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Lucie Baudouin, Pamela Godoy, Alberto Mercado. Carleman estimates for the wave equation in heterogeneous media with non-convex interface. *Journal of Differential Equations*, 2022, 311, pp.1-28. 10.1016/j.jde.2021.12.001 . hal-03211176

HAL Id: hal-03211176

<https://hal.science/hal-03211176v1>

Submitted on 28 Apr 2021

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Carleman estimates for the wave equation in heterogeneous media with non-convex interface.

Lucie Baudouin* Pamela Godoy† Alberto Mercado‡

April 28, 2021

Abstract

A wave equation whose main coefficient is discontinuous models the evolution of waves amplitude in a media composed of at least two different materials, in which the propagation speed is different. In our mathematical setting, the spatial domain where the partial differential equation evolves is an open bounded subset of \mathbb{R}^2 and the wave speed is assumed to be constant in each one of two sub-domains, separated by a smooth and possibly non-convex interface. This article is concerned with the construction of Carleman weights for this wave operator, allowing generalizations of previous results to the case of an interface that is not necessarily the boundary of a convex set. Indeed, using the orthogonal projection onto this interface, we define convex functions satisfying the transmission conditions imposed by the equation, such that, under usual hypothesis on the sign of the jump of the wave speed, can be used as Carleman weights.

Keywords: Wave equation. Carleman estimates. Transmission system.

1 Introduction

In the last two decades, controllability and inverse problems for partial differential equations have been extensively studied using Carleman estimates. The books [5], [13] and [21] allow an overview of this tool and its application in those fields. More recently, equations in heterogeneous media have gained a lot of attention in the literature. In particular, we mention the works [19], [18], [9] concerning elliptic equations, and [10], [20], [17], [12], dealing with parabolic equations.

In the present work, we are interested in systems of transmission wave equations. More precisely, let us consider here the system given by

$$\begin{cases} \partial_t^2 u(x, t) - \operatorname{div}(a(x)\nabla u(x, t)) = 0, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u^0(x), \quad \partial_t u(x, 0) = v^0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

*LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France. E-mail: lbaudouin@laas.fr.

†Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaiso, Chile. E-mail: pamela.godoy@alumnos.usm.cl. Partially supported by PIIC - DGIIP UTFSM.

‡Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaiso, Chile. E-mail: alberto.mercado@usm.cl. Partially supported by the ANR Labex CIMI (grant ANR-11-LABX-0040) within the French State Programme “Investissements d’Avenir” and by Fondecyt grant 1171712.

where $T > 0$, $\Omega \subset \mathbb{R}^2$ is an open set with a Lipschitz boundary $\partial\Omega$, and the coefficient a is piecewise constant and given by

$$a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2. \end{cases} \quad (1.2)$$

Here, a_1, a_2 are positive constants and Ω_1, Ω_2 define a partition of Ω , in the sense that they are non-empty open sets with Lipschitz boundaries and such that

$$\Omega_1 \cap \Omega_2 = \emptyset \quad \text{and} \quad \Omega = \text{Int}(\overline{\Omega_1 \cup \Omega_2}). \quad (1.3)$$

The interface between Ω_1 and Ω_2 is denoted Γ_* and given by

$$\Gamma_* = \overline{\Omega_1} \cap \overline{\Omega_2} = \partial\Omega_1 \cap \partial\Omega_2. \quad (1.4)$$

Remark 1.1. *A particular example of a partition is given by the case of embedded domains: $\overline{\Omega_1} \subset \Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. In this case we have $\Gamma_* = \partial\Omega_1$.*

The objective of this work is to prove a Carleman inequality for equation (1.1) under suitable geometric hypothesis.

1.1 Context of the problem

In the seminal book [22], one of the setting that is studied concerning the wave equation is the case of embedded domains and a main coefficient $a(x)$ of the wave operator which is constant in each one of the subdomains. Using the multiplier method, the author proves the controllability of the system, assuming that the interior domain is star-shaped and the value of a at the exterior domain is smaller than the interior one. Moreover, references [6], [7], [23] even proved that if the monotonicity of the jump of the coefficient across the interface is inverted, then there exist solutions whose energy is arbitrarily concentrated in a neighborhood of the interface, allowing to deduce a lack of controllability. But on the other hand, it seems possible to aim at dealing with less restrictions on the shape of the interface (which is the boundary of the interior domain here).

Up to our knowledge, [4] is the first work where a global Carleman inequality for a wave equation with a jump at an interface is obtained. In this article, the case of embedded domains is studied, under hypotheses of monotonicity of velocities and strict convexity of the inner domain. The Carleman inequalities obtained are the main tool for the study of the Lipschitz stability of an inverse problem, the one of recovering the potential (a zero-order coefficient term) in the equation by means of an observation given by the trace of the normal derivative of the solution on some subset of the external boundary. Those Carleman estimates were also used in [24] in order to obtain Hölder stability of the related main-coefficient inverse problem. An analogous topic for the Schrödinger equation is investigated in [3], where the construction of the weight function is generalized for convex inner domains in \mathbb{R}^n .

We also want to mention the recent work [14], where the case of an embedded strictly convex domain is considered. In that article, micro-local analysis techniques are used to obtain sufficient conditions for the observation zone, given by a proper subset of the boundary, in order to get observability.

The main objective of the present work is to generalize the Carleman estimate obtained in [4], taking into account the gap between the inner domain being ‘strictly convex’ and ‘star-shaped’. Actually, we will prove a Carleman estimate for a setting where the interface of discontinuity of the main coefficient is not necessarily the boundary of a convex domain, but rather the boundary of a domain that can be seen as a perturbation of a convex set. The main step is the construction of the weight function, performed using the properties of the orthogonal projection. From there, using standard techniques, one can deduce properties of controllability and stability for coefficient inverse problems (see [2] for example). Nevertheless, we expect that the method of construction of the Carleman weight performed here could be useful in other studies of wave propagation in discontinuous media.

1.2 Notations and main results

Throughout the paper we will denote $Q = \Omega \times (-T, T)$, $\Sigma = \partial\Omega \times (-T, T)$ and $\Sigma_* = \Gamma_* \times (-T, T)$. Also, for each $j = 1, 2$, we denote by n_j the unitary exterior normal vector of Ω_j , we set $Q_j = \Omega_j \times (-T, T)$, $u_j = u\mathbf{1}_{Q_j}$ for $u \in L^2(\Omega)$ and we define the normal derivative by $\frac{\partial u_j}{\partial n_j} = \nabla u_j \cdot n_j$.

Taking into account (1.3), (1.4) and the fact that the boundaries of Ω_1 and Ω_2 are Lipschitz, we directly have that equation (1.1) is equivalent to the system

$$\begin{cases} \partial_t^2 u_1 - a_1 \Delta u_1 = 0, & \text{in } Q_1, \\ \partial_t^2 u_2 - a_2 \Delta u_2 = 0, & \text{in } Q_2, \\ u = 0, & \text{on } \Sigma, \\ u(\cdot, 0) = u^0, \quad \partial_t u(\cdot, 0) = u^1, & \text{in } \Omega, \end{cases}$$

coupled with the transmission conditions

$$\begin{cases} u_1 = u_2, & \text{on } \Sigma_*, \\ a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} = 0, & \text{on } \Sigma_*. \end{cases} \quad (1.5)$$

This work relies strongly on the construction of a function $\mu = \mu(x)$ satisfying the transmission conditions (1.5) and such that it can be used as the spatial part of a Carleman weight function (see Section 2 below). Assuming that such a function μ is given, in a quite usual way we define

$$\phi(x, t) = \mu(x) - \beta t^2 + M, \quad (x, t) \in \Omega \times \mathbb{R}, \quad (1.6)$$

and, for any $\lambda > 0$, we set

$$\varphi(x, t) = e^{\lambda \phi(x, t)}, \quad (x, t) \in \Omega \times \mathbb{R}. \quad (1.7)$$

Remark 1.2. *One may already know that a Carleman estimate can be described as a weighted energy estimate. It is, as already mentioned, a specific tool accurately built to prove the stability of an inverse problem or an observability estimate for a controllability goal, situations that are both defined for a given observation term. We would like to highlight the fact that the domain of this observation is intrinsically linked with the weight function used to design the Carleman estimate. Therefore one should bear in mind that our approach aims at building weight functions that makes interface terms vanishing in the process of the proof, allowing the observations to be performed only at the exterior boundary of the domain.*

The following Carleman estimates for equation (1.1) are the main result of this work.

Theorem 1.3. *Let Ω be a bounded domain with a partition Ω_1, Ω_2 and a C^3 interface Γ_* . Suppose that there exists a function $\rho \in C^3(\overline{\Omega})$ satisfying*

$$\nabla \rho \neq 0 \quad \text{in} \quad \overline{\Omega}, \quad (1.8)$$

$$\rho \text{ is strictly convex in } \overline{\Omega}, \quad (1.9)$$

and

$$|\theta(\nabla \rho(x), n_1(x))| < \pi/4, \quad \forall x \in \Gamma_*, \quad (1.10)$$

where $\theta(\nabla \rho(x), n_1(x))$ denotes the angle between $\nabla \rho(x)$ and $n_1(x)$.

