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On the analyticity of electronic reduced densities for molecules.

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Abstract

We consider an electronic bound state of the usual, non-relativistic, molecular Hamiltonian with Coulomb interactions and fixed nuclei. Away from appropriate collisions, we prove the real analyticity of all the reduced densities and density matrices, that are associated to this bound state. We provide a similar result for the associated reduced current density.

Keywords: Analytic elliptic regularity, molecular Hamiltonian, electronic reduced densities, electronic reduced density matrices, Coulomb potential, twisted differential calculus.

1 Introduction.

The notions of density and density matrices are quite old tools in the treatment of physical N -body problems. For instance, the Thomas-Fermi theory goes back to the year 1927 (cf. [Li]). Nowadays these tools do play a central rôle in one important approach of the N -body problem, namely the Density Functional Theory (DFT) (cf. [E, LiSe]). Regularity properties of the related reduced density and density matrices are useful to develop the DFT. For instance, the real analyticity of a density was recently used for Hohenberg-Kohn Theorems in [G]. The purpose of the present paper is to prove, in a large region of the configuration space, the real analyticity of all reduced densities and density matrices of a pure state for a molecular Hamiltonian with fixed nuclei.

We consider a molecule with N moving electrons, with $N > 1$, and L fixed nuclei, with $L \geq 1$ (Born-Oppenheimer idealization). While the L distinct vectors $R_1, \dots, R_L \in \mathbb{R}^3$ denote the positions of the nuclei, the positions of the electrons are given by $x_1, \dots, x_N \in \mathbb{R}^3$. The charges of the nuclei are respectively given by the positive Z_1, \dots, Z_L and the electronic charge is -1 . In this picture, the Hamiltonian of the electronic system is

$$H := \sum_{j=1}^N \left(-\Delta_{x_j} - \sum_{k=1}^L Z_k |x_j - R_k|^{-1} \right) + \sum_{1 \leq j < j' \leq N} |x_j - x_{j'}|^{-1} + E_0, \quad (1.1)$$

where $E_0 = \sum_{1 \leq k < k' \leq L} Z_k Z_{k'} |R_k - R_{k'}|^{-1}$

and $-\Delta_{x_j}$ stands for the Laplacian in the variable x_j . Here we denote by $|\cdot|$ the euclidian norm on \mathbb{R}^3 . Setting $\Delta := \sum_{j=1}^N \Delta_{x_j}$, we define the potential V of the system as the multiplication operator satisfying $H = -\Delta + V$. Thanks to Hardy's inequality

$$\exists c > 0; \forall f \in W^{1,2}(\mathbb{R}^3), \int_{\mathbb{R}^3} |t|^{-2} |f(t)|^2 dt \leq c \int_{\mathbb{R}^3} |\nabla f(t)|^2 dt, \quad (1.2)$$

one can show that V is Δ -bounded with relative bound 0. Therefore the Hamiltonian H is self-adjoint on the domain of the Laplacian Δ , namely $W^{2,2}(\mathbb{R}^{3N})$ (see Kato's theorem in [RS2], p. 166-167).

From now on, we fix an electronic bound state $\psi \in W^{2,2}(\mathbb{R}^{3N}) \setminus \{0\}$ such that, for some real E , $H\psi = E\psi$. We point out (cf. [S, Z]) that such a bound state exists at least for appropriate $E \leq E_0$ (cf. [FH]) and for

$$N < 1 + 2 \sum_{k=1}^L Z_k.$$

Associated to that bound state ψ , we consider the following objects. Let k be an integer such that $0 < k < N$. Let $\rho_k : (\mathbb{R}^3)^k \rightarrow \mathbb{R}$ be the almost everywhere defined, $L^1(\mathbb{R}^{3k})$ -function given by, for $\underline{x} = (x_1; \dots; x_k) \in \mathbb{R}^{3k}$,

$$\rho_k(\underline{x}) = \int_{\mathbb{R}^{3(N-k)}} |\psi(\underline{x}; y)|^2 dy. \quad (1.3)$$

Define $\gamma_k : (\mathbb{R}^3)^{2k} \rightarrow \mathbb{C}$ as the almost everywhere defined function given by, for $\underline{x} = (x_1; \dots; x_k) \in \mathbb{R}^{3k}$ and $\underline{x}' = (x'_1; \dots; x'_k) \in \mathbb{R}^{3k}$,

$$\gamma_k(\underline{x}; \underline{x}') = \int_{\mathbb{R}^{3(N-k)}} \overline{\psi(\underline{x}; y)} \psi(\underline{x}'; y) dy. \quad (1.4)$$

Note that, almost everywhere, $\rho_k(\underline{x}) = \gamma_k(\underline{x}; \underline{x})$. It turns out that γ_k may be seen as the kernel (“Green function”) of a trace class operator (see [Le, LiSe]). We call the function ρ_k the *k-particle reduced density* and the kernel γ_k the *k-particle reduced density matrix*. When $k = 1$, $\rho := \rho_1$ is often simply called the (electronic) density. In this case, we also introduce the *reduced current density*, defined as the almost everywhere defined, $L^1(\mathbb{R}^3)$ -function $C : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by, for $x \in \mathbb{R}^3$, the following imaginary part

$$C(x) = \Im \int_{\mathbb{R}^{3(N-1)}} \overline{\nabla_x \psi(x; y)} \psi(x; y) dy. \quad (1.5)$$

From a physical point of view, the previous objects differ from the true physical ones by some prefactor (see [E, Le, LiSe, LSc]).

It is useful to introduce the following subsets of \mathbb{R}^{3k} . Denoting for a positive integer p by $\llbracket 1; p \rrbracket$ the set of the integers j satisfying $1 \leq j \leq k$, the closed set

$$\mathcal{C}_k := \{ \underline{x} = (x_1; \dots; x_k) \in \mathbb{R}^{3k}; \exists (j; j') \in \llbracket 1; k \rrbracket^2; j \neq j' \text{ and } x_j = x_{j'} \}$$

gathers all possible collisions between the first k electrons while the closed set

$$\mathcal{R}_k := \{ \underline{x} = (x_1; \dots; x_k) \in \mathbb{R}^{3k}; \exists j \in \llbracket 1; k \rrbracket, \exists \ell \in \llbracket 1; L \rrbracket; x_j = R_\ell \}$$

groups together all possible collisions of these k electrons with the nuclei. We set $\mathcal{U}_k^{(1)} := \mathbb{R}^{3k} \setminus (\mathcal{C}_k \cup \mathcal{R}_k)$, which is an open subset of \mathbb{R}^{3k} .

