda_1 & x \in \mathbb{R}, \\ M_{1,2}^p(v(x), [u])(x) |p + v_x| = \lambda_1 & x \in \mathbb{R}, \\ u, v \text{ are bounded.} \end{cases} \quad (5.2)$$

2) For every $p \leq 0$, there exists a unique $\lambda_2 \in \mathbb{R}$ such that there exists a viscosity solution w of

$$\begin{cases} M_{2,3}^p(w(x), [w])(x) |p + w_x| = \lambda_2 & x \in \mathbb{R}, \\ w \text{ is bounded.} \end{cases} \quad (5.3)$$

3) For every $p \leq 0$, there exists a unique $\lambda_3 \in \mathbb{R}$ such that there exists a viscosity solution (u, v, w) of

$$\begin{cases} M_{3,1}^p(u(x), [v])(x) |p + u_x| = \lambda_3 & x \in \mathbb{R}, \\ M_{3,1}^p(v(x), [w])(x) |p + v_x| = \lambda_3 & x \in \mathbb{R}, \\ M_{3,1}^p(w(x), [u])(x) |p + w_x| = \lambda_3 & x \in \mathbb{R}, \\ u, v, w \text{ are bounded.} \end{cases} \quad (5.4)$$

In particular, $\lambda_i = \overline{H}_i(p)$ with \overline{H}_i defined in (2.10), (2.11) and (2.12).

Proof. We claim that $(u, v) = \left(0, \frac{1}{2}\right)$ is a solution of (5.2) with $\lambda_1 = -|p|V_1\left(\frac{-3}{2p}\right)$. We have

$$\begin{aligned} M_{1,1}^p(u(x), [v])(x) + V_{max} &= \int_0^{\frac{-3}{2p}} V'_1(z) E_{-1}\left(\frac{1}{2} + pz\right) dz + \int_{\frac{-3}{2p}}^{+\infty} V'_1(z) E_{-1}\left(\frac{1}{2} + pz\right) dz \\ &= \int_{\frac{-3}{2p}}^{+\infty} V'_1(z) dz = V_{max} - V_1\left(\frac{-3}{2p}\right), \end{aligned}$$

where we used that if $z < \frac{-3}{2p}$, then $\frac{1}{2} + pz > -1$. Similarly, we have

$$\begin{aligned} M_{1,2}^p(v(x), [u])(x) + V_{max} &= \int_0^{\frac{-3}{2p}} V'_1(z) E_{-2}\left(-\frac{1}{2} + pz\right) dz + \int_{\frac{-3}{2p}}^{+\infty} V'_1(z) E_{-2}\left(-\frac{1}{2} + pz\right) dz \\ &= \int_{\frac{-3}{2p}}^{+\infty} V'_1(z) dz = V_{max} - V_1\left(\frac{-3}{2p}\right) \end{aligned}$$

where we used that if $z < \frac{-3}{2p}$, then $-\frac{1}{2} + pz > -2$. The above computations implies (5.2).

To prove (5.3) and (5.4), we proceed in the same way taking respectively the corrector $w = 0$ and $(u, v, w) = (0, 0, 0)$. The uniqueness of λ_i is a classical result and the reader can refer to [31] for the proof.

□

5.2 Correctors at the junction point

In this subsection, we will construct correctors near the junction point. To do this, we will consider the following equation: for $\lambda \in \mathbb{R}$, we consider a viscosity solution $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of

$$\begin{cases} R_{1,1}(x, \mathbf{u}(x), [\mathbf{v}](x))|\mathbf{u}_x| = \lambda & x \in \mathbb{R}, \\ R_{1,2}(x, \mathbf{v}(x), [\mathbf{u}], [\mathbf{w}](x))|\mathbf{v}_x| = \lambda & x \in \mathbb{R}, \\ R_{2,1}(x, \mathbf{w}(x), [\mathbf{u}](x))|\mathbf{w}_x| = \lambda & x \in \mathbb{R}, \end{cases} \quad (5.5)$$

where the non-local operators $R_{1,1}, R_{1,2}$ and $R_{2,1}$ are defined in (3.6), (3.7) and (3.8). We now prove the existence result of the corrector. For $\bar{A} \geq H_0$ with H_0 defined in (2.14), we introduce the real numbers \bar{p}_-^1, \bar{p}_-^2 and \bar{p}_+^3 satisfying

$$\bar{H}_1(\bar{p}_-^1) = \bar{H}_1^-(\bar{p}_-^1) = \bar{A}, \quad \bar{H}_2(\bar{p}_-^2) = \bar{H}_2^-(\bar{p}_-^2) = \bar{A}, \quad \bar{H}_3(\bar{p}_+^3) = \bar{H}_3^+(\bar{p}_+^3) = \bar{A}.$$

Theorem 5.2 (Existence of a global corrector for the junction). *Assume (A).*

i) (General properties) There exists a constant $\bar{A} \in [H_0, 0]$ such that there exists a solution $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of (5.5) with $\lambda = \bar{A}$ and such that there exists a constant C and a globally Lipschitz continuous function m such that

$$|\mathbf{u}(x) - m(x)| \leq C, \quad |\mathbf{v}(x) - m(x)| \leq C, \quad |\mathbf{w}(x) - m(x)| \leq C. \quad (5.6)$$

ii) (Rescaling) For $\varepsilon > 0$, we set

$$\mathbf{u}^\varepsilon(x) = \varepsilon \mathbf{u}\left(\frac{x}{\varepsilon}\right), \quad \mathbf{v}^\varepsilon(x) = \varepsilon \mathbf{v}\left(\frac{x}{\varepsilon}\right), \quad \mathbf{w}^\varepsilon(x) = \varepsilon \mathbf{w}\left(\frac{x}{\varepsilon}\right).$$

Then (along a subsequence $\varepsilon_n \rightarrow 0$), we have the following convergence locally uniformly: $\mathbf{u}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} U$, $\mathbf{v}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} V$ and $\mathbf{w}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} W$ with U and W satisfying

$$\begin{cases} |U(x) - U(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_1(U_x) = \bar{A} & \text{for all } x < 0, \\ \bar{H}_3(U_x) = \bar{A} & \text{for all } x > 0, \end{cases} \quad (5.7)$$

and

$$\begin{cases} |W(x) - W(y)| \leq C|x - y| & \text{for all } x, y \in \mathbb{R}, \\ \bar{H}_2(W_x) = \bar{A} & \text{for all } x < 0, \\ \bar{H}_3(W_x) = \bar{A} & \text{for all } x > 0. \end{cases} \quad (5.8)$$

In particular, we have (with $U(0) = W(0) = 0$)

$$U(x) = \bar{p}_+^3 x 1_{\{x > 0\}} + \bar{p}_-^1 x 1_{\{x < 0\}}, \quad (5.9)$$

$$W(x) = \bar{p}_+^3 x 1_{\{x > 0\}} + \bar{p}_-^2 x 1_{\{x < 0\}}. \quad (5.10)$$

iii) (Uniqueness of the flux limiter \bar{A}) We define the following set of functions

$$\mathcal{S} = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \text{ s.t. } \exists \text{ Lipschitz function } m \text{ and } C \geq 0 \text{ satisfying (5.6)}\}.$$

Then we have

$$\bar{A} = \inf \{ \lambda \in \mathbb{R} : \exists (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{S} \text{ solution of (5.5)} \}.$$

5.3 Proof of Theorem 5.2

This subsection contains the proof of Theorem 5.2. In [20, 18], authors considered a bounded domain $(-l, l)$ and a continuous function depending on a parameter $R > 0$ which allows a continuous transition between the non-local operators and the Hamiltonians then respectively took l and R to $+\infty$. In this work, we remarked that we can shorten this proof by considering a bounded domain depending on R and a continuous transition function between the non-local operators and the increasing and decreasing parts of the Hamiltonians. Let $R > R^1$ (R^1 the constant appearing in (2.4)), we want to find λ_R , such that there exists a solution (u^R, v^R, w^R) of

$$\begin{cases} G_R^1(x, u^R(x), [v^R], u_x^R) = \lambda_R & \text{if } x \in [-l_R, l_R], \\ G_R^2(x, v^R(x), [u^R], [w^R], v_x^R) = \lambda_R & \text{if } x \in [-l_R, l_R], \\ G_R^3(x, w^R(x), [u^R], w_x^R) = \lambda_R & \text{if } x \in [-l_R, l_R], \end{cases} \quad (5.11)$$

with

$$\begin{aligned} G_R^1(x, U, [V], q) &= \psi_R^-(x)\psi_R^+(x)\phi(x, M_{1,1}(U, [V])(x), M_{3,1}(U, [V])(x))|q| + \\ &+ (1 - \psi_R^-(x)) \cdot \overline{H}_1^-(q) + (1 - \psi_R^+(x)) \cdot \overline{H}_3^+(q), \end{aligned} \quad (5.12)$$

$$\begin{aligned} G_R^2(x, V, [U], [W], q) &= \psi_R^-(x)\psi_R^+(x)\phi(x, M_{1,2}(V, [U])(x), N(V, [W])(x))K(V, [W])(x)|q| + \\ &+ (1 - \psi_R^-(x)) \cdot \overline{H}_1^-(q) + (1 - \psi_R^+(x)) \cdot \overline{H}_3^+(q), \end{aligned} \quad (5.13)$$

$$\begin{aligned} G_R^3(x, W, [U], q) &= \psi_R^-(x)\psi_R^+(x)\phi(x, M_{2,3}(W, [W])(x), N(W, [U])(x))K(W, [U])(x)|q| + \\ &+ (1 - \psi_R^-(x)) \cdot \overline{H}_2^-(q) + (1 - \psi_R^+(x)) \cdot \overline{H}_3^+(q), \end{aligned} \quad (5.14)$$

and $\psi_R^+, \psi_R^- \in C^\infty$, $\psi_R^\pm : \mathbb{R} \rightarrow [0, 1]$, with $\psi_R^-(x) = \psi_R^+(-x)$,

$$\psi_R^+ \equiv \begin{cases} 1 & \text{on } [-\infty, R] \\ 0 & \text{outside } [R+1, +\infty), \end{cases} \quad \text{and } \psi_R^+ \text{ non-increasing,}$$

and $l_R \gg R+1 \gg \max(h_{max}, R^3+r)$. To each operator G_R^i , we associate \tilde{G}_R^i which is defined in the same way but replacing the non-local operators $M_{i,a}, N$ and K by $\tilde{M}_{i,a}, \tilde{N}$ and \tilde{K} .

Proposition 5.3 (Comparison principle). *Let $\lambda \in \mathbb{R}$,*

1) *Let us consider the following problem,*

$$\begin{cases} \tilde{G}_R^1(x, \hat{u}(x), [\hat{v}], \hat{u}_x) \geq \lambda & \text{if } x \in (0, l_R], \\ \tilde{G}_R^2(x, \hat{v}(x), [\hat{u}], [\hat{w}], \hat{v}_x) \geq \lambda & \text{if } x \in (0, l_R], \\ \tilde{G}_R^3(x, \hat{w}(x), [\hat{u}], \hat{w}_x) \geq \lambda & \text{if } x \in (0, l_R], \end{cases}$$

and for $\varepsilon > 0$

$$\begin{cases} G_R^1(x, u(x), [v], u_x) \leq \lambda - \varepsilon & \text{if } x \in (0, l_R], \\ G_R^2(x, v(x), [u], [w], v_x) \leq \lambda - \varepsilon & \text{if } x \in (0, l_R], \\ G_R^3(x, w(x), [u], w_x) \leq \lambda - \varepsilon & \text{if } x \in (0, l_R]. \end{cases}$$

If

$$u(0) \leq \hat{u}(0), \quad v(0) \leq \hat{v}(0), \quad w(0) \leq \hat{w}(0),$$

then for all $x \in [0, l_R]$, we have

$$u(x) \leq \hat{u}(x), \quad v(x) \leq \hat{v}(x), \quad w(x) \leq \hat{w}(x).$$

2) Let us consider the following problem,

$$\begin{cases} \tilde{G}_R^1(x, \hat{u}(x), [\hat{v}], \hat{u}_x) \geq \lambda & \text{if } x \in [-l_R, -R^1], \\ \tilde{G}_R^2(x, \hat{v}(x), [\hat{u}], [\hat{w}], \hat{v}_x) \geq \lambda & \text{if } x \in [-l_R, -R^1], \end{cases}$$

and for $\varepsilon > 0$

$$\begin{cases} G_R^1(x, u(x), [v], u_x) \leq \lambda - \varepsilon & \text{if } x \in [-l_R, -R^1], \\ G_R^2(x, v(x), [u], [w], v_x) \leq \lambda - \varepsilon & \text{if } x \in [-l_R, -R^1]. \end{cases}$$

If

$$u(x) \leq \hat{u}(x), \quad v(x) \leq \hat{v}(x) \quad \text{for } x \in [-R^1 - h_{max}, -R^1],$$

then for all $x \in [-l_R, -R^1]$, we have

$$u(x) \leq \hat{u}(x), \quad v(x) \leq \hat{v}(x).$$

3) Let us consider the following problem,

$$\tilde{G}_R^3(x, \hat{w}(x), [\hat{v}], \hat{w}_x) \geq \lambda \quad \text{if } x \in [-l_R, -R^1],$$

and for $\varepsilon > 0$

$$G_R^3(x, w(x), [v], w_x) \leq \lambda - \varepsilon \quad \text{if } x \in [-l_R, -R^1].$$