Then there exists $\varepsilon > 0$ such that, for every pair a_1, a_2 satisfying

$$1 \leq \frac{a_1}{a_2} < 1 + \varepsilon, \quad (1.11)$$

there exist $C, s_0, \lambda_0 > 0$ and a convex function $\mu = \mu(x)$, such that, for φ defined as in (1.7), we have

$$\begin{aligned} s\lambda \iint_Q (|u_t|^2 + a|\nabla u|^2) \varphi e^{2s\varphi} dxdt + s^3 \lambda^3 \iint_Q |u|^2 \varphi^3 e^{2s\varphi} dxdt \\ \leq C \iint_Q |\partial_t^2 u - \operatorname{div}(a(x)\nabla u)|^2 e^{2s\varphi} dxdt + s\lambda C \iint_{\Sigma^+} a^2 \left| \frac{\partial u}{\partial n} \right|^2 \varphi e^{2s\varphi} d\sigma dt \end{aligned} \quad (1.12)$$

for any $u \in X, \lambda \geq \lambda_0, s \geq s_0$, where $\Sigma^+ := \left\{ x \in \partial\Omega \mid \frac{\partial \mu}{\partial n}(x) > 0 \right\} \times (-T, T)$ and

$$X = \{ u \in H_0^1(Q) \mid u_j \in H^2(Q_j) \text{ for } j = 1, 2, u \text{ satisfies (1.5), } \partial_t u(\pm T, \cdot) = 0 \}.$$

Remark 1.4. *The function μ will be constructed modifying ρ in order to impose the transmission conditions (1.5) (see (2.3) and (2.4)). Hypothesis (1.10) is a generalization of the particular case of Γ_* being part of a curve $\{x \in \mathbb{R}^2 : \rho(x) = c\}$ which is the boundary of a strictly convex domain including Ω_1 , that is precisely the hypothesis considered in [4]. Indeed, in that case we would have n_1 collinear with $\nabla \rho$ on Γ_* and then (1.10) is trivially satisfied. Hypothesis (1.10) can be seen as the requirement of Γ_* being a perturbation of the boundary of a convex set. See Figure 1.*

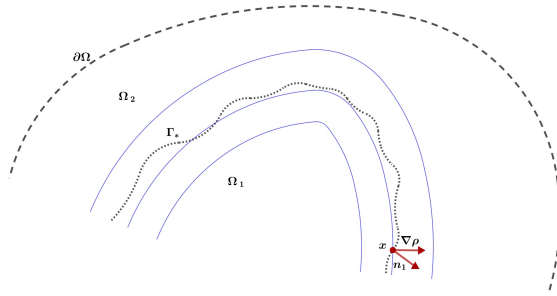


Figure 1: The thin continuous lines represent the level curves of the convex function ρ . Hypothesis (1.10) means that the unitary normal vectors of the interface Γ_* are close to the corresponding normal vectors of those curves.

In order to conclude this introductory section, we highlight that the contribution of this article is a Carleman estimate for the wave equation that has the noteworthy quality of holding under extended assumptions where the main coefficient is constant by pieces across a possibly non-convex interface. Up to our knowledge, this result is completely new, and paves the way for the usual applications of Carleman estimates in controllability results (e.g. [2]) and stability issues for inverse problems (e.g. [4], [5]).

The rest of the paper is organized as follows: In Section 2 we perform the geometric construction of the spatial part of the weight function, modifying a function which is smooth through the boundary, in order to satisfy the transmission conditions. Section 3 is devoted to present some examples of weight functions allowing estimates for domains with non-convex or flat interfaces. Finally, we prove the Carleman estimate in Section 4.

2 Construction of weight functions

We will denote, here and throughout the paper, the ratio between the two wave speeds as

$$\xi = \frac{a_1}{a_2}. \quad (2.1)$$

The key of the construction of an appropriate weight functional for the Carleman estimate is the careful choice of its spatial dependence. Let us assume that a function $\rho \in C^2(\overline{\Omega})$ satisfies (1.8) and (1.9). Those are the main properties needed for ρ in order to be the spatial part of a Carleman weight for the wave equation (1.1) with $a = 1$, and they are related to the usual pseudoconvexity condition (see Section 4.2 of [5]). A typical example is given by $\rho(x) = |x - x_0|^2$ for some $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ (see [16], [1]), but other choices of convex functions are suitable.

The key point of this constructive argument, in order to deal with the jump of the main coefficient of the equation, is the following. We will modify the function ρ in order to fulfill the

transmission conditions (1.5). Indeed, we leave ρ unchanged in Ω_1 , and we redefine it in the subdomain Ω_2 in such a way that the obtained function satisfies (1.8), (1.9) and (1.5). We will denote by μ the function constructed this way.

Finally, the function defined by (1.6)-(1.7) will be used as a Carleman weight in the entire domain Ω for equation (1.1).

2.1 Definition of $\mu = \mu(x)$

In order to define μ , we will use the properties of the orthogonal projection P onto $\overline{\Omega}_1$, and μ will be defined using the parameter $\xi \in (1, 1 + \varepsilon)$. We recall that the projection $P(x)$ of $x \in \mathbb{R}^2$ in $\overline{\Omega}_1$ is defined by being the only point $P(x) \in \overline{\Omega}_1$ satisfying

$$|x - P(x)| = \min\{|x - y| : y \in \overline{\Omega}_1\}. \quad (2.2)$$

It is natural to consider two cases.

Case 1: Ω_1 is convex.

Then the closed set $\overline{\Omega}_1$ is also convex, and therefore P is well defined in all \mathbb{R}^2 . We set, for each $x \in \overline{\Omega}$,

$$\mu(x) = \begin{cases} \rho(x), & \text{if } x \in \overline{\Omega}_1, \\ \xi\rho(x) + (1 - \xi)\rho(P(x)), & \text{if } x \in \overline{\Omega}_2 \setminus \overline{\Omega}_1. \end{cases} \quad (2.3)$$

Case 2: General case: Ω_1 is not necessarily convex.

Even if Ω_1 is not convex, given that the interface Γ_* is regular, there exists a neighborhood U of Γ_* in \mathbb{R}^2 (called the *tubular neighborhood*, see for instance [15]) such that

$$P : U \cap \Omega_2 \longrightarrow \overline{\Omega}_1$$

is well defined. Hence, given an open set V such that $\Gamma_* \subset V \subset \overline{V} \subset U$, we take a cut-off function $\eta \in C_0^\infty(U)$ such that $\eta = 1$ in \overline{V} . We define, for each $x \in \overline{\Omega}$,

$$\mu(x) = \begin{cases} \rho(x), & \text{if } x \in \overline{\Omega}_1, \\ \xi\rho(x) + (1 - \xi)\eta(x)\rho(P(x)), & \text{if } x \in U \setminus \overline{\Omega}_1, \\ \xi\rho(x), & \text{if } x \in \overline{\Omega}_2 \setminus U. \end{cases} \quad (2.4)$$

2.2 Properties of the function μ

We will prove the properties satisfied by μ which will be used in the proof of the Carleman estimate. In this work, the non-convex case is the most interesting, so that we will consider μ defined by (2.4). However, we recall also that the definition (2.3) will be used in Subsection 3.2 in order to construct a weight function corresponding to a flat interface, which is also a case not covered by previous results in the literature.

The following proposition proves that the function μ constructed here satisfies the transmission conditions, that are local properties near the interface Γ_* .

Proposition 2.1. *If $\rho \in C^1(\overline{\Omega})$, then the function μ defined by (2.4) is well-defined, regular in each one of the sets $\overline{\Omega}_1$ and $\overline{\Omega}_2$ and it satisfies the transmission conditions (1.5) on Γ_* .*

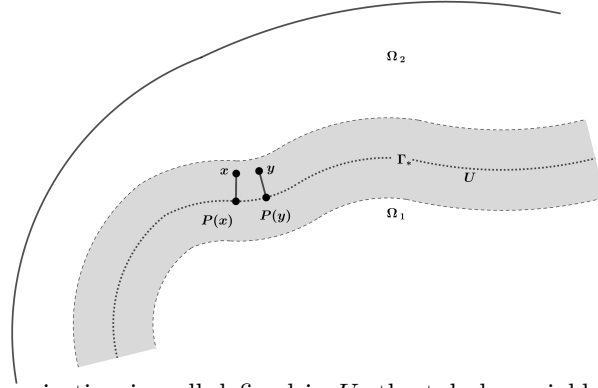


Figure 2: The orthogonal projection is well-defined in U , the tubular neighborhood (denoted in gray) of the interface Γ_* .

Proof. It is known that P is regular in the tubular neighborhood U (see [11] for projections on convex sets, [8] for the general case). For each $x \in \Gamma_* = \overline{\Omega}_1 \cap \overline{\Omega}_2 \subset U$, we have $P(x) = x$ and $\eta(x) = 1$. Hence

$$\mu_2(x) = \rho(x) = \mu_1(x).$$

On the other hand, for each $x \in \Gamma_*$, P is constant in the direction of $n(x)$, the unitary normal vector of Γ_* at $P(x)$. Then

$$\nabla(\rho(P(x))) \cdot n(x) = 0, \quad \forall x \in \Gamma_*.$$

As $\eta = 1$ in a neighborhood of Γ_* , we can conclude that

$$\frac{\partial \mu_2}{\partial n} = \nabla [\xi \rho + (1 - \xi) \eta \rho(P(\cdot))] \cdot n = \xi \nabla \rho \cdot n = \xi \frac{\partial \mu_1}{\partial n} \text{ on } \Gamma_*.$$

□

The role of the cut-off function η corresponds to the fact that the orthogonal projection in $\overline{\Omega}_1$ is not well defined in all $\mathbb{R}^2 \setminus \overline{\Omega}_1$. We will handle its effect in the estimates of the weight function by regarding the term $(1 - \xi) \eta(x) \rho(P(x))$ in (2.4) as a perturbation of $\xi \rho$. In order to do this, we will assume that $1 - \xi$ is small enough. This is showed in the next result.