Now, we consider two sets of positions for the first k electrons and introduce the set of all possible collisions between positions of different sets, namely

$$\mathcal{C}_k^{(2)} := \left\{ \begin{array}{l} (\underline{x}; \underline{x}') \in (\mathbb{R}^{3k})^2; \underline{x} = (x_1; \dots; x_k), \underline{x}' = (x'_1; \dots; x'_k), \\ \exists (j; j') \in \llbracket 1; k \rrbracket^2; x_j = x'_{j'} \end{array} \right\}.$$

We introduce the open subset of $(\mathbb{R}^{3k})^2$ defined by

$$\mathcal{U}_k^{(2)} = (\mathcal{U}_k^{(1)} \times \mathcal{U}_k^{(1)})^2 \setminus \mathcal{C}_k^{(2)}.$$

In the $k = 1$ case, we note that $\mathcal{C}_1 = \emptyset$, $\mathcal{U}_1^{(1)} = \mathbb{R}^3 \setminus \{R_1, \dots, R_L\}$ and

$$\mathcal{U}_1^{(2)} = (\mathbb{R}^3 \setminus \{R_1, \dots, R_L\})^2 \setminus D,$$

where $D = \{(x; x') \in (\mathbb{R}^3)^2; x = x'\}$ is the diagonal of $(\mathbb{R}^3)^2$. We have the following

Theorem 1.1. [FHHS1, FHHS2, J].

The one-particle reduced density $\rho := \rho_1$ is real analytic on $\mathcal{U}_1^{(1)} = \mathbb{R}^3 \setminus \{R_1, \dots, R_L\}$.

The original proof was given in [FHHS1, FHHS2] and relies on a clever decomposition of the integration domain of the integral defining ρ into pieces, on which one can perform an appropriate change of variables. A different proof of Theorem 1.1 is provided in [J]. We refer to [FHHS3, FS] for a refined information on ρ . In [FHHS2], a similar analyticity result on γ_1 was announced, without proof. More precisely, it was claimed there that γ_1 should be real analytic on $\mathcal{U}_1^{(1)} \times \mathcal{U}_1^{(1)}$. Adapting the method of [FHHS2], a slightly weaker result on γ_1 was proved in [HS], namely

Theorem 1.2. [HS].

The one-particle reduced density matrix γ_1 is real analytic on $\mathcal{U}_1^{(2)}$.

Note that $\mathcal{U}_1^{(2)}$ is indeed a strict subset of $\mathcal{U}_1^{(1)} \times \mathcal{U}_1^{(1)}$.

In [FHHS2], it was also announced without proof that ρ_2 should be real analytic on

$$\mathcal{U}_2^{(1)} = \mathbb{R}^6 \setminus (\mathcal{C}_2 \cup \mathcal{R}_2) = (\mathbb{R}^3 \setminus \{R_1, \dots, R_L\})^2 \setminus D = \mathcal{U}_1^{(2)}.$$

In the present paper, we work with the tools and the method, used in [J] to get Theorem 1.1, to prove the following three results.

Theorem 1.3. *The reduced current density C is real analytic on $\mathcal{U}_1^{(1)} = \mathbb{R}^3 \setminus \{R_1, \dots, R_L\}$.*

Theorem 1.4. *For all integer k with $0 < k < N$, the k -particle reduced density ρ_k is real analytic on $\mathcal{U}_k^{(1)}$.*

Theorem 1.5. *For all integer k with $0 < k < N$, the k -particle reduced density matrix γ_k is real analytic on $\mathcal{U}_k^{(2)} = (\mathcal{U}_k^{(1)} \times \mathcal{U}_k^{(1)}) \setminus \mathcal{C}_k^{(2)}$.*

Theorem 1.4 for $k = 1$ coincides with Theorem 1.1. The proof of Theorem 1.4 is, when restricted to the $k = 1$ case, just a rewriting of the proof of Theorem 1.1 in [J] and provides a direct justification of Theorem 1.3. We note that Theorem 1.5 for $k = 1$ is exactly Theorem 1.2 and that Theorem 1.4 with $k = 2$ fits precisely to the announced result on ρ_2 in [FHHS2].

The set of all possible collisions between particles is $\mathcal{C}_N \cup \mathcal{R}_N$ and the potential V is real analytic precisely on $\mathbb{R}^{3N} \setminus (\mathcal{C}_N \cup \mathcal{R}_N)$. Classical elliptic regularity applied to the equation $H\psi = E\psi$ shows that ψ is also real analytic on $\mathbb{R}^{3N} \setminus (\mathcal{C}_N \cup \mathcal{R}_N)$. We refer to [ACN, FHHS4, FS, K] for more information on the behaviour of ψ near collisions. It turns out that, at least at some places in $\mathcal{C}_N \cup \mathcal{R}_N$, ψ is not real analytic (cf. [FHHS4]). Thus, in the definition (1.3) of ρ_k , for an integer k with $0 < k < N$, the domain of integration does contain such singularities. Therefore it is not clear a priori that ρ_k is real analytic somewhere. This is however true, by Theorem 1.4, on the complement of the set of all possible collisions of the “external” electrons, namely on $\mathcal{U}_k^{(1)}$.

The proofs of Theorems 1.1 and 1.2 do not directly use the analyticity of ψ . This is also the case of the proofs of Theorems 1.3, 1.4, and 1.5, below.

It turns out that our proofs of Theorems 1.4 and 1.5 have a common structure that we want to describe now. Using an appropriate, \underline{x} -dependent, unitary operator $U_{\underline{x}}$, that acts on $L^2(\mathbb{R}^{3(N-k)})$, we locally transform the equation $H\psi = E\psi$ into an elliptic differential equation $(U_{\underline{x}} H U_{\underline{x}}^{-1}) U_{\underline{x}} \psi = E U_{\underline{x}} \psi$ (the “twisted” equation), the coefficients of which have

nice analytic properties. Applying elliptic regularity to the latter equation, we obtain the real analyticity of $\underline{x} \mapsto U_{\underline{x}}\psi \in L^2(\mathbb{R}^{3(N-k)})$ that yields the real analyticity of $\rho_k : \underline{x} \mapsto \|U_{\underline{x}}\psi\|^2$ ($\|\cdot\|$ being the norm on $L^2(\mathbb{R}^{3(N-k)})$). For γ_k , we follow the same lines but with a $(\underline{x}; \underline{x}')$ -dependent, unitary operator $U_{(\underline{x}; \underline{x}')}$, and two “twisted” differential equations in the variables $(\underline{x}; \underline{x}'; y)$. These proofs are performed in Section 4.

Let us now compare the two available methods to prove such results, namely the one mentioned just above and the one used in [FHHS2, HS] to get Theorems 1.1 and 1.2. We point out that elliptic regularity was also an important tool in [FHHS2, HS]. Another common tool is the use of a \underline{x} -dependent change of variables on the “internal” variables $y \in \mathbb{R}^{3(N-k)}$, since the above unitary operators are both the implementation in $L^2(\mathbb{R}^{3(N-k)})$ of such a change of variables. However, we use here global changes of variables in contrast to the local ones in [FHHS2, HS]. While, there, the integrals on $\mathbb{R}^{3(N-1)}$ defining ρ_1 and γ_1 are split in several pieces, on which an appropriate change of variables is performed to make the real analyticity apparent, we view here ρ_k (resp. γ_k) as the $L^2(\mathbb{R}^{3(N-k)})$ -norm (resp. the $L^2(\mathbb{R}^{3(N-k)})$ -scalar product) of one (resp. two) $L^2(\mathbb{R}^{3(N-k)})$ -valued, real analytic function(s).