If

$$w(x) \leq \hat{w}(x) \quad \text{for } x \in [-R^1 - h_{max}, -R^1],$$

then for all $x \in [-l_R, -R^1]$, we have

$$w(x) \leq \hat{w}(x).$$

Proof. The proof of this proposition can be done using classical arguments of viscosity solutions theory, (see [5, 11]). Let us remark that the initial comparison between sub-solution and super-solution in points 2)-3) is given on the interval $[-R^1 - h_{max}, -R^1]$. This condition will ensure, when doing the proof, that the maximum points (x, y) satisfy $x + z, y + z < -R^1$ for $z \in [0, h_{max}]$. \square

Proposition 5.4 (Existence of correctors). *There exists a unique $\lambda_R \in \mathbb{R}$ such that there exists a solution (u^R, v^R, w^R) of (5.11). Moreover, there exists a constant C , and a Lipschitz continuous function m^R such that*

$$\begin{cases} H_0 \leq \lambda_R \leq 0, \\ |m^R(x) - m^R(y)| \leq C|x - y| & \text{for } x \in [-l_R, l_R], \\ |u^R(x) - m^R(x)| \leq C & \text{for } x \in [-l_R, l_R], \\ |v^R(x) - m^R(x)| \leq C & \text{for } x \in [-l_R, l_R], \\ |w^R(x) - m^R(x)| \leq C & \text{for } x \in [-l_R, l_R]. \end{cases} \quad (5.15)$$

12

Proof. In order to construct a corrector on the truncated domain, we will consider the approximated problem

$$\begin{cases} \delta u^\delta + G_R^1(x, u^\delta(x), [v^\delta], u_x^\delta) = 0 & \text{if } x \in [-l_R, l_R], \\ \delta v^\delta + G_R^2(x, v^\delta(x), [u^\delta], [w^\delta], v_x^\delta) = 0 & \text{if } x \in [-l_R, l_R], \\ \delta w^\delta + G_R^3(x, w^\delta(x), [u^\delta], w_x^\delta) = 0 & \text{if } x \in [-l_R, l_R]. \end{cases} \quad (5.16)$$

1 **Step 1: Construction of the solution.**

2 **Lemma 5.5.** *There exists a unique viscosity solution $(u^\delta, v^\delta, w^\delta)$ of (5.16) such that*

$$\begin{cases} 0 \leq u^\delta(x) \leq \frac{|H_0^1|}{\delta}, \\ 0 \leq v^\delta(x) \leq \frac{|H_0^1|}{\delta}, \\ 0 \leq w^\delta(x) \leq \frac{|H_0^2|}{\delta}. \end{cases} \quad (5.17)$$

Proof. We can easily show that,

$$(u^-, v^-, w^-) = (0, 0, 0) \text{ and } (u^+, v^+, w^+) = \left(\frac{|H_0^1|}{\delta}, \frac{|H_0^1|}{\delta}, \frac{|H_0^2|}{\delta} \right)$$

3 are respectively sub-solution and super-solution of (5.16). Using the comparison principle and the
4 Perron's method, we construct a unique viscosity solution $(u^\delta, v^\delta, w^\delta)$ satisfying (5.17). □

6 **Step 2: A new super-solution**

Lemma 5.6. *Let $b_0 = \max(g_0, k_0)$ and $g_0 = 1/d_0$ with d_0 defined in (2.8). Let*

$$(u^{++}, v^{++}, w^{++}) = \left(-3b_0(x - l_R) + \frac{|H_0^3|}{\delta}, -3b_0(x - l_R) + \frac{|H_0^3|}{\delta}, -3b_0(x - l_R) + \frac{|H_0^3|}{\delta} \right).$$

7 *Then (u^{++}, v^{++}, w^{++}) is a super-solution of (5.16).*

Proof. Since $x \in [-l_R, l_R]$, we have $u^{++}, v^{++}, w^{++} \geq \frac{|H_0^3|}{\delta}$. In addition, using the definition of H_k^- , we remark that for $k = 1, 2, 3$,

$$H_k(-3b_0) \geq H_k(-3k_0) = H_k^-(-3k_0) \geq 0.$$

We recall that for $i = 1, 2, 3$, we have

$$V_i(h) = 0 \text{ if } h \leq h_0, \quad \text{and } \omega(x, y, p) = 0 \text{ for } x \in [-R^3, 0], y < 0 \text{ and } p \leq d_0. \quad (5.18)$$

8 We claim that

$$\begin{aligned} M_{1,1}(u^{++}(x), [v^{++}]) &= 0 & M_{1,2}(v^{++}(x), [u^{++}]) &= 0 & \text{if } x \leq -R^3, \\ M_{2,3}(w^{++}(x), [w^{++}]) &= 0 & & & \text{if } x \leq -R^3, \\ K(v^{++}(x), [w^{++}]) &= 0 & K(w^{++}(x), [u^{++}]) &= 0 & \text{if } x \in (-R^3, -d_0], \\ N(v^{++}(x), [w^{++}]) &= 0 & N(w^{++}(x), [u^{++}]) &= 0 & \text{if } x \geq 0. \end{aligned}$$

In addition, on $(-d_0, 0)$, we have

$$\begin{aligned} K(v^{++}(x), [w^{++}]) &= 0 & K(w^{++}(x), [u^{++}]) &= 0 & \text{if } \frac{h_0}{2} < d_0, \\ N(v^{++}(x), [w^{++}]) &= 0 & N(w^{++}(x), [u^{++}]) &= 0 & \text{if } \frac{h_0}{2} \geq d_0. \end{aligned}$$

9 We prove only four equalities since others are done in the same way. First, let $x \leq -R^3$,

$$\begin{aligned} M_{1,1}(u^{++}(x), [v^{++}]) &+ V_{max} = \int_0^{+\infty} V_1'(z) E_{-1}(v^{++}(x+z) - u^{++}(x)) dz \\ &= \int_{h_0}^{+\infty} V_1'(z) E_{-1}(-3b_0z) dz. \end{aligned}$$

For $z \geq h_0$, $-3b_0z \leq -3 < -1$. Using the definition of E_{-1} (see (3.1)), we get

$$M_{1,1}(u^{++}(x), [v^{++}])(x) = \int_{h_0}^{h_{max}} V_1'(z) dz - V_{max} = 0.$$

1 Second, let $x \in (-R^3, -d_0]$,

$$\begin{aligned} K(v^{++}(x), [w^{++}])(x) - 1 &= \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F(w^{++}(z) - v^{++}(x)) dz \\ &= \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F(-3b_0(z - x)) dz \\ &= \int_{\mathbb{R}^-} \omega_z(x, z, |x| - |z|) F(-3b_0(z - x)) dz \\ &\quad + \int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) F(-3b_0(z - x)) dz. \end{aligned}$$

2 Let $z \leq 0$: using (5.18), we remark that we can assume that $z \geq x + d_0$. In that case, we have
 3 $-3b_0(z - x) \leq -3b_0d_0 = -3 < -1$. The last inequality is also true if we take $z > 0$. Using the
 4 definition of F (see (3.2)), we obtain

$$\begin{aligned} K(v^{++}(x), [w^{++}])(x) &= - \int_{x+d_0}^0 \omega_z(x, z, |x| - |z|) dz - \int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) dz + 1 \\ &= -\omega(x, 0, |x|) + \omega(x, x + d_0, d_0) - 1 + \omega(x, 0, |x|) + 1 = 0 \end{aligned}$$

5 where we used in the last line the fact that $\omega(x, z, p) = 1$ if $z \gg 0$. Finally, let $x \in (-d_0, 0)$: if
 6 $\frac{h_0}{2} \leq d_0$, then using (5.18), we evaluate $K(v^{++}(x), [w^{++}])(x)$ only for $z \in \mathbb{R}^+$ since $x + d_0 > 0 > z$.
 7 If $z \geq 0$, then $-3b_0(z - x) \leq 3b_0x \leq -3b_0\frac{h_0}{2} \leq -\frac{3}{2} < -1$, and we get

$$K(v^{++}(x), [w^{++}])(x) = - \int_{\mathbb{R}^+} \omega_z(x, z, |x| - |z|) dz = -1 + \omega(x, 0, |x|) + 1 = 0$$

8 where we used that $|x| = -x < d_0$. Assume now $\frac{h_0}{2} > d_0$. We have

$$\begin{aligned} N(v^{++}(x), [w^{++}])(x) + V_{max} &= \int_{z \geq x} V_3'(|z| - x) E_{-1}(w^{++}(z) - v^{++}(x)) dz \\ &= \int_{z \geq x} V_3'(|z| - x) E_{-1}(-3b_0(z - x)) dz \\ &= \int_{0 > z \geq x} V_3'(-z - x) E_{-1}(-3b_0(z - x)) dz \\ &\quad + \int_{z \geq 0} V_3'(z - x) E_{-1}(-3b_0(z - x)) dz. \end{aligned}$$

9 We evaluate the above integral only if $z \geq 0$ since if $x \leq z < 0$, we get $-z - x \leq -2x < 2d_0 < h_0$
 10 and so $V_3'(-z - x) = 0$. If $z \geq 0$ and $z - x \geq h_0$, we get that $-3b_0(z - x) \leq 3b_0h_0 < -1$ and using
 11 the definition of E_{-1} , we get

$$N(v^{++}(x), [w^{++}])(x) = \int_{z \geq x+h_0} V_3'(z - x) dz - V_{max} = 0.$$

1 Recalling the definition of G_R^1, G_R^2 and G_R^3 (see (5.12),(5.13) and (5.14)) and using the lower
 2 bound of u^{++}, v^{++} and w^{++} , we deduce that (u^{++}, v^{++}, w^{++}) is a super-solution of the approx-
 3 imated problem for $x \in (-l_R, l_R)$. Let us now treat the boundary conditions. Consider a test
 4 function φ touching u^{++} from below at $x = -l_R$ (resp. $x = l_R$). We have that

$$\begin{aligned} \delta\varphi(-l_R) + H_1^-(\varphi_x(-l_R)) &\geq H_1^-(u_x^{++}(-l_R)) = H_1^-(-3b_0) \geq 0, \\ \text{(resp. } \delta\varphi(l_R) + H_3^+(\varphi_x(l_R)) &\geq \delta u^{++}(l_R) + H_0^3 \geq 0). \end{aligned}$$

5 We can argue in the same way for the function v^{++} and w^{++} . This implies that (u^{++}, v^{++}, w^{++})
 6 is a super-solution of the approximated problem for $x \in [-l_R, l_R]$. Using the stability of viscosity
 7 super-solution, we can construct a new super-solution,

$$(\min(u^+, u^{++}), \min(v^+, v^{++}), \min(w^+, w^{++})).$$

8 The comparison principle then implies that

$$\begin{cases} 0 \leq u^\delta(x) \leq \min(u^+, u^{++}), \\ 0 \leq v^\delta(x) \leq \min(v^+, v^{++}), \\ 0 \leq w^\delta(x) \leq \min(w^+, w^{++}). \end{cases} \quad (5.19)$$

9

□

10 **Step 3: bounds for the solution.**

11 **Lemma 5.7.** *Let $(u^\delta, v^\delta, w^\delta)$ be the solution of (5.16). The functions u^δ, v^δ and w^δ are non-*
 12 *increasing and we have the following bounds,*

$$\begin{cases} u^\delta(x) - v^\delta(x) \leq 1 & \text{for all } x \in [-l_R, l_R], \\ -2 \leq u^\delta(x) - v^\delta(x) & \text{if } -l_R \leq x \leq -R_3, \end{cases} \quad (5.20)$$

13 and

$$u^\delta(x) - w^\delta(x) \geq -1 \quad \text{and} \quad w^\delta(x) - v^\delta(x) \geq -1 \quad \text{if } 0 \leq x \leq l_R. \quad (5.21)$$

14 In addition,

$$u^\delta(x) - w^\delta(x) \geq -1 \quad \text{and} \quad w^\delta(x) - v^\delta(x) \geq -1 \quad \text{if } -R^3 \leq x \leq 0. \quad (5.22)$$

15 In particular, we deduce that

$$\begin{cases} -2 \leq u^\delta(x) - v^\delta(x) \leq 1 & \text{for all } x \in [-l_R, l_R], \\ -1 \leq u^\delta(x) - w^\delta(x) \leq 2 & \text{if } 0 \leq x \leq l_R. \end{cases}$$

16 *Proof.* The proof of monotony is similar to that in [20, Lemma 5.5] so we skip it.

17 We will only prove the second bound in (5.20). The first one is simpler since we do not need
 18 a localization procedure. To prove (5.21), we proceed in the same way.