Proposition 2.2. *Suppose that $\rho \in C^2(\overline{\Omega})$ satisfies (1.8) and (1.9). Then there exists $\varepsilon > 0$ such that, if $|\xi - 1| < \varepsilon$, then the function μ defined by (2.4) satisfies the following properties:*

- *There exists $\delta > 0$ such that μ is strictly positive and*

$$|\nabla \mu| \geq \delta > 0 \quad \text{in } \Omega_1 \cup \Omega_2, \quad (2.5)$$

- *There exists $c > 0$ such that the hessian matrix of μ satisfies*

$$D^2 \mu(\zeta, \zeta) \geq c |\zeta|^2, \quad \text{in } \Omega_1 \cup \Omega_2, \quad \forall \zeta \in \mathbb{R}^2 \setminus \{0\}. \quad (2.6)$$

Proof. From hypotheses (1.9) and (1.8) on ρ and the definition (2.4), properties (2.5) and (2.6) are fulfilled by $\mu = \mu_1$ in Ω_1 . On the other hand, we can see μ_2 as a perturbation of the function $\xi\rho$. Indeed, in $U \cap \Omega_2$ we have

$$\nabla\mu_2(x) = \xi\nabla\rho(x) + (1 - \xi)\nabla(\eta\rho(P(x)))$$

and

$$D^2\mu_2(x) = \xi D^2\rho(x) + (1 - \xi)D^2(\eta\rho(P(x))).$$

Then, taking into account the hypothesis on ρ , we get that $\nabla\mu_2 \neq 0$ and $D^2\mu_2$ is positive definite if $|1 - \xi|$ is small enough. \square

Remark 2.3. *The definition (2.3) comes naturally, at least locally near the interface, from the transmission conditions (1.5). Indeed, given ρ regular across the interface, and $\mu_1 = \rho$ already defined in $\bar{\Omega}_1$, it is not difficult to see that μ_1, μ_2 satisfy (1.5) if and only if*

$$\mu_2(x) = \rho(P(x)) + \xi\nabla\rho(P(x)) \cdot (x - P(x)) + g(x), \quad x \in \Omega_2, \quad (2.7)$$

where g is some regular function in $\bar{\Omega}_2$ satisfying

$$g = \frac{\partial g}{\partial n_1} = 0 \text{ on } \Gamma_*. \quad (2.8)$$

Our construction in (2.3) corresponds to (2.7) with

$$g(x) = \xi [\rho(x) - \rho(P(x)) - \nabla\rho(P(x)) \cdot (x - P(x))].$$

Using other functions g satisfying (2.8) would allow us to obtain more precise estimates. In particular, a construction where the orthogonal projection $P(x)$ is only used for x near the interface would be useful in order to avoid the use of the cut-off function η in (2.4) and then to consider more general geometries.

Remark 2.4. *In the particular case where the interface Γ_* is contained in the boundary of a strictly convex set U containing Ω_1 , then we can take ρ as the square of the Minkowski functional (also called gauge function) of U , and then $\rho(P(x)) = 1$ for any $x \in \Omega_2$. In that case μ_2 is just a multiple of ρ plus a constant. This is precisely the case of the weight function considered in [4] (see Subsection 3.1 below). The construction of a more general weight function developed here allows us to deal with two cases not covered in [4]: the case of a flat interface (for any $a_2 < a_1$, see Subsection 3.2) and also the case of non-convex interface (provided a_2 is not too far away from a_1 , see Theorem 2.4 above and Subsection 3.3).*

In the following we explicitly state some inequalities which will be very useful for estimate the traces at the interface in the proof of the Carleman inequality.

Proposition 2.5. *Hypotheses (1.8) and (1.10) imply that μ_1 , which is equal to ρ on $\bar{\Omega}_1$, satisfies*

$$\left| \frac{\partial\mu_1}{\partial\tau} \right| < \frac{\partial\mu_1}{\partial n_1} \text{ on } \Gamma_*, \quad (2.9)$$

and

$$\frac{\partial\mu_1}{\partial n_1} \geq \delta > 0 \text{ on } \Gamma_*. \quad (2.10)$$

for some $\delta > 0$, where $\tau = \tau(x)$ stands for the unitary tangent vector on Γ_* at each point $x \in \Gamma_*$ and $\frac{\partial\mu_1}{\partial\tau} := \nabla\mu_1 \cdot \tau$ is the tangential derivative.

Proof. First recall that $\mu_1 = \rho$ in Ω_1 . Directly from properties of scalar product, we have that (1.10) is equivalent to

$$\frac{\partial \mu_1}{\partial n_1} = \cos(\theta) |\nabla \mu_1| > \frac{1}{\sqrt{2}} |\nabla \mu_1| \text{ on } \Gamma_*, \quad (2.11)$$

where $\theta = \theta(\nabla \rho(x), n_1(x))$ denotes the angle between $\nabla \rho(x)$ and $n_1(x)$.

On the other hand, at each point of Γ_* , the orthogonal decomposition $\nabla \mu_1 = \frac{\partial \mu_1}{\partial \tau} \tau + \frac{\partial \mu_1}{\partial n_1} n_1$ brings, using (2.11),

$$\left| \frac{\partial \mu_1}{\partial n_1} \right|^2 + \left| \frac{\partial \mu_1}{\partial \tau} \right|^2 = |\nabla \mu_1|^2 < 2 \left| \frac{\partial \mu_1}{\partial n_1} \right|^2 \text{ on } \Gamma_*.$$

Taking into account that $\frac{\partial \mu_1}{\partial n_1}$ is non-negative, we obtain (2.9).

Finally, from (1.8), (2.11) and the regularity of ρ , we obtain (2.10). \square

Remark 2.6. *Inequality (2.9) will play a crucial role in the proof that, at the computation of the Carleman estimate for the whole domain, the sum of the resulting trace terms at the interface is non-negative (see Proposition 4.4 and also Remark 4.5). We recall that this inequality, which in fact is equivalent to (1.10), can be regarded as a generalization of the case where ρ is constant on the interface Γ_* .*

Finally, given the function μ defined in (2.4), the weight function we will use for the Carleman estimate is given, as usual, by

$$\phi(x, t) = \mu(x) - \beta t^2 + M, \quad (x, t) \in \Omega \times \mathbb{R}, \quad (2.12)$$

where $\beta > 0$ will be fixed later (see (4.8)) and $M \geq 0$ is such that $\phi \geq 0$. Finally, for each $\lambda > 0$, we set

$$\varphi = e^{\lambda \phi}. \quad (2.13)$$

3 Examples of interfaces

We give here explicit examples of weight functions allowing the general constructions stated in (2.4) and (2.3) corresponding to various media interfaces that can be covered by our approach.

3.1 Convex set

Here we consider the particular case of the interface Γ_* being part of the boundary of a strictly (or *strongly*) convex set. This case is essentially covered by previous results in the literature (see [4]), but it is included here in the sake of completeness.

We assume that there exists an open strictly convex set $C \subset \mathbb{R}^2$ such that $\Omega_1 = \Omega \cap C$, and we take $x_0 \in C$. With a translation argument we can assume that $x_0 = 0$. Then, we take p as the Minkowsky functional (also called the *gauge* function) of the strictly convex set C , which is defined for each $x \in \mathbb{R}^2$, by

$$p(x) = \inf\{t > 0 \mid x \in tC\}. \quad (3.1)$$

We recall that $tC = \{tx \mid x \in C\}$. Finally, we set

$$\rho = p^2.$$

By construction, Γ_* is contained in the level curve $\{\rho = 1\}$, and then (1.10) is trivially satisfied. Indeed, the weight function defined in (2.3) is equivalent to the function used in [4], where it is proved that ρ is smooth and properties (1.8) and (1.9) are satisfied outside a neighborhood of x_0 . Therefore, if we take $x_0 \in C \setminus \bar{\Omega}$ we can guarantee that those properties are valid in $\bar{\Omega}$.

In the case $\bar{\Omega}_1 \subset \Omega$ and $C := \Omega_1$ then we cannot take such x_0 , but instead we take two different points $x_1, x_2 \in \Omega_1$ and then we use the corresponding functions ρ_1 and ρ_2 . Therefore, each function ρ_k satisfies (1.8) and (1.9) in $\Omega \setminus B_\varepsilon(x_k)$ for $k = 1, 2$ with $\varepsilon > 0$. Taking ε small enough, we can combine the resulting estimates from both weights functions, as is performed in [4], in order to obtain an estimate in the whole domain (see Subsection 2.3.5 of [4]).

3.2 Flat interface

In the particular case where the interface Γ_* is given by a segment of a straight line, we can explicitly define a weight function. We recall that this case does not fulfill the hypothesis of [4], where the only admissible interface is the boundary of a strictly convex set.

In this example the points of the domain $\Omega \subset \mathbb{R}^2$ are denoted by an ordered pair (x, y) of real numbers. Let us assume that Γ_* is contained in the set $\{y = b\}$, meaning that Ω_1 is convex (and not strictly convex). We choose the typical function

$$\rho(x, y) = |(x, y) - (x_0, y_0)|^2,$$

so that the function μ defined by (2.3) is given by

$$\mu(x, y) = \begin{cases} (x - x_0)^2 + (y - y_0)^2, & \text{if } y < b, \\ (x - x_0)^2 + \xi(y - y_0)^2 + (1 - \xi)(b - y_0)^2, & \text{if } y > b. \end{cases} \quad (3.2)$$

It is not difficult to obtain the following result.

Proposition 3.1. *Given an open and bounded set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary, an interface Γ_* given by a straight segment and $\xi \geq 1$, there exists $(x_0, y_0) \in \mathbb{R}^2$ such that function μ defined in (3.2) satisfies hypothesis (1.10) and also the conclusions of Propositions 2.1 and 2.2, with no restrictions on the value of $\varepsilon > 0$.*

Figure 3 shows an example of a flat interface given by a straight segment, $\xi = 3$, corresponding to $\rho(x, y) = |(x, y) - (5, -3)|^2$.

3.3 Perturbation of a convex set

This is the most important example considered in this article, since it allows the case of non-convex geometries. We will use again the Minkowski functional introduced in Subsection 3.1. Suppose that Γ_* can be approximated by the boundaries of some strictly convex sets in the following sense: there exists a strictly convex domain C (without loss of generality, we can assume that C contains the origin), such that, for each $x \in \Gamma_*$, if $t > 0$ is the only positive number such that $x \in t\partial C$, then

$$\theta(n_C(x), n_1(x)) < \pi/4, \quad (3.3)$$

where $\theta(n_C(x), n_1(x))$ is the angle between the unit normal vectors exterior to $tC = \{tz : z \in C\}$ and Ω_1 at the point x . Then, we take the Minkowsky functional p defined by C (see (3.1)), and $\rho = p^2$.