In view of the behaviour of ψ near some points in $\mathcal{C}_N \cup \mathcal{R}_N$, that was established in [FHHS4], it is quite natural to expect that, for an integer k with $0 < k < N$, neither ρ_k nor γ_k is real analytic everywhere. Their exact domain of real analyticity is still an open question. The domains obtained in Theorems 1.4 and 1.5 are the largest ones that can be reached by the method used in this paper, as explained in Remarks 4.2 and 4.4.

As already pointed out in [J], the twisted calculus, that we use here, allows us to treat more general Hamiltonians than H . We introduce, in Section 5, a class of elliptic, second order differential operators with Coulomb singularities. We observe that the conjugation by the twist actually preserves the ellipticity. Assuming that there exists a bound state for such an operator, we show in Theorem 5.1 that our previous, main results hold true for it. Moreover, we expect that this should extend to a class of elliptic, *pseudo*-differential operators with Coulomb singularities, since the used tools are available for such operators. We refer to [MS] for more information on the twisted (pseudo-)differential calculus.

Maybe it would be useful for the DFT to extend the present results on pure states to general states (see Sections 3.1.4 and 3.1.5 in [LiSe] for details). We expect that Theorem 3.9 below could be a good starting point for this purpose.

The paper is organized as follow. In Section 2, we provide a general notation, basic facts on real analyticity, and a result on elliptic regularity, that was essentially present in [J]. In Section 3, we review the twisted calculus based on Hunziker’s twist, show how it can be used to “desingularize” a differential equation with Coulomb singularities, and apply elliptic regularity to the corresponding twisted equation. Section 4 is devoted to the proof of our main results, namely Theorems 1.3, 1.4, and 1.5. The extension of these results to a larger class of Hamiltonians is performed in Section 5. Finally, we gathered in Appendix A some computations that are used in the main text and provide in Appendix B basic material on global pseudo-differential operators.

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2 Basic tools and elliptic regularity.

To prepare the proof of the main results, we recall basic tools and state a general result on the analytic regularity of solutions of some elliptic equations.

2.1 Notation and basic tools.

We start with a general notation. We denote by \mathbb{C} the field of complex numbers. For $z \in \mathbb{C}$, $\Re z$ (resp. $\Im z$) stands for the real part (resp. imaginary part) of z .

Let p be a positive integer. Recall that, for $u \in \mathbb{R}^p$, we write $|u|$ for the euclidian norm of u . Given such a vector $u \in \mathbb{R}^p$ and a nonnegative real number r , we denote by $B(u; r[$ (resp. $B(u; r]$) the open (resp. closed) ball of radius r and centre u , for the euclidian norm $|\cdot|$ in \mathbb{R}^p .

In the one dimensional case, we use the following convention for (possibly empty) intervals: for $(a; b) \in \mathbb{R}^2$, let $[a; b] = \{t \in \mathbb{R}; a \leq t \leq b\}$, $[a; b[= \{t \in \mathbb{R}; a \leq t < b\}$, $]a; b] = \{t \in \mathbb{R}; a < t \leq b\}$, and $]a; b[= \{t \in \mathbb{R}; a < t < b\}$.

We denote by \mathbb{N} the set of nonnegative integers and set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If $p \leq q$ are non negative integers, we set $\llbracket p; q \rrbracket := [p; q] \cap \mathbb{N}$, $\llbracket p; q \llbracket = [p; q[\cap \mathbb{N}$, $\llbracket p; q \rrbracket =]p; q] \cap \mathbb{N}$, and $\llbracket p; q \llbracket =]p; q[\cap \mathbb{N}$.

Given an open subset O of \mathbb{R}^p and $n \in \mathbb{N}$, we denote by $W^{n,2}(O)$ the standard Sobolev space of those L^2 -functions on O such that, for $n' \in \llbracket 0; n \rrbracket$, their distributional derivatives of order n' belong to $L^2(O)$. We denote by $\langle \cdot, \cdot \rangle_n$ (resp. $\|\cdot\|_n$) the right linear scalar product (resp. the norm) on $W^{n,2}(O)$. In particular, $W^{0,2}(O) = L^2(O)$. Without reference to p and O , we denote by $\|\cdot\|$ (resp. $\langle \cdot, \cdot \rangle$) the L^2 -norm (resp. the right linear scalar product) on $L^2(O)$.

For two Banach spaces $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, the space $\mathcal{L}(\mathcal{A}; \mathcal{B})$ of continuous linear maps from \mathcal{A} to \mathcal{B} is also a Banach space for the operator norm $\|\cdot\|_{\mathcal{L}(\mathcal{A}; \mathcal{B})}$ defined by

$$\forall M \in \mathcal{L}(\mathcal{A}; \mathcal{B}), \quad \|M\|_{\mathcal{L}(\mathcal{A}; \mathcal{B})} := \sup_{a \in \mathcal{A} \setminus \{0\}} \frac{\|M(a)\|_{\mathcal{B}}}{\|a\|_{\mathcal{A}}}.$$

We simply denote $\mathcal{L}(\mathcal{A}; \mathcal{A})$ by $\mathcal{L}(\mathcal{A})$.

Let p be a positive integer and O an open subset of \mathbb{R}^p . Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach space and $\theta : O \ni x = (x_1; \dots; x_p) \mapsto \theta(x) \in \mathcal{A}$. For $j \in \llbracket 1; p \rrbracket$, we denote by $\partial_j \theta$ or $\partial_{x_j} \theta$ the j 'th first partial derivative of θ . For $\alpha \in \mathbb{N}^p$ and $x \in \mathbb{R}^d$, we set $D_x^\alpha := (-i\partial_x)^\alpha := (-i\partial_{x_1})^{\alpha_1} \dots (-i\partial_{x_p})^{\alpha_p}$, $D_x = -i\nabla_x$, $x^\alpha := x_1^{\alpha_1} \dots x_p^{\alpha_p}$, $|\alpha| := \alpha_1 + \dots + \alpha_p$, $\alpha! := (\alpha_1!) \dots (\alpha_p!)$, $|x|^2 = x_1^2 + \dots + x_p^2$, and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We choose the same notation for the length $|\alpha|$ of a multiindex $\alpha \in \mathbb{N}^p$ and for the euclidian norm $|x|$ of a vector $x \in \mathbb{R}^p$ but the context should avoid any confusion.

We denote by $\mathcal{C}^\infty(O; \mathcal{A})$ (resp. $\mathcal{C}_b^\infty(O; \mathcal{A})$, resp. $\mathcal{C}_c^\infty(O; \mathcal{A})$, resp. $\mathcal{C}^\omega(O; \mathcal{A})$) the vector space of functions from O to \mathcal{A} which are smooth (resp. smooth with bounded derivatives, resp. smooth with compact support, resp. real analytic). If $\mathcal{A} = \mathbb{C}$, we simply write $\mathcal{C}^\infty(O)$ (resp. $\mathcal{C}_b^\infty(O)$, resp. $\mathcal{C}_c^\infty(O)$, resp. $\mathcal{C}^\omega(O)$).