1 **Case 1:** $-R < x < -R^3$. Let $-R < z_0 < -R^3$ and $a > 0$ small enough. We will prove that for
 2 $x \in B_a(z_0)$,

$$v^\delta(x) - u^\delta(x) \leq 2 + L_a(x - z_0)^2 \quad (5.23)$$

3 with $B_a(z_0) = (z_0 - a, z_0 + a)$ and $L_a > \frac{|H_0^1|}{\delta a^2}$. Taking $x = z_0$ in (5.23), we obtain the second
 4 inequality in (5.20). We consider

$$M = \sup_{x \in B_a(z_0)} \{v^\delta(x) - u^\delta(x) - 2 - L_a(x - z_0)^2\}$$

and we assume by contradiction that $M > 0$. We then introduce

$$M_\varepsilon = \sup_{x, y \in B_a(z_0)} \left\{ v^\delta(x) - u^\delta(y) - 2 - L_a(x - z_0)^2 - \frac{(x - y)^2}{2\varepsilon} \right\}.$$

5 The supremum M_ε is reached at a point (x, y) with $|x - y| \rightarrow 0$ as $\varepsilon \rightarrow 0$. If $|x - z_0| = a$, then
 6 we obtain

$$0 < M_\varepsilon \leq \frac{|H_0^1|}{\delta} - L_a a^2 < 0,$$

7 which implies that for $\varepsilon > 0$ small, $|x - z_0| \neq a$ and $|y - z_0| \neq a$. Writing the viscosity
 8 inequalities, we get

$$\begin{aligned} 0 < \delta v^\delta(x) - \delta u^\delta(y) &\leq \tilde{G}_R^1 \left(y, u^\delta(y), [v^\delta], \frac{x - y}{\varepsilon} \right) \\ &\quad - G_R^2 \left(x, v^\delta(x), [u^\delta], [w^\delta], \frac{x - y}{\varepsilon} + 2L_a(x - z_0) \right) \\ &\leq -G_R^2 \left(x, v^\delta(x), [u^\delta], [w^\delta], \frac{x - y}{\varepsilon} + 2L_a(x - z_0) \right) \end{aligned}$$

9 where we used that $\tilde{G}_R^1 \leq 0$. Since $-R < x < -R^3$ and by definition of G_R^2 (see (5.13)) and ϕ
 10 (see (2.4)), we remark that we obtain a contradiction if we prove that

$$M_{1,2}(v^\delta(x), [u^\delta])(x) = 0. \quad (5.24)$$

11 Using the supremum M_ε , we obtain that

$$u^\delta(y) - v^\delta(x) < -2.$$

12 Using the fact that $|x - y| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we deduce that for ε small enough,

$$u^\delta(x) - v^\delta(x) < -2.$$

13 Since u^δ is decreasing, we get for $z \geq h_0$

$$u^\delta(x + z) - v^\delta(x) \leq u^\delta(x) - v^\delta(x) < -2$$

14 which implies $E_{-2}(u^\delta(x + z) - v^\delta(x)) = 1$ and therefore (5.24). We deduce that (5.23) is true
 15 and in particular by continuity that the second inequality of (5.20) is proved for $x \in [-R, -R^3]$.

1 **Case 2:** $-l_R \leq x < -R$. We introduce the following supremum

$$M_\varepsilon = \sup_{-l_R \leq x, y < -R} \left\{ v^\delta(x) - u^\delta(y) - 2 - \frac{(x-y)^2}{2\varepsilon} \right\}.$$

2 Classically, we have that

$$\begin{cases} M_\varepsilon \geq M > 0 \text{ and } M_\varepsilon \text{ is reached at a point } (x, y), \\ x, y \rightarrow \bar{x} \text{ as } \varepsilon \rightarrow 0. \end{cases} \quad (5.25)$$

3 In addition, using step 1, we have $x, y < -R$. for ε small enough. Writing the viscosity
4 inequalities, we get

$$\begin{aligned} \delta v^\delta(x) - \delta u^\delta(y) &\leq \tilde{G}_R^1 \left(y, u^\delta(y), [v^\delta], \frac{x-y}{\varepsilon} \right) \\ &\quad - G_R^2 \left(x, v^\delta(x), [u^\delta], [w^\delta], \frac{x-y}{\varepsilon} \right). \end{aligned}$$

5 As in step 1, we can prove that

$$M_{1,2}(v^\delta(x), u^\delta)(x) = 0 \quad (5.26)$$

6 and this implies

$$\delta v^\delta(x) - \delta u^\delta(y) \leq (1 - \psi_R^-(y)) \overline{H}_1^- \left(\frac{x-y}{\varepsilon} \right) - (1 - \psi_R^-(x)) \overline{H}_1^- \left(\frac{x-y}{\varepsilon} \right). \quad (5.27)$$

7 In particular, the sub-solution viscosity inequality and (5.26) implies

$$(1 - \psi_R^-(x)) \overline{H}_1^- \left(\frac{x-y}{\varepsilon} \right) \leq -\delta v^\delta(x) \leq 0.$$

Recalling the definition of H_1^- (see (2.18)) and since $(1 - \psi_R^-(x)) > 0$, we deduce that

$$H_0^1 \leq \overline{H}_1^- \left(\frac{x-y}{\varepsilon} \right) \leq 0. \quad (5.28)$$

8 Finally, using (5.28), we deduce that inequality (5.27) yields to

$$\begin{aligned} 2\delta < \delta v^\delta(x) - \delta u^\delta(y) &\leq (\psi_R^-(x) - \psi_R^-(y)) \overline{H}_1^- \left(\frac{x-y}{\varepsilon} \right) \\ &\leq |H_0^1| \cdot \|(\psi_R^-)'\|_\infty \cdot |x-y|. \end{aligned}$$

9 Sending ε to zero, we get a contradiction and this ends the proof of (5.20).

10 We turn now to the proof of the first inequality in (5.22) (the second inequality is treated in
11 the same way). Let $-R^3 < z_0 < 0$, $a > 0$ small and $L_a > \frac{|H_0^2|}{\delta a^2}$. We will prove that for $x \in B_a(z_0)$,

$$u^\delta(x) - w^\delta(x) \geq -1 - L_a(x - z_0)^2.$$

1 If the above inequality is true, then taking $x = z_0$, we obtain the first inequality in (5.22). For
 2 $\varepsilon > 0$, we define

$$M_\varepsilon = \sup_{x, y \in B_a(z_0)} \left\{ w^\delta(y) - u^\delta(x) - 1 - L_a(x - z_0)^2 - \frac{(x - y)^2}{2\varepsilon} \right\}.$$

We have that $M_\varepsilon \geq M > 0$ and since the functions w^δ and u^δ are continuous, M_ε is reached at a point (\bar{x}, \bar{y}) . Classically, we have

$$0 < M_\varepsilon < \frac{|H_0^2|}{\delta} - \frac{(\bar{x} - \bar{y})^2}{2\varepsilon}$$

3 which implies $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$. If $|\bar{x} - z_0| = a$, and using the definition of L_a , we obtain

$$0 < \frac{|H_0^2|}{\delta} - L_a a^2 \leq 0$$

4 which gives a contradiction and implies that $|\bar{x} - z_0| \neq a$ and $|\bar{y} - z_0| \neq a$ for ε small enough.
 5 Writing the sub-solution viscosity inequality, we get

$$\delta < \delta w^\delta(\bar{y}) \leq -G_R^3 \left(\bar{y}, w^\delta(\bar{y}), [u^\delta], \frac{\bar{x} - \bar{y}}{\varepsilon} \right).$$

6 Using the definition of G_R^3 for $\bar{y} \in (-R^3, 0)$, we can see that we obtain a contradiction if we prove
 7 that

$$K(w^\delta(\bar{y}), [u^\delta])(\bar{y}) = \int_{\mathbb{R}} \omega_z(\bar{y}, z, |\bar{y}| - |z|) F(u^\delta(z) - w^\delta(\bar{y})) dz + 1 = 0. \quad (5.29)$$

8 Recalling that $\omega(x, y, p) = 0$ if $x \in (-R^3, 0)$, $y < 0$ and $p < 0$, we can evaluate the above
 9 integral for $z \geq 0$. If we prove $u^\delta(z) - w^\delta(\bar{y}) < -1$ for all $z \geq \bar{y}$, we will get

$$\int_{\bar{y}}^{+\infty} \omega_z(\bar{y}, z, |\bar{y}| - |z|) F(u^\delta(z) - w^\delta(\bar{y})) dz = -1$$

which implies (5.29). Using the supremum M_ε , we have

$$u^\delta(\bar{x}) - w^\delta(\bar{y}) < -1$$

10 and since $|\bar{x} - \bar{y}| \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have for ε small enough

$$u^\delta(\bar{y}) - w^\delta(\bar{y}) < -1.$$

11 Using the fact that u^δ is non-increasing, we have for $z \geq \bar{y}$,

$$u^\delta(z) - w^\delta(\bar{y}) \leq u^\delta(\bar{y}) - w^\delta(\bar{y}) < -1$$

12 which gives the desired result.

13 □

1 **Step 4: control of oscillations.**

2 **Lemma 5.8.** *Let $(u^\delta, v^\delta, w^\delta)$ be a solution of (5.16). Let $g_0 = 1/d_0$ and $b_0 = \max(g_0, k_0)$. We*
 3 *have that*

$$\begin{cases} -3b_0(x-y) - 3 \leq u^\delta(x) - u^\delta(y) \leq 0 & \text{for } -l_R \leq y \leq x \leq l_R, \\ -3b_0(x-y) - 3 \leq v^\delta(x) - v^\delta(y) \leq 0 & \text{for } -l_R \leq y \leq x \leq l_R, \\ -3b_0(x-y) - 3 \leq w^\delta(x) - w^\delta(y) \leq 0 & \text{for } -l_R \leq y \leq x \leq l_R. \end{cases} \quad (5.30)$$

4 *Proof.* We will only prove the second inequality in (5.30). To prove the other inequalities, we
 5 proceed similarly (the proof is even simpler).

6 We define

$$M = \sup_{-l_R \leq y \leq x \leq l_R} \{v^\delta(y) - v^\delta(x) - 3b_0(x-y) - 3\}.$$

7 We want to prove that $M \leq 0$. Assume by contradiction that $M > 0$. Classically, M is reached
 8 at a point (\bar{x}, \bar{y}) with $\bar{x} \geq \bar{y}$. Moreover $\bar{x} \neq \bar{y}$ since $M > 0$. Writing the viscosity inequalities, we
 9 get

$$\begin{aligned} \delta v^\delta(\bar{y}) - \delta v^\delta(\bar{x}) &\leq \tilde{G}_R^2(\bar{x}, v^\delta(\bar{x}), [u^\delta], [w^\delta], -3b_0) \\ &\quad - G_R^2(\bar{y}, v^\delta(\bar{y}), [u^\delta], [w^\delta], -3b_0) \\ &\leq (1 - \psi_R^-(\bar{x}))\overline{H}_1^-(-3b_0) + (1 - \psi_R^+(\bar{x}))\overline{H}_3^+(-3b_0) \\ &\quad - G_R^2(\bar{y}, v^\delta(\bar{y}), [u^\delta], [w^\delta], -3b_0). \end{aligned}$$

10 We claim that

$$\phi(\bar{y}, M_{1,2}(v^\delta(\bar{y}), [u^\delta])(\bar{y}), N(v^\delta(\bar{y}), [w^\delta])(\bar{y})) K(v^\delta(\bar{y}), [w^\delta])(\bar{y}) = 0. \quad (5.31)$$

11 If we prove (5.31), using the definition of G_R^2 , we get

$$\begin{aligned} \delta v^\delta(\bar{y}) - \delta v^\delta(\bar{x}) &\leq (1 - \psi_R^-(\bar{x}))\overline{H}_1^-(-3b_0) + (1 - \psi_R^+(\bar{x}))\overline{H}_3^+(-3b_0) \\ &\quad - (1 - \psi_R^-(\bar{y}))\overline{H}_1^-(-3b_0) - (1 - \psi_R^+(\bar{y}))\overline{H}_3^+(-3b_0) \\ &= (\psi_R^-(\bar{y}) - \psi_R^-(\bar{x}))\overline{H}_1^-(-3b_0) + (\psi_R^+(\bar{y}) - \psi_R^+(\bar{x}))\overline{H}_3^+(-3b_0). \end{aligned}$$

12 Using that $H_1^-(-3b_0) \geq 0, H_3^+(-3b_0) \leq 0$ and that ψ^- and ψ^+ are respectively non-decreasing
 13 and non-increasing, we get a contradiction.

