

Technical Report Associated with the Paper “Singularity Conditions for Continuum Parallel Robots”

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I. INTRODUCTION

In the present technical report, associated with the paper [1], we provide the expressions of the gradient and Hessian of the Lagrangian, and of the constraints, with respect to the variables \mathbf{q}_{pu} as well as the expressions of the gradient of the constraints and the derivative of the gradient of the Lagrangian with respect to the variables \mathbf{q}_a .

Sections II and III are brief recalls of the equations provided in Sections II to IV of the paper [1]. These recalls are necessary in order to have a self-contained report. Sections IV and V provide the equations for computing all gradients and Hessian necessary for writing the geometric-static and kinemato-static models of the continuum parallel robot.

II. POTENTIAL ENERGY OF THE CONTINUUM PARALLEL ROBOT

This Section is almost identical to Section II of [1], but it introduces all necessary variables reused in Sections IV and V for the computation of all gradients and Hessian necessary for writing the geometric-static and kinemato-static models of the continuum parallel robot.

A. Description of the continuum parallel robot

Let us describe the generic continuum parallel robot architecture that we consider in this report (Fig. 1): it is a robot which is made of n slender flexible rods (called legs). An extremity of each rod is connected to a motor at one end (points A_i , $i = 1, \dots, n$), the other extremity to the moving platform via a joint (at points B_i , $i = 1, \dots, n$), which is either a passive revolute joint, a passive spherical joint or a fixed joint, as proposed in [2], [3]. Other types of legs could be considered, by modifying the equations of the constraints and the Jacobian matrices associated to the motions of the motors, which are displayed further.

The legs can be either of constant length and connected at points A_i to an active revolute (Fig. 2(a)) or prismatic joint (Fig. 2(b)) as it was done in [2], or of variable length, i.e. acting like a soft cylinder fixed on the ground at point A_i as proposed in [3] (Fig. 2(c)). The robot moving platform, on which is located the end-effector, is considered to be rigid.

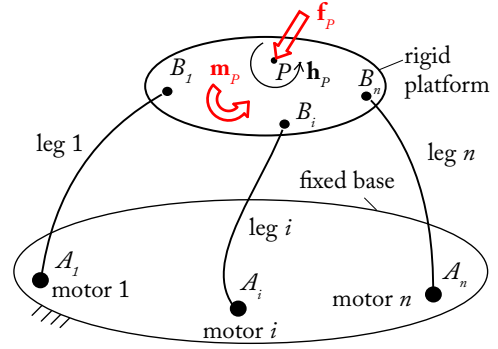


Fig. 1. A general parallel continuum robot

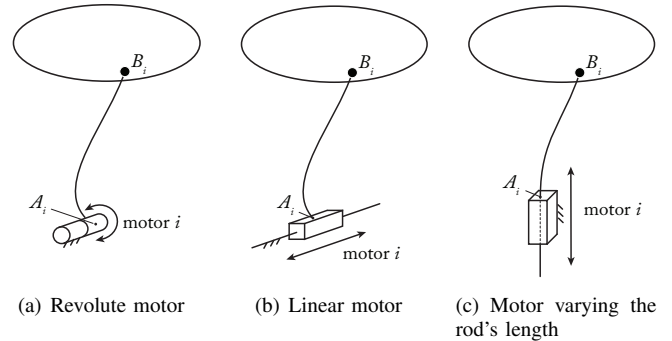


Fig. 2. Three different types of motors of legs for continuum parallel robots.

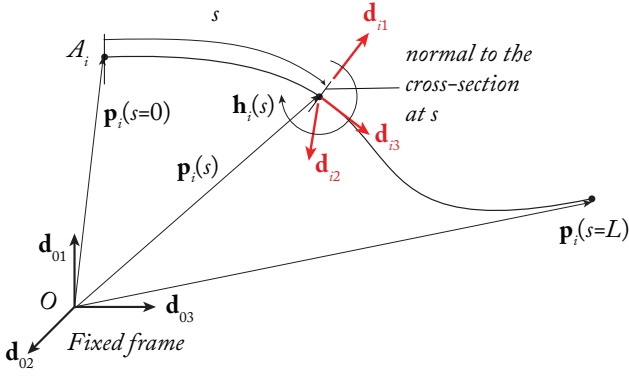
The variable associated with the motor displacement for the leg i is denoted by q_{ai} . All motor variables are grouped in the vector \mathbf{q}_a . The variables associated with the platform motion are \mathbf{p}_P , the location of the platform center of mass P , and $\mathbf{h}_P = h_{P1} \hat{\mathbf{i}} + h_{P2} \hat{\mathbf{j}} + h_{P3} \hat{\mathbf{k}}$ the unit quaternion (i.e. $\mathbf{h}_P^T \mathbf{h}_P = 1$) characterizing the platform orientation (Fig. 1).

Let us now compute the potential energy of the robot legs, which can be modeled as flexible rods of constant length clamped at points A_1 to A_n when the motors are fixed.

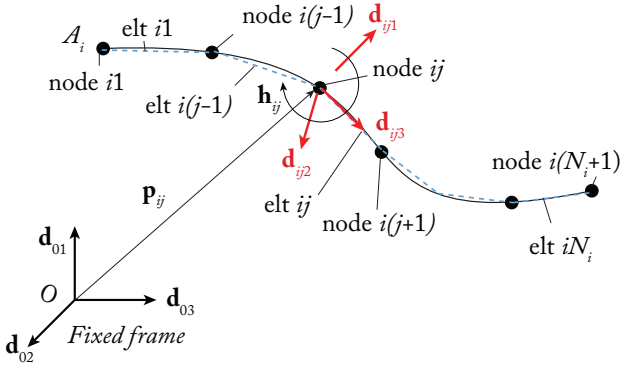
B. Potential energy of a single flexible rod

In this paper, we take the same hypotheses as in [2]: the rods are modeled using the Kirchoff assumptions (shear is neglected) and the rod elongation is considered to be negligible. More general assumptions, like the Cosserat ones [3], could

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(a) Parameterization of the continuous rod



(b) Parameterization of the discrete rod

Fig. 3. Parameterization of the continuum clamped-free slender rod

be used without changing the definitions of the singularity conditions further.

We derive here the equations for the 3D problem. Equations for the 2D case can be found in [4].

Let us consider the leg i of the robot. It can be modeled as a slender rod of length L_i (Fig. 3(a)) clamped into the ground at point A_i , made of an isotropic material and having a constant cross-section along its longitudinal direction. The robot base frame is denoted as $\mathcal{F}_0 : (O, \mathbf{d}_{01}, \mathbf{d}_{02}, \mathbf{d}_{03})$. The local configuration of a rod cross-section located at the curvilinear abscissa s is defined by the frame $\mathcal{F}_c : (S, \mathbf{d}_{i1}(s), \mathbf{d}_{i2}(s), \mathbf{d}_{i3}(s))$, in which $\mathbf{d}_{i3}(s)$ is the vector tangent to the cross-section at s . The configuration of the frame \mathcal{F}_c wrt the frame \mathcal{F}_0 is provided by the homogeneous transformation matrix ${}^0\mathbf{T}_s$ defined by:

$${}^0\mathbf{T}_s = \begin{bmatrix} {}^0\mathbf{R}_s & \mathbf{p}_i(s) \\ \mathbf{0} & 1 \end{bmatrix} \quad (1)$$

in which

$${}^0\mathbf{R}_s = [{}^0\mathbf{d}_{i1}(s) \quad {}^0\mathbf{d}_{i2}(s) \quad {}^0\mathbf{d}_{i3}(s)] \quad (2)$$

is the rotation matrix between \mathcal{F}_0 and \mathcal{F}_c and $\mathbf{p}_i(s)$ is the position of S in \mathcal{F}_0 . The superscript “0” before the vectors \mathbf{d}_{ik} means that they are expressed in the frame \mathcal{F}_0 .

The curvature and torsion of the rod can be represented by the Darboux vector $\mathbf{u}_i(s)$, defined as:

$$\mathbf{u}_i = \sum_{k=1}^3 u_{ik} \mathbf{d}_{ik} \quad (3)$$

with [5]

$$u_{i1} = -\mathbf{d}_{i2}^T \mathbf{d}'_{i3}, \quad u_{i2} = -\mathbf{d}_{i3}^T \mathbf{d}'_{i1}, \quad u_{i3} = -\mathbf{d}_{i1}^T \mathbf{d}'_{i2} \quad (4)$$

in which $(\cdot)'$ denotes the derivative wrt the variable s , i.e. $(\cdot)' = \partial(\cdot)/\partial s$.

In order to avoid any singularity of representation, we used the unit quaternion $\mathbf{h}_i(s) = h_{i1}(s) \hat{\mathbf{i}} + h_{i2}(s) \hat{\mathbf{j}} + h_{i3}(s) \hat{\mathbf{k}}$ in order to parameterize the orientation of the cross-section at s . As a result, the columns of the matrix ${}^0\mathbf{R}_s$ are given by the well-known expression (omitting the variable s):

$${}^0\mathbf{d}_{i1} = \begin{bmatrix} h_{i1}^2 + h_{i2}^2 - h_{i3}^2 - h_{i4}^2 \\ 2(h_{i1} h_{i4} + h_{i2} h_{i3}) \\ 2(h_{i2} h_{i4} - h_{i1} h_{i3}) \end{bmatrix} \quad (5)$$

$${}^0\mathbf{d}_{i2} = \begin{bmatrix} 2(h_{i2} h_{i3} - h_{i1} h_{i4}) \\ h_{i1}^2 - h_{i2}^2 + h_{i3}^2 - h_{i4}^2 \\ 2(h_{i1} h_{i2} + h_{i3} h_{i4}) \end{bmatrix} \quad (6)$$

$${}^0\mathbf{d}_{i3} = \begin{bmatrix} 2(h_{i1} h_{i3} + h_{i2} h_{i4}) \\ 2(h_{i3} h_{i4} - h_{i1} h_{i2}) \\ h_{i1}^2 - h_{i2}^2 - h_{i3}^2 + h_{i4}^2 \end{bmatrix} \quad (7)$$