Now we recall basic facts on analytic functions. Take again a Banach space $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, a positive integer p , and an open subset O of \mathbb{R}^p . For a smooth function $\theta : O \rightarrow \mathcal{A}$, the

following properties are equivalent: $\theta \in \mathcal{C}^\omega(O; \mathcal{A})$;

$$\exists A > 0, \forall \text{ compact } K \subset O, \forall \alpha \in \mathbb{N}^p, \sup_{x \in K} \|(D_x^\alpha \theta)(x)\|_{\mathcal{A}} \leq A^{|\alpha|+1} \cdot (\alpha!); \quad (2.1)$$

$$\exists A > 0, \forall \text{ compact } K \subset O, \forall \alpha \in \mathbb{N}^p, \sup_{x \in K} \|(D_x^\alpha \theta)(x)\|_{\mathcal{A}} \leq A^{|\alpha|+1} \cdot (|\alpha|!); \quad (2.2)$$

$$\exists A > 0, \forall \text{ compact } K \subset O, \forall \alpha \in \mathbb{N}^p, \sup_{x \in K} \|(D_x^\alpha \theta)(x)\|_{\mathcal{A}} \leq A^{|\alpha|+1} \cdot (1 + |\alpha|)^{|\alpha|}. \quad (2.3)$$

We refer to [Hö3, HS, J] for details.

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be another Banach space and $f : \mathcal{A}^2 \rightarrow \mathcal{B}$ be a continuous bilinear map. Then, for any $(\theta_1; \theta_2) \in (\mathcal{C}^\omega(O; \mathcal{A}))^2$, the map $f(\theta_1; \theta_2) : O \ni x \mapsto f(\theta_1(x); \theta_2(x)) \in \mathcal{B}$ is real analytic.

Consider the case where $\mathcal{A} = W^{n,2}(O)$, for some integer n , and recall that $\langle \cdot, \cdot \rangle_n$ (resp. $\|\cdot\|_n$) is the right linear scalar product (resp. the norm) on \mathcal{A} . Take $(\theta_1; \theta_2) \in (\mathcal{C}^\omega(O; \mathcal{A}))^2$. Then the map $\langle \theta_1, \theta_2 \rangle_n : O \rightarrow \mathbb{C}$, defined by $\langle \theta_1, \theta_2 \rangle_n(x) = \langle \theta_1(x), \theta_2(x) \rangle_n$, is real analytic. So is also the real valued map $\|\theta_1\|_n^2 : O \ni x \mapsto \|\theta_1(x)\|_n^2$. In particular, taking $n = 0$, the maps $\langle \theta_1, \theta_2 \rangle : O \ni x \mapsto \langle \theta_1(x), \theta_2(x) \rangle \in \mathbb{C}$ and $\|\theta_1\|^2 : O \ni x \mapsto \|\theta_1(x)\|^2 \in \mathbb{R}^+$ are also real analytic.

Finally we need some specific notation to describe the structure of the considered Hamiltonian and of our results.

For a \mathcal{A} -valued function $\theta : \mathbb{R}^3 \ni z = (z^{(1)}; z^{(2)}; z^{(3)}) \mapsto \theta(z) \in \mathcal{A}$, we denote by $d\theta$ or $d_z\theta$ the total derivative of θ . According to the previous notation, we denote, for $j \in \{1; 2; 3\}$, by $\partial_j\theta$ or $\partial_{z^{(j)}}\theta$ the j 'th first partial derivative. For a multiindex $\alpha = (\alpha_1; \alpha_2; \alpha_3) \in \mathbb{N}^3$, we set $D_z^\alpha := (-i\partial_z)^\alpha := (-i\partial_1)^{\alpha_1}(-i\partial_2)^{\alpha_2}(-i\partial_3)^{\alpha_3}$, $D_z = -i\nabla_z$, $z^\alpha = (z^{(1)})^{\alpha_1}(z^{(2)})^{\alpha_2}(z^{(3)})^{\alpha_3}$, and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Let m be a positive integer. We write a point $x \in \mathbb{R}^{3m}$ as $x = (x_1; \dots; x_m)$. For any multiindex $\alpha = (\alpha_1; \dots; \alpha_m) \in \mathbb{N}^{3m}$, we set $D_x^\alpha := (-i\partial_x)^\alpha := (-i\partial_{x_1})^{\alpha_1} \dots (-i\partial_{x_m})^{\alpha_m}$, $D_x = -i\nabla_x$, $x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}$, and $|\alpha| = |\alpha_1| + \dots + |\alpha_m|$.

Let m and p be positive integers. We write $x \in \mathbb{R}^{3m}$ (resp. $y \in \mathbb{R}^{3p}$) as $x = (x_1; \dots; x_m)$ (resp. $y = (y_1; \dots; y_p)$). For a \mathcal{A} -valued function $\theta : \mathbb{R}^{3m} \times \mathbb{R}^{3p} \ni (x; y) \mapsto \theta(x; y) \in \mathcal{A}$, let $d_x\theta$ (resp. $d_y\theta$) be the total derivative of θ w.r.t. x (resp. y).

For $n \in \mathbb{N}$, $W^{n,2}(\mathbb{R}^{3p})$ denotes the standard Sobolev space of L^2 -functions on \mathbb{R}^{3p} admitting, up to order n , L^2 distributional derivatives. In particular, $W^{0,2}(\mathbb{R}^{3p}) = L^2(\mathbb{R}^{3p})$. We set $\mathcal{W}_n = W^{n,2}(\mathbb{R}^{3p})$ and let $\mathcal{B}_n = \mathcal{L}(\mathcal{W}_n; \mathcal{W}_0)$ be the Banach space of linear continuous maps from \mathcal{W}_n to \mathcal{W}_0 .

2.2 Elliptic regularity.

In this subsection, we extend, essentially with the same proof, a regularity result that was proved but not stated in [J].

Let $(m; p) \in (\mathbb{N}^*)^2$. Let Ω be an open subset of \mathbb{R}^{3m} . We consider second order differential

operators P on $\Omega \times \mathbb{R}^{3p}$ given by

$$P = \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| \leq 2}} c_{\alpha\beta}(\mathbf{x}; \mathbf{y}) D_{\mathbf{x}}^{\alpha} D_{\mathbf{y}}^{\beta}, \quad (2.4)$$

where the coefficients $c_{\alpha\beta}$ belong to $\mathcal{C}_b^{\infty}(\Omega \times \mathbb{R}^{3p}; \mathbb{C})$. In particular, for $\mathbf{x} \in \Omega$, the multiplication operator $C_{\alpha\beta}(\mathbf{x}; \cdot)$ by $c_{\alpha\beta}(\mathbf{x}; \cdot)$ belongs to $\mathcal{L}(\mathcal{W}_n)$, for all $n \in \mathbb{N}$.