14 Let us now prove (5.31). Using the definition of $\bar{\phi}$ (see (2.4)), we obtain (5.31) if we prove:

$$\begin{cases} M_{1,2}(v^\delta(\bar{y}), [u^\delta])(\bar{y}) = 0 & \text{if } \bar{y} < -R^3, \\ N(v^\delta(\bar{y}), [w^\delta])(\bar{y}) = 0 & \text{if } \bar{y} > 0, \\ K(v^\delta(\bar{y}), [w^\delta])(\bar{y}) = 0 & \text{if } \bar{y} \in [-R^3, 0]. \end{cases} \quad (5.32)$$

15 We will only show the third equality of (5.32) since the others can be treated in the same way
 16 and are even simpler. Recalling the definition of ω , and assuming that $\bar{y} + d_0 < 0$, we can write,

$$K(v^\delta(\bar{y}), [w^\delta])(\bar{y}) - 1 = \int_{\bar{y}+d_0}^{+\infty} \omega_z(\bar{y}, z, |\bar{y}| - |z|) F(w^\delta(z) - v^\delta(\bar{y})) dz.$$

1 We claim that $w^\delta(z) - v^\delta(\bar{y}) < -1$ for all $z > \bar{y} + d_0$. If the preceding inequality is true, we
 2 get $F(w^\delta(z) - v^\delta(\bar{y})) = -1$ and then

$$K(v^\delta(\bar{y}), [w^\delta])(\bar{y}) = \omega(\bar{y}, \bar{y} + d_0, d_0) - 1 + 1 = 0.$$

3 We distinguish two cases:

4 **Case 1:** $\bar{x} < z$. Using the supremum, we have

$$v^\delta(\bar{x}) - v^\delta(\bar{y}) < -3.$$

5 The fact that v^δ is non-increasing implies

$$v^\delta(z) - v^\delta(\bar{y}) < -3. \quad (5.33)$$

6 Combining (5.33) and Lemma 5.7, we get $w^\delta(z) - v^\delta(\bar{y}) < -1$.

7 **Case 2:** $\bar{x} \geq z$. Using the supremum, we have $M \geq v^\delta(z) - v^\delta(\bar{x}) - 3b_0(\bar{x} - z) - 3$, which implies

$$v^\delta(z) - v^\delta(\bar{y}) \leq -3b_0(z - \bar{y}) \leq -3b_0d_0 \leq -3. \quad (5.34)$$

8 Combining (5.34) and Lemma 5.7, we get $w^\delta(z) - v^\delta(\bar{y}) < -1$ which gives us the desired result.

9 If $\bar{y} + d_0 \geq 0$, in that case, we only evaluate the integral for $z \in [0, r]$ since the other one is zero.
 10 We proceed as above assuming that $\bar{x} < z$ and $\bar{x} \geq z$. We deduce that $K(v^\delta(\bar{y}), [w^\delta])(\bar{y}) = 0$ and
 11 similarly (5.32). This ends the proof of the lemma.
 12 □

13 Step 5: Lipschitz estimate.

14 **Lemma 5.9.** *There exists a Lipschitz continuous function m^δ , such that there exists a constant*
 15 *C , (independent of R and δ) such that for $x, y \in [-l_R, l_R]$,*

$$\begin{cases} |m^\delta(x) - m^\delta(y)| \leq C|x - y|, \\ |u^\delta(x) - m^\delta(x)| \leq C, \\ |v^\delta(x) - m^\delta(x)| \leq C, \\ |w^\delta(x) - m^\delta(x)| \leq C. \end{cases} \quad (5.35)$$

16

17 *Proof of Lemma 5.9.* Let $a_0 = \min(h_0, d_0)$. Let us define m^δ as an affine function in each interval
 18 of the form $[ia_0, (i+1)a_0]$, with $i \in \mathbb{Z}$, such that

$$m^\delta(ia_0) = u^\delta(ia_0) \quad \text{and} \quad m^\delta((i+1)a_0) = u^\delta((i+1)a_0).$$

19 Since m^δ, u^δ are non-increasing and $|u^\delta((i+1)a_0) - u^\delta(ia_0)| \leq 3b_0a_0 + 3 = 6$, we deduce that
 20 $\forall x \in [ia_0, (i+1)a_0]$,

$$-6 \leq u^\delta((i+1)a_0) - m^\delta(ia_0) \leq u^\delta(x) - m^\delta(x) \leq u^\delta(ia_0) - m^\delta((i+1)a_0) \leq 6,$$

21 and

$$|m^\delta(x) - m^\delta(y)| \leq 6b_0|x - y|.$$

22 Lemma 5.7 provides the last two inequality in (5.35).
 23 □

1 **Step 6: passing to the limit as δ goes to 0.** Using Lemma 5.7 and (5.35), we deduce that
 2 there exists $\delta_n \rightarrow 0$, $\lambda_R \in \mathbb{R}$ and a Lipschitz continuous function m^R such that

$$\begin{aligned} \delta_n u^{\delta_n}(0) &\rightarrow -\lambda_R && \text{as } n \rightarrow +\infty, \\ \delta_n v^{\delta_n}(0) &\rightarrow -\lambda_R && \text{as } n \rightarrow +\infty, \\ \delta_n w^{\delta_n}(0) &\rightarrow -\lambda_R && \text{as } n \rightarrow +\infty, \\ m^{\delta_n} - m^{\delta_n}(0) &\rightarrow m^R && \text{as } n \rightarrow +\infty, \end{aligned}$$

3 the last convergence being locally uniform. Thanks to Lemma 5.8, we define

$$\bar{u}^R = \limsup_{\delta_n \rightarrow 0}^* (u^{\delta_n} - u^{\delta_n}(0)) \quad \text{and} \quad \underline{u}^R = \liminf_{\delta_n \rightarrow 0}^* (u^{\delta_n} - u^{\delta_n}(0)),$$

4

$$\bar{v}^R = \limsup_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)) \quad \text{and} \quad \underline{v}^R = \liminf_{\delta_n \rightarrow 0}^* (v^{\delta_n} - v^{\delta_n}(0)),$$

5 and

$$\bar{w}^R = \limsup_{\delta_n \rightarrow 0}^* (w^{\delta_n} - w^{\delta_n}(0)) \quad \text{and} \quad \underline{w}^R = \liminf_{\delta_n \rightarrow 0}^* (w^{\delta_n} - w^{\delta_n}(0)).$$

6 Therefore, we have that $\lambda_R, \bar{u}^R, \underline{u}^R, \bar{v}^R, \underline{v}^R, \bar{w}^R, \underline{w}^R$ and m^R satisfy

$$\begin{aligned} H_0 &\leq \lambda_R \leq 0, \\ |\bar{u}^R - m^R| &\leq C, \quad |\underline{u}^R - m^R| \leq C, \\ |\bar{v}^R - m^R| &\leq C, \quad |\underline{v}^R - m^R| \leq C, \\ |\bar{w}^R - m^R| &\leq C, \quad |\underline{w}^R - m^R| \leq C, \\ |(m^R)_x| &\leq C. \end{aligned} \tag{5.36}$$

7 By stability of the solutions we have that $(\bar{u}^R - 2C, \bar{v}^R - 2C, \bar{w}^R - 2C)$ and $(\underline{u}^R, \underline{v}^R, \underline{w}^R)$ are
 8 respectively a sub-solution and a super-solution of (5.11) and

$$\bar{u}^R - 2C \leq \underline{u}^R, \quad \bar{v}^R - 2C \leq \underline{v}^R, \quad \bar{w}^R - 2C \leq \underline{w}^R.$$

9 By Perron's method we can construct a solution (u^R, v^R, w^R) of (5.11) and thanks to Lemma 5.9,
 10 m^R, λ_R and (u^R, v^R, w^R) satisfy (5.15).

11 The uniqueness of λ_R is classical so we skip it. This ends the proof of Proposition 5.4.
 12 □

13 **Proposition 5.10** (First definition of the flux limiter). *The following limit exists (up to a subse-*
 14 *quence)*

$$\left\{ \bar{A} = \lim_{R \rightarrow +\infty} \lambda_R. \right. \tag{5.37}$$

15 Moreover, we have

$$H_0 \leq \lambda_R, \bar{A} \leq 0,$$

16

17 with H_0 defined in (2.14).

18 *Proof.* This results is a direct consequence of the following bound on λ_R which is independent of
 19 R (see the barriers in (5.19)),

$$H_0 \leq \lambda_R \leq 0.$$

20 □

¹ **Proposition 5.11** (Control of the slopes). *Assume that R is big enough. Let (u^R, v^R, w^R) be*
² *the solution of (5.11) given by Proposition 5.4. We also assume that up to a sub-sequence $\bar{A} =$*
³ $\lim_{R \rightarrow +\infty} \lambda_R > H_0$. *Then there exists $\gamma_0 > 0$ such that for every $\gamma \in (0, \gamma_0)$, there exists a constant*
⁴ \bar{C} *such that for $x > 0$ and $h \geq 0$,*

$$\begin{cases} u^R(x+h) - u^R(x) \geq (\bar{p}_+^3 - \gamma)h - \bar{C}, \\ v^R(x+h) - v^R(x) \geq (\bar{p}_+^3 - \gamma)h - \bar{C}, \\ w^R(x+h) - w^R(x) \geq (\bar{p}_+^3 - \gamma)h - \bar{C}. \end{cases} \quad (5.38)$$

⁵ *If $x < -R^1$ and $h \geq 0$, then*

$$\begin{cases} u^R(x-h) - u^R(x) \geq (-\bar{p}_-^1 - \gamma)h - \bar{C}, \\ v^R(x-h) - v^R(x) \geq (-\bar{p}_-^1 - \gamma)h - \bar{C}, \end{cases} \quad (5.39)$$

⁶ *and*

$$w^R(x-h) - w^R(x) \geq (-\bar{p}_-^2 - \gamma)h - \bar{C}. \quad (5.40)$$

⁷ *Proof.* We only prove (5.39) since the proof for (5.38) and (5.40) is similar (and even simpler).
⁸ For $\mu > 0$ small enough, we denote by p_μ^1 the real number such that

$$\bar{H}_1(p_\mu^1) = \bar{H}_1^-(p_\mu^1) = \lambda_R - \mu.$$

⁹ Using that $H_0 < \lambda_R \leq 0$, we deduce that p_μ^1 exists, is unique and satisfies $\frac{-3k_0}{2} \leq p_\mu^1 \leq 0$ for μ
¹⁰ small enough. We define for $-l_R \leq x < -R^1$,

$$(u^-, v^-) = \left(p_\mu^1 x, p_\mu^1 x + \frac{1}{2} \right).$$

¹¹ We have that

$$\bar{H}_1^-(u_x^-) = \bar{H}_1^-(v_x^-) = \lambda_R - \mu.$$

We also have for $-l_R < x < -R^1$,

$$\begin{aligned} M_{1,1}(u^-(x), [v^-])(x) + V_{max} &= \int_{\mathbb{R}} V_1'(z) E_{-1} \left(p_\mu^1 z + \frac{1}{2} \right) dz \\ &= \int_{\frac{-3}{2p_\mu^1}}^{+\infty} V_1'(z) dz \\ &= V_{max} - V_1 \left(\frac{-3}{2p_\mu^1} \right), \end{aligned}$$

and

$$\begin{aligned} M_{1,2}(v^-(x), [u^-])(x) + V_{max} &= \int_{\mathbb{R}} V_1'(z) E_{-2} \left(p_\mu^1 z - \frac{1}{2} \right) dz \\ &= \int_{\frac{-3}{p_\mu^1}}^{+\infty} V_1'(z) dz \\ &= V_{max} - V_1 \left(\frac{-3}{2p_\mu^1} \right). \end{aligned}$$

1 We deduce that (u^-, v^-) satisfies for $-l_R \leq x < -R^1$,

$$\begin{cases} \psi_R^-(x)M_{1,1}(u^-(x), [v^-])(x)|u_x^-| + (1 - \psi_R^-(x)) \cdot \overline{H}_1^-(u_x^-) = \lambda_R - \mu, \\ \psi_R^-(x)M_{1,2}(v^-(x), [u^-])(x)|v_x^-| + (1 - \psi_R^-(x)) \cdot \overline{H}_1^-(v_x^-) = \lambda_R - \mu. \end{cases}$$

2 Let $x_0 < -R^1$ and $\bar{C} = (3b_0 - p_\mu^1)h_{max} + 3$. Using Proposition 5.4, we have for $x \in [x_0 - h_{max}, x_0]$,

$$u^R(x) - u^R(x_0) \geq p_\mu^1(x - x_0) - \bar{C}, \quad v^R(x) - v^R(x_0) \geq p_\mu^1(x - x_0) - \bar{C}.$$

3 Using the comparison principle (Proposition 5.3, point 2), we deduce that the above inequalities
4 are true for all $x \leq x_0$. Finally, if we choose $\gamma_0 < |p_1 - \bar{p}_-^1|$ (with p_1 defined in (2.14)), then

$$\overline{H}_1(\bar{p}_-^1 + \gamma) = \overline{H}_1^-(\bar{p}_-^1 + \gamma),$$

5 and we can choose $\mu > 0$ such that

$$p_1^\mu = \bar{p}_-^1 + \gamma$$

6 and this ends the proof. □

8 *Proof of Theorem 5.2.* The proof is performed in three steps.