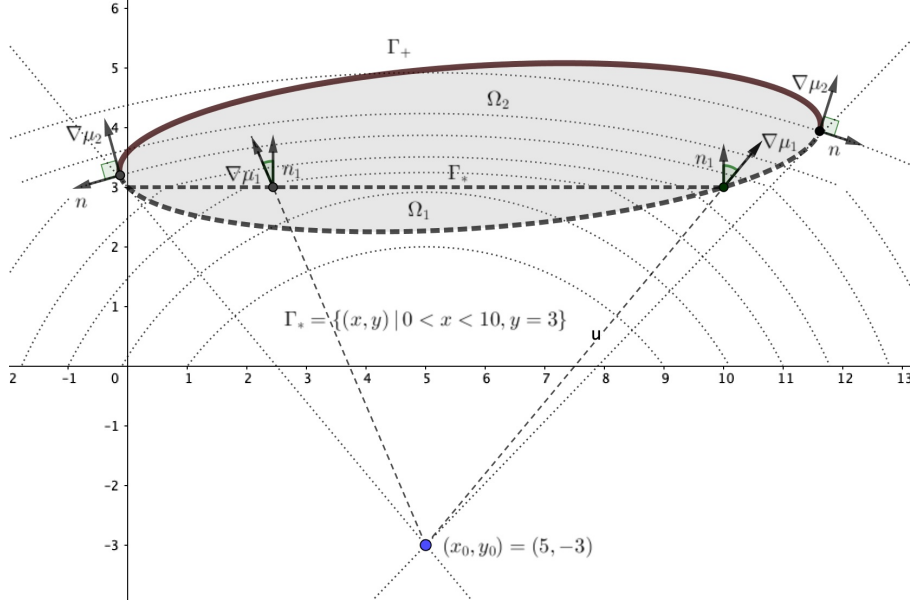


Figure 3: An example of flat interface, where $\xi = 3$. Here is used $\rho(x, y) = |(x, y) - (5, -3)|^2$. Level curves of the function μ given by (2.3) are represented by thin dotted lines. We recall that the observation zone Γ_+ (represented by the continuous line) is contained in $\partial\Omega_2$.

Since each level curve of ρ is given by the boundary of tC for some $t > 0$, we have that hypothesis (1.10) is satisfied. Similarly to the previous cases, hypotheses (1.8) and (1.9) are satisfied in $\Omega \setminus B_\varepsilon(x_0)$ for any $\varepsilon > 0$ (see Figure 4).

Also, if we cannot take $x_0 \in C \setminus \bar{\Omega}$, we can combine the weight functions defined by two points $x_1, x_2 \in \Omega$, as was explained also in the subsection 3.1 (see Figure 5).

4 Proof of the Carleman inequality

This section is devoted to the proof of Theorem 1.3 and is divided into three parts. The first one presents the usual setting and computations concerning a Carleman estimate for a wave operator in a generic domain without boundary conditions, taking into account all the traces at the boundary resulting from the integration by parts. Then, we derive the previous computations in the specific situation studied in this work. The terms at the interior and at the boundary of the whole domain are treated in a classical way, and we carefully study the resulting terms at the interface. Finally, we gather all the informations and conclude the proof.

4.1 Carleman inequality with traces at the boundary

We denote the wave operator $L_a := \partial_t^2 - a\Delta$, and we define

$$F_a(z) := |z_t|^2 - a|\nabla z|^2. \quad (4.1)$$

As usual for the proof of Carleman estimates, we perform the following changes of variables

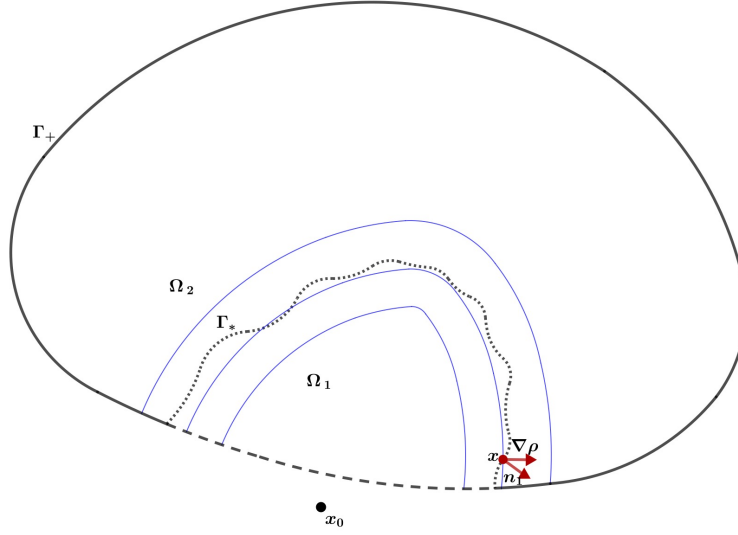


Figure 4: If the interface Γ_* is given by the dotted black line, we can take ρ as the square of the Minkowski functional of the domain bounded by the thin blue curve.

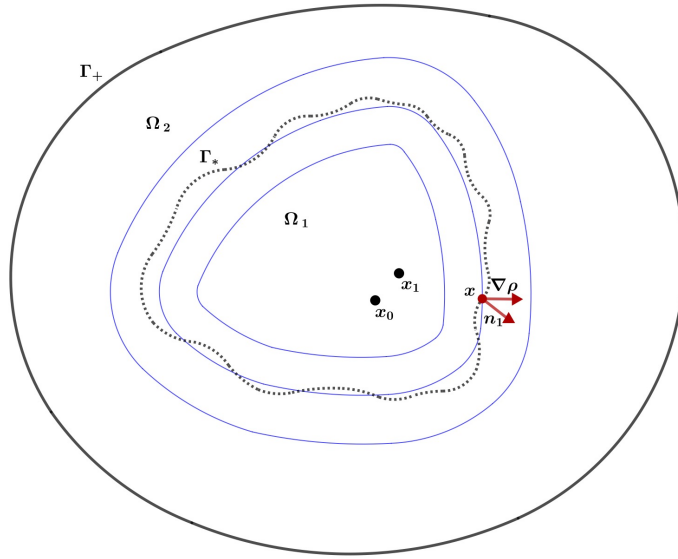


Figure 5: If the interface Γ_* is given by the dotted black line, we can take ρ as the square of the Minkowski functional of the domain bounded by the thin blue curve approximating Γ_* , centered at x_0 or x_1 . Then both weight functions can be combined in order to obtain the Carleman inequality. See Subsection 3.1

parametrized by $s > 0$ and $\lambda > 0$:

$$\varphi = e^{\lambda\phi}, \quad w = e^{s\varphi}u, \quad \mathcal{L}(w) = e^{s\varphi}L_a(e^{-s\varphi}w),$$

and we write

$$\mathcal{L}(w) = \mathcal{L}_1(w) + \mathcal{L}_2(w) + R(w), \quad (4.2)$$

where, for some $\gamma \in (0, 1)$ which we will fix later (see (4.9)),

$$\begin{aligned} \mathcal{L}_1(w) &= w_{tt} - a\Delta w + s^2\lambda^2\varphi^2 F_a(\phi)w, \\ \mathcal{L}_2(w) &= (\gamma - 1)s\lambda\varphi L_a(\phi)w - s\lambda^2\varphi F_a(\phi)w - 2s\lambda\varphi(\phi_t w_t - a\nabla\phi \cdot \nabla w), \\ R(w) &= -\gamma s\lambda\varphi L_a(\phi)w. \end{aligned}$$

We perform the following computations in a generic set $Q = U \times (-T, T)$ without assuming any boundary condition for w on the boundary $\Sigma = \partial U \times (-T, T)$, and for a constant. Later, we will apply the obtained results to each one of the subdomains Ω_1 and Ω_2 instead of U .

We set

$$\langle \mathcal{L}_1(w), \mathcal{L}_2(w) \rangle_{L^2(Q)} = \sum_{i,j=1}^3 I_{i,j},$$

where $I_{i,j}$ denotes the product of the i -th term in $\mathcal{L}_1(w)$ with the j -th term in $\mathcal{L}_2(w)$. Standard computations and integrations by parts allow to obtain (see e.g. [4]):

$$\begin{aligned} I_{1,1} &= -s\lambda(\gamma - 1) \iint_Q L_a(\phi)|w_t|^2\varphi \, dxdt + \frac{s\lambda^2(\gamma - 1)}{2} \iint_Q |w|^2(\phi_{tt} + \lambda|\phi_t|^2)L_a(\phi)\varphi \, dxdt, \\ I_{1,2} &= s\lambda^2 \iint_Q |w_t|^2 F_a(\phi)\varphi \, dxdt - s\lambda^2 \iint_Q |w|^2|\phi_{tt}|^2\varphi \, dxdt - \frac{5s\lambda^3}{2} \iint_Q |w|^2\phi_{tt}|\phi_t|^2\varphi \, dxdt \\ &\quad + \frac{s\lambda^3}{2}a \iint_Q |w|^2\phi_{tt}|\nabla\phi|^2\varphi \, dxdt - \frac{s\lambda^4}{2} \iint_Q |w|^2|\phi_t|^2 F_a(\phi)\varphi \, dxdt, \\ I_{1,3} &= s\lambda \iint_Q |w_t|^2(\phi_{tt} + \lambda|\phi_t|^2)\varphi \, dxdt - 2s\lambda^2a \iint_Q w_t\phi_t\nabla w \cdot \nabla\phi\varphi \, dxdt \\ &\quad + s\lambda a \iint_Q |w_t|^2(\Delta\phi + \lambda|\nabla\phi|^2)\varphi \, dxdt - s\lambda \iint_{\Sigma} a|w_t|^2 \frac{\partial\phi}{\partial n}\varphi \, d\sigma dt, \\ I_{2,1} &= -s\lambda(\gamma - 1) \iint_{\Sigma} aL_a(\phi)w \frac{\partial w}{\partial n}\varphi \, d\sigma dt + s\lambda(\gamma - 1)a \iint_Q |\nabla w|^2 L_a(\phi)\varphi \, dxdt \\ &\quad - s\lambda^2 \frac{\gamma - 1}{2}a \iint_Q |w|^2 L_a(\phi)(\lambda|\nabla\phi|^2 + \Delta\phi)\varphi \, dxdt - s\lambda \frac{\gamma - 1}{2}a \iint_Q |w|^2 \Delta(L_a(\phi))\varphi \, dxdt \\ &\quad - s\lambda^2(\gamma - 1)a \iint_Q |w|^2(\nabla\phi \cdot \nabla L_a(\phi))\varphi \, dxdt + s\lambda \frac{\gamma - 1}{2} \iint_{\Sigma} a|w|^2 \nabla L_a(\phi) \cdot n\varphi \, d\sigma dt \\ &\quad + s\lambda^2 \frac{\gamma - 1}{2} \iint_{\Sigma} a|w|^2 L_a(\phi) \frac{\partial\phi}{\partial n}\varphi \, d\sigma dt, \end{aligned}$$