The principal symbol of those operators P is the smooth, complex-valued function σ_P , that is defined on $\Omega \times \mathbb{R}^{3p} \times \mathbb{R}^{3m} \times \mathbb{R}^{3p}$ by

$$\sigma_P(\mathbf{x}; \mathbf{y}; \xi; \eta) = \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| = 2}} c_{\alpha\beta}(\mathbf{x}; \mathbf{y}) \xi^{\alpha} \eta^{\beta}. \quad (2.5)$$

We assume that P is (globally) elliptic in the following sense: there exists $C_P > 0$ such that, for all $(\mathbf{x}; \mathbf{y}; \xi; \eta) \in \Omega \times \mathbb{R}^{3p} \times \mathbb{R}^{3m} \times \mathbb{R}^{3p}$,

$$|\xi|^2 + |\eta|^2 \geq 1 \implies |\Re \sigma_P(\mathbf{x}; \mathbf{y}; \xi; \eta)| \geq C_P (|\xi|^2 + |\eta|^2). \quad (2.6)$$

Furthermore, we require that, for $(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p}$ with $|\alpha| + |\beta| \leq 2$, the map $\Omega \ni \mathbf{x} \mapsto C_{\alpha\beta}(\mathbf{x}; \cdot) \in \mathcal{B}_0$ is well-defined and real analytic.

Under these assumptions on P , we note that, for all $\alpha \in \mathbb{N}^{3m}$ with $|\alpha| \leq 2$, the function $a_{\alpha} : \Omega \longrightarrow \mathcal{B}_{2-|\alpha|}$, which maps each $\mathbf{x} \in \Omega$ to the differential operator

$$a_{\alpha}(\mathbf{x}) = \sum_{\substack{\beta \in \mathbb{N}^{3p} \\ |\beta| \leq 2 - |\alpha|}} c_{\alpha\beta}(\mathbf{x}; \cdot) D_{\mathbf{y}}^{\beta} \quad (2.7)$$

on \mathbb{R}^{3p} , is well-defined and real analytic.

Definition 2.1. For $(m; p) \in (\mathbb{N}^*)^2$ and Ω an open subset of \mathbb{R}^{3m} . We denote by $\text{Diff}_2(\Omega)$ the set of all the differential operators P on $\Omega \times \mathbb{R}^{3p}$ satisfying the previous requirements.

Theorem 2.2. [J].

Let P be differential operator in the class $\text{Diff}_2(\Omega)$. Let $W : \Omega \longrightarrow \mathcal{B}_1$ be a real analytic map. Take $\varphi \in W^{2,2}(\Omega \times \mathbb{R}^{3p})$ such that $(P + W)\varphi = 0$. Then, the map $\Omega \ni \mathbf{x} \mapsto \varphi(\mathbf{x}; \cdot) \in \mathcal{W}_2$ is real analytic.

Remark 2.3. If, in Theorem 2.2, one replaces $3m$ by some positive integer n and one sets $p = 0$ and $W = 0$ with the convention that $\Omega \times \mathbb{R}^0 = \Omega$ and $\mathcal{W}_2 = \mathbb{C}$, then one recovers exactly Hörmander's Theorem 7.5.1 in [Hö1]. We assign Theorem 2.2 to [J] since, although it was not stated there, it was proved in a particular case and the corresponding proof directly extends to the present, general case. In [FHHS1, FHHS2, HS], one can find similar results in a slightly narrower framework. Therefore, Theorem 2.2 is more or less known in the literature.

Nevertheless, we provide below, for completeness, a proof of Theorem 2.2. To make it more transparent and facilitate the comparison with known results of this kind, we state

intermediate results in some lemmata.

For convenience, we choose to use an appropriate pseudo-differential calculus. Since we are concerned with a local result (in Ω), it is natural to take a local one (see, for instance, in [Hö2] on page 83 or in [T], section 7.10). However, we prefer to use the basic global calculus presented in the beginning of Chapter 18 in [Hö2], since we think that it is more accessible.

Let $P \in \text{Diff}_2(\Omega)$. To prove Theorem 2.2, it suffices to show the required real analyticity near each point in Ω . Let $x^0 \in \Omega$ and let Ω'_0 be a bounded neighbourhood of x^0 such that $\Omega'_0 \subset \Omega$. As a first step, we construct an extension $\hat{P} \in \text{Diff}_2(\mathbb{R}^{3m})$ of the restriction of P to Ω'_0 .

In view of (2.6), we observe that the real part $\Re \sigma_P$ of σ_P does not vanish, except for $(\xi; \eta) = (0; 0)$. Since it is continuous, it must keep the same sign σ^0 on

$$\Omega \times \mathbb{R}^{3p} \times (\mathbb{R}^{3(m+p)} \setminus \{(0; 0)\}).$$

Furthermore, (2.6) also implies that, for $(x; y; \xi; \eta) \in \Omega \times \mathbb{R}^{3p} \times \mathbb{R}^{3(m+p)}$,

$$\sigma^0 \cdot \Re \sigma_P(x; y; \xi; \eta) \geq C_P(|\xi|^2 + |\eta|^2). \quad (2.8)$$

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^{3m})$ be a cut-off function with support in Ω such that $0 \leq \chi \leq 1$ and, on Ω'_0 , $\chi = 1$. Recalling that P is given by (2.4), we set

$$\hat{P} = \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| \leq 2}} \hat{c}_{\alpha\beta}(x; y) D_x^\alpha D_y^\beta,$$

where $\hat{c}_{\alpha\beta}(x; y) = \chi(x) c_{\alpha\beta}(x; y)$, for all $(x; y) \in \mathbb{R}^{3(m+p)}$ and $|\alpha| + |\beta| < 2$, and where, for all $(x; y; \xi; \eta) \in \mathbb{R}^{6(m+p)}$,

$$\begin{aligned} \sigma_{\hat{P}}(x; y; \xi; \eta) &:= \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| = 2}} \hat{c}_{\alpha\beta}(x; y) \xi^\alpha \eta^\beta \\ &:= \chi(x) \cdot \sigma_P(x; y; \xi; \eta) + \sigma^0 C_P \cdot (1 - \chi(x)) \cdot (|\xi|^2 + |\eta|^2). \end{aligned} \quad (2.9)$$

We observe that (2.8) with σ_P replaced by $\sigma_{\hat{P}}$ holds true on $\mathbb{R}^{6(m+p)}$. In particular, \hat{P} is elliptic. We immediately verify that $\hat{P} \in \text{Diff}_2(\mathbb{R}^{3m})$. We note that \hat{P} coincides with P as differential operator on Ω'_0 .