9 **Step 1: proof of i)** The goal is to pass to the limit as $R \rightarrow +\infty$. We consider (up to some
10 subsequence)

$$\left\{ \begin{array}{l} \overline{u}(x) = \limsup_{R \rightarrow +\infty}^* (u^R - u^R(0)), \quad \underline{u}(x) = \liminf_{R \rightarrow +\infty_*} (u^R - u^R(0)) \\ \overline{v}(x) = \limsup_{R \rightarrow +\infty}^* (v^R - v^R(0)), \quad \underline{v}(x) = \liminf_{R \rightarrow +\infty_*} (v^R - v^R(0)), \\ \overline{w}(x) = \limsup_{R \rightarrow +\infty}^* (w^R - w^R(0)), \quad \underline{w}(x) = \liminf_{R \rightarrow +\infty_*} (w^R - w^R(0)), \\ \overline{A} = \lim_{R \rightarrow +\infty} \lambda_R, \\ m = \lim_{R \rightarrow +\infty} (m^R - m^R(0)). \end{array} \right.$$

11 The last convergence being locally uniform. Thanks to Proposition 5.4, we know that

$$\begin{aligned} m - C &\leq \underline{u} \leq \overline{u} \leq m + C, \\ m - C &\leq \underline{v} \leq \overline{v} \leq m + C, \\ m - C &\leq \underline{w} \leq \overline{w} \leq m + C. \end{aligned}$$

12 By stability of viscosity solutions, $(\overline{u} - 2C, \overline{v} - 2C, \overline{w} - 2C)$ and $(\underline{u}, \underline{v}, \underline{w})$ are respectively a sub and
13 a super-solution of (5.5) with $\lambda = \overline{A}$. Using Perron's method, we can then construct a solution
14 $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of (5.5) with $\lambda = \overline{A}$ that satisfies (5.6). In addition, (5.38), (5.39) and (5.40) hold for
15 $(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Step 2: proof of ii). We are now interested in the rescaled functions

$$\mathbf{u}^\varepsilon(x) = \varepsilon \mathbf{u}\left(\frac{x}{\varepsilon}\right), \quad \mathbf{v}^\varepsilon(x) = \varepsilon v\left(\frac{x}{\varepsilon}\right), \quad \mathbf{w}^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right).$$

1 Using (5.6), we have that

$$\mathbf{u}^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \quad \mathbf{v}^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon) \quad \mathbf{w}^\varepsilon(x) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + O(\varepsilon).$$

Therefore, we can find a sequence $\varepsilon_n \rightarrow 0$, such that locally uniformly as $n \rightarrow +\infty$,

$$\mathbf{u}^{\varepsilon_n} \rightarrow U, \quad \mathbf{v}^{\varepsilon_n} \rightarrow V, \quad \mathbf{w}^{\varepsilon_n} \rightarrow W$$

with $U(0) = V(0) = W(0) = 0$. Like in [27], arguing as in the proof of convergence away from the junction point, we have that (U, V, W) satisfies

$$\begin{cases} \overline{H}_1(U_x) = \overline{A} & \text{for } x < 0 & \text{and } \overline{H}_3(U_x) = \overline{A} & \text{for } x > 0, \\ \overline{H}_1(V_x) = \overline{A} & \text{for } x < 0 & \text{and } \overline{H}_3(V_x) = \overline{A} & \text{for } x > 0, \\ \overline{H}_2(W_x) = \overline{A} & \text{for } x < 0 & \text{and } \overline{H}_3(W_x) = \overline{A} & \text{for } x > 0. \end{cases}$$

For all $\gamma \in (0, \gamma_0)$, we have that if $\overline{A} > H_0$ and $x > 0$,

$$U_x \geq \overline{p}_+^3 - \gamma, \quad V_x \geq \overline{p}_+^3 - \gamma, \quad W_x \geq \overline{p}_+^3 - \gamma,$$

where we have used that $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ satisfies (5.38). Therefore we get for $x > 0$,

$$U_x = \overline{p}_+^3, \quad V_x = \overline{p}_+^3, \quad W_x = \overline{p}_+^3,$$

this result remains valid even if $\overline{A} = H_0$ (in this particular case $U_x = V_x = W_x = p_3$). Similarly, using (5.39), we get for $x < 0$,

$$U_x = \overline{p}_-^1, \quad V_x = \overline{p}_-^1,$$

and using (5.40) for $x < 0$,

$$W_x = \overline{p}_-^2,$$

2 which implies that $U = V$.

3 **Step 3: proof of iii).** The proof of this point can be done as an easy adaptation of the proof
4 in [20]. This ends the proof of Theorem 5.2.

5

□

6 Proof of Convergence

In this section, we will show how to obtain our main result Theorem 2.5. Let $\varepsilon > 0$ and

$$u^\varepsilon(t, x) = \varepsilon u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad v^\varepsilon(t, x) = \varepsilon v\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad w^\varepsilon(t, x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

7 with (u, v, w) the unique viscosity solution of (4.1). Then $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ is the unique viscosity
8 solution for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ of

$$\begin{cases} u_t^\varepsilon + R_{1,1}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{u^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{v^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x) |u_x^\varepsilon| = 0, \\ v_t^\varepsilon + R_{1,2}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{v^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{w^\varepsilon(t, \cdot)}{\varepsilon}\right]\right]\right)(x) |v_x^\varepsilon| = 0, \\ w_t^\varepsilon + R_{2,1}^\varepsilon\left(\frac{x}{\varepsilon}, \frac{w^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x) |w_x^\varepsilon| = 0, \\ u^\varepsilon(0, x) = u_0(x), \\ v^\varepsilon(0, x) = u_0(x), \\ w^\varepsilon(0, x) = w_0(x). \end{cases} \quad (6.1)$$

9 First, we will prove the following convergence result.

Theorem 6.1 (Junction condition by homogenization). *Assume (A) and (A0). For $\varepsilon > 0$, let $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ be the solution of (6.1). Let χ^ε defined by*

$$\chi^\varepsilon(t, x) = \begin{cases} u^\varepsilon(t, -d(0, x)) & \text{if } x \in J_1^*, \\ w^\varepsilon(t, -d(0, x)) & \text{if } x \in J_2^*, \\ v^\varepsilon(t, d(0, x)) & \text{if } x \in J_3. \end{cases} \quad (6.2)$$

1 Then there exists $\bar{A} \in [H_0, 0]$ such that χ^ε converges locally uniformly to the unique viscosity
2 solution u^0 of (2.16) with the initial condition \bar{u}_0 defined by

$$\bar{u}_0(x) = \begin{cases} u_0(-d(0, x)) & \text{if } x \in J_1^*, \\ w_0(-d(0, x)) & \text{if } x \in J_2^*, \\ u_0(d(0, x)) & \text{if } x \in J_3. \end{cases}$$

3 **Remark 6.2.** *The same result remains true if we take v^ε on J_1^* and $(u^\varepsilon$ or $w^\varepsilon)$ on J_3 .*

4 Theorem 6.1 gives also gradient estimates for the solution of (2.16).

Corollary 6.3. *Assume (A0)-(A). Let u^0 be the unique solution of (2.16) then u^0 is Lipschitz continuous and we have,*

$$\begin{cases} -\frac{3k_0}{2} \leq u_x^0 \leq 0 & \text{if } x \in J_1^*, \\ -3k_0 \leq u_x^0 \leq 0 & \text{if } x \in J_2^*, \\ -k_0 \leq u_x^0 \leq 0 & \text{if } x \in J_3^*. \end{cases}$$

5

6 *Proof of Corollary 6.3.* The Lipschitz continuity of u^0 was proved in [27].

Let $x, y \in J_1^*$ s.t $x = x_1 \cdot e_1$ and $y = (x_1 + h) \cdot e_1$ with $x_1 + h < 0$ and $h > 0$ small enough. For $\varepsilon > 0$ small enough, we have $x_1 + h < -\varepsilon R^1$. Using Theorem 4.5, we have

$$\frac{-3k_0}{2}h - \varepsilon \leq v^\varepsilon(t, x_1 + h) - u^\varepsilon(t, x_1) \leq 0.$$

7 Using Theorem 6.1 and passing to the limit as $\varepsilon \rightarrow 0$, we obtain the desired result. We proceed
8 similarly on J_2^* and J_3^* by using respectively (4.5) and (4.6).

9

□

10 We will do first the Proof of Theorem 6.1 and then deduce Theorem 2.5.

Proof of Theorem 6.1. We first define for a given point $(t_0, x_0) \in (0, T) \times J$ and for $\bar{r}, \bar{R} > 0$, the set

$$\mathcal{Q}_{\bar{r}, \bar{R}}(t_0, x_0) = \begin{cases} \{(t, x) \in (0, T) \times J_i^* \text{ s.t } |t - t_0| \leq \bar{r} \text{ and } d(x, x_0) \leq \bar{R}\} & \text{if } x_0 \in J_i^*, \\ \{(t, x) \in (0, T) \times J \text{ s.t } |t - t_0| \leq \bar{r} \text{ and } d(x, x_0) \leq \bar{R}\} & \text{if } x_0 = 0. \end{cases}$$

We introduce

$$\bar{u}(t, x) = \limsup_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} u^\varepsilon(s, y) \quad \text{and} \quad \underline{u}(t, x) = \liminf_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} u^\varepsilon(s, y), \quad (6.3)$$

$$\bar{v}(t, x) = \limsup_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} v^\varepsilon(s, y) \quad \text{and} \quad \underline{v}(t, x) = \liminf_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} v^\varepsilon(s, y), \quad (6.4)$$

$$\bar{w}(t, x) = \limsup_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} w^\varepsilon(s, y) \quad \text{and} \quad \underline{w}(t, x) = \liminf_{(s, y) \rightarrow (t, x), \varepsilon \rightarrow 0} w^\varepsilon(s, y). \quad (6.5)$$

11 We then define the following functions

$$\bar{z}(t, x) = \begin{cases} \max(\bar{u}(t, d(0, x)), \bar{v}(t, d(0, x)), \bar{w}(t, d(0, x))) & \text{if } x \in J_3, \\ \max(\bar{u}(t, -d(0, x)), \bar{v}(t, -d(0, x))) & \text{if } x \in J_1^*, \\ \bar{w}(t, -d(0, x)) & \text{if } x \in J_2^*, \end{cases} \quad (6.6)$$

1 and

$$\underline{z}(t, x) = \begin{cases} \min(\underline{u}(t, d(0, x)), \underline{v}(t, d(0, x)), \underline{w}(t, d(0, x))) & \text{if } x \in J_3, \\ \min(\underline{u}(t, -d(0, x)), \underline{v}(t, -d(0, x))) & \text{if } x \in J_1^*, \\ \underline{w}(t, -d(0, x)) & \text{if } x \in J_2^*. \end{cases} \quad (6.7)$$

2 We will prove that \bar{z} and \underline{z} are respectively sub and super solutions of (2.16). In this case, the
3 comparison principle will imply that $\bar{z} \leq \underline{z}$ and by construction, we have that $\underline{z} \leq \bar{z}$. Hence, we
4 will get that

$$\bar{z}(t, x) = \underline{z}(t, x) = u^0(t, x)$$

5 where u^0 is the unique viscosity solution of (2.16). Let us prove that \bar{z} is a sub-solution of
6 (2.16) (the proof for \underline{z} is similar and we skip it). We argue by contradiction and assume that
7 there exists a test function $\varphi \in \mathcal{C}^1((0, T) \times J)$ (in the sense of Theorem 2.7 in [27]), and a point
8 $(\bar{t}, \bar{x}) \in (0, +\infty) \times J$ such that

$$\begin{cases} \bar{z}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x}) \\ \bar{z} \leq \varphi \\ \bar{z} \leq \varphi - 2\eta \\ \varphi_t(\bar{t}, \bar{x}) + \bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) = \theta \end{cases} \quad \begin{array}{ll} \text{on } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \bar{r} > 0 \\ \text{outside } \mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}) & \text{with } \eta > 0 \\ \text{with } \theta > 0, \end{array} \quad (6.8)$$

9 where

$$\bar{H}(\bar{x}, \varphi_x(\bar{t}, \bar{x})) := \begin{cases} \bar{H}_k(\partial_k \varphi(\bar{t}, \bar{x})) & \text{if } \bar{x} \in J_k^*, \\ \bar{F}_{\frac{-}{A}}(\partial_1 \varphi(\bar{t}, 0), \partial_2 \varphi(\bar{t}, 0), \partial_3 \varphi(\bar{t}, 0)) & \text{if } \bar{x} = 0. \end{cases}$$

10 We denote by $\bar{y} = \text{sign}(\bar{x})d(0, \bar{x})$ with $\text{sign} : J \rightarrow \mathbb{R}$, defined by

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \in J_3^*, \\ -1 & \text{if } x \in J_1^* \cup J_2^*, \\ 0 & \text{if } x = 0. \end{cases}$$

Up to changing φ at infinity, we can assume that for ε small enough, we have

$$\begin{cases} u^\varepsilon(t, \text{sign}(x)d(0, x)) \leq \varphi(t, x) - \eta & \text{for } (t, x) \in (\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}))^c \cap (J_1 \cup J_3), \\ v^\varepsilon(t, \text{sign}(x)d(0, x)) \leq \varphi(t, x) - \eta & \text{for } (t, x) \in (\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}))^c \cap (J_1 \cup J_3), \\ w^\varepsilon(t, \text{sign}(x)d(0, x)) \leq \varphi(t, x) - \eta & \text{for } (t, x) \in (\mathcal{Q}_{\bar{r}, \bar{r}}(\bar{t}, \bar{x}))^c \cap (J_2 \cup J_3). \end{cases} \quad (6.9)$$

11 **First case:** $\bar{x} \neq 0$. We only consider the case where $\bar{x} \in J_1^*$ since the other cases can be treated
12 in the same way. Let $p = \partial_1 \varphi(\bar{t}, \bar{x})$ which according to Corollary 6.3 satisfies $p \leq 0$ and

$$\varphi_t(\bar{t}, \bar{x}) + \bar{H}_1(p) = \theta.$$

13 Let \bar{r} small enough so that $\bar{y} + \bar{r} < 0$. We define a new test function $\tilde{\varphi} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\tilde{\varphi}(t, y) = \varphi(t, ye_1) \quad \text{for } y < 0.$$

1 For $(t, y) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$ (see (4.2)), ε small enough and by definition of ϕ and ω , we have that

$$\phi\left(\frac{y}{\varepsilon}, a, b\right) = a \quad \text{and} \quad \omega\left(\frac{y}{\varepsilon}, z, \left|\frac{y}{\varepsilon}\right| - |z|\right) = 1. \quad (6.10)$$

2 Using (6.10), and the definition of the non-local operator K^ε (see (3.12)) implies that on the
3 ball $P_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$, the function $(u^\varepsilon, v^\varepsilon)$ is a solution in the viscosity sense of

$$\begin{cases} u_t^\varepsilon + M_{1,1}^\varepsilon\left(\frac{u^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{v^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) |u_y^\varepsilon| = 0, \\ v_t^\varepsilon + M_{1,2}^\varepsilon\left(\frac{v^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{u^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) |v_y^\varepsilon| = 0. \end{cases}$$

4 We define the following function

$$\psi^\varepsilon(t, y) = \tilde{\varphi}(t, y) + \frac{\varepsilon}{2}.$$

5 We will prove that for $(t, y) \in P_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$, we have

$$\begin{aligned} \tilde{\varphi}_t + \tilde{M}_{1,1}^\varepsilon\left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) |\varphi_y| &\geq 0, \\ \psi_t^\varepsilon + \tilde{M}_{1,2}^\varepsilon\left(\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}\right]\right)(y) |\psi_y^\varepsilon| &\geq 0. \end{aligned}$$

For all $(t, y) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$, we have for \bar{r} small enough

$$\begin{aligned} \tilde{\varphi}_t(t, y) + \tilde{M}_{1,1}^\varepsilon\left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) \cdot |\tilde{\varphi}_y^\varepsilon| &= \tilde{\varphi}_t(\bar{t}, \bar{y}) + o_{\bar{r}}(1) \\ &+ \tilde{M}_{1,1}^\varepsilon\left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) \cdot |\tilde{\varphi}_y| \\ &= \theta + o_{\bar{r}}(1) + \tilde{M}_{1,1}^\varepsilon\left(\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(y) \cdot |p| - \overline{H}_1(p) =: \Delta, \end{aligned} \quad (6.11)$$

6 where we used that $\tilde{\varphi}_y(\bar{t}, \bar{y}) = \partial_1 \varphi(\bar{t}, \bar{x}) = p$ and (6.8).