$$\begin{aligned}
I_{2,2} &= s\lambda^2 \iint_{\Sigma} aF_a(\phi)w \frac{\partial w}{\partial n} \varphi \, d\sigma dt - \frac{s\lambda^2}{2} \iint_{\Sigma} a|w|^2 \nabla(\varphi F_a(\phi)) \cdot n \, d\sigma dt \\
&\quad + \frac{s\lambda^3}{2} a \iint_Q |w|^2 F_a(\phi) (\Delta\phi + \lambda|\nabla\phi|^2) \varphi \, dxdt - 2s\lambda^3 a^2 \iint_Q |w|^2 D^2(\phi) (\nabla\phi, \nabla\phi) \varphi \, dxdt \\
&\quad + \frac{s\lambda^2}{2} a \iint_Q |w|^2 \Delta(F_a(\phi)) \varphi \, dxdt - s\lambda^2 a \iint_Q |\nabla w|^2 F_a(\phi) \varphi \, dxdt,
\end{aligned}$$

$$\begin{aligned}
I_{2,3} &= s\lambda a \iint_Q |\nabla w|^2 L_a(\phi) \varphi \, dxdt + s\lambda^2 a \iint_Q |\nabla w|^2 F_a(\phi) \varphi \, dxdt + 2s\lambda^2 a^2 \iint_Q |\nabla\phi \cdot \nabla w|^2 \varphi \, dxdt \\
&\quad - 2s\lambda^2 a \iint_Q \phi_t w_t \nabla w \cdot \nabla\phi \varphi \, dxdt + 2s\lambda a^2 \iint_Q D^2(\phi) (\nabla w, \nabla w) \varphi \, dxdt \\
&\quad + s\lambda \iint_{\Sigma} a^2 |\nabla w|^2 \frac{\partial\phi}{\partial n} \varphi \, d\sigma dt + 2s\lambda \iint_{\Sigma} a(\phi_t w_t - a\nabla\phi \cdot \nabla w) \frac{\partial w}{\partial n} \varphi \, d\sigma dt,
\end{aligned}$$

$$I_{3,1} = s^3 \lambda^3 (\gamma - 1) \iint_Q |w|^2 L_a(\phi) F_a(\phi) \varphi^3 \, dxdt,$$

$$I_{3,2} = -s^3 \lambda^4 \iint_Q |w|^2 F_a(\phi)^2 \varphi^3 \, dxdt,$$

and

$$\begin{aligned}
I_{3,3} &= s^3 \lambda^3 \iint_Q |w|^2 F_a(\phi) L_a(\phi) \varphi^3 \, dxdt + 2s^3 \lambda^3 \iint_Q |w|^2 (|\phi_t|^2 \phi_{tt} + a^2 D^2(\phi) (\nabla\phi, \nabla\phi)) \varphi^3 \, dxdt \\
&\quad + 3s^3 \lambda^4 \iint_Q |w|^2 F_a(\phi)^2 \varphi^3 \, dxdt + s^3 \lambda^3 \iint_{\Sigma} a|w|^2 F_a(\phi) \frac{\partial\phi}{\partial n} \varphi^3 \, d\sigma dt.
\end{aligned}$$

With all the previous computations, recalling $Q = U \times (-T, T)$ and $\Sigma = \partial U \times (-T, T)$ we write

$$\langle \mathcal{L}_1(w), \mathcal{L}_2(w) \rangle_{L^2(Q)} = A_U + Y_U + B_{\partial U}, \quad (4.3)$$

where A_U gathers the so-called ‘dominating’ internal terms

$$\begin{aligned}
A_U &= 2s\lambda \iint_Q |w_t|^2 \phi_{tt} \varphi \, dxdt - \gamma s\lambda \iint_Q |w_t|^2 L_a(\phi) \varphi \, dxdt \\
&\quad + 2s\lambda^2 \iint_Q (|w_t|^2 |\phi_t|^2 - 2w_t \phi_t a \nabla w \cdot \nabla\phi + a^2 |\nabla\phi \cdot \nabla w|^2) \varphi \, dxdt \\
&\quad + 2s\lambda a^2 \iint_Q D^2(\phi) (\nabla w, \nabla w) \varphi \, dxdt \\
&\quad + \gamma s\lambda a \iint_Q |\nabla w|^2 L_a(\phi) \varphi \, dxdt + 2s^3 \lambda^4 \iint_Q |w|^2 F_a(\phi)^2 \varphi^3 \, dxdt \\
&\quad + 2s^3 \lambda^3 \iint_Q |w|^2 (|\phi_t|^2 \phi_{tt} + a^2 D^2(\phi) (\nabla\phi, \nabla\phi)) \varphi^3 \, dxdt \\
&\quad + \gamma s^3 \lambda^3 \iint_Q |w|^2 L_a(\phi) F_a(\phi) \varphi^3 \, dxdt,
\end{aligned} \quad (4.4)$$

$B_{\partial U}$ is the sum of all the boundary terms

$$B_{\partial U} = \sum_{i=1}^9 B_i(U, \partial U)$$

with $\Sigma = \partial U \times (-T, T)$ and

$$\begin{aligned} B_1(U, \partial U) &= -s\lambda \iint_{\Sigma} a|w_t|^2 \frac{\partial \phi}{\partial n} \varphi \, d\sigma dt, \\ B_2(U, \partial U) &= -(\gamma - 1)s\lambda \iint_{\Sigma} aL_a(\phi)w \frac{\partial w}{\partial n} \varphi \, d\sigma dt, \\ B_3(U, \partial U) &= \frac{(\gamma - 1)}{2}s\lambda^2 \iint_{\Sigma} a|w|^2 \frac{\partial \phi}{\partial n} L_a(\phi) \varphi \, d\sigma dt, \\ B_4(U, \partial U) &= \frac{(\gamma - 1)}{2}s\lambda \iint_{\Sigma} a|w|^2 \frac{\partial L_a(\phi)}{\partial n} \varphi \, d\sigma dt, \\ B_5(U, \partial U) &= s\lambda^2 \iint_{\Sigma} aw \frac{\partial w}{\partial n} F_a(\phi) \varphi \, d\sigma dt, \\ B_6(U, \partial U) &= -\frac{s\lambda^2}{2} \iint_{\Sigma} a|w|^2 \frac{\partial(\varphi F_a(\phi))}{\partial n} \, d\sigma dt, \\ B_7(U, \partial U) &= 2s\lambda \iint_{\Sigma} a \frac{\partial w}{\partial n} (\phi_t w_t - a \nabla \phi \nabla w) \varphi \, d\sigma dt, \\ B_8(U, \partial U) &= s\lambda \iint_{\Sigma} a^2 |\nabla w|^2 \frac{\partial \phi}{\partial n} \varphi \, d\sigma dt, \\ B_9(U, \partial U) &= s^3 \lambda^3 \iint_{\Sigma} a F_a(\phi) \frac{\partial \phi}{\partial n} |w|^2 \varphi^3 \, d\sigma dt, \end{aligned} \tag{4.5}$$

and Y_U is the sum of the remaining interior terms. In particular, since $\lambda < M\varphi = Me^{\lambda\phi}$, we have that

$$|Y_U| \leq Ms\lambda^3 \iint_Q \varphi^3 |w|^2 \, dx dt. \tag{4.6}$$

4.2 Carleman inequality for the transmission system

Assuming the hypothesis of Theorem 1.3, we will fix the constants of the Carleman weight. From hypothesis (2.6) we have that

$$m_0 := \frac{1}{2} \inf\{a(x)D^2\mu(x)(\zeta, \zeta) : x \in \Omega_1 \cup \Omega_2, \zeta \in \mathbb{R}^2, |\zeta| = 1\} > 0. \tag{4.7}$$

Also, the parameter β of the weight function can be chosen such that it satisfies

$$0 < \beta < m_0 \frac{\inf\{a(x)\Delta\mu(x) : x \in \Omega_1 \cup \Omega_2\} + 2\beta}{\sup\{a(x)\Delta\mu(x) : x \in \Omega_1 \cup \Omega_2\} + 2\beta}. \tag{4.8}$$

Indeed, the expression at the right hand side of (4.8) is a continuous function on β and it takes a positive value at $\beta = 0$; therefore (4.8) is fulfilled if β is small enough. Hence, from (4.8) it is clear that we can chose γ such that

$$\sup_{\Omega_1 \cup \Omega_2} \frac{4\beta}{a\Delta\mu + 2\beta} < \gamma < \inf_{\Omega_1 \cup \Omega_2} \frac{4m_0}{a\Delta\mu + 2\beta}. \tag{4.9}$$

Remark 4.1. We mention that, the precise value of β from (4.8) is usually important in the applications of Carleman estimate. For instance, in order to apply the Bukgeim-Klibanov method, it would be necessary to take additional hypothesis: typically, we have to take $T > T_0$ with

$$0 < \delta \leq \mu(x) \leq \beta T_0^2 \quad (4.10)$$

for all $x \in \Omega$. Therefore, a very interesting further problem would be to construct precise examples of the weight function μ adapted to particular cases of (non-convex) interfaces such that the choice of $\beta > 0$ is optimal.

We apply the computations developed in the previous part of the work to each one of the subdomains Ω_1 and Ω_2 , with the main coefficient given by a_1 and a_2 respectively, and then we add up the resulting terms. Therefore, from (4.3) we can write

$$\langle \mathcal{L}_1(w), \mathcal{L}_2(w) \rangle_{L^2(\Omega \times (-T, T))} = A_{\Omega_1} + A_{\Omega_2} + Y_{\Omega_1} + Y_{\Omega_2} + B_{\partial\Omega_1} + B_{\partial\Omega_2}. \quad (4.11)$$

Here, the terms gathered in $A_{\Omega_1} + A_{\Omega_2}$ are integrals over $\Omega \times (-T, T)$, denoted again by Q , and can be treated as usual, e.g. [4] or [1].

In the following result we state the norm we obtain from the minimization of this interior integrals.