We observe that the symbol of \hat{P} belongs to the global class $S^2(\mathbb{R}^{6N})$ of symbols, for $N = m + p$, and, since it is elliptic, we can construct a parametrix (cf. Appendix B). Thus there exist two pseudo-differential operators $Q = q(x; y; D_x; D_y)$ and $R = r(x; y; D_x; D_y)$ with symbols $q, r \in S^{-2}(\mathbb{R}^{6N})$ such that $Q\hat{P} = I - R$ (where I is the identity differential operator) and

$$Q, R \in \mathcal{L}(W^{k,2}(\mathbb{R}^{3N}); W^{k+2,2}(\mathbb{R}^{3N})). \quad (2.10)$$

Proof of Theorem 2.2: We take $\varphi \in W^{2,2}(\Omega \times \mathbb{R}^{3p})$ such that $(P + W)\varphi = 0$. Let $x^0 \in \Omega$. We use the above construction. Let Ω_0 a neighbourhood of x^0 such that $\Omega_0 \subset \Omega'_0$. Let $\chi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with $\chi_0 = 1$ near Ω_0 and with support in Ω'_0 . Thus $\chi\chi_0 = \chi_0$, where χ is the cut-off function appearing in the definition of \hat{P} .

Lemma 2.4. *We have $\chi_0\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{3m}; \mathcal{W}_2)$.*

Proof: Since $Q\hat{P} = I - R$, $\hat{P}\chi_0 = \chi P\chi_0$, and $(P + W)\varphi = 0$, we obtain

$$\chi_0\varphi = R\chi_0\varphi + Q\chi[P, \chi_0]\varphi - Q\chi_0W\varphi, \quad (2.11)$$

where the commutator $[P, \chi_0]$ is a first order differential operator in x . Since $\varphi \in W^{2,2}(\Omega \times \mathbb{R}^{3p})$, $\varphi \in W^{1,2}(\Omega; \mathcal{W}_1)$ and $\chi_0\varphi \in W^{1,2}(\mathbb{R}^{3m}; \mathcal{W}_1)$. We successively observe that

$$W\varphi \in W^{1,2}(\Omega; \mathcal{W}_0), \quad \chi_0W\varphi \in W^{1,2}(\mathbb{R}^{3m}; \mathcal{W}_0), \quad \text{and} \quad Q\chi W\chi_0\varphi \in W^{2,2}(\mathbb{R}^{3m}; \mathcal{W}_1).$$

Similarly, we have

$$[P, \chi_0]\varphi \in W^{0,2}(\Omega; \mathcal{W}_1), \quad \chi[P, \chi_0]\varphi \in W^{0,2}(\mathbb{R}^{3m}; \mathcal{W}_1),$$

$$Q\chi[P, \chi_0]\varphi \in W^{2,2}(\mathbb{R}^{3m}; \mathcal{W}_1) \quad \text{and} \quad R\chi_0\varphi \in W^{3,2}(\mathbb{R}^{3m}; \mathcal{W}_1).$$

Thus $\chi_0\varphi \in W^{2,2}(\mathbb{R}^{3m}; \mathcal{W}_1)$, by (2.11). Now, we apply the same argument again but with this new information, yielding $\chi_0\varphi \in W^{2,2}(\mathbb{R}^{3m}; \mathcal{W}_2)$. In this way, we prove by induction that, for all $k \in \mathbb{N}$, $\chi_0\varphi \in W^{k,2}(\mathbb{R}^{3m}; \mathcal{W}_2)$, yielding $\chi_0\varphi \in \mathcal{C}^\infty(\mathbb{R}^{3m}; \mathcal{W}_2)$. \square

Lemma 2.5. *There exists $C_e > 0$ such that, for all $v \in \mathcal{C}_c^\infty(\mathbb{R}^{3m}; \mathcal{W}_2)$ with support in Ω'_0 , for all $r \in \{0; 1; 2\}$, for all $\alpha \in \mathbb{N}^{3m}$,*

$$|\alpha| + r \leq 2 \implies \|D_x^\alpha v\|_{L^2(\Omega_0; \mathcal{W}_r)} \leq C_e \|(P + W)v\|_{L^2(\Omega_0; \mathcal{W}_0)}. \quad (2.12)$$

Proof: Let $P_2 := \sigma_P(x; y; D_x; D_y)$ and $P_1 = P - P_2$. We consider the following operators as operators in $L^2(\Omega_0; \mathcal{W}_0)$ with domain $W^{2,2}(\Omega_0 \times \mathbb{R}^{3p})$. By ellipticity, $\Delta = \Delta_x + \Delta_y$ is P_2 -bounded. By assumption, W is $(-\Delta_x)^{1/2}$ -bounded. Since the operators $(-\Delta_x)^{1/2}$ and $D_x^\alpha D_y^\beta$, for $|\alpha| + |\beta| \leq 1$, are Δ -bounded with relative bound less than one, so are also W and, thanks to the boundedness assumption on its coefficients, P_1 . Thus W and P_1 are P_2 -bounded with relative bound less than one. This implies that W is P -bounded with relative bound less than one. For $|\alpha| + |\beta| \leq 2$, $D_x^\alpha D_y^\beta$ is Δ -bounded. The above properties imply that it is also $(P + W)$ -bounded, yielding (2.12). \square

For $\epsilon \geq 0$, let $\Omega_\epsilon := \{x \in \Omega_0; d(x; \mathbb{R}^{3m} \setminus \Omega_0) > \epsilon\}$ and, for $r \in \mathbb{N}$, denote the $L^2(\Omega_\epsilon; \mathcal{W}_r)$ -norm of v by $N_{\epsilon;r}(v)$. Note that $\chi_0\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{3m}; \mathcal{W}_2)$ with support in Ω'_0 , by Lemma 2.4.

Lemma 2.6. *There exists $B > 0$ such that, for all $\epsilon > 0$, $j \in \mathbb{N}$, $r \in \{0; 1; 2\}$, and $\alpha \in \mathbb{N}^{3m}$,*

$$r + |\alpha| < 2 + j \implies \epsilon^{r+|\alpha|} N_{j\epsilon;r}(D_x^\alpha \varphi) \leq B^{r+|\alpha|+1}. \quad (2.13)$$

Proof: Let D_0 be the diameter of Ω_0 . Since $P \in \text{Diff}_2(\Omega)$, there exists $C_a > 0$ such that, for all $\alpha \in \mathbb{N}^3$, $0 \leq \epsilon_1 \leq D_0$,

$$\epsilon_1^{|\alpha|} \sum_{|\beta| \leq 2} \sup_{x \in \Omega_{\epsilon_1}} \|\partial_x^\alpha a_\beta\|_{\mathcal{B}_{2-|\beta|}} \leq C_a^{|\alpha|+1} \cdot (|\alpha|!). \quad (2.14)$$

by (2.2). Using that $(P + W)\varphi = 0$, it suffices to follow the proof of (7.5.9) in [Hö1], p. 179-180, to get the desired result. \square

To complete the proof of Theorem 2.2, it suffices, thanks to (2.13), to follow the argument on page 180 in [Hö1] and use (2.2). \square

Remark 2.7. *We point out that one can prove Theorem 2.2 without pseudodifferential technics. We used then in the proof of Lemma 2.4. In fact, it turns out that the arguments used in [FHHS2] (see the proof of Lemma 3.1) and in [HS] (see Section 3.3) can be adapted to get a different proof of those lemma.*

3 Twisted differential calculus.

In this section, we recall the notion of “twist” used in [Hu, KMSW, MS] to “desingularize” a class of Hamiltonians with Coulomb-like singularities.