7 If $p = 0$, then using that $\overline{H}_1(0) = 0$, we obtain that $\Delta = \theta + o_{\bar{r}}(1) > 0$ for \bar{r} small enough.

8 Assume now that $p < 0$. For all $z \in [h_0, h_{max}]$, and for ε and \bar{r} small enough we have that

$$\frac{\psi^\varepsilon(t, y + \varepsilon z) - \tilde{\varphi}(t, y)}{\varepsilon} = \frac{\tilde{\varphi}(t, y + \varepsilon z) - \tilde{\varphi}(t, y)}{\varepsilon} + \frac{1}{2} \leq pz + o_{\bar{r}}(1) + c\varepsilon + \frac{1}{2},$$

9 where we used the fact that $\tilde{\varphi} \in \mathcal{C}^1$ and that $z \in [h_0, h_{max}]$. Now using the fact that \tilde{E}_{-1} is
10 decreasing we have

$$\tilde{E}_{-1}\left(pz + c\varepsilon + o_{\bar{r}}(1) + \frac{1}{2}\right) \leq \tilde{E}_{-1}\left(\frac{\tilde{\varphi}(t, y + \varepsilon z) - \tilde{\varphi}(t, y)}{\varepsilon} + \frac{1}{2}\right).$$

We deduce that

$$\begin{aligned}
 \tilde{M}_{1,1}^\varepsilon \left(\frac{\tilde{\varphi}}{\varepsilon}(t, \cdot), \left[\frac{\psi^\varepsilon}{\varepsilon}(t, \cdot) \right] \right) (y) + V_{max} &\geq \int_0^{+\infty} V_1'(z) \tilde{E}_{-1} \left(pz + o_{\bar{r}}(1) + c\varepsilon + \frac{1}{2} \right) dz \\
 &= \int_0^{(-3/2 - c\varepsilon - o_{\bar{r}}(1))/p} V_1'(z) \tilde{E}_{-1} \left(pz + o_{\bar{r}}(1) + c\varepsilon + \frac{1}{2} \right) dz \\
 &+ \int_{(-3/2 - c\varepsilon - o_{\bar{r}}(1))/p}^{+\infty} V_1'(z) \tilde{E}_{-1} \left(pz + o_{\bar{r}}(1) + c\varepsilon + \frac{1}{2} \right) dz
 \end{aligned} \tag{6.12}$$

$$= 0 + \int_{(-3/2 - c\varepsilon - o_{\bar{r}}(1))/p}^{+\infty} V_1'(z) = V_{max} - V_1 \left(\frac{-3}{2} - c\varepsilon - o_{\bar{r}}(1) \right) \tag{6.13}$$

- 1 where we used in the last line the fact that $pz + o_{\bar{r}}(1) + c\varepsilon + \frac{1}{2} < -1$ if $z > \frac{-3 - c\varepsilon - o_{\bar{r}}(1)}{p}$. Injecting
 2 (6.13) in (6.11) and choosing ε and \bar{r} small enough, we obtain

$$\begin{aligned}
 \Delta &\geq \theta + o_{\bar{r}}(1) + |p| \cdot \left[-V_1 \left(\frac{-3}{2} - c\varepsilon - o_{\bar{r}}(1) \right) + V_1 \left(\frac{-3}{2p} \right) \right] \\
 &\geq \theta + o_{\bar{r}}(1) - \|V_1'\|_\infty \cdot (c\varepsilon + o_{\bar{r}}(1)) \\
 &\geq \frac{\theta}{2},
 \end{aligned}$$

- 3 where we used assumption (A1) for the second line. In the same way, we prove that in $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$

$$\psi_t^\varepsilon + \tilde{M}_{1,2}^\varepsilon \left(\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}, \left[\frac{\tilde{\varphi}(t, \cdot)}{\varepsilon} \right] \right) (y) |\psi_y^\varepsilon| \geq \frac{\theta}{2}$$

- 4 using that

$$\frac{\tilde{\varphi}^\varepsilon(t, y + \varepsilon z) - \psi(t, y)}{\varepsilon} = \frac{\tilde{\varphi}(t, y + \varepsilon z) - \tilde{\varphi}(t, y)}{\varepsilon} - \frac{1}{2} \leq pz + o_{\bar{r}}(1) + c\varepsilon - \frac{1}{2}$$

- 5 and then we do the same computations using the definition of \tilde{E}_{-2} .

- 6 **Getting a contradiction.** By definition, we have for ε small enough and using (6.9),

$$\begin{aligned}
 u^\varepsilon &\leq \tilde{\varphi} - \eta \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}), \\
 v^\varepsilon &\leq \psi^\varepsilon - \eta \quad \text{outside } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}).
 \end{aligned} \tag{6.14}$$

- 7 Using the comparison principle on bounded subsets for (6.1) (Theorem 4.2), we get

$$\begin{aligned}
 u^\varepsilon &\leq \tilde{\varphi} - \eta \quad \text{on } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}), \\
 v^\varepsilon &\leq \psi^\varepsilon - \eta \quad \text{on } \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y}).
 \end{aligned} \tag{6.15}$$

- 8 Passing to the limit as $\varepsilon \rightarrow 0$, we get $\bar{u} \leq \tilde{\varphi} - \eta$ and $\bar{v} \leq \tilde{\varphi} - \eta$ on $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, \bar{y})$ and this contradicts
 9 the fact that $\bar{z}(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$.

- 10 **Second case: $\bar{x} = 0$:** In that case, we have that

$$\varphi_t(\bar{t}, \bar{x}) + \bar{F}_{\bar{A}}(\partial_1 \varphi(\bar{t}, 0), \partial_2 \varphi(\bar{t}, 0), \partial_3 \varphi(\bar{t}, 0)) = \theta \tag{6.16}$$

- 11 and using Theorem 2.7 in [27], we can assume that φ has a specific form given by

$$\varphi(t, x) = g(t) + \bar{p}_-^1 x 1_{\{x \in J_1^*\}} + \bar{p}_-^2 x 1_{\{x \in J_2^*\}} + \bar{p}_+^3 x 1_{\{x \in J_3^*\}} \quad (6.17)$$

with $g \in C^1(0, T)$. In particular, (6.16) can be replaced by

$$g_t(\bar{t}) + \bar{A} = \theta. \quad (6.18)$$

1 Let us consider the solution $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of (5.5) provided by Theorem 5.2, and let us denote by

$$\varphi^\varepsilon(t, y) = \begin{cases} g(t) + \mathbf{u}^\varepsilon(t, y) & \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, ye_1 1_{\{y \leq 0\}} + ye_3 1_{\{y \geq 0\}}) & \text{outside } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \end{cases} \quad (6.19)$$

2 and

$$\psi^\varepsilon(t, y) = \begin{cases} g(t) + \mathbf{v}^\varepsilon(t, y) & \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, ye_1 1_{\{y \leq 0\}} + ye_3 1_{\{y \geq 0\}}) & \text{outside } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \end{cases} \quad (6.20)$$

3 and

$$\Psi^\varepsilon(t, y) = \begin{cases} g(t) + \mathbf{w}^\varepsilon(t, y) & \text{on } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0), \\ \varphi(t, ye_2 1_{\{y \leq 0\}} + ye_3 1_{\{y \geq 0\}}) & \text{outside } \mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0). \end{cases} \quad (6.21)$$

4 We will prove that $(\varphi^\varepsilon, \psi^\varepsilon, \Psi^\varepsilon)$ is a viscosity super-solution on $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$ of (6.1). Let h be a
5 test function touching φ^ε from below at $(t_1, x_1) \in \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$, so we have

$$\mathbf{u}\left(\frac{x_1}{\varepsilon}\right) = \frac{1}{\varepsilon} (h(t_1, x_1) - g(t_1)),$$

6 and

$$\mathbf{u}(y) \geq \frac{1}{\varepsilon} (h(\varepsilon s, \varepsilon y) - g(\varepsilon s)),$$

7 for (s, y) in a neighbourhood of $\left(\frac{t_1}{\varepsilon}, \frac{x_1}{\varepsilon}\right)$. Therefore, we have

$$h_t(t_1, x_1) - g'(t_1) + \tilde{R}_{1,1}\left(\frac{x_1}{\varepsilon}, \mathbf{u}\left(\frac{x_1}{\varepsilon}\right), [\mathbf{v}]\right)(x_1) \cdot |h_x(t_1, x_1)| \geq \bar{A}.$$

This implies that (using (6.18) and taking \bar{r} small enough)

$$h_t(t_1, x_1) + \tilde{R}_{1,1}\left(\frac{x_1}{\varepsilon}, \mathbf{u}\left(\frac{x_1}{\varepsilon}\right), [\mathbf{v}]\right)(x_1) \cdot |h_x(t_1, x_1)| \geq \bar{A} + g'(t_1) \geq \frac{\theta}{2}.$$

8 Now for ε small enough such that $\varepsilon h_{max} \leq \bar{r}$, we deduce from the previous inequality and using
9 the fact that the non-local operators are of bounded support, that

$$h_t(t_1, x_1) + \tilde{R}_{1,1}^\varepsilon\left(\frac{x_1}{\varepsilon}, \frac{\varphi^\varepsilon(t, x)}{\varepsilon}, \left[\frac{\psi^\varepsilon(t, \cdot)}{\varepsilon}\right]\right)(x) \cdot |h_x(t_1, x_1)| \geq \frac{\theta}{2}.$$

10 The two last super-solution inequalities are obtained in same way using the correctors and
11 (6.18).

12 **Getting the contradiction.** Using (6.9) and (6.17), we have for ε small enough on $\mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$,

$$\begin{aligned} u^\varepsilon &\leq g(t) + \bar{p}_-^1 y 1_{\{y < 0\}} + \bar{p}_+^3 y 1_{\{y > 0\}} - \eta, \\ v^\varepsilon &\leq g(t) + \bar{p}_-^1 y 1_{\{y < 0\}} + \bar{p}_+^3 y 1_{\{y > 0\}} - \eta, \\ w^\varepsilon &\leq g(t) + \bar{p}_-^2 y 1_{\{y < 0\}} + \bar{p}_+^3 y 1_{\{y > 0\}} - \eta. \end{aligned}$$

Using the fact that $u^\varepsilon, v^\varepsilon \rightarrow U$ and $w^\varepsilon \rightarrow W$ with U, W defined in (5.10) we have for ε small enough on $\mathcal{P}_{\bar{r}, 2\bar{r}}(\bar{t}, 0) \setminus \mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$

$$u^\varepsilon \leq \varphi^\varepsilon - \frac{\eta}{2} \quad \text{and} \quad v^\varepsilon \leq \psi^\varepsilon - \frac{\eta}{2} \quad \text{and} \quad w^\varepsilon \leq \Psi^\varepsilon - \frac{\eta}{2}.$$

Combining this with the particular form of the test function given in (6.17), we get that outside $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$,

$$u^\varepsilon \leq \varphi^\varepsilon - \frac{\eta}{2} \quad \text{and} \quad v^\varepsilon \leq \psi^\varepsilon - \frac{\eta}{2} \quad \text{and} \quad w^\varepsilon \leq \Psi^\varepsilon - \frac{\eta}{2}.$$

- 1 By the comparison principle on bounded subsets (Theorem 4.2) the previous inequality holds in
 2 $\mathcal{P}_{\bar{r}, \bar{r}}(\bar{t}, 0)$. Passing to the limit as $\varepsilon \rightarrow 0$ and evaluating the inequality in $(\bar{t}, 0)$, we obtain

$$\bar{u}(\bar{t}, 0) \leq \varphi(\bar{t}, 0) - \frac{\eta}{2} \quad \text{and} \quad \bar{v}(\bar{t}, 0) \leq \psi(\bar{t}, 0) - \frac{\eta}{2} \quad \text{and} \quad \bar{w}(\bar{t}, 0) \leq \Psi(\bar{t}, 0) - \frac{\eta}{2}.$$

- 3 which is a contradiction.