Proposition 4.2. *If $\rho \in C^3(\bar{\Omega})$, μ is defined by (2.4) and ε is given by Proposition 2.2, then, for each pair $a_1, a_2 > 0$ satisfying*

$$\frac{a_1}{a_2} \in (1, 1 + \varepsilon),$$

there exist $\delta, \lambda_0, s_0 > 0$ such that

$$A = A_{\Omega_1} + A_{\Omega_2} \geq \delta \left[s\lambda \iint_Q (|w_t|^2 + a|\nabla w|^2) \varphi \, dxdt + s^3\lambda^3 \iint_Q |w|^2 \varphi^3 \, dxdt \right]$$

for all $s \geq s_0$, $\lambda \geq \lambda_0$.

Proof. We denote by A_j , $j = 1, \dots, 8$ the eight integrals coming from the L^2 product of $\mathcal{L}_1(w)$ and $\mathcal{L}_2(w)$. Hence, from (4.9) we get that there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} A_1 + A_2 &= 2s\lambda \iint_Q |w_t|^2 \phi_{tt} \varphi \, dxdt - \gamma s\lambda \iint_Q |w_t|^2 L_a(\phi) \varphi \, dxdt \\ &= s\lambda \iint_Q |w_t|^2 (-4\beta - \gamma(\phi_{tt} - a\Delta\phi)) \varphi \, dxdt = s\lambda \iint_Q |w_t|^2 (-4\beta + \gamma(2\beta + a\Delta\mu)) \varphi \, dxdt \\ &\geq \delta_1 s\lambda \iint_Q |w_t|^2 \varphi \, dxdt \end{aligned}$$

and

$$\begin{aligned} A_4 + A_5 &= 2s\lambda \iint_Q a^2 D^2(\phi)(\nabla w, \nabla w) \varphi \, dxdt + \gamma s\lambda \iint_Q a|\nabla w|^2 L_a(\phi) \varphi \, dxdt \\ &\geq s\lambda \iint_Q a|\nabla w|^2 [4m_0 - \gamma(2\beta + a\Delta\mu)] \varphi \, dxdt \\ &\geq \delta_2 s\lambda \iint_Q a\varphi |\nabla w|^2 \, dxdt. \end{aligned}$$

Also, we have

$$\begin{aligned} A_3 &= 2s\lambda^2 \iint_Q (|w_t|^2 |\phi_t|^2 - 2w_t \phi_t a \nabla w \cdot \nabla \phi + a^2 |\nabla \phi \cdot \nabla w|^2) \varphi dxdt \\ &= 2s\lambda^2 \iint_Q (w_t \phi_t - a \nabla w \cdot \nabla \phi)^2 \varphi dxdt \geq 0. \end{aligned}$$

Concerning the remaining interior terms, recalling that $F_a(\phi) = |\phi_t|^2 - a|\nabla \phi|^2$, we obtain

$$\begin{aligned} \sum_{j=6}^8 A_j &= 2s^3 \lambda^4 \iint_Q |w|^2 \varphi^3 F_a(\phi)^2 dxdt + 2s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 (|\phi_t|^2 \phi_{tt} + a^2 D^2(\phi)(\nabla \phi, \nabla \phi)) dxdt \\ &\quad + \gamma s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 L_a(\phi) F_a(\phi) dxdt \\ &= s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 [2\lambda F_a(\phi)^2 - 4\beta |\phi_t|^2 + 2a^2 D^2(\mu)(\nabla \mu, \nabla \mu) + \gamma L_a(\phi) F_a(\phi)] dxdt \\ &= s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 [2\lambda F_a(\phi)^2 + (\gamma L_a(\phi) - 4\beta) F_a(\phi) + 2a^2 D^2(\mu)(\nabla \mu, \nabla \mu) - 4\beta a |\nabla \mu|^2] dxdt. \end{aligned}$$

Therefore, denoting $b = \|\gamma L_a(\phi) - 4\beta\|_{L^\infty}$, using (2.5) and recalling that (4.8) implies $\beta < m_0$, where m_0 is defined by (4.7), we deduce that there exists $d_0 > 0$ such that

$$\sum_{j=6}^8 A_j \geq s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 [2\lambda F_a(\phi)^2 - b |F_a(\phi)| + d_0] dxdt. \quad (4.12)$$

Denoting $g_\lambda(x) = 2\lambda x^2 - b|x| + d_0$ for $x \in \mathbb{R}$, it is not difficult to see that there exists $\delta_3 > 0$ and $\lambda_0 > 0$ such that

$$\min_{x \in \mathbb{R}} g_\lambda(x) \geq \delta_3 > 0 \quad (4.13)$$

for each $\lambda \geq \lambda_0$. From (4.12) and (4.13), it is deduced that

$$\sum_{j=6}^8 A_j \geq s^3 \lambda^3 \delta_3 \iint_Q |w|^2 \varphi^3 dxdt$$

for all $\lambda \geq \lambda_0$, and the Proposition is proved. \square

On the other hand, in order to continue dealing with the terms listed in (4.11), concerning the negligible terms, directly from (4.6) we have

$$|Y_{\Omega_1}| + |Y_{\Omega_2}| \leq Ms\lambda^3 \iint_Q \varphi^3 |w|^2 dxdt. \quad (4.14)$$

Finally, regarding the boundary terms coming from Ω_1 and Ω_2 , we recall that $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega \cup \Gamma_*$ and $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma_*$. Hence we can write

$$B_{\partial\Omega_1} + B_{\partial\Omega_2} = B_{\partial\Omega} + [B_{\Gamma_*}], \quad (4.15)$$

where $B_{\partial\Omega}$ denotes the sum of the integrals from (4.5) supported in $\partial\Omega$, and by $[B_{\Gamma_*}]$ we mean the sum of the corresponding integrals supported by the interface Γ_* , coming from the integration by parts on both domains Ω_1 and Ω_2 , i.e.

$$[B_{\Gamma_*}] = \sum_{i=1}^9 (B_i(\Omega_1, \Gamma_*) + B_i(\Omega_2, \Gamma_*)).$$

Thanks to the Dirichlet homogeneous boundary condition of the equation, we have

$$\nabla w = \frac{\partial w}{\partial n} n \quad \text{on } \partial\Omega$$

and then we can deal with the terms at the boundary $\partial\Omega$ as usual, obtaining the observation term stated in the following result.

Lemma 4.3. *There exist positive constants M, s_0, λ_0 such that*

$$B_{\partial\Omega} = \sum_{i=1}^9 B_i(\Omega, \partial\Omega) \geq -Ms\lambda \iint_{\Sigma^+} \varphi \left| a \frac{\partial w}{\partial n} \right|^2 d\sigma dt.$$

for any $s \geq s_0$ and $\lambda \geq \lambda_0$, where $\Sigma^+ := \left\{ x \in \partial\Omega, \frac{\partial\phi}{\partial n}(x) > 0 \right\}$.

Finally, dealing with the terms at the interface will be crucial in this work. The rest of this section is devoted to prove that, under the hypothesis we are assuming, we actually have $[B_{\Gamma_*}] \geq 0$ if the parameter s is large enough. See Proposition 4.4 below.

4.3 Carleman terms at the interface Γ_* between Ω_1 and Ω_2

Recall that the exterior unitary normal vectors of Ω_1 and Ω_2 satisfy $n_2 = -n_1$ on the interface Γ_* . Using the notation presented in (4.5), we get the following result.

Proposition 4.4. *Assuming the hypotheses of Theorem 1.3, a_1, a_2 satisfying $a_2 < a_1$ and taking μ and ϕ defined by (2.4) and (2.12) respectively, there exist $s_0 > 0$ such that*

$$[B_{\Gamma_*}] := \sum_{i=1}^9 (B_i(\Omega_1, \Gamma_*) + B_i(\Omega_2, \Gamma_*)) \geq 0 \tag{4.16}$$

for all $s \geq s_0$.

Proof. Step 1 - Preliminaries about properties at the interface Γ_ .*

By construction of μ and ϕ , using ρ that satisfies the assumptions of Theorem 1.3, the hypotheses of propositions 2.1 and 2.2 are verified. Hence, the function μ (and therefore ϕ) satisfy their conclusions. These properties, together with inequality (2.9) stated in Remark 2.5 will be key ingredients of this proof. We recall that, for any function f defined in Ω we denote $f_j := f|_{\Omega_j}$ for $j = 1, 2$.

Indeed, from Propositions 2.1 we get that ϕ satisfies the transmission conditions (1.5) at the interface $\Sigma_* = \Gamma_* \times (-T, T)$, meaning

$$\phi_1 = \phi_2 \quad \text{and} \quad a_1 \frac{\partial\phi_1}{\partial n_1} + a_2 \frac{\partial\phi_2}{\partial n_2} = 0 \quad \text{on } \Sigma_*,$$

so that φ and $w = e^{s\varphi}u$ satisfy them as well since $u \in X$. Therefore, on Σ_* , one has $\varphi_1 = \varphi_2$, $w_1 = w_2$, but also $w_{1t} = w_{2t}$ and $\phi_{1t} = \phi_{2t}$, and similarly for any other time-derivative, like w_{tt} or ϕ_{tt} . This allows to write simply φ or w . Moreover, if we denote by τ the unitary tangential vector of Γ_* , there is no ambiguity in the notation

$$\frac{\partial \phi}{\partial \tau} := \frac{\partial \phi_2}{\partial \tau} = \frac{\partial \phi_1}{\partial \tau} = \nabla \phi_1 \cdot \tau \quad \text{on } \Sigma_*. \quad (4.17)$$

Hence, at the interface we have $\nabla \phi_j = \frac{\partial \phi_j}{\partial n_j} n_j + \frac{\partial \phi}{\partial \tau} \tau$ and

$$|\nabla \phi_j|^2 = \left| \frac{\partial \phi_j}{\partial n_j} \right|^2 + \left| \frac{\partial \phi}{\partial \tau} \right|^2, \quad \text{for } j = 1, 2. \quad (4.18)$$