3.1 Framework.

We need some more notation. Denoting by \mathbb{S}^2 the unit euclidian sphere of \mathbb{R}^3 , let $\tilde{v}_0 : \mathbb{S}^2 \rightarrow \mathbb{C}$ and $\eta :]0; +\infty[\rightarrow \mathbb{R}^+$ be two functions such that, for some $\eta_0 \in]0; 1[$ and $B_0 > 0$, $(1/B_0) \leq |\tilde{v}_0| \leq B_0$ and

$$\forall t > 0, \quad \sup_{t(1-\eta_0) \leq s \leq t(1+\eta_0)} \eta(s) \leq B_0 \cdot \eta(t). \quad (3.1)$$

We assume further that the multiplication operator by the function $\eta(|\cdot|) : \mathbb{R}^3 \ni z \mapsto \eta(|z|) \in \mathbb{R}^+$ belongs to $\mathcal{L}(\mathcal{W}^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$. The multiplication operator by the function $v_0 : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{C}$, defined by $v_0(z) = \eta(|z|)\tilde{v}_0(z/|z|)$, also belongs to $\mathcal{L}(\mathcal{W}^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$.

Definition 3.1. *Let \mathcal{V} be the class of functions $v : \mathbb{R}^3 \rightarrow \mathbb{C}$ such that $v \in \mathcal{C}^\omega(\mathbb{R}^3 \setminus \{0\})$ and there exists $C > 0$ such that*

$$\forall \alpha \in \mathbb{N}^3, \forall z \in \mathbb{R}^3 \setminus \{0\}, |z|^{|\alpha|} \cdot |D^\alpha v(z)| \leq C^{1+|\alpha|} \cdot (|\alpha|!) \cdot |v_0(z)|. \quad (3.2)$$

We observe that, if $v \in \mathcal{V}$, then the multiplication operator by v belongs to the space $\mathcal{L}(\mathcal{W}^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$.

We consider a system of $m + p$ electrons, interacting to one another and moving under the influence of L fixed nuclei. Let $\mathcal{I}_e := \{1; \dots; m\}$ be the set of indices for the “external” electronic variables $x \in \mathbb{R}^{3m}$, let $\mathcal{I}_i := \{1; \dots; p\}$ be the one of “internal” electronic variables $y \in \mathbb{R}^{3p}$, and let $\mathcal{I}_n := \{1; \dots; L\}$ be the one of the nuclear variables $R_\ell \in \mathbb{R}^3$. It is convenient to introduce the disjoint union of those sets, namely

$$\mathcal{I}_e \sqcup \mathcal{I}_i \sqcup \mathcal{I}_n := \{(a; j); a \in \{e; i; n\}, j \in \mathcal{I}_a\}.$$

For $c = (a; j) \in \mathcal{I}_e \sqcup \mathcal{I}_i \sqcup \mathcal{I}_n$, let $\pi_1(c) = a$ and $\pi_2(c) = j$. We define $X_c = x_j$ if $\pi_1(c) = e$, $X_c = y_j$ if $\pi_1(c) = i$, and $X_c = R_j$ if $\pi_1(c) = n$. Let

$$\mathcal{P} := \{ \{c; c'\}; (c; c') \in (\mathcal{I}_e \sqcup \mathcal{I}_i \sqcup \mathcal{I}_n)^2 \text{ and } c \neq c' \}$$

be the set of all possible particle pairings. Let $\underline{\Omega}$ be an open subset of \mathbb{R}^{3m} . We consider potentials \underline{V} where

$$\forall (x; y) \in \underline{\Omega} \times \mathbb{R}^{3p}, \quad \underline{V}(x; y) = \sum_{\{c; c'\} \in \mathcal{P}} V_{\{c; c'\}}(X_c - X_{c'}), \quad (3.3)$$

with $V_{\{c; c'\}} \in \mathcal{V}$ (cf. Definition 3.1), for all $\{c; c'\} \in \mathcal{P}$.

Recall that $\mathcal{W}_n = W^{n,2}(\mathbb{R}^{3p})$, for $n \in \mathbb{N}$, and $\mathcal{B}_1 = \mathcal{L}(\mathcal{W}_1; \mathcal{W}_0)$. We note that, if $v \in \mathcal{V}$ and $\{c; c'\} \in \mathcal{P}$, then, for any fixed x , the multiplication by the almost everywhere defined function $v(X_c - X_{c'})$ of y belongs to \mathcal{B}_1 , thanks to (3.2) and the assumption on v_0 .

Recall that we introduced in Definition 2.1 a class of differential operators. The class of Hamiltonians, that we want to treat, is defined in the following:

Definition 3.2. For $(m; p) \in (\mathbb{N}^*)^2$ and $\underline{\Omega}$ an open subset of \mathbb{R}^{3m} . We denote by $\text{Ham}_2(\underline{\Omega})$ the set of all the Hamiltonians \underline{H} on $\underline{\Omega} \times \mathbb{R}^{3p}$ such that $\underline{H} = \underline{P} + \underline{V}$, where $\underline{P} \in \text{Diff}_2(\underline{\Omega})$ (cf. Definition 2.1) and \underline{V} satisfies (3.3).

3.2 Heuristic.

In this subsection, we want to motivate the notion of “twist”, that is going to play a crucial rôle in the proof of the main results. To explain the basic idea, let us consider the following situation.

Take the Hamiltonian H in (1.1) with $N = 2$. We are not able to apply Theorem 2.2 with $x = x_1$ and $y = x_2$ to the equation $H\psi = E\psi$, since the map $x_1 \mapsto (x_2 \mapsto |x_1 - x_2|^{-1}) \in \mathcal{B}_1 = \mathcal{L}(W^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$ is admittedly well defined by (1.2) but is not analytic. The problem lies in the dependency of the Coulomb singularity on x_1 .

Assume that, for some open subset Ω of \mathbb{R}^3 and some $x^0 \in \mathbb{R}^3$, we have a smooth map $f : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that, for all $x_1 \in \Omega$, $f(x_1; \cdot)$ is a smooth diffeomorphism of \mathbb{R}^3 satisfying $f(x_1; x^0) = x_1$. For such x_1 , let U_{x_1} be the unitary implementation of $f(x_1; \cdot)$ on $L^2(\mathbb{R}^3)$ (the “twist”). For any function $\varphi \in W^{1,2}(\mathbb{R}^3)$ and for $x_1 \in \Omega$,

$$U_{x_1}(|x_1 - \cdot|^{-1}\varphi)(y) = |f(x_1; x^0) - f(x_1; y)|^{-1} \cdot (U_{x_1}\varphi)(y).$$

Now, since $f(x_1; \cdot)$ is a diffeomorphism, the singularity occurs when $y = x^0$. Therefore, it does not depend on x_1 anymore. We can write

$$(U_{x_1}|x_1 - \cdot|^{-1}U_{x_1}^{-1}\varphi)(y) = |x^0 - y| \cdot |f(x_1; x^0) - f(x_1; y)|^{-1} \cdot |x^0 - y|^{-1} \cdot \varphi(y).$$

Assuming further that the map

$$\Omega \ni x_1 \mapsto |x^0 - \cdot| \cdot |f(x_1; x^0) - f(x_1; \cdot)|^{-1} \in L^\infty(\mathbb{R}^3) \quad (3.4)$$

is smooth, we see that $\Omega \ni x_1 \mapsto U_{x_1}|x_1 - \cdot|^{-1}U_{x_1}^{-1} \in \mathcal{B}_1 = \mathcal{L}(W^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$ is also smooth, thanks to (1.2).