under the constraint $\mathbf{h}_i^T \mathbf{h}_i = 1$, and their derivatives wrt s are:

$${}^0\mathbf{d}'_{i1} = \mathbf{D}_{i1} \mathbf{h}'_i, \quad {}^0\mathbf{d}'_{i2} = \mathbf{D}_{i2} \mathbf{h}'_i, \quad {}^0\mathbf{d}'_{i3} = \mathbf{D}_{i3} \mathbf{h}'_i \quad (8)$$

where \mathbf{h}'_i is the derivative of \mathbf{h}_i wrt s and the matrices $\mathbf{D}_{\alpha k}$ are defined, for any quaternion $\mathbf{h}_\alpha = [h_{\alpha1} \ h_{\alpha2} \ h_{\alpha3} \ h_{\alpha4}]^T$, as

$$\mathbf{D}_{\alpha 1} = 2 \begin{bmatrix} h_{\alpha 1} & h_{\alpha 2} & -h_{\alpha 3} & -h_{\alpha 4} \\ h_{\alpha 4} & h_{\alpha 3} & h_{\alpha 2} & h_{\alpha 1} \\ -h_{\alpha 3} & h_{\alpha 4} & -h_{\alpha 1} & h_{\alpha 2} \end{bmatrix} \quad (9)$$

$$\mathbf{D}_{\alpha 2} = 2 \begin{bmatrix} -h_{\alpha 4} & h_{\alpha 3} & h_{\alpha 2} & -h_{\alpha 1} \\ h_{\alpha 1} & -h_{\alpha 2} & h_{\alpha 3} & -h_{\alpha 4} \\ h_{\alpha 2} & h_{\alpha 1} & h_{\alpha 4} & h_{\alpha 3} \end{bmatrix} \quad (10)$$

$$\mathbf{D}_{\alpha 3} = 2 \begin{bmatrix} h_{\alpha 3} & h_{\alpha 4} & h_{\alpha 1} & h_{\alpha 2} \\ -h_{\alpha 2} & -h_{\alpha 1} & h_{\alpha 4} & h_{\alpha 3} \\ h_{\alpha 1} & -h_{\alpha 2} & -h_{\alpha 3} & h_{\alpha 4} \end{bmatrix} \quad (11)$$

Introducing (5) to (8) into (4) leads to, for $k = 1, 2, 3$:

$$u_{ik} = 2 \mathbf{h}_i^T \mathbf{B}_k^T \mathbf{h}'_i \quad (12)$$

with

$$\mathbf{B}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (13)$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

For an isotropic material, the rod deformation energy is then given by [6]:

$$V_{ei} = \frac{1}{2} \int_0^{L_i} ({}^c \mathbf{u}_i - {}^c \hat{\mathbf{u}}_i)^T \mathbf{K}_i ({}^c \mathbf{u}_i - {}^c \hat{\mathbf{u}}_i) ds \quad (14)$$

where $\hat{\mathbf{u}}_i = [\hat{u}_{i1} \ \hat{u}_{i2} \ \hat{u}_{i3}]^T$ is the (constant) pre-curvature of the rod, L_i is the rod length, and the matrix \mathbf{K}_i is defined by:

$$\mathbf{K}_i = \begin{bmatrix} K_{i1} & 0 & 0 \\ 0 & K_{i2} & 0 \\ 0 & 0 & K_{i3} \end{bmatrix}, \quad (15)$$

$$K_{i1} = E_i I_{i1}, \quad K_{i2} = E_i I_{i2}, \quad K_{i3} = G_i I_{i0}$$

in which E_i is the material Young's modulus, G_i is its shear modulus, I_{i1} (I_{i2} , resp.) is the area moment of inertia of the cross-section around \mathbf{d}_{i1} (\mathbf{d}_{i2} , resp.), and $I_{i0} = I_{i1} + I_{i2}$. The superscript “ c ” before the vectors \mathbf{u}_i and $\hat{\mathbf{u}}_i$ means that they are expressed in the frame \mathcal{F}_c . Introducing (12) into (14) leads to:

$$V_{ei} = \frac{1}{2} \int_0^{L_i} \left(\sum_{k=1}^3 K_{ik} (2 \mathbf{h}_i^T \mathbf{B}_k^T \mathbf{h}'_i - \hat{u}_{ik})^2 \right) ds \quad (16)$$

It is possible to approximate the expression of \mathbf{h}' by using finite differences [6]. Thus, let us discretize the rod into N rigid elements of equal length $\ell_{ei} = L_i/N_j$ (Fig. 3(b)). The orientation of the element i is defined by the quaternion \mathbf{h}_{ij} . The expression of the deformation energy becomes:

$$V_{ei} \approx \sum_{j=1}^{N_i} V_{eij}, \quad (17)$$

where

$$V_{eij} = \frac{1}{2} \left(\sum_{k=1}^3 K_{ik} \int_0^{\ell_{ei}} (2 \mathbf{h}_{ij}^T \mathbf{B}_k^T \mathbf{h}'_{ij} - \hat{u}_{ijk})^2 ds \right) \quad (18)$$

with \mathbf{h}'_{ij} being the derivative of the quaternion \mathbf{h}_{ij} of the element j , and \hat{u}_{ijk} is k th component of the precurvature vector ${}^c \hat{\mathbf{u}}_{ij}$ of the element j .

The expression of \mathbf{h}'_{ij} can be then approximated as follows:

$$\mathbf{h}'_{ij} = \frac{\mathbf{h}_{ij} - \mathbf{h}_{i(j-1)}}{\ell_{ei}}, \quad (19)$$

where, if the rod is assumed to be clamped at $s = 0$, \mathbf{h}_{i1} is a known constant and $\mathbf{h}_{i0} = \mathbf{h}_{i1}$ (i.e. $\mathbf{h}'_{i1} = \mathbf{0}$). Then, introducing (19) into (18), and integrating, leads to:

$$V_{eij} = \frac{1}{2} \left(\sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} (2 \mathbf{h}_{ij}^T \mathbf{B}_k^T (\mathbf{h}_{ij} - \mathbf{h}_{i(j-1)}) - \ell_{ei} \hat{u}_{ijk})^2 \right) \quad (20)$$

Let us now compute the potential energy due to the external wrenches. We assume that these wrenches are conservative and that only two types of external effects are applied on the robot:

- gravitational effects due to the gravity field \mathbf{g} and
- nodal wrenches.

The potential energy due to gravity is given by:

$$V_{gi} = -\mathbf{g}^T \rho_i \int_0^{L_i} \mathbf{p}_i(s) ds \quad (21)$$

with ρ_i the material linear density. Again,

- discretizing the rod into N_i rigid elements, and
- assuming that for the element j , $\mathbf{p}_i(s) \approx \mathbf{p}_{ij} + s \mathbf{d}_{ij3}$, where \mathbf{p}_{ij} is the position of the node j while \mathbf{d}_{ij3} is the third column of the matrix ${}^0 \mathbf{R}_{ij}$ representing the rotation of the element j wrt the frame \mathcal{F}_0 ,

the gravity potential energy is approximated by:

$$V_{gi} \approx \sum_{j=1}^{N_i} V_{gij}, \quad (22)$$

where

$$\begin{aligned} V_{gij} &= -\mathbf{g}^T \rho_i \int_0^{\ell_{ei}} (\mathbf{p}_{ij} + s \mathbf{d}_{ij3}) ds \\ &= -m_{ij} \mathbf{g}^T (\mathbf{p}_{ij} + \mathbf{d}_{ij3} \ell_{ei}/2) \end{aligned} \quad (23)$$

m_{ij} being the mass of the element j . It should be mentioned that the expression of \mathbf{p}_{ij} can be computed thanks to the following recursive formula:

$$\mathbf{p}_{ij} = \mathbf{p}_{i(j-1)} + \ell_{ei} \mathbf{d}_{i(j-1)3} \quad (24)$$

starting with $\mathbf{p}_{i0} = \mathbf{p}_{A_i}$ which is the known location of the point A_i . Thus, \mathbf{p}_{ij} is a function of \mathbf{h}_{i1} to $\mathbf{h}_{i(j-1)}$ and V_{gij} a function of \mathbf{h}_{i1} to \mathbf{h}_{ij} .

Let us finally denote as \mathbf{f}_{ij} the constant force applied at node j . The potential energy due to this force is given by:

$$V_{wij} = -\mathbf{f}_{ij}^T (\mathbf{p}_{ij} - \hat{\mathbf{p}}_{ij}) \quad (25)$$

where $(\hat{\cdot})$ corresponds to the state without deformation. Thus, V_{wij} a function of \mathbf{h}_{i1} to \mathbf{h}_{ij} .

As a result, the total potential energy is equal to:

$$V_{\text{rod}_i}(\mathbf{h}_i, \mathbf{h}_{i1}, \mathbf{p}_{A_i}, L_i) = \sum_{j=1}^{N_i} (V_{eij} + V_{gij} + V_{wij}) \quad (26)$$

It is a function of all orientation variables \mathbf{h}_{i1} and $\mathbf{h}_i = [\mathbf{h}_{i2}^T \dots \mathbf{h}_{iN_i}^T]^T$, of the rod length L_i and of the position of the point \mathbf{p}_{A_i} . Note that we deliberately exited \mathbf{h}_{i1} from the vector \mathbf{h}_i grouping all element orientation because \mathbf{h}_{i1} is a known constant when the motor position is fixed. If the motor is a revolute actuator located at A_i , then this constant depends on the motor position q_{ai} , i.e. $\mathbf{h}_{i1} = \mathbf{h}_{i1}(q_{ai})$. If the motor is a prismatic joint, then this is \mathbf{p}_{A_i} which depends on the motor position q_{ai} , i.e. $\mathbf{p}_{A_i} = \mathbf{p}_{A_i}(q_{ai})$; \mathbf{h}_{i1} is then a constant defined by design. Finally, if this is the length L_i of the rod which varies through actuation, L_i is a function of q_{ai} , i.e. $L_i = L_i(q_{ai})$. Thus, $V_{\text{rod}_i}(\mathbf{h}_i, \mathbf{h}_{i1}, \mathbf{p}_{A_i}, L_i) = V_{\text{rod}_i}(\mathbf{h}_i, q_{ai})$, and \mathbf{h}_{i1} is again a constant defined by design.