Within the framework of the previous notation, we begin by proving a useful identity for the traces of any pair of functions satisfying the transmission conditions on the interface. If f and g satisfy (1.5) we have, on the interface Γ_* :

$$\begin{aligned} a_2 \nabla f_2 \cdot \nabla g_2 - a_1 \nabla f_1 \cdot \nabla g_1 &= a_2 \left(\frac{\partial f_2}{\partial n_2} \frac{\partial g_2}{\partial n_2} + \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau} \right) - a_1 \left(\frac{\partial f_1}{\partial n_1} \frac{\partial g_1}{\partial n_1} + \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau} \right) \\ &= \frac{1}{a_2} \left(a_2 \frac{\partial f_2}{\partial n_2} a_2 \frac{\partial g_2}{\partial n_2} \right) + a_2 \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau} - \frac{1}{a_1} \left(a_1 \frac{\partial f_1}{\partial n_1} a_1 \frac{\partial g_1}{\partial n_1} \right) - a_1 \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau} \\ &= a_1^2 \frac{\partial f_1}{\partial n_1} \frac{\partial g_1}{\partial n_1} \left(\frac{1}{a_2} - \frac{1}{a_1} \right) + (a_2 - a_1) \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau}, \end{aligned}$$

and then we conclude that

$$a_2 \nabla f_2 \cdot \nabla g_2 - a_1 \nabla f_1 \cdot \nabla g_1 = (a_1 - a_2) \left(\frac{a_1}{a_2} \frac{\partial f_1}{\partial n_1} \frac{\partial g_1}{\partial n_1} - \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \tau} \right) \quad \text{on } \Gamma_*. \quad (4.19)$$

Identity (4.19), which will be applied to functions ϕ and w , is an essential ingredient of the proof of the estimate of $[B_{\Gamma_*}]$.

In order to prove Proposition 4.4, we now split the calculation of the boundary terms in two separate steps, one devoted to the negligible terms and one to the dominant positive traces at the interface that can absorb the others.

Step 2 - Negligible boundary bounded terms.

It is not difficult to prove that the first six terms

$$[B_k] = B_k(\Omega_1, \Gamma_*) + B_k(\Omega_2, \Gamma_*) \quad \text{for } k = 1, \dots, 6$$

can be bounded in the following way. We claim that there exists $M > 0$ such that

$$\sum_{k=1}^6 |[B_k]| \leq M \iint_{\Sigma_*} \left(s^2 \lambda^3 |w|^2 + \lambda \left| \frac{\partial w_1}{\partial n_1} \right|^2 \right) \varphi \, d\sigma dt. \quad (4.20)$$

We detail the computations needed to prove (4.20), using the transmission conditions satisfied by the functions ϕ and w and Young's inequality.

- $B_1(\Omega_i, \Gamma_*) = -s\lambda \iint_{\Sigma} a|w_t|^2 \frac{\partial \phi}{\partial n} \varphi \, d\sigma dt$, and using the above introductive comments, we have

$$[B_1] := B_1(\Omega_1, \Gamma_*) + B_1(\Omega_2, \Gamma_*) = -s\lambda \iint_{\Sigma_*} |w_t|^2 \left(a_1 \frac{\partial \phi_1}{\partial n_1} + a_2 \frac{\partial \phi_2}{\partial n_2} \right) \varphi \, d\sigma dt = 0.$$

- $B_2(\Omega_i, \Gamma_*) = -(\gamma - 1)s\lambda \iint_{\Sigma} aL_a(\phi)w \frac{\partial w}{\partial n} \varphi \, d\sigma dt$, and using the transmission conditions on w and ϕ_{tt} , and $L_a(\phi) = \phi_{tt} - a\Delta\phi$, we get,

$$\begin{aligned} [B_2] &:= B_2(\Omega_1, \Gamma_*) + B_2(\Omega_2, \Gamma_*) = -(\gamma - 1)s\lambda \iint_{\Sigma_*} w \left[L_{a_1}(\phi_1)a_1 \frac{\partial w_1}{\partial n_1} + L_{a_2}(\phi_2)a_2 \frac{\partial w_2}{\partial n_2} \right] \varphi \, d\sigma dt \\ &= -(\gamma - 1)s\lambda \iint_{\Sigma_*} wa_1 \frac{\partial w_1}{\partial n_1} [L_{a_1}(\phi_1) - L_{a_2}(\phi_2)] \varphi \, d\sigma dt \\ &= -(\gamma - 1)s\lambda \iint_{\Sigma_*} wa_1 \frac{\partial w_1}{\partial n_1} [a_2\Delta\phi_2 - a_1\Delta\phi_1] \varphi \, d\sigma dt \end{aligned}$$

and then

$$|[B_2]| \leq Ms^2\lambda \iint_{\Sigma_*} |w|^2 \varphi \, d\sigma dt + M\lambda \iint_{\Sigma_*} \left| \frac{\partial w_1}{\partial n_1} \right|^2 \varphi \, d\sigma dt.$$

- $B_3(\Omega_i, \Gamma_*) = \frac{(\gamma - 1)}{2}s\lambda^2 \iint_{\Sigma} a|w|^2 \frac{\partial \phi}{\partial n} L_a(\phi) \varphi \, d\sigma dt$, and for the same reasons, we have

$$\begin{aligned} [B_3] &:= B_3(\Omega_1, \Gamma_*) + B_3(\Omega_2, \Gamma_*) = \frac{(\gamma - 1)}{2}s\lambda^2 \iint_{\Sigma_*} |w|^2 \left[a_1 \frac{\partial \phi_1}{\partial n_1} L_{a_1}(\phi_1) + a_2 \frac{\partial \phi_2}{\partial n_2} L_{a_2}(\phi_2) \right] \varphi \, d\sigma dt \\ &= a_1 \frac{(1 - \gamma)}{2}s\lambda^2 \iint_{\Sigma_*} |w|^2 (a_1\Delta\phi_1 - a_2\Delta\phi_2) \frac{\partial \phi_1}{\partial n_1} \varphi \, d\sigma dt \end{aligned}$$

and then

$$|[B_3]| \leq Ms\lambda^2 \iint_{\Sigma_*} |w|^2 \varphi \, d\sigma dt.$$

- $B_4(\Omega_i, \Gamma_*) = \frac{(\gamma - 1)}{2}s\lambda \iint_{\Sigma} a|w|^2 \frac{\partial L_a(\phi)}{\partial n} \varphi \, d\sigma dt$, so that the regularity of ϕ assumed in Theorem 1.3 allows to write

$$[B_4] := B_4(\Omega_1, \Gamma_*) + B_4(\Omega_2, \Gamma_*) = \frac{(\gamma - 1)}{2}s\lambda \iint_{\Sigma_*} |w|^2 \left(a_1 \frac{\partial L_{a_1}(\phi_1)}{\partial n_1} + a_2 \frac{\partial L_{a_2}(\phi_2)}{\partial n_2} \right) \varphi \, d\sigma dt$$

and then

$$|[B_4]| \leq Ms\lambda \iint_{\Sigma_*} |w|^2 \varphi \, d\sigma dt.$$

• $B_5(\Omega_i, \Gamma_*) = s\lambda^2 \iint_{\Sigma} aw \frac{\partial w}{\partial n} F_a(\phi) \varphi \, d\sigma dt$, and recalling that $F_a(\phi) = |\phi_t|^2 - a|\nabla\phi|^2$, using the transmission conditions and Young's inequality, we get

$$\begin{aligned} [B_5] &:= B_5(\Omega_1, \Gamma_*) + B_5(\Omega_2, \Gamma_*) \\ &= s\lambda^2 \iint_{\Sigma_*} w \left(|\phi_t|^2 \left[a_1 \frac{\partial w_1}{\partial n_1} + a_2 \frac{\partial w_2}{\partial n_2} \right] - \left[a_1 \frac{\partial w_1}{\partial n_1} a_1 |\nabla\phi_1|^2 + a_2 \frac{\partial w_2}{\partial n_2} a_2 |\nabla\phi_2|^2 \right] \right) \varphi \, d\sigma dt \\ &= -s\lambda^2 \iint_{\Sigma_*} wa_1 \frac{\partial w_1}{\partial n_1} (a_1 |\nabla\phi_1|^2 - a_2 |\nabla\phi_2|^2) \varphi \, d\sigma dt \end{aligned}$$

and then

$$|[B_5]| \leq Ms^2\lambda^3 \iint_{\Sigma_*} |w|^2 \varphi \, d\sigma dt + M\lambda \iint_{\Sigma_*} \left| \frac{\partial w_1}{\partial n_1} \right|^2 \varphi \, d\sigma dt.$$

• $B_6(\Omega_i, \Gamma_*) = -\frac{s\lambda^2}{2} \iint_{\Sigma} a|w|^2 \frac{\partial(\varphi F_a(\phi))}{\partial n} \, d\sigma dt$, so that the regularity of ϕ allows to write

$$|[B_6]| = |B_6(\Omega_1, \Gamma_*) + B_6(\Omega_2, \Gamma_*)| \leq Ms\lambda^3 \iint_{\Sigma_*} |w|^2 \varphi \, d\sigma dt.$$

Therefore, gathering all these estimates, (4.20) is obtained.

Step 3 - Boundary dominant terms.