What can we do with the other singular terms in H ? Since the terms $|x_1 - R_\ell|^{-1}$ do not depend on x_2 , the previous conjugation has no effect on them. But, if x_1 stays far away from all R_ℓ , they are not singular. We are left with the terms $|x_2 - R_\ell|^{-1}$, which are a constant function of x_1 with values in \mathcal{B}_1 by (1.2). Since we want to perform the above conjugation to treat the term $|x_1 - x_2|^{-1}$, we need to verify that this conjugation does not destroy the nice property of these terms. If we require that, for all ℓ and all $x_1 \in \Omega$, $f(x_1; R_\ell) = R_\ell$, and the smoothness of the map (3.4) with x^0 replaced by R_ℓ , the above computation shows that the map $\Omega \ni x_1 \mapsto U_{x_1}|x_2 - R_\ell|^{-1}U_{x_1}^{-1} \in \mathcal{B}_1$ is also smooth.

With some more work, we can prove real analyticity of the previous terms. We just have seen that the “twisted” potential $U_{x_1}VU_{x_1}^{-1}$ is a real analytic, \mathcal{B}_1 -valued function. What about the “twisted” kinetic energy? A simple computation shows that the operators ∇_{x_1} and ∇_{x_2} are transformed by the twist into differential operators in the $(x_1; x_2)$ variables with explicit coefficients in terms of the derivatives of f (cf. (3.16) and (3.18)). As we shall see below, the ellipticity of H is preserved by the “twist” and the coefficients of the “twisted” operator have some property of real analyticity.

Coming back to the original problem, we do not apply Theorem 2.2 with $x = x_1$ and $y = x_2$ to the equation $H\psi = E\psi$ but to the “twisted” equation $(U_{x_1}HU_{x_1}^{-1})U_{x_1}\psi = EU_{x_1}\psi$. Therefore, we obtain some analytic regularity of the “twisted” bound state, namely $U_{x_1}\psi$, not on ψ itself. But, if we are interested in the regularity of ρ_1 , we are done. Indeed, since we can write $\rho_1(x_1) = \|\psi(x_1; \cdot)\|^2$ ($\|\cdot\|$ being the L^2 -norm on \mathbb{R}^3) and since U_{x_1} is unitary on $L^2(\mathbb{R}^3)$, $\rho_1(x_1) = \|U_{x_1}\psi(x_1; \cdot)\|^2$ and ρ_1 is real analytic on Ω by composition.

We just have sketched, in the $N = 2$ case, the proof of the smoothness of ρ_1 , which was performed in [J], provided there exists a map f satisfying the above requirements. Actually it does (see Subsection 3.3 below).

3.3 Hunziker’s twist.

With the above strategy in mind, we recall Hunziker’s twist (cf. [Hu, KMSW]) and provide basic properties of it.

We take a real-valued cut-off function $\tau \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ such that $\tau(x) = 0$ if $|x| \geq 1$, and $\tau(0) = 1$. Let \mathcal{U} be the open subset of \mathbb{R}^{3m} defined by

$$\begin{aligned} \mathcal{U} &= \{x \in \mathbb{R}^{3m}; \forall (c; c') \in \mathcal{P}; \pi_1(c) = e \text{ and } \pi_1(c') \in \{e; n\}, \quad X_c \neq X_{c'}\} \\ &= \{x \in \mathbb{R}^{3m}; \forall (j; j') \in \llbracket 1; m \rrbracket^2, \forall \ell \in \llbracket 1; L \rrbracket, x_j \neq R_\ell \\ &\quad \text{and } j \neq j' \implies x_j \neq x_{j'}\}. \end{aligned} \quad (3.5)$$

Let $x^0 = (x_1^0; \dots; x_m^0) \in \mathcal{U}$. We can find some $r_0 > 0$ such that, for $(j; j') \in \llbracket 1; m \rrbracket^2$, for $\ell \in \llbracket 1; L \rrbracket$,

$$|x_j^0 - R_\ell| \geq r_0 \quad \text{and} \quad j \neq j' \implies |x_j^0 - x_{j'}^0| \geq r_0. \quad (3.6)$$

Let $f : \mathbb{R}^{3m} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the smooth function defined by

$$f(\mathbf{x}; z) = z + \sum_{j=1}^m \tau(r_0^{-1}(z - \mathbf{x}_j^0)) \cdot (\mathbf{x}_j - \mathbf{x}_j^0). \quad (3.7)$$

Notice that, by (3.6), we have, for all $\mathbf{x} \in \mathbb{R}^{3m}$,

$$\forall j \in \llbracket 1; m \rrbracket, \ell \in \llbracket 1; L \rrbracket, \quad f(\mathbf{x}; \mathbf{x}_j^0) = \mathbf{x}_j \quad \text{and} \quad f(\mathbf{x}; R_\ell) = R_\ell. \quad (3.8)$$

For all $(\mathbf{x}; z) \in \mathbb{R}^{3m} \times \mathbb{R}^3$, the total differential of f w.r.t. \mathbf{x} is given by

$$(d_{\mathbf{x}}f)(\mathbf{x}; z) = \sum_{j=1}^m \tau(r_0^{-1}(z - \mathbf{x}_j^0)) d\mathbf{x}_j \quad (3.9)$$

where $d\mathbf{x}_j : \mathbb{R}^{3m} \ni \mathbf{x}' \mapsto \mathbf{x}'_j \in \mathbb{R}^3$, for $1 \leq j \leq m$. In particular, the map $\mathbb{R}^{3m} \ni \mathbf{x} \mapsto (d_{\mathbf{x}}f)(\mathbf{x}; \cdot) \in \mathcal{C}_b^\infty(\mathbb{R}^3; \mathcal{L}(\mathbb{R}^{3m}; \mathbb{R}^3))$ is well-defined and constant. Using (3.9) and setting

$$C(\tau) = \sup_{z \in \mathbb{R}^3} \|d\tau(z)\|_{\mathcal{L}(\mathbb{R}^3)} < +\infty,$$

we can write, for all $j \in \llbracket 1; m \rrbracket$, for all $(z; z') \in (\mathbb{R}^3)^2$, and for all $\mathbf{x} \in \mathbb{R}^{3m}$,

$$\begin{aligned} \|(d_{\mathbf{