C. Potential energy of the closed-loop mechanism

Considering now the full continuum parallel robot, its total potential energy is given by:

$$V_{\text{tot}} = V_{\text{platform}} + \sum_{i=1}^n V_{\text{rod}_i}(\mathbf{h}_i, q_{ai}) \quad (27)$$

where V_{platform} is the potential energy of the platform considered here as rigid. Its expression is given by

$$V_{\text{platform}} = -\mathbf{f}_P^T (\mathbf{p}_P - \hat{\mathbf{p}}_P) \quad (28)$$

\mathbf{f}_P is a constant force exerted on the robot platform at point P (\mathbf{f}_P includes the effect of the gravity field – recall that P is considered to the platform center of mass). A force \mathbf{f}_Q applied on the platform at any point Q with location \mathbf{p}_Q could be added as well, using the potential energy $-\mathbf{f}_Q^T (\mathbf{p}_Q - \hat{\mathbf{p}}_Q) = \mathbf{f}_Q^T (\mathbf{p}_P + {}^0\mathbf{R}_P \overrightarrow{PQ} - \hat{\mathbf{p}}_Q)$, but we disregard it in the following of the computations. Moreover, 3D moment being non conservative [7], we do not consider them for the computation of the energy. However, they could easily be added later in Section III-A by using the Principle of Virtual Works. However, for reasons of writing clarity, we disregard them in the following of the report.

Obviously, due to the closed kinematic chain geometric constraints (and of the quaternion constraints), variables \mathbf{h} , \mathbf{p}_P and \mathbf{h}_P are related by the following expressions (for $j = 2, \dots, N_i$, $i = 1, \dots, n$):

$$c_{hij}(\mathbf{h}_{ij}) = \mathbf{h}_{ij}^T \mathbf{h}_{ij} - 1 = 0, \quad (29)$$

$$c_{hp}(\mathbf{h}_P) = \mathbf{h}_P^T \mathbf{h}_P - 1 = 0 \quad (30)$$

$$\begin{aligned} \mathbf{c}_{Bi}(\mathbf{p}_P, \mathbf{h}_P, \mathbf{h}_i, q_{ai}) &= \mathbf{p}_{B_i}(\mathbf{p}_P, \mathbf{h}_P) \\ &\quad - \mathbf{p}_{B_i}(\mathbf{h}_i, q_{ai}) = \mathbf{0} \end{aligned} \quad (31)$$

where $\mathbf{p}_{B_i}(\mathbf{h}_i, q_{ai})$ is the position of the rod i extremity that can be computed thanks to the expression (24) while $\mathbf{p}_{B_i}(\mathbf{p}_P, \mathbf{h}_P)$ is given by:

$$\mathbf{p}_{B_i}(\mathbf{p}_P, \mathbf{h}_P) = \mathbf{p}_P + {}^0\mathbf{R}_P \mathbf{r}_{PB_i} \quad (32)$$

in which \mathbf{r}_{PB_i} is the vector $\overrightarrow{PB_i}$ expressed in the platform frame, and ${}^0\mathbf{R}_P$ is the rotation matrix between the robot fixed frame \mathcal{F}_0 and the moving platform frame.

If the joint at B_i is spherical, Eqs. (29) to (31) are enough in order to characterize the constraints. This is no more the case if the joint is of another type:

- for a fixed joint, the orientation ${}^0\mathbf{R}_P$ of the platform frame and the orientation ${}^0\mathbf{R}_{iN_i}$ of the extremity of the link i (that depends on \mathbf{h}_{iN_i} only) are related by a predefined constant offset orientation, \mathbf{R}_{off_i} (depending on the platform design). This constant offset rotation leads to [3]

$${}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i} = \mathbf{R}_{off_i} \Rightarrow \mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i} = \mathbf{1}_3 \quad (33)$$

where $\mathbf{1}_3$ is the identity matrix of dimension 3. Constraint (33) has nine equations for constraining three rotations. Keeping in mind that the result of the product of several rotation matrices must be an orthogonal matrix, it is enough to check that three extra-diagonal terms of the product $\mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i}$ defined by $\mathbf{e}_j^T (\mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i}) \mathbf{e}_k$ (with $\mathbf{e}_1 = [1 \ 0 \ 0]^T$, $\mathbf{e}_2 =$

$[0 \ 1 \ 0]^T$ and $\mathbf{e}_3 = [0 \ 0 \ 1]^T$, for $\{j, k\} = \{2, 1\}$, $\{j, k\} = \{3, 2\}$ and $\{j, k\} = \{1, 3\}$) are all equal to 0, i.e. that:

$$\mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) = \begin{bmatrix} \mathbf{e}_2^T (\mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i}) \mathbf{e}_1 \\ \mathbf{e}_3^T (\mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i}) \mathbf{e}_2 \\ \mathbf{e}_1^T (\mathbf{R}_{off_i}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i}) \mathbf{e}_3 \end{bmatrix} = \mathbf{0} \quad (34)$$

- for a revolute joint of axis \mathbf{a}_i ($\mathbf{a}_i^T \mathbf{a}_i = 1$), it is necessary to have:

$${}^0\mathbf{R}_P^T \mathbf{a}_i = {}^0\mathbf{R}_{iN_i}^T {}^{iN_i} \mathbf{a}_i \quad (35)$$

where ${}^P \mathbf{a}_i = \mathbf{R}_{off_{i1}} \mathbf{e}_3$ is the (constant) value of the vector \mathbf{a}_i in the platform frame and ${}^{iN_i} \mathbf{a}_i = \mathbf{R}_{off_{i2}} \mathbf{e}_3$ is the (constant) expression of the same vector but in the frame of the last element of the rod i , matrices $\mathbf{R}_{off_{ij}}$ being two constant rotation matrices characterizing the orientation of the joint axis in the platform and leg extremity frames, respectively. We can thus rewrite the expression (35) under the form:

$$\begin{aligned} {}^0 \mathbf{a}_i &= {}^0\mathbf{R}_P \mathbf{R}_{off_{i1}} \mathbf{e}_3 = {}^0\mathbf{R}_{iN_i} \mathbf{R}_{off_{i2}} \mathbf{e}_3 \\ &\Rightarrow \mathbf{e}_3 = \mathbf{R}_{off_{i2}}^T {}^0\mathbf{R}_{iN_i}^T {}^0\mathbf{R}_P^T \mathbf{R}_{off_{i1}} \mathbf{e}_3 \end{aligned} \quad (36)$$

Constraint (36) has three equations for constraining two rotations. It is indeed enough to check that the two first components of the vector $\mathbf{R}_{off_{i1}}^T {}^0\mathbf{R}_P^T {}^0\mathbf{R}_{iN_i} \mathbf{R}_{off_{i2}} \mathbf{e}_3$ are all equal to 0, i.e. that:

$$\mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) = \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{bmatrix} \mathbf{R}_{off_{i2}}^T {}^0\mathbf{R}_{iN_i}^T {}^0\mathbf{R}_P^T \mathbf{R}_{off_{i1}} \mathbf{e}_3 = \mathbf{0} \quad (37)$$

In what follows, we stack all constraints in the vector Φ such that:

$$\Phi(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P, \mathbf{q}_a) = [c_{hl}^T \ c_{hp}^T \ \mathbf{c}_B^T \ \mathbf{c}_r^T]^T \quad (38)$$

in which

$$\mathbf{c}_{hl}(\mathbf{h}) = [c_{h1}^T \ \dots \ c_{hn}^T]^T \quad (39)$$

with $\mathbf{c}_{hi}(\mathbf{h}_i) = [c_{hi2} \ \dots \ c_{hiN_i}]^T$,

$$\mathbf{c}_B(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P, \mathbf{q}_a) = [c_{B1}^T \ \dots \ c_{Bn}^T]^T \quad (40)$$

and \mathbf{c}_r stacks the additional constraints (34) or (37), depending if the robot is made with either revolute or fixed joints on the platform. \mathbf{c}_r does not exist if all joints are spherical.

III. GEOMETRICO-STATIC AND KINEMATO-STATIC PROBLEMS

This Section is a brief summary of the equations defining the geometrico-static and kinemato-static models, whose detailed expressions are provided in Sections III and IV of [1].

A. Geometrico-static problem

In the static context, configurations are acceptable provided that for fixed motor positions \mathbf{q}_a they are stable with respect to internal and external forces. This leads to the definitions of the geometrico-static model, which relates fixed motor positions and stable configurations, and of the forward and inverse problems. As expected, we will see below that the number of

coordinates of the platform position and orientation that can be controlled is exactly the number $n = \dim \mathbf{q}_a$ of actuated motors. Therefore, we define $\mathbf{q}_p \in \mathbb{R}^n$ as the vector containing the coordinates to be controlled. Usually, it contains a subset of $(\mathbf{p}_p, \mathbf{h}_p)$ of the platform position and orientation coordinates, but it may contain other types of variables. We also stack all remaining coordinates amongst $(\mathbf{p}_p, \mathbf{h}_p)$ and \mathbf{h} to obtain the vector $\mathbf{q}_u \in \mathbb{R}^m$ of uncontrolled coordinates. For notational convenience, we furthermore define $\mathbf{q}_{pu} = (\mathbf{q}_p, \mathbf{q}_u) \in \mathbb{R}^{n+m}$ and $\mathbf{q} = (\mathbf{q}_a, \mathbf{q}_p, \mathbf{q}_u) \in \mathbb{R}^{n+n+m}$. These coordinates are related by the system of equations $\Phi(\mathbf{q}) = \mathbf{0}$. The number of equations in $\Phi(\mathbf{q}) = \mathbf{0}$ is denoted by $n_\Phi \leq n + m$.