We now aim at proving that the remaining terms can provide positive traces on the boundary, allowing to absorb the right hand side of (4.20) by means of taking the parameter s large enough. More precisely, we claim that there exists $\delta > 0$ such that

$$[B_7] + [B_8] + [B_9] \geq \delta \iint_{\Sigma_*} \left(s^3 \lambda^3 \varphi^3 |w|^2 + s\lambda \varphi \left| \frac{\partial w_1}{\partial n_1} \right|^2 \right) \, d\sigma dt. \quad (4.21)$$

In order to prove (4.21), we carefully analyze the result of adding the contribution of each term. We begin by showing that $[B_7] + [B_8]$ provide a positive expression of the normal derivative. Indeed, recalling that

$$\bullet B_7(\Omega_i, \Gamma_*) = 2s\lambda \iint_{\Sigma_*} a \frac{\partial w}{\partial n} (\phi_t w_t - a \nabla\phi \cdot \nabla w) \varphi \, d\sigma dt,$$

we have, using the transmission conditions,

$$\begin{aligned} [B_7] &:= B_7(\Omega_1, \Gamma_*) + B_7(\Omega_2, \Gamma_*) \\ &= 2s\lambda \iint_{\Sigma_*} a_1 \frac{\partial w_1}{\partial n_1} (\phi_t w_t - a_1 \nabla\phi_1 \cdot \nabla w_1) \varphi \, d\sigma dt + 2s\lambda \iint_{\Sigma_*} a_2 \frac{\partial w_2}{\partial n_2} (\phi_t w_t - a_2 \nabla\phi_2 \cdot \nabla w_2) \varphi \, d\sigma dt \\ &= -2s\lambda \iint_{\Sigma_*} \left(a_1 \frac{\partial w_1}{\partial n_1} a_1 \nabla\phi_1 \cdot \nabla w_1 + a_2 \frac{\partial w_2}{\partial n_2} a_2 \nabla\phi_2 \cdot \nabla w_2 \right) \varphi \, d\sigma dt \\ &= 2s\lambda \iint_{\Sigma_*} a_1 \frac{\partial w_1}{\partial n_1} (a_2 \nabla\phi_2 \cdot \nabla w_2 - a_1 \nabla\phi_1 \cdot \nabla w_1) \varphi \, d\sigma dt, \end{aligned}$$

and then, using (4.19) with $f = w$ and $g = \phi$, we get

$$[B_7] = 2s\lambda(a_1 - a_2) \iint_{\Sigma_*} a_1 \frac{\partial w_1}{\partial n_1} \left(\frac{a_1}{a_2} \frac{\partial\phi_1}{\partial n_1} \frac{\partial w_1}{\partial n_1} - \frac{\partial\phi}{\partial\tau} \frac{\partial w}{\partial\tau} \right) \varphi \, d\sigma dt. \quad (4.22)$$

In the same way,

- $B_8(\Omega_i, \Gamma_*) = s\lambda \iint_{\Sigma_*} a^2 |\nabla w|^2 \frac{\partial \phi}{\partial n} \varphi \, d\sigma dt$, so that

$$\begin{aligned} [B_8] &:= B_8(\Omega_1, \Gamma_*) + B_8(\Omega_2, \Gamma_*) = s\lambda \iint_{\Sigma_*} a_1^2 |\nabla w_1|^2 \frac{\partial \phi_1}{\partial n_1} \varphi \, d\sigma dt + s\lambda \iint_{\Sigma_*} a_2^2 |\nabla w_2|^2 \frac{\partial \phi_2}{\partial n_2} \varphi \, d\sigma dt \\ &= -s\lambda \iint_{\Sigma_*} a_1 \frac{\partial \phi_1}{\partial n_1} (a_2 |\nabla w_2|^2 - a_1 |\nabla w_1|^2) \varphi \, d\sigma dt. \end{aligned}$$

Thus, from (4.19) with $f = g = w$, we get

$$[B_8] = -s\lambda(a_1 - a_2) \iint_{\Sigma_*} a_1 \frac{\partial \phi_1}{\partial n_1} \left(\frac{a_1}{a_2} \left| \frac{\partial w_1}{\partial n_1} \right|^2 - \left| \frac{\partial w}{\partial \tau} \right|^2 \right) \varphi \, d\sigma dt. \quad (4.23)$$

Therefore, from (4.22) and (4.23) we have

$$[B_7] + [B_8] = s\lambda(a_1 - a_2)a_1 \iint_{\Sigma_*} \left(\frac{a_1}{a_2} \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w_1}{\partial n_1} \right|^2 + \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w}{\partial \tau} \right|^2 - 2 \frac{\partial w_1}{\partial n_1} \frac{\partial \phi}{\partial \tau} \frac{\partial w}{\partial \tau} \right) \varphi \, d\sigma dt. \quad (4.24)$$

Moreover, from hypothesis (2.9) on μ , we directly have

$$\left| \frac{\partial \phi}{\partial \tau} \right| \leq \frac{\partial \phi_1}{\partial n_1} \quad \text{on } \Sigma_*. \quad (4.25)$$

This estimate allow us to bound by below the non-positive term in the right-hand side of (4.24). Indeed, we have

$$2 \left| \frac{\partial w_1}{\partial n_1} \frac{\partial \phi}{\partial \tau} \frac{\partial w}{\partial \tau} \right| \leq 2 \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w_1}{\partial n_1} \frac{\partial w}{\partial \tau} \right| \leq \left(\frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w_1}{\partial n_1} \right|^2 + \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w}{\partial \tau} \right|^2 \right). \quad (4.26)$$

Thus, taking into account that $1 < \frac{a_1}{a_2}$, from (4.24) and (4.26) we deduce that $\exists \delta_1 > 0$ such that

$$[B_7] + [B_8] = s\lambda(a_1 - a_2)a_1 \iint_{\Sigma_*} \left(\frac{a_1}{a_2} - 1 \right) \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial w_1}{\partial n_1} \right|^2 \varphi \, d\sigma dt \geq \delta_1 s\lambda \iint_{\Sigma_*} \left| \frac{\partial w_1}{\partial n_1} \right|^2 \varphi \, d\sigma dt. \quad (4.27)$$

For the last term, we proceed as follows:

- $B_9(\Omega_i, \Gamma_*) = s^3 \lambda^3 \iint_{\Sigma_*} F_a(\phi) a \frac{\partial \phi}{\partial n} |w|^2 \varphi^3 \, d\sigma dt$, so that

$$\begin{aligned} [B_9] &:= B_9(\Omega_1, \Gamma_*) + B_9(\Omega_2, \Gamma_*) \\ &= s^3 \lambda^3 \left(\iint_{\Sigma_*} (|\phi_t|^2 - a_1 |\nabla \phi_1|^2) a_1 \frac{\partial \phi_1}{\partial n_1} |w|^2 \varphi^3 \, d\sigma dt + \iint_{\Sigma_*} (|\phi_t|^2 - a_2 |\nabla \phi_2|^2) a_2 \frac{\partial \phi_2}{\partial n_2} |w|^2 \varphi^3 \, d\sigma dt \right) \\ &= -s^3 \lambda^3 \iint_{\Sigma_*} |w|^2 \left[a_1 |\nabla \phi_1|^2 a_1 \frac{\partial \phi_1}{\partial n_1} + a_2 |\nabla \phi_2|^2 a_2 \frac{\partial \phi_2}{\partial n_2} \right] \varphi^3 \, d\sigma dt \\ &= s^3 \lambda^3 \iint_{\Sigma_*} |w|^2 a_1 \frac{\partial \phi_1}{\partial n_1} (a_2 |\nabla \phi_2|^2 - a_1 |\nabla \phi_1|^2) \varphi^3 \, d\sigma dt. \end{aligned}$$

Using (4.19) for $f = g = \phi$ and (4.25), we have

$$a_2 |\nabla \phi_2|^2 - a_1 |\nabla \phi_1|^2 = (a_1 - a_2) \left(\frac{a_1}{a_2} \left| \frac{\partial \phi_1}{\partial n_1} \right|^2 - \left| \frac{\partial \phi}{\partial \tau} \right|^2 \right) \geq (a_1 - a_2) \left| \frac{\partial \phi_1}{\partial n_1} \right|^2 \left(\frac{a_1}{a_2} - 1 \right),$$

and therefore, there exists a $\delta_2 > 0$ such that

$$[B_9] \geq s^3 \lambda^3 a_1 (a_1 - a_2) \left(\frac{a_1}{a_2} - 1 \right) \iint_{\Sigma_*} |w|^2 \frac{\partial \phi_1}{\partial n_1} \left| \frac{\partial \phi_1}{\partial n_1} \right|^2 \varphi^3 d\sigma dt \geq \delta_2 s^3 \lambda^3 \iint_{\Sigma_*} \varphi^3 |w|^2 d\sigma dt. \quad (4.28)$$

From (4.27) and (4.28) we obtain (4.21).

Finally, from (4.20) and (4.21) we deduce that there exists s_0 such that

$$[B_{\Gamma_*}] \geq 0, \quad \forall s \geq s_0,$$

which concludes the proof of Proposition 4.4. \square

Remark 4.5. *In the proof of Proposition 4.4 we used (4.25), which is implied by (1.10) (see Proposition 2.5). But, actually, it is enough to assume inequality*

$$\left| \frac{\partial \phi}{\partial \tau} \right| < \sqrt{\frac{a_1}{a_2}} \frac{\partial \phi_1}{\partial n_1} \quad \text{on } \Sigma_*, \quad (4.29)$$

which, recalling that $a_2 < a_1$, is a more general hypothesis than (4.25). However, in this work we are not able to take advantage of (4.29) since we are taking a_1, a_2 satisfying (1.11) where ε depends on ρ and the geometry of the interface (via the orthogonal projection, see Proposition 2.2).

It would be very interesting to set a more precise geometrical construction of weight functions taking advantage of (4.29), in order to obtain more general and more precise results.

4.4 Concluding the proof

The proof of Theorem 1.3 is a direct consequence of the previous results of the article. Indeed, given ρ satisfying hypotheses (1.8), (1.9) and (1.10), we define μ by (2.4) and then ϕ and φ by (2.12) and (2.13) respectively. We take $\varepsilon > 0$ given by Proposition 2.2, and then we assume that a_1, a_2 satisfy (1.11). We develop the decomposition (4.2) in each subdomain Ω_1 and Ω_2 , obtaining (4.11). From Propositions 4.2 and 4.4 and Lemma 4.3, we obtain that there exists M such that

$$\begin{aligned} \|\mathcal{L}_1(w)\|_{L^2(Q)}^2 + \|\mathcal{L}_2(w)\|_{L^2(Q)}^2 + s\lambda \iint_Q (|w_t|^2 + a|\nabla w|^2) \varphi dxdt + s^3 \lambda^3 \iint_Q |w|^2 \varphi^3 dxdt \\ \leq M \|\mathcal{L}(w)\|_{L^2(Q)}^2 + Ms\lambda \iint_{\Sigma^+} \varphi \left| a \frac{\partial w}{\partial n} \right|^2 d\sigma dt. \end{aligned} \quad (4.30)$$

As usual, taking $u = e^{-s\varphi} w$ we obtain the Carleman inequality (1.12), and Theorem 1.3 is proved.

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