4

□

- 5 We can now turn to the proof of our main result, Theorem 2.5.

Proof of Theorem 2.5. We recall that the initial conditions defined in Theorem 2.5 are given by

$$\begin{aligned} u_0(x) &= -\frac{3}{2h_1}x1_{\{x < 0\}} - \frac{1}{h_3}x1_{\{x \geq 0\}}, \\ w_0(x) &= -\frac{3}{h_2}x1_{\{x < 0\}} - \frac{1}{h_3}x1_{\{x \geq 0\}}. \end{aligned}$$

- 6 We claim that there exists f, g and h such that

$$|\rho^\varepsilon(0, x) - u_0(x)| \leq f(\varepsilon), \quad |\sigma^\varepsilon(0, x) - u_0(x)| \leq g(\varepsilon), \quad |\tau^\varepsilon(0, x) - w_0(x)| \leq h(\varepsilon) \quad (6.22)$$

- 7 with $f(\varepsilon), g(\varepsilon), h(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We prove the last one in (6.22) since the others are done in
 8 the same way. We consider a piece-wise function v satisfying

$$\tau(0, x) = v(x) \text{ for } x = U_i(0) \text{ and } i \in \mathcal{L}_3 \cup \mathbb{N}.$$

Since $U_{i+3}(0) - U_i(0) \geq h_0$ for $i \in \mathcal{L}_3$ and $U_{i+1}(0) - U_i(0) \geq h_0$ for $i \in \mathbb{N}$, we get that

$$\begin{aligned} |v(x) - v(y)| &\leq \frac{3}{h_0}|x - y|, \\ |v(x) - \tau(x)| &\leq 3. \end{aligned}$$

Let us consider the integers $i_0, i_1 \in \mathbb{Z}$ defined by

$$\begin{aligned} i_0 &= \sup\{i \in \mathcal{L}_3, \text{ s.t. } U_i(0) \leq -R\}, \\ i_1 &= \inf\{i \in \mathbb{N}, \text{ s.t. } U_i(0) \geq R\}. \end{aligned}$$

- 9 Using respectively the third and fourth line in (2.23), we have for $x \leq U_{i_0}(0)$,

$$v(x) = \frac{3(U_{i_0}(0) - x)}{h_2} - i_0,$$

- 10 and for $x \geq U_{i_1}(0)$,

$$v(x) = \frac{U_{i_1}(0) - x}{h_3} - i_1.$$

1 We recall that $\tau^\varepsilon(0, x) = \varepsilon\tau\left(0, \frac{x}{\varepsilon}\right)$. If $x \notin [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$, we have

$$\begin{aligned} |\tau^\varepsilon(0, x) - w_0(x)| &\leq \left| \tau^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - w_0(x) \right| \\ &\leq 3\varepsilon + \varepsilon \max\left(\left| \frac{3U_{i_0}(0)}{h_2} - i_0 \right|, \left| \frac{U_{i_1}(0)}{h_3} - i_1 \right| \right) \end{aligned}$$

where we used the definition of w_0 . If $x \in [\varepsilon U_{i_0}(0), \varepsilon U_{i_1}(0)]$, we have

$$\begin{aligned} |\tau^\varepsilon(0, x) - w_0(x)| &\leq \left| \tau^\varepsilon(0, x) - \varepsilon v\left(\frac{x}{\varepsilon}\right) \right| + \left| \varepsilon v\left(\frac{x}{\varepsilon}\right) - w_0(x) \right| \\ &\leq 3\varepsilon + \varepsilon \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|\tau(y) - w(y)|). \end{aligned}$$

Finally, we can define h by

$$h(\varepsilon) = 3\varepsilon + \varepsilon \max\left(\max\left(\left| \frac{3U_{i_0}(0)}{h_2} - i_0 \right|, \left| \frac{U_{i_1}(0)}{h_3} - i_1 \right| \right), \max_{y \in [U_{i_0}(0), U_{i_1}(0)]} (|\tau(y) - w(y)|) \right).$$

2 In the same way we construct the functions f and g . Therefore we have

$$\begin{aligned} u_0(x) - m(\varepsilon) &\leq \rho^\varepsilon(0, x) \leq (\rho^\varepsilon)^*(0, x) \leq u_0(x) + m(\varepsilon) + \varepsilon, \\ u_0(x) - m(\varepsilon) &\leq \sigma^\varepsilon(0, x) \leq (\sigma^\varepsilon)^*(0, x) \leq u_0(x) + m(\varepsilon) + \varepsilon, \\ w_0(x) - m(\varepsilon) &\leq \tau^\varepsilon(0, x) \leq (\tau^\varepsilon)^*(0, x) \leq w_0(x) + m(\varepsilon) + \varepsilon, \end{aligned}$$

3 with $m(\varepsilon) = \max(f(\varepsilon), g(\varepsilon), h(\varepsilon))$. Using the comparison principle provided in Proposition 4.1,
4 and since by Theorem 3.3, $(\rho^\varepsilon, \sigma^\varepsilon, \tau^\varepsilon)$ is a viscosity solution of (3.16), we get that

$$\begin{aligned} u^\varepsilon(t, x) - m(\varepsilon) &\leq \rho^\varepsilon(t, x) \leq (\rho^\varepsilon)^*(t, x) \leq u^\varepsilon(t, x) + m(\varepsilon) + \varepsilon, \\ v^\varepsilon(t, x) - m(\varepsilon) &\leq \sigma^\varepsilon(t, x) \leq (\sigma^\varepsilon)^*(t, x) \leq v^\varepsilon(t, x) + m(\varepsilon) + \varepsilon, \\ w^\varepsilon(t, x) - m(\varepsilon) &\leq \tau^\varepsilon(t, x) \leq (\tau^\varepsilon)^*(t, x) \leq w^\varepsilon(t, x) + m(\varepsilon) + \varepsilon, \end{aligned}$$

5 with $(u^\varepsilon, v^\varepsilon, w^\varepsilon)$ the solution of (6.1). Taking ε to zero and using Theorem 6.1, we obtain the
6 desired result.

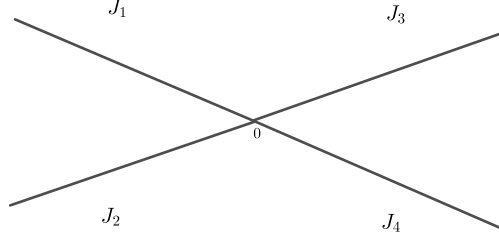
7

□

8 A Appendix: the junction case.

9 We will now show how to extend our result to the case of a general junction. We consider a junction
10 with two incoming roads and two outgoing roads. The junction is defined by $J = J_1 \cup J_2 \cup J_3 \cup J_4$
11 where J_1 and J_2 are the incoming roads and isometric to \mathbb{R}^- and J_3 and J_4 are the outgoing roads
12 and isometric to \mathbb{R}^+ . The branches of the junction are glued at the point 0. The point 0 is called
13 junction point and for all $k = 1, 2, 3, 4$, we define $J_k^* = J_k \setminus \{0\}$. The definition of the junction is
14 sketched in the figure below.

15 We will consider a microscopic model with the same periodic distribution before the junction
16 point (i.e, two cars enter from J_1 and then one car from J_2). Concerning the distribution after
17 the junction point, we will assume that cars with odd labels goes to the branch J_3 and others
18 (even) goes to the branch J_4 . Obviously, more general (periodic) distributions and junctions can



1 be considered. We recall the definition of the sets $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 , (see (2.1)). Since we have two
 2 different road choice depending on the label (even or odd), we should divide these sets as follows,

$$\begin{cases} \mathcal{L}_1^o = \{-5 - 6k, k \in \mathbb{N}\} & \text{and } \mathcal{L}_1^e = \{-2 - 6k, k \in \mathbb{N}\}, \\ \mathcal{L}_2^o = \{-1 - 6k, k \in \mathbb{N}\} & \text{and } \mathcal{L}_2^e = \{-4 - 6k, k \in \mathbb{N}\}, \\ \mathcal{L}_3^o = \{-3 - 6k, k \in \mathbb{N}\} & \text{and } \mathcal{L}_3^e = \{-6 - 6k, k \in \mathbb{N}\}, \\ \mathbb{N}^o = \{2k + 1, k \in \mathbb{N}\} & \text{and } \mathbb{N}^e = \{2k, k \in \mathbb{N}\}. \end{cases}$$

3 We will consider the following model on the junction,

$$\begin{cases} U_i'(t) = \bar{\phi}(V_1(U_{i+1}(t) - U_i(t)), V_3(|U_{i+2}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_1^o \cup \mathbb{N}^o, \\ U_i'(t) = \bar{\phi}(V_1(U_{i+1}(t) - U_i(t)), V_4(|U_{i+2}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_1^e \cup \mathbb{N}^e, \\ U_i'(t) = F(U_i(t), U_{i+1}(t), V_1(U_{i+2}(t) - U_i(t)), V_3(U_{i+2}(t) - U_i(t))) & \text{if } i \in \mathcal{L}_2^o, \\ U_i'(t) = F(U_i(t), U_{i+1}(t), V_1(U_{i+2}(t) - U_i(t)), V_4(U_{i+2}(t) - U_i(t))) & \text{if } i \in \mathcal{L}_2^e, \\ U_i'(t) = F(U_i(t), U_{i+1}(t), V_1(U_{i+3}(t) - U_i(t)), V_3(|U_{i+2}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_3^o, \\ U_i'(t) = F(U_i(t), U_{i+1}(t), V_1(U_{i+3}(t) - U_i(t)), V_4(|U_{i+2}(t)| - U_i(t))) & \text{if } i \in \mathcal{L}_3^e, \end{cases} \quad (\text{A.1})$$

4 with

$$F(x, y, a, b) = \bar{\phi}(x, a, b) \cdot \omega(x, y, |x| - |y|).$$

5 The transition function $\bar{\phi}(x, a, b)$ is defined as in (2.4) and is equal to b on J_4 . The functions
 6 $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined in the same way as in (2.5). For this model, we should define six cu-
 7 mulative distribution functions ρ^o (resp. $\rho^e, \sigma^o, \sigma^e, \tau^o$ and τ^e) such that $\rho^o(t, U_i(t)) = -i$ (resp.
 8 $\rho^e(t, U_i(t)) = -i, \sigma^o(t, U_i(t)) = -i, \sigma^e(t, U_i(t)) = -i, \tau^o(t, U_i(t)) = -i, \tau^e(t, U_i(t)) = -i$ for
 9 $i \in \mathcal{L}_1^o \cup \mathbb{N}^o$ (resp. $\mathcal{L}_1^e \cup \mathbb{N}^e, \mathcal{L}_2^o \cup \mathbb{N}^o, \mathcal{L}_2^e \cup \mathbb{N}^e, \mathcal{L}_3^o \cup \mathbb{N}^o$ and $\mathcal{L}_3^e \cup \mathbb{N}^e$).

10 We then inject these functions into the following non-local PDE, for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$,

$$\begin{cases} \rho_t^o(t, x) + R_{1,2}^o(x, \rho^o(t, x), [\sigma^e(t, \cdot)], [\tau^o(t, \cdot)])(x) |\rho_x^o(t, x)| = 0, \\ \rho_t^e(t, x) + R_{1,2}^e(x, \rho^e(t, x), [\sigma^o(t, \cdot)], [\tau^e(t, \cdot)])(x) |\rho_x^e(t, x)| = 0, \\ \sigma_t^o(t, x) + R_{1,1}^o(x, \sigma^o(t, x), [\rho^o(t, \cdot)], [\tau^e(t, \cdot)])(x) |\sigma_x^o(t, x)| = 0, \\ \sigma_t^e(t, x) + R_{1,1}^e(x, \sigma^e(t, x), [\rho^e(t, \cdot)], [\tau^o(t, \cdot)])(x) |\sigma_x^e(t, x)| = 0, \\ \tau_t^o(t, x) + R_{2,1}^o(x, \tau^o(t, x), [\tau^e(t, \cdot)], [\sigma^o(t, \cdot)], [\rho^e(t, \cdot)])(x) |\tau_x^o(t, x)| = 0, \\ \tau_t^e(t, x) + R_{2,1}^e(x, \tau^e(t, x), [\tau^o(t, \cdot)], [\sigma^e(t, \cdot)], [\rho^o(t, \cdot)])(x) |\tau_x^e(t, x)| = 0. \end{cases} \quad (\text{A.2})$$

11 The non-local operators are defined by,

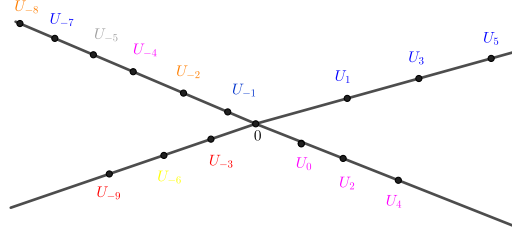


Figure 4: In blue (resp. orange, pink, grey, red and yellow), vehicles with label in $\mathcal{L}_2^o \cup \mathbb{N}^o$ (resp. \mathcal{L}_1^e , $\mathcal{L}_2^e \cup \mathbb{N}^e$, \mathcal{L}_1^o , \mathcal{L}_3^o and \mathcal{L}_3^e).

$$\begin{cases} R_{1,2}^o(x, U, [V], [W])(x) &= \phi(x, M_{1,1}(U, [V])(x), N_3(U, [W])(x)), \\ R_{1,2}^e(x, U, [V], [W])(x) &= \phi(x, M_{1,1}(U, [V])(x), N_4(U, [W])(x)), \\ R_{1,1}^o(x, V, [U], [W])(x) &= \phi(x, M_{1,2}(V, [U])(x), M_{3,2}(V, [U])(x)) \cdot K(V, [W])(x), \\ R_{1,1}^e(x, V, [U], [W])(x) &= \phi(x, M_{1,2}(V, [U])(x), M_{4,2}(V, [U])(x)) \cdot K(V, [W])(x), \\ R_{2,1}^o(x, Z, [U], [V], [W])(x) &= \phi(x, M_{2,3}(Z, [U])(x), N_3(Z, [V])(x)) \cdot K(Z, [W])(x), \\ R_{2,1}^e(x, Z, [U], [V], [W])(x) &= \phi(x, M_{2,3}(Z, [U])(x), N_4(Z, [V])(x)) \cdot K(Z, [W])(x), \end{cases}$$

1 where $\phi(x, a, b) = -\bar{\phi}(x, -a, -b)$ and,

$$M_{i,a}(U, [V])(x) = \int_0^{+\infty} V_i'(z) E_a(V(x+z) - U(x)) dz - V_{max} \quad \text{for } i = 1, 2, 3, 4,$$

2

$$N_i(U, [V])(x) = \int_{z \geq x} V_i'(|z| - x) E_{-2}(V(z) - U(x)) dz - V_{max} \quad \text{for } i = 3, 4,$$

3 and

$$K(U, [V])(x) = \int_{\mathbb{R}} \omega_z(x, z, |x| - |z|) F(V(z) - U(x)) dz + 1,$$

4 with E_a and F defined in (3.1) and (3.2). We can prove the following theorem.