Lagrange conditions provide the following characterization of local extrema: Under the condition that the gradients of the equality constraint are linearly independent, i.e., $\nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q})$ is full rank, \mathbf{q} is a local extrema if and only if there exist multipliers $\lambda \in \mathbb{R}^{n_\Phi}$ such that

$$\nabla_{\mathbf{q}_{pu}} V_{\text{tot}}(\mathbf{q}) + \nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q}) \lambda = \mathbf{0} \quad (41)$$

$$\Phi(\mathbf{q}) = \mathbf{0}. \quad (42)$$

This system of equations represents the implicit geometrico-static model and is valid as long as $\nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q})$ is of full rank. The explicit expressions of these equations can be obtained in an analytical way by using the expressions of the section IV. Note that the expression in the left-hand side of Equation (41) is the gradient $\nabla_{\mathbf{q}_{pu}} \mathcal{L}(\mathbf{q}, \lambda)$ of the so-called Lagrangian function

$$\mathcal{L}(\mathbf{q}, \lambda) = V_{\text{tot}}(\mathbf{q}) + \Phi(\mathbf{q})^T \lambda \quad (43)$$

associated to the equality constrained optimization problem.

B. Kinemato-static Model

The kinemato-static model of the robot can be obtained by differentiating the equations (41) and (42) wrt all variables $\mathbf{q}_a \in \mathbb{R}^n$, $\mathbf{q}_p \in \mathbb{R}^n$ and $\mathbf{q}_u \in \mathbb{R}^m$, and with respect to multipliers $\lambda \in \mathbb{R}^{n_\Phi}$ as well. We obtain the following linear relations relating the variations $\Delta \mathbf{q}_a$, $\Delta \mathbf{q}_p$, $\Delta \mathbf{q}_u$ and $\Delta \lambda$ of variables and multipliers:

$$\mathbf{A}_{\mathcal{L}} \Delta \mathbf{q}_a + \mathbf{P}_{\mathcal{L}} \Delta \mathbf{q}_p + \mathbf{U}_{\mathcal{L}} \Delta \mathbf{q}_u + \Lambda_{\mathcal{L}} \Delta \lambda = \mathbf{0} \quad (44)$$

$$\mathbf{A}_{\Phi} \Delta \mathbf{q}_a + \mathbf{P}_{\Phi} \Delta \mathbf{q}_p + \mathbf{U}_{\Phi} \Delta \mathbf{q}_u = \mathbf{0} \quad (45)$$

where

- $\mathbf{A}_{\mathcal{L}} = D_{\mathbf{q}_a} \nabla_{\mathbf{q}_{pu}} \mathcal{L}(\mathbf{q}, \lambda) \in \mathbb{R}^{(n+m) \times n}$,
- $\mathbf{P}_{\mathcal{L}} = D_{\mathbf{q}_p} \nabla_{\mathbf{q}_{pu}} \mathcal{L}(\mathbf{q}, \lambda) \in \mathbb{R}^{(n+m) \times n}$,
- $\mathbf{U}_{\mathcal{L}} = D_{\mathbf{q}_u} \nabla_{\mathbf{q}_{pu}} \mathcal{L}(\mathbf{q}, \lambda) \in \mathbb{R}^{(n+m) \times m}$,
- $\Lambda_{\mathcal{L}} = D_{\lambda} \nabla_{\mathbf{q}_{pu}} \mathcal{L}(\mathbf{q}, \lambda) = \nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q}) \in \mathbb{R}^{(n+m) \times n_\Phi}$,
- $\mathbf{A}_{\Phi} = D_{\mathbf{q}_a} \Phi(\mathbf{q}) \in \mathbb{R}^{n_\Phi \times n}$,
- $\mathbf{P}_{\Phi} = D_{\mathbf{q}_p} \Phi(\mathbf{q}) \in \mathbb{R}^{n_\Phi \times n}$, and
- $\mathbf{U}_{\Phi} = D_{\mathbf{q}_u} \Phi(\mathbf{q}) \in \mathbb{R}^{n_\Phi \times m}$.

The expressions of the gradients $\nabla_{\mathbf{q}_a} \Phi$, $\nabla_{\mathbf{q}_p} \Phi$ and $\nabla_{\mathbf{q}_u} \Phi$ are given in Section IV, while the expressions of the $\mathbf{A}_{\mathcal{L}}$, $\mathbf{P}_{\mathcal{L}}$ and $\mathbf{U}_{\mathcal{L}}$ are provided in Section V.

The system of equation (44), (45) is composed of $n+m+n_\Phi$ equations and $2n + m + n_\Phi$ unknowns, which are $\Delta \mathbf{q}_a$, $\Delta \mathbf{q}_p$, $\Delta \mathbf{q}_u$, $\Delta \lambda$. For the kinemato-static analysis, there is little interest to characterize the variations $\Delta \lambda$ wrt to the

others. Furthermore, as explained previously, the degeneracy of the matrix $\Lambda_{\mathcal{L}} = \nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q})$ is unlikely and uninteresting in practice. Therefore, it is worthy to eliminate the multipliers variations $\Delta \lambda$ from the system of equations, reaching this way a kinemato-static model with $n + m$ equations and $2n + m$ unknowns closer to the physical intuition. This can be done by left-multiplying the equation (44) by the transposed of the nullspace \mathbf{Z} of the matrix $\Lambda_{\mathcal{L}}^T = \nabla_{\mathbf{q}_{pu}} \Phi(\mathbf{q})^T$, thus leading to the wanted kinemato-static model, which relates directly the variations $\Delta \mathbf{q}_a$, $\Delta \mathbf{q}_p$ and $\Delta \mathbf{q}_u$:

$$\mathbf{Z}^T \mathbf{A}_{\mathcal{L}} \Delta \mathbf{q}_a + \mathbf{Z}^T \mathbf{P}_{\mathcal{L}} \Delta \mathbf{q}_p + \mathbf{Z}^T \mathbf{U}_{\mathcal{L}} \Delta \mathbf{q}_u = \mathbf{0} \quad (46)$$

$$\mathbf{A}_{\Phi} \Delta \mathbf{q}_a + \mathbf{P}_{\Phi} \Delta \mathbf{q}_p + \mathbf{U}_{\Phi} \Delta \mathbf{q}_u = \mathbf{0} \quad (47)$$

Grouping equations (46) and (47), we obtain the final expression of the kinemato-static model

$$\mathbf{A} \Delta \mathbf{q}_a + \mathbf{P} \Delta \mathbf{q}_p + \mathbf{U} \Delta \mathbf{q}_u = \mathbf{0}, \quad (48)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{Z}^T \mathbf{A}_{\mathcal{L}} \\ \mathbf{A}_{\Phi} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} \mathbf{Z}^T \mathbf{P}_{\mathcal{L}} \\ \mathbf{P}_{\Phi} \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} \mathbf{Z}^T \mathbf{U}_{\mathcal{L}} \\ \mathbf{U}_{\Phi} \end{bmatrix}. \quad (49)$$

Next Sections deals with the computations of all gradients and Hessian necessary for writing the geometric-static and kinemato-static models of the continuum parallel robot.

IV. GRADIENTS OF THE POTENTIAL ENERGY AND OF THE CONSTRAINTS

In this section, we provide the expressions of the gradients of the potential energy and of the constraints, based on the expressions used in Section II. These expressions are used for computing the geometrico-static model of the robot. We start with the gradients of the potential energy.

A. Gradients of the potential energy

The gradients of the potential energy wrt the variables $(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$ are given by:

$$\nabla_{\mathbf{p}_P} V_{\text{tot}} = -\mathbf{f}_P \quad (50)$$

$$\nabla_{\mathbf{h}_P} V_{\text{tot}} = -\mathbf{0} \quad (51)$$

$$\nabla_{\mathbf{h}} V_{\text{tot}} = \begin{bmatrix} \nabla_{\mathbf{h}_1} (V_{e1} + V_{g1} + V_{w1}) \\ \vdots \\ \nabla_{\mathbf{h}_n} (V_{en} + V_{gn} + V_{wn}) \end{bmatrix} \quad (52)$$

The different terms in $\nabla_{\mathbf{h}} V_{\text{tot}}$ are detailed below. First, the gradient $\nabla_{\mathbf{h}_i} V_{ei}$ takes the following form:

$$\nabla_{\mathbf{h}_i} V_{ei} = \begin{bmatrix} a_{i1} + b_{i2} \\ \vdots \\ a_{i(N_i-2)} + b_{i(N_i-1)} \\ a_{i(N_i-1)} \end{bmatrix} \quad (53)$$

in which

$$a_{ij} = 2 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} \mathbf{B}_k \mathbf{h}_{ij} A_{ijk} \quad (54)$$

$$b_{ij} = -2 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} \mathbf{B}_k \mathbf{h}_{i(j+1)} A_{ijk} \quad (55)$$

for

$$A_{ijk} = -2 \mathbf{h}_{i(j+1)}^T \mathbf{B}_k^T \mathbf{h}_{ij} - \hat{u}_{ijk}, \quad (56)$$

where matrices \mathbf{B}_k are defined in (13).

The gradient $\nabla_{\mathbf{h}_i} V_{gi}$ in (52) is given by

$$\nabla_{\mathbf{h}_i} V_{gi} = - \sum_{j=2}^{N_i} m_{ij} (\nabla_{\mathbf{h}_i} \mathbf{p}_{ij}^T + \frac{\ell_{ei}}{2} \nabla_{\mathbf{h}_i} \mathbf{d}_{ij3}^T) \mathbf{g} \quad (57)$$

where

$$(\nabla_{\mathbf{h}_i} \mathbf{d}_{ij3}^T)^T = \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} \\ \mathbf{0}_{3 \times 4} & \dots & \mathbf{D}_{ij3} & \dots & \mathbf{0}_{3 \times 4} \end{bmatrix} \quad (58)$$

and

$$(\nabla_{\mathbf{h}_i} \mathbf{p}_{ij}^T)^T = \ell_{ei} \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{i(j-1)} & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} \\ \mathbf{D}_{i2,3} & \dots & \mathbf{D}_{i(j-1),3} & \mathbf{0}_{3 \times 4} & \dots & \mathbf{0}_{3 \times 4} \end{bmatrix} \quad (59)$$

where the expressions $\mathbf{D}_{i(j-1),3}$ can be obtained from (11). Finally, the gradient $\nabla_{\mathbf{h}_i} V_{wi}$ in (52) can be written as

$$\nabla_{\mathbf{h}_i} V_{wi} = - \sum_{j=2}^{N_i} ((\nabla_{\mathbf{h}_i} \mathbf{p}_{ij}^T)^T \mathbf{f}_{ij}) \quad (60)$$

B. Gradients of the constraints with respect to the variables $(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$

The gradients of the constraints Φ wrt the variables $(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$ can be decomposed as follows:

$$\nabla_{\mathbf{h}} \Phi = [\nabla_{\mathbf{h}} \mathbf{c}_{hl} \quad \nabla_{\mathbf{h}} \mathbf{c}_{hp} \quad \nabla_{\mathbf{h}} \mathbf{c}_B \quad \nabla_{\mathbf{h}} \mathbf{c}_r] \quad (61)$$

$$\nabla_{\mathbf{p}_P} \Phi = [\nabla_{\mathbf{p}_P} \mathbf{c}_{hl} \quad \nabla_{\mathbf{p}_P} \mathbf{c}_{hp} \quad \nabla_{\mathbf{p}_P} \mathbf{c}_B \quad \nabla_{\mathbf{p}_P} \mathbf{c}_r] \quad (62)$$

$$\nabla_{\mathbf{h}_P} \Phi = [\nabla_{\mathbf{h}_P} \mathbf{c}_{hl} \quad \nabla_{\mathbf{h}_P} \mathbf{c}_{hp} \quad \nabla_{\mathbf{h}_P} \mathbf{c}_B \quad \nabla_{\mathbf{h}_P} \mathbf{c}_r] \quad (63)$$

where $\nabla_{\mathbf{h}} \mathbf{c}_{hp} = \mathbf{0}$, $\nabla_{\mathbf{p}_P} \mathbf{c}_{hl} = \mathbf{0}$, $\nabla_{\mathbf{p}_P} \mathbf{c}_{hp} = \mathbf{0}$, $\nabla_{\mathbf{p}_P} \mathbf{c}_r = \mathbf{0}$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_{hl} = \mathbf{0}$, \mathbf{c}_{hl} , \mathbf{c}_{hp} , \mathbf{c}_B and \mathbf{c}_r being defined in Section II-C. The other gradients appearing in these expressions are detailed below. First, the expression of $\nabla_{\mathbf{h}} \mathbf{c}_{hl}$ is given by

$$\nabla_{\mathbf{h}} \mathbf{c}_{hl} = \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{h1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{h2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{hn} \end{bmatrix} \quad (64)$$

with

$$\nabla_{\mathbf{h}_i} \mathbf{c}_{hi} = 2 \begin{bmatrix} \mathbf{h}_{i2} & \mathbf{0}_{4 \times 1} & \dots & \mathbf{0}_{4 \times 1} \\ \mathbf{0}_{4 \times 1} & \mathbf{h}_{i3} & \dots & \mathbf{0}_{4 \times 1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{4 \times 1} & \mathbf{0}_{4 \times 1} & \dots & \mathbf{h}_{iN_i} \end{bmatrix} \quad (65)$$

$\nabla_{\mathbf{h}} \mathbf{c}_B$ can be decomposed as follows:

$$\nabla_{\mathbf{h}} \mathbf{c}_B = \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bn} \end{bmatrix} \quad (66)$$

with

$$(\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi})^T = -\ell_{ei} [\mathbf{D}_{i2,3} \quad \dots \quad \mathbf{D}_{iN_i,3}] \quad (67)$$

The expressions of $\nabla_{\mathbf{p}_P} \mathbf{c}_B$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_B$ are:

$$\nabla_{\mathbf{p}_P} \mathbf{c}_B = [\mathbf{1}_3 \quad \dots \quad \mathbf{1}_3] \quad (68)$$

$$\nabla_{\mathbf{h}_P} \mathbf{c}_B = [\nabla_{\mathbf{h}_P} \mathbf{c}_{B1} \quad \dots \quad \nabla_{\mathbf{h}_P} \mathbf{c}_{Bn}] \quad (69)$$

with $\mathbf{1}_3$ the identity matrix of dimension 3. $\nabla_{\mathbf{h}_P} \mathbf{c}_{hp}$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_{Bi}$ take the following form:

$$\nabla_{\mathbf{h}_P} \mathbf{c}_{hp} = 2 \mathbf{h}_P, \quad (70)$$

$$\nabla_{\mathbf{h}_P} \mathbf{c}_{Bi} = \sum_{k=1}^3 r_{ik} \mathbf{D}_{pk}^T \quad (71)$$

where $\mathbf{r}_{PB_i} = [r_{i1} \ r_{i2} \ r_{i3}]^T$ and \mathbf{D}_{pk} can be found by using the definitions of the matrices in (9) to (11) applied to the quaternion \mathbf{h}_P .

Finally, the gradients $\nabla_{\mathbf{h}} \mathbf{c}_r$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_r$ exist if only if the joint at B_i is not spherical.

- In case the joint at the tip of leg i is rigid, then \mathbf{c}_{ri} can be rewritten as

$$\mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) = \begin{bmatrix} \mathbf{e}_2^T \left(\mathbf{R}_{offi}^T \mathbf{0} \mathbf{R}_P^T \mathbf{d}_{1,iN_i} \right) \\ \mathbf{e}_3^T \left(\mathbf{R}_{offi}^T \mathbf{0} \mathbf{R}_P^T \mathbf{d}_{2,iN_i} \right) \\ \mathbf{e}_1^T \left(\mathbf{R}_{offi}^T \mathbf{0} \mathbf{R}_P^T \mathbf{d}_{3,iN_i} \right) \end{bmatrix} = \mathbf{0} \quad (72)$$

where $\mathbf{0} \mathbf{R}_{iN_i} = [\mathbf{d}_{1,iN_i} \ \mathbf{d}_{2,iN_i} \ \mathbf{d}_{3,iN_i}]$. Moreover, denoting the result of the product $\mathbf{e}_k^T \mathbf{R}_{offi}^T$ as $\mathbf{e}_k^T \mathbf{R}_{offi}^T = [\beta_{1ki} \ \beta_{2ki} \ \beta_{3ki}]$, and taking into account that $\mathbf{0} \mathbf{R}_P = [\mathbf{d}_{1,P} \ \mathbf{d}_{2,P} \ \mathbf{d}_{3,P}]$, then

$$\mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) = \begin{bmatrix} (\beta_{12i} \mathbf{d}_{1,P} + \beta_{22i} \mathbf{d}_{2,P} + \beta_{32i} \mathbf{d}_{3,P})^T \mathbf{d}_{1,iN_i} \\ (\beta_{13i} \mathbf{d}_{1,P} + \beta_{23i} \mathbf{d}_{2,P} + \beta_{33i} \mathbf{d}_{3,P})^T \mathbf{d}_{2,iN_i} \\ (\beta_{11i} \mathbf{d}_{1,P} + \beta_{21i} \mathbf{d}_{2,P} + \beta_{31i} \mathbf{d}_{3,P})^T \mathbf{d}_{3,iN_i} \end{bmatrix} \quad (73)$$

Finally, using (9) to (10), and noticing that the only non null gradients for \mathbf{c}_{ri} are $\nabla_{\mathbf{h}_{iN_i}} \mathbf{c}_{ri}$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_{ri}$, we found that:

$$\nabla_{\mathbf{h}_{iN_i}} \mathbf{c}_{ri} = [\mathbf{D}_{iN_i,1}^T \gamma_{2i} \quad \mathbf{D}_{iN_i,2}^T \gamma_{3i} \quad \mathbf{D}_{iN_i,3}^T \gamma_{1i}] \quad (74)$$

with $\gamma_{ki} = \beta_{1ki} \mathbf{d}_{1,P} + \beta_{2ki} \mathbf{d}_{2,P} + \beta_{3ki} \mathbf{d}_{3,P}$. Then, noticing that $\gamma_{ki} = \mathbf{d}_{k,iN_i}$ (because, from (33), $\mathbf{e}_k^T (\mathbf{R}_{offi}^T \mathbf{0} \mathbf{R}_P^T \mathbf{R}_{iN_i}) \mathbf{e}_k = \gamma_{ki}^T \mathbf{d}_{k,iN_i} = 1$ and $\gamma_{ki} = \mathbf{e}_k^T \mathbf{R}_{offi}^T \mathbf{0} \mathbf{R}_P^T$ is necessarily a unit vector), then

$$\begin{aligned} \nabla_{\mathbf{h}_{iN_i}} \mathbf{c}_{ri} &= [\mathbf{D}_{iN_i,1}^T \mathbf{d}_{2,iN_i} \quad \mathbf{D}_{iN_i,2}^T \mathbf{d}_{3,iN_i} \quad \mathbf{D}_{iN_i,3}^T \mathbf{d}_{1,iN_i}] \\ &= 2 \begin{bmatrix} -h_{iN_i,4} & -h_{iN_i,2} & -h_{iN_i,3} \\ h_{iN_i,3} & h_{iN_i,1} & -h_{iN_i,4} \\ -h_{iN_i,2} & h_{iN_i,4} & h_{iN_i,1} \\ h_{iN_i,1} & -h_{iN_i,3} & h_{iN_i,2} \end{bmatrix} \end{aligned} \quad (75)$$

Similarly, it also comes that

$$\nabla_{\mathbf{h}_P} \mathbf{c}_{ri} = [\Psi_{i1} \quad \Psi_{i2} \quad \Psi_{i3}] \quad (76)$$

with $\Psi_{ij} = (\beta_{1ki} \mathbf{D}_{P,1} + \beta_{2ki} \mathbf{D}_{P,2} + \beta_{3ki} \mathbf{D}_{P,3})^T \mathbf{d}_{j,iN_i}$, where $\{k, j\} = \{2, 1\}$, $\{k, j\} = \{3, 2\}$ or $\{k, j\} = \{1, 3\}$.