Theorem A.1 (Junction condition by homogenization: application to traffic flow). *Assume (A1)-(A2)-(A3) for V_i with $i = 1, 2, 3, 4$ and that at the initial time, we have,*

$$\begin{cases} U_i(0) \leq U_{i+1}(0) - h_0 & \text{if } i \in \mathcal{L}_1^o \cup \mathcal{L}_1^e, \\ U_i(0) \leq U_{i+2}(0) - h_0 & \text{if } i \in \mathcal{L}_2^o \cup \mathcal{L}_2^e \cup \mathbb{N}, \\ U_i(0) \leq U_{i+3}(0) - h_0 & \text{if } i \in \mathcal{L}_3^o \cup \mathcal{L}_3^e. \end{cases}$$

We also assume that there exists $R > 0$ such that

$$\begin{cases} U_{i+1}(0) - U_i(0) = h_1 & \text{if } i \in \mathcal{L}_1^o \cup \mathcal{L}_1^e \text{ and } U_i(0) < -R, \\ U_{i+2}(0) - U_i(0) = h_1 & \text{if } i \in \mathcal{L}_2^o \cup \mathcal{L}_2^e \text{ and } U_i(0) < -R, \\ U_{i+3}(0) - U_i(0) = h_2 & \text{if } i \in \mathcal{L}_3^o \cup \mathcal{L}_3^e \text{ and } U_i(0) < -R, \\ U_{i+2}(0) - U_i(0) = h_3 & \text{if } i \in \mathbb{N}^o \text{ and } U_i(0) \geq R, \\ U_{i+2}(0) - U_i(0) = h_4 & \text{if } i \in \mathbb{N}^e \text{ and } U_i(0) \geq R \end{cases}$$

1 with $h_1, h_2, h_3, h_4 \geq h_0$. We define four functions u_0^o, u_0^e, w_0^o and w_0^e by

$$\begin{aligned} u_0^o(x) &= -\frac{3}{2h_1}x1_{\{x<0\}} - \frac{2}{h_3}x1_{\{x\geq 0\}}, \\ u_0^e(x) &= -\frac{3}{2h_1}x1_{\{x<0\}} - \frac{2}{h_4}x1_{\{x\geq 0\}}, \\ w_0^o(x) &= -\frac{3}{h_2}x1_{\{x<0\}} - \frac{2}{h_3}x1_{\{x\geq 0\}}, \\ w_0^e(x) &= -\frac{3}{h_2}x1_{\{x<0\}} - \frac{2}{h_4}x1_{\{x\geq 0\}}. \end{aligned}$$

2 Let $\varepsilon > 0$ and $\chi^\varepsilon : \mathbb{R}^+ \times J \rightarrow \mathbb{R}$ be the function defined by

$$\chi^\varepsilon(t, x) = \begin{cases} (\rho^o)^\varepsilon(t, -d(0, x)) & \text{if } x \in J_1^*, \\ (\tau^e)^\varepsilon(t, -d(0, x)) & \text{if } x \in J_2^*, \\ (\rho^o)^\varepsilon(t, d(0, x)) & \text{if } x \in J_3, \\ (\sigma^e)^\varepsilon(t, d(0, x)) & \text{if } x \in J_4, \end{cases}$$

3 with

$$(\rho^o)^\varepsilon(t, x) = \varepsilon \rho^o\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad (\tau^e)^\varepsilon(t, x) = \varepsilon \tau^e\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \quad (\sigma^e)^\varepsilon(t, x) = \varepsilon \sigma^o\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

4 We define

$$\bar{u}_0(x) = \begin{cases} u_0^o(-d(0, x)) & \text{if } x \in J_1^*, \\ w_0^e(-d(0, x)) & \text{if } x \in J_2^*, \\ u_0^o(d(0, x)) & \text{if } x \in J_3^*, \\ w_0^e(d(0, x)) & \text{if } x \in J_4. \end{cases}$$

5 There exists $\bar{A} \in [H_0, 0]$ such that the function χ^ε converges towards the unique solution u^0 of

$$\begin{cases} u_t^0 + \bar{H}_k(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times J_k^*, \\ u_t^0 + F_A^-(u_x^0) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\}, \\ u^0(0, x) = \bar{u}_0(x) & \text{for } x \in J, \end{cases}$$

6 where \bar{H}_1 and \bar{H}_2 are the same Hamiltonians defined in (2.10) and (2.11). For $k = 3, 4$, the
7 Hamiltonian \bar{H}_k is given by

$$\bar{H}_k(p) = \begin{cases} -p - 2k_0 & \text{for } p < -2k_0, \\ -V_k\left(\frac{-2}{p}\right)|p| & \text{for } -2k_0 \leq p \leq 0, \\ p & \text{for } p > 0, \end{cases}$$

8 and F_A^- is defined by

$$F_A^-(p_1, p_2, p_3, p_4) = \max\left(\bar{A}, \bar{H}_1^+(p_1), \bar{H}_2^+(p_2), \bar{H}_3^-(p_3), \bar{H}_4^-(p_4)\right).$$

9 **Remark A.2.** The above macroscopic model differs from the one in (2.16) after the point zero.
10 The difference comes from the fact that after the junction point, the leader is now U_{i+2} and we
11 have two different optimal velocities on branches J_3 and J_4 .

1 The path to obtain our convergence result is the same as in the merge case: we consider the
 2 non-local PDE with suitable initial conditions and prove the convergence of its solution. The
 3 control of oscillations for this PDE can be done as in Theorem 4.5 using a localization procedure.
 4 For the reader convenience, and since we proved a similar result before $-R^1$, we will prove one of
 5 these oscillations after zero.

6 **Theorem A.3** (Control of the oscillations). *Let $T > 0$ and $(u^o, u^e, v^o, v^e, w^o, w^e)$ be the solution*
 7 *of (A.2) on $(0, T)$. For all $t > 0$, the functions $u^o(t, \cdot), u^e(t, \cdot), v^o(t, \cdot), v^e(t, \cdot), w^o(t, \cdot)$ and $w^e(t, \cdot)$*
 8 *are non-increasing. Let $x < -R^1$ and $h_0 > h > 0$ small enough such that $x + h < -R^1$. For $t > 0$,*
 9 *we consider the following oscillations:*

$$\begin{cases} -\frac{3k_0}{2}h - 1 \leq v^e(t, x + h) - u^o(t, x), \\ -3k_0h - 3 \leq v^e(t, x + h) - w^e(t, x). \end{cases} \quad (\text{A.3})$$

10 For $x > 0$ and $h_0 > h > 0$, we consider for $t > 0$,

$$\begin{cases} -2k_0h - 2 \leq w^o(t, x + h) - u^o(t, x), \\ -2k_0h - 2 \leq w^e(t, x + h) - u^e(t, x). \end{cases} \quad (\text{A.4})$$

11 Assume (A.3) and (A.4) for $t = 0$. Then, they hold for $t > 0$.

Proof. We will only do the proof of the first inequality in (A.4). Let $x_0 > 0$. Take $\delta > 0$ small
 enough such that $x_0 > \delta$. We will prove for all $t > 0$, $y \in B_\delta(x_0)$ and $x \in B_\delta(x_0 + h)$ that

$$-2k_0(x - y) - 2 - L_\delta(y - x_0)^2 - L_\delta(x - x_0 - h)^2 \leq w^o(t, x) - u^o(t, y) \quad (\text{A.5})$$

with $L_\delta = \frac{KT}{\delta^2}$ and K the constant is constant obtained by the barriers construction. In particular,
 taking $y = x_0$ and $x = x_0 + h$, we obtain the desired result. To prove (A.5), we introduce

$$\Delta = [0, T) \times B_\delta(x_0 + h) \times B_\delta(x_0)$$

12 and consider the following supremum

$$M = \sup_{(t, x, y) \in \Delta} \{u^o(t, y) - w^o(t, x) - 2k_0(x - y) - 2 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2\}.$$

13 We want to prove that $M \leq 0$. We argue by contradiction and assume that $M > 0$.

14 **Step 1: the test function.** For $\eta > 0$ small, we define

$$\varphi(t, x, y) = u^o(t, y) - w^o(t, x) - 2k_0(x - y) - 2 - L_\delta(x - x_0 - h)^2 - L_\delta(y - x_0)^2 - \frac{\eta}{T - t}.$$

15 Since φ is continuous, it reaches a maximum on Δ at a point that we denote by $(\bar{t}, \bar{x}, \bar{y})$. Classically
 16 we have for η small enough,

$$0 < \frac{M}{2} \leq \varphi(\bar{t}, \bar{x}, \bar{y}).$$

17 As in the proof of Theorem 4.5, we prove that $\bar{t} \neq 0$ and $|\bar{x} - x_0 - h| \neq \delta$ and $|\bar{y} - x_0| \neq \delta$.

1 **Step 2: utilization of the equation.** By doing a duplication of the time variable and passing
 2 to the limit in this duplication parameter, we get that

$$\frac{\eta}{(T - \bar{t})^2} \leq -N_3 (u^\circ(\bar{t}, \bar{y}), [w^\circ(\bar{t}, \cdot)]) (\bar{y}) \cdot |-2k_0 + 2L_\delta(\bar{y} - x_0)| \quad (\text{A.6})$$

3 where we used the fact that $\tilde{R}_{2,1}^o \leq 0$ and $\phi(x, a, b) = b$ for $x > 0$. We claim that

$$N_3 (u^\circ(\bar{t}, \bar{y}), [w^\circ(\bar{t}, \cdot)]) (\bar{y}) = 0. \quad (\text{A.7})$$

4 In fact, if (A.7) is true, we will obtain a contradiction in (A.6) since $\tilde{N}_3 \leq 0$. We consider only
 5 $z \geq \bar{y} + h_0$ because, we have $V_3'(h) = 0$ for $h \leq h_0$. We have

$$N_3 (u^\circ(\bar{t}, \bar{y}), [w^\circ(\bar{t}, \cdot)]) (\bar{y}) = \int_{h_0 + \bar{y}}^{+\infty} V_3'(z - \bar{y}) E_{-2}(w^\circ(\bar{t}, z) - u^\circ(\bar{t}, \bar{y})) dz - V_{max}.$$

Using that $\varphi(\bar{t}, \bar{x}, \bar{y}) > 0$, we have that

$$w^\circ(\bar{t}, \bar{x}) - u^\circ(\bar{t}, \bar{y}) < -2.$$

If $z > \bar{y} + h_0$ and $h > 0$ we have

$$z > x_0 - \delta + h_0 > x_0 + \delta + h > \bar{x}.$$

6 Since $w^\circ(t, \cdot)$ is non-increasing,

$$w^\circ(\bar{t}, z) - u^\circ(\bar{t}, \bar{y}) \leq w^\circ(\bar{t}, \bar{x}) - u^\circ(\bar{t}, \bar{y}) < -2$$

7 which implies $E_{-2}(w^\circ(\bar{t}, \bar{y} + z) - u^\circ(\bar{t}, \bar{y})) = 1$ and then (A.7). This ends the proof.

8

□

9 As in the Theorem above, all others results use the same ideas of the merge model, the dif-
 10 ferences can be resumed in two points: the number of equations is doubled (for example when
 11 constructing correctors near the junction point, we should consider six equations). The second
 12 point is the definition of Hamiltonians after the junction point.

13 **Remark A.4** (General periodic case). *We can consider a microscopic model with a very general*
 14 *periodic distribution before and after the junction point and our micro-macro connection still*
 15 *works. The only difference that we should consider more "cumulative distribution functions".*

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