- In case the joint at the tip of leg i is revolute, then \mathbf{c}_{ri} can be rewritten as

$$\begin{aligned} \mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) &= \begin{bmatrix} \mathbf{e}_3^T \mathbf{R}_{offi1}^T & {}^0 \mathbf{R}_P^T {}^0 \mathbf{R}_{iN_i} & \mathbf{R}_{offi2} \mathbf{e}_1 \\ \mathbf{e}_3^T \mathbf{R}_{offi1}^T & {}^0 \mathbf{R}_P^T {}^0 \mathbf{R}_{iN_i} & \mathbf{R}_{offi2} \mathbf{e}_2 \end{bmatrix} \\ &= \begin{bmatrix} {}^0 \mathbf{a}_i^T {}^0 \mathbf{R}_{iN_i} & \mathbf{R}_{offi2} \mathbf{e}_1 \\ {}^0 \mathbf{a}_i^T {}^0 \mathbf{R}_{iN_i} & \mathbf{R}_{offi2} \mathbf{e}_2 \end{bmatrix} \end{aligned} \quad (77)$$

Denoting the result of the product $\mathbf{e}_k^T \mathbf{R}_{offi1}^T$ as $\mathbf{e}_k^T \mathbf{R}_{offi2}^T = [\beta_{1ki2} \quad \beta_{2ki2} \quad \beta_{3ki2}] = \beta_{k2}^T$ (which is a constant vector), and taking into account that ${}^0 \mathbf{R}_{iN_i} = [\mathbf{d}_{1,iN_i} \quad \mathbf{d}_{2,iN_i} \quad \mathbf{d}_{3,iN_i}]$, then

$$\mathbf{c}_{ri}(\mathbf{h}_P, \mathbf{h}_{iN_i}) = \begin{bmatrix} {}^0 \mathbf{a}_i^T {}^0 \mathbf{R}_{iN_i} \beta_{12} \\ {}^0 \mathbf{a}_i^T {}^0 \mathbf{R}_{iN_i} \beta_{22} \end{bmatrix} \quad (78)$$

Using (9) to (10), and noticing that the only non null gradients for \mathbf{c}_{ri} are $\nabla_{\mathbf{h}_{iN_i}} \mathbf{c}_{ri}$ and $\nabla_{\mathbf{h}_P} \mathbf{c}_{ri}$, we found that:

$$\nabla_{\mathbf{h}_{iN_i}} \mathbf{c}_{ri} = [\eta_{i1} \quad \eta_{i2}] \quad (79)$$

with $\eta_{ij} = (\beta_{1ki2} \mathbf{D}_{iN_i,1} + \beta_{2ki2} \mathbf{D}_{iN_i,2} + \beta_{3ki2} \mathbf{D}_{iN_i,3})^T {}^0 \mathbf{a}_i$.

Similarly, it also comes that

$$\nabla_{\mathbf{h}_P} \mathbf{c}_{ri} = [\mu_{i1} \quad \mu_{i2}] \quad (80)$$

with $\mu_{ik} = (a_{i1} \mathbf{D}_{P,1} + a_{i2} \mathbf{D}_{P,2} + a_{i3} \mathbf{D}_{P,3})^T \beta_{k2}$, where ${}^0 \mathbf{a}_i = [a_{i1} \quad a_{i2} \quad a_{i3}]^T$.

C. Gradients of the constraints with respect to the variables \mathbf{q}_a

The robot being composed of one motor per leg, the gradient $\nabla_{\mathbf{q}_a} \Phi$ takes the following form

$$\nabla_{\mathbf{q}_a} \Phi = [\nabla_{\mathbf{q}_a} \mathbf{c}_{hl} \quad \nabla_{\mathbf{q}_a} \mathbf{c}_{hp} \quad \nabla_{\mathbf{q}_a} \mathbf{c}_B \quad \nabla_{\mathbf{q}_a} \mathbf{c}_r] \quad (81)$$

where $\nabla_{\mathbf{q}_a} \mathbf{c}_{hl} = \mathbf{0}$, $\nabla_{\mathbf{q}_a} \mathbf{c}_{hp} = \mathbf{0}$ and $\nabla_{\mathbf{q}_a} \mathbf{c}_r = \mathbf{0}$. $\nabla_{\mathbf{q}_a} \mathbf{c}_B$ can be decomposed as follows:

$$\nabla_{\mathbf{q}_a} \mathbf{c}_B = \begin{bmatrix} \nabla_{q_{a1}} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{q_{a2}} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{q_{an}} \mathbf{c}_{Bn} \end{bmatrix} \quad (82)$$

The expression of $\nabla_{q_{ai}} \mathbf{c}_{Bi}$ will take different forms, depending on the choice of the motor for the leg:

- If the leg i is connected to a prismatic joint (Fig. 2(b))

$$\nabla_{q_{ai}} \mathbf{c}_{Bi} = -\nabla_{q_{ai}} \mathbf{p}_{A_i}^T \quad (83)$$

- If the leg i is connected to a revolute joint (Fig. 2(a))

$$\nabla_{q_{ai}} \mathbf{c}_{Bi} = -\ell_{ei} \nabla_{q_{ai}} \mathbf{d}_{i13}^T \quad (84)$$

- If the motor changes the length of the leg i (Fig. 2(c))

$$\nabla_{q_{ai}} \mathbf{c}_{Bi} = -\frac{1}{N_i} \sum_{j=1}^{N_i} \mathbf{d}_{i13}^T \quad (85)$$

V. HESSIAN OF THE LAGRANGIAN, AND DERIVATIVE OF $\nabla_{\mathbf{q}_{pu}} \mathcal{L}$ WITH RESPECT TO \mathbf{q}_a

In this section, we provide the expressions of the Hessian of the Lagrangian wrt variables in $(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$, and of the derivative of $\nabla_{\mathbf{q}_{pu}} \mathcal{L}$ with respect to \mathbf{q}_a , based on the expressions used in Section II. These expressions are used for computing the geometrico-static model of the robot. We start with the Hessian of the potential energy.

A. Hessian of the potential energy

The Hessian \mathbf{H}_V of the potential energy wrt the variables $(\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$ can be decomposed as follows:

$$\mathbf{H}_V = \begin{bmatrix} \nabla_{\mathbf{h}, \mathbf{h}}^2 V_{\text{tot}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{p}_P, \mathbf{p}_P}^2 V_{\text{tot}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \nabla_{\mathbf{h}_P, \mathbf{h}_P}^2 V_{\text{tot}} \end{bmatrix} \quad (86)$$

in which

$$\nabla_{\mathbf{p}_P, \mathbf{p}_P}^2 V_{\text{tot}} = \mathbf{0} \quad (87)$$

$$\nabla_{\mathbf{h}_P, \mathbf{h}_P}^2 V_{\text{tot}} = \mathbf{0} \quad (88)$$

$$\nabla_{\mathbf{h}, \mathbf{h}}^2 V_{\text{tot}} = \begin{bmatrix} \nabla_{\mathbf{h}_1, \mathbf{h}_1}^2 V_{\text{tot}} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \nabla_{\mathbf{h}_n, \mathbf{h}_n}^2 V_{\text{tot}} \end{bmatrix} \quad (89)$$

where $\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{\text{tot}} = \nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (V_{ei} + V_{gi} + V_{wi})$.

The Hessian $\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{ei}$ is given by

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{ei} = \begin{bmatrix} \mathbf{E}_{i1} & \mathbf{F}_{i2}^T & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{i2} & \mathbf{E}_{i2} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{E}_{i(N-2)} & \mathbf{F}_{i(N-1)}^T \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{F}_{i(N-1)} & \mathbf{E}_{i(N-1)} \end{bmatrix} \quad (90)$$

in which

$$\mathbf{E}_{ij} = \mathbf{M}_{ij} + \mathbf{N}_{i(j+1)} \quad (91)$$

$$\mathbf{F}_{ij} = -4 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} ((\mathbf{B}_k \mathbf{h}_{ij})(\mathbf{B}_k \mathbf{h}_{i(j+1)})^T - \mathbf{B}_k A_{ijk}/2) \quad (92)$$

$$\mathbf{M}_{ij} = -4 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} (\mathbf{B}_k \mathbf{h}_{ij})(\mathbf{B}_k^T \mathbf{h}_{ij})^T \quad (93)$$

$$\mathbf{N}_{ij} = 4 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} (\mathbf{B}_k \mathbf{h}_{i(j+1)})(\mathbf{B}_k \mathbf{h}_{i(j+1)})^T \quad (94)$$

The Hessian $\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{gi}$ takes the expression

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{gi} = - \sum_{j=2}^{N_i} m_{ij} (\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{p}_{ij}^T \mathbf{g}) + \frac{\ell_e}{2} \nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ij3}^T \mathbf{g})) \quad (95)$$

where

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ij3}^T \mathbf{g}) = \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{\Gamma} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{h}_{i2} \\ \mathbf{h}_{ij} \\ \mathbf{h}_{iN_i} \end{matrix} \quad (96)$$

with

$$\Gamma = 2 \begin{bmatrix} g_z & -g_y & g_x & 0 \\ -g_y & -g_z & 0 & g_x \\ g_x & 0 & -g_z & g_y \\ 0 & g_x & g_y & g_z \end{bmatrix} \quad (97)$$

in which g_x , g_y and g_z are the three components of \mathbf{g} in the world frame, the gravity field, defined by $\mathbf{g} = [g_x \ g_y \ g_z]^T$, and,

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{p}_{ij}^T \mathbf{g}) = \ell_{ei} \sum_{k=2}^{j-1} \nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ik3}^T \mathbf{g}) \quad (98)$$

Finally, the Hessian $\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{wi}$ in (89) can be written as

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 V_{wi} = - \sum_{j=2}^{N_i} (\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{p}_{ij}^T \mathbf{f}_{ij})) \quad (99)$$

where

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{p}_{ij}^T \mathbf{f}_{ij}) = \ell_{ei} \sum_{k=1}^{j-1} \nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ik3}^T \mathbf{f}_{ij}) \quad (100)$$

with

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ij3}^T \mathbf{f}_{ij}) = \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \Gamma_{ij} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{h}_{i2} \\ \mathbf{h}_{ij} \\ \mathbf{h}_{iN_i} \end{matrix} \quad (101)$$

in which

$$\Gamma_{ij} = 2 \begin{bmatrix} f_{ijz} & -f_{ijy} & f_{ijx} & 0 \\ -f_{ijy} & -f_{ijz} & 0 & f_{ijx} \\ f_{ijx} & 0 & -f_{ijz} & f_{ijy} \\ 0 & f_{ijx} & f_{ijy} & f_{ijz} \end{bmatrix} \quad (102)$$

B. Hessian of the Lagrangian

The Hessian \mathbf{H} of the Lagrangian wrt the variables $\mathbf{q}_r = (\mathbf{h}, \mathbf{p}_P, \mathbf{h}_P)$ is equal to

$$\mathbf{H} = \mathbf{H}_V + \sum_{k=1}^{m_\phi} \lambda_k \nabla_{\mathbf{q}_r, \mathbf{q}_r}^2 \Phi_k \quad (103)$$

where Φ_k is the k th component of Φ . Let us compute the Hessians of the different constraints. The Hessian of the constraint c_{hij} in (29) is given by:

$$\nabla_{\mathbf{q}_r, \mathbf{q}_r}^2 c_{hij} = 2 \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} & \mathbf{p}_P & \mathbf{h}_P \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{1}_4 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{h}_{i2} \\ \mathbf{h}_{ij} \\ \mathbf{h}_{iN_i} \\ \mathbf{p}_P \\ \mathbf{h}_P \end{matrix} \quad (104)$$

where $\mathbf{1}_4$ is the identity matrix of dimension 4. The Hessian of the constraint c_{hp} in (30) is given by:

$$\nabla_{\mathbf{q}_r, \mathbf{q}_r}^2 c_{hp} = 2 \begin{bmatrix} \mathbf{h} & \mathbf{p}_P & \mathbf{h}_P \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_4 \end{bmatrix} \begin{matrix} \mathbf{h} \\ \mathbf{p}_P \\ \mathbf{h}_P \end{matrix} \quad (105)$$

The Hessian of the constraint c_{Bi} in (31) is given by:

$$\nabla_{\mathbf{q}_r, \mathbf{q}_r}^2 \mathbf{c}_{Bi}^{(k)} = \begin{bmatrix} \mathbf{h}_1 & \dots & \mathbf{h}_i & \dots & \mathbf{h}_n & \mathbf{p}_P & \mathbf{h}_P \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{C}_{bij}^{(k)} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{C}_p^{(k)} \end{bmatrix} \begin{matrix} \mathbf{h}_1 \\ \mathbf{h}_i \\ \mathbf{h}_n \\ \mathbf{p}_P \\ \mathbf{h}_P \end{matrix} \quad (106)$$

$\mathbf{c}_{Bi}^{(k)}$ being the k th component of \mathbf{c}_{Bi} , with

$$\mathbf{C}_{bij}^{(k)} = -\ell_{ei} \sum_{t=2}^{N_i} \nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{it3}^T \mathbf{l}_k) \quad (107)$$

where \mathbf{l}_k is defined by $\mathbf{l}_k = [l_{1k} \ l_{2k} \ l_{3k}]^T$, with $l_{mk} = \delta_{mk}$, δ_{mk} being the Kronecker symbol, and

$$\nabla_{\mathbf{h}_i, \mathbf{h}_i}^2 (\mathbf{d}_{ij3}^T \mathbf{l}_k) = \begin{bmatrix} \mathbf{h}_{i2} & \dots & \mathbf{h}_{ij} & \dots & \mathbf{h}_{iN_i} \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \Omega_{k3} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{h}_{i2} \\ \mathbf{h}_{ij} \\ \mathbf{h}_{iN_i} \end{matrix} \quad (108)$$

in which

$$\Omega_{k3} = 2 \begin{bmatrix} l_{3k} & -l_{2k} & l_{1k} & 0 \\ -l_{2k} & -l_{3k} & 0 & l_{1k} \\ l_{1k} & 0 & -l_{3k} & l_{2k} \\ 0 & l_{1k} & l_{2k} & l_{3k} \end{bmatrix} \quad (109)$$

and

$$\mathbf{C}_p^{(k)} = \sum_{t=1}^3 r_{it} \Omega_{kt} \quad (110)$$

in which Ω_{k3} is provided in (109) and

$$\Omega_{k1} = 2 \begin{bmatrix} l_{1k} & 0 & -l_{3k} & l_{2k} \\ 0 & l_{1k} & l_{2k} & l_{3k} \\ -l_{3k} & l_{2k} & -l_{1k} & 0 \\ l_{2k} & l_{3k} & 0 & -l_{1k} \end{bmatrix} \quad (111)$$

$$\Omega_{k2} = 2 \begin{bmatrix} l_{2k} & l_{3k} & 0 & -l_{1k} \\ l_{3k} & -l_{2k} & l_{1k} & 0 \\ 0 & l_{1k} & l_{2k} & l_{3k} \\ -l_{1k} & 0 & l_{3k} & -l_{2k} \end{bmatrix} \quad (112)$$

Finally, the Hessians $\nabla_{\mathbf{q}_r, \mathbf{q}_r}^2 \mathbf{c}_{ri}^{(k)}$ (k being the k th component of \mathbf{c}_{ri}) do not exist if the joint at B_i is spherical. We do

not provide their expressions for the other types of joints, but they can be straightforwardly derived from expressions (34) and (37).

C. Derivative of $\nabla_{\mathbf{q}_{pu}} \mathcal{L}$ with respect to \mathbf{q}_a

The robot being composed of one motor per leg, the derivative of $\nabla_{\mathbf{q}_{pu}} \mathcal{L}$ with respect to \mathbf{q}_a can be decomposed as follows:

$$D_{\mathbf{q}_a} \nabla_{\mathbf{q}_{pu}} \mathcal{L} = \begin{bmatrix} \frac{\partial}{\partial q_{a1}} \nabla_{\mathbf{h}_1} \mathcal{L} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \frac{\partial}{\partial q_{an}} \nabla_{\mathbf{h}_n} \mathcal{L} \\ \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} \quad (113)$$

The expression of $\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \mathcal{L}$ will take different forms, depending on the choice of the motor for the leg:

- If the leg i is connected to a prismatic joint (Fig. 2(b))

$$\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \mathcal{L} = \mathbf{0} \quad (114)$$

- If the leg i is connected to a revolute joint (Fig. 2(a))

$$\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \mathcal{L} = \begin{bmatrix} \mathbf{F}_{i1} \\ \mathbf{0} \end{bmatrix} \frac{\partial \mathbf{h}_{i1}}{\partial q_{ai}} \quad (115)$$

\mathbf{F}_{i1} being defined in (92);

- If the motor changes the length of the leg i (Fig. 2(c))

$$\begin{aligned} \frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \mathcal{L} &= \frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} V_{ei} + \frac{1}{N_i} \nabla_{\mathbf{h}_i} V_{gi} \\ &- \frac{1}{N_i} \sum_{j=2}^{N_i} (\nabla_{\mathbf{h}_i} \mathbf{p}_{ij})^T \mathbf{f}_{ij} + \frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \Phi \lambda \end{aligned} \quad (116)$$

where

$$\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} V_{ei} = \begin{bmatrix} \frac{\partial a_{i1}}{\partial q_{ai}} + \frac{\partial b_{i2}}{\partial q_{ai}} \\ \vdots \\ \frac{\partial a_{i(N_i-2)}}{\partial q_{ai}} + \frac{\partial b_{i(N_i-1)}}{\partial q_{ai}} \\ \frac{\partial a_{i(N_i-1)}}{\partial q_{ai}} \end{bmatrix} \quad (117)$$

in which

$$\frac{\partial a_{ij}}{\partial q_{ai}} = 2 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} \mathbf{B}_k \mathbf{h}_{ij} \left(-\frac{N_i}{\ell_{ei}} A_{ijk} + \hat{u}_{ijk} \right) \quad (118)$$

$$\frac{\partial b_{ij}}{\partial q_{ai}} = 2 \sum_{k=1}^3 \frac{K_{ik}}{\ell_{ei}} \mathbf{B}_k \mathbf{h}_{i(j+1)} \left(\frac{N_i}{\ell_{ei}} A_{ijk} - \hat{u}_{ijk} \right) \quad (119)$$

and the only non null term in $\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \Phi$ is $\frac{\partial}{\partial q_{ai}} \nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$ which is equal to $\frac{1}{N_i} \nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$. The expressions of the other terms in (116) can be found in the previous sections.

VI. STUDY OF THE LOSS OF RANK OF THE GRADIENT OF THE CONSTRAINTS

A. Gradient of the constraints \mathbf{c}_{hl} and \mathbf{c}_{hp}

Due to its block-diagonal structure, $\nabla_{\mathbf{h}} \mathbf{c}_{hl}$ if and only if one block on the diagonal is degenerated. However, those blocks are given in (65) are made of a stacking by blocks on the diagonal the quaternions $\mathbf{h}_i j$. The only possibility for $\nabla_{\mathbf{h}} \mathbf{c}_{hl}$

to be rank deficient is thus that at least a quaternion $\mathbf{h}_i j$ is null, which can of course never appear.

A similar condition can be found for \mathbf{c}_{hp} . Thus, the gradients of the constraints \mathbf{c}_{hl} and \mathbf{c}_{hp} can never be rank deficient. This is not the case of the gradient of the constraints \mathbf{c}_B which is studied in the next Section.

B. Gradient of the constraint \mathbf{c}_B

Let us rewrite the gradient of the constraint \mathbf{c}_B below:

$$\begin{bmatrix} \nabla_{\mathbf{h}} \mathbf{c}_B \\ \nabla_{\mathbf{p}_P} \mathbf{c}_B \\ \nabla_{\mathbf{h}_P} \mathbf{c}_B \end{bmatrix} = \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bn} \\ \mathbf{1}_3 & \mathbf{1}_3 & \dots & \mathbf{1}_3 \\ \nabla_{\mathbf{h}_P} \mathbf{c}_{B1} & \nabla_{\mathbf{h}_P} \mathbf{c}_{B2} & \dots & \nabla_{\mathbf{h}_P} \mathbf{c}_{Bn} \end{bmatrix} \quad (120)$$

First, let us remark the following property for the transpose of the gradient of the constraint \mathbf{c}_{Bi} :

$$-\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi} \dot{\mathbf{h}}_i = \nabla_{\mathbf{p}_P} \mathbf{c}_{Bi} \dot{\mathbf{p}}_P + \nabla_{\mathbf{h}_P} \mathbf{c}_{Bi} \dot{\mathbf{h}}_P = \frac{d}{dt} \overrightarrow{OB}_i \quad (121)$$

i.e. they relate the velocities (in quasi-static motions) of \mathbf{h}_i or \mathbf{p}_P and \mathbf{h}_P to the velocity of point B_i . As a result, the right-hand side of the previous equation can also be rewritten as:

$$\nabla_{\mathbf{p}_P} \mathbf{c}_{Bi} \dot{\mathbf{p}}_P + [\overrightarrow{PB}_i]_{\times}^T \boldsymbol{\omega}_P = \frac{d}{dt} \overrightarrow{OB}_i \quad (122)$$

where $[\overrightarrow{PB}_i]_{\times}$ is the skew symmetric matrix associated with the vector \overrightarrow{PB}_i and $\boldsymbol{\omega}_P$ is the platform rotational velocity. Because the quaternions are a singularity-free representation of the orientation (and, as a result, their derivatives are singularity-free representation of the angular velocities), then the loss of rank of the matrix in (120) appears for the same condition as the following matrix:

$$\Xi = \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bn} \\ [\overrightarrow{PB}_1]_{\times} & [\overrightarrow{PB}_2]_{\times} & \dots & [\overrightarrow{PB}_n]_{\times} \end{bmatrix} \quad (123)$$

Due to the presence of the identity matrix, the columns of Ξ (and of the matrix in (120)) corresponding to the the constraint \mathbf{c}_{Bi} ($i = 1, \dots, n$) can never degenerate. Therefore, the degeneracy of Ξ and, thus, of the gradient of the constraints \mathbf{c}_B , appears if and only if there is a linear combinations between the block-columns corresponding to several legs. Moreover, because of the block-diagonal structure of $\nabla_{\mathbf{h}} \mathbf{c}_B$, degeneracy of Ξ can be obtained if and only if at least two blocks $\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$ and $\nabla_{\mathbf{h}_j} \mathbf{c}_{Bj}$ ($i \neq j$) are rank deficient.

Before going further, let us first remark from (122) that, at a sign, $\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$ is the transposed Jacobian of the leg i with respect to the angular variations represented by the variations of the leg's quaternions. This Jacobian can thus be assimilated to the Jacobian of an hyperredundant serial robot controlled by a series of active spherical joints. As a result, $\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$ is not

full rank if and only if the leg i is fully stretched. Let us denote as \mathbf{u}_i a vector in the nullspace of $\nabla_{\mathbf{h}_i} \mathbf{c}_{Bi}$ appearing when the leg i is fully stretched. By definition, \mathbf{u}_i gives the direction along which the motions of the leg are no more feasible: they provide the directions along which the leg i are fully stretched.

In what follows, we consider without loss of generality that legs 1 to k ($k \leq n$) are fully stretched, i.e. $\nabla_{\mathbf{h}_i}$ is rank deficient for $i = 1, \dots, k$. The direction along which leg i is stretched is given by the unit vector $\underline{\mathbf{u}}_i$. Let us then group in matrix Ξ_k the columns of Ξ corresponding to legs 1 to k :

$$\Xi_k = \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bk} \\ \mathbf{1}_3 & \mathbf{1}_3 & \dots & \mathbf{1}_3 \\ \overrightarrow{[PB_1]}_{\times} & \overrightarrow{[PB_2]}_{\times} & \dots & \overrightarrow{[PB_k]}_{\times} \end{bmatrix} \quad (124)$$

By multiplying Ξ_k by the vector $\mathbf{u} = [\alpha_1 \underline{\mathbf{u}}_1^T \dots \alpha_k \underline{\mathbf{u}}_k^T]^T$, we get

$$\begin{aligned} \Xi_k \mathbf{u} &= \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bk} \\ \mathbf{1}_3 & \mathbf{1}_3 & \dots & \mathbf{1}_3 \\ \overrightarrow{[PB_1]}_{\times} & \overrightarrow{[PB_2]}_{\times} & \dots & \overrightarrow{[PB_k]}_{\times} \end{bmatrix} \begin{bmatrix} \alpha_1 \underline{\mathbf{u}}_1 \\ \alpha_2 \underline{\mathbf{u}}_2 \\ \vdots \\ \alpha_k \underline{\mathbf{u}}_k \end{bmatrix} \\ &= \begin{bmatrix} \nabla_{\mathbf{h}_1} \mathbf{c}_{B1} \underline{\mathbf{u}}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_{\mathbf{h}_2} \mathbf{c}_{B2} \underline{\mathbf{u}}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \nabla_{\mathbf{h}_n} \mathbf{c}_{Bk} \underline{\mathbf{u}}_k \\ \underline{\mathbf{u}}_1 & \underline{\mathbf{u}}_2 & \dots & \underline{\mathbf{u}}_k \\ \overrightarrow{[PB_1]}_{\times} \underline{\mathbf{u}}_1 & \overrightarrow{[PB_2]}_{\times} \underline{\mathbf{u}}_2 & \dots & \overrightarrow{[PB_k]}_{\times} \underline{\mathbf{u}}_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \underline{\mathbf{u}}_1 & \underline{\mathbf{u}}_2 & \dots & \underline{\mathbf{u}}_k \\ \overrightarrow{[PB_1]}_{\times} \underline{\mathbf{u}}_1 & \overrightarrow{[PB_2]}_{\times} \underline{\mathbf{u}}_2 & \dots & \overrightarrow{[PB_k]}_{\times} \underline{\mathbf{u}}_k \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} \end{aligned} \quad (125)$$

As a result, Ξ_k is rank-deficient if and only if there is a linear combination between the columns of

$$\mathcal{L}_k = \begin{bmatrix} \underline{\mathbf{u}}_1 & \underline{\mathbf{u}}_2 & \dots & \underline{\mathbf{u}}_k \\ \overrightarrow{[PB_1]}_{\times} \underline{\mathbf{u}}_1 & \overrightarrow{[PB_2]}_{\times} \underline{\mathbf{u}}_2 & \dots & \overrightarrow{[PB_k]}_{\times} \underline{\mathbf{u}}_k \end{bmatrix} \quad (126)$$

Noticing that $\begin{bmatrix} \underline{\mathbf{u}}_i \\ \overrightarrow{[PB_i]}_{\times} \underline{\mathbf{u}}_i \end{bmatrix}$ is the Plücker representation of a line with direction $\underline{\mathbf{u}}_i$ passing through point B_i , then \mathcal{L}_k will degenerate if the system of lines composing it degenerates. These conditions are well known, and can be found in [8], [9]. For instance, if only two legs are stretched, singularity will appear if the lines passing through them are superposed.

As a general result, we can conclude that the gradient of the constraint \mathbf{c}_B will degenerate if and only if:

- several legs are fully stretched

- and the system of lines passing through them is degenerated.

C. Gradient of the constraint \mathbf{c}_r

Based on the same analysis as previously, a necessary condition of rank-deficiency for the constraints due to the fixed and revolute joints is that, for at least a leg i , the gradient $\nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}$ (in both types of joint) is rank deficient. Indeed, it can be proven that

- for the fixed joint, due to its quaternionic nature, the determinant of the matrix $\nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}^T \nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}$ is constant and equal to 64.
- for the revolute joint, due to its quaternionic nature, the determinant of the matrix $\nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}^T \nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}$ is equal to $16^0 \mathbf{a}_i^T \mathbf{0} \mathbf{a}_i (\beta_{12} \times \beta_{22})^T (\beta_{12} \times \beta_{22})$. Because β_{12} and β_{22} are unit vectors that must be orthogonal, and also because $\mathbf{0} \mathbf{a}_i$ is also a unit vector, the determinant of the matrix $\nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}^T \nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}$ is constant and equal to 16.

As a result, $\nabla_{\mathbf{h}_{Ni}} \mathbf{c}_{ri}$ can never be rank-deficient, whatever the type of joints considered

REFERENCES

- [1] S. Briot and A. Goldsztejn, "Singularity conditions for continuum parallel robots," *IEEE Transactions on Robotics*, 2020, submitted to.
- [2] O. Altuzarra, D. Caballero, F.J.Campa, and C. Pinto, "Position analysis in planar parallel continuum mechanisms," *Mechanism and Machine Theory*, vol. 132, pp. 13–29, 2019.
- [3] C. Black, J. Till, and D. Rucker, "Parallel continuum robots: Modeling, analysis, and actuation-based force sensing," *IEEE Transactions on Robotics*, vol. 34, no. 1, pp. 29–47, 2017.
- [4] F. Zaccaria, S. Briot, M. Chikhaoui, E. Idà, and M. Carricato, "An analytical formulation for the geometrico-static problem of continuum planar parallel robots," in *Proceedings of the 23rd CISM IFTOMM Symposium on Robot Design, Dynamics and Control (RoManSy 2020)*, 2020.
- [5] A. Lazarus, J. Miller, and P. Reis, "Continuation of equilibria and stability of slender elastic rods using an asymptotic numerical method," *Journal of the Mechanics and Physics of Solids*, vol. 61, no. 8, pp. 1712–1736, 2013.
- [6] Q. Peyron, K. Rabenoroso, N. Andreff, and P. Renaud, "A numerical framework for the stability and cardinality analysis of concentric tube robots: Introduction and application to the follow-the-leader deployment," *Mechanism and Machine Theory*, vol. 132, pp. 176–192, 2019.
- [7] H. Ziegler, *Principles of Structural Stability*. Cambridge, MA, USA: Birkhäuser, 1977.
- [8] K. Hunt, *Kinematic geometry of mechanisms*. Oxford: Clarendon Press, 1978.
- [9] J.-P. Merlet, *Parallel Robots*, 2nd ed. Springer, 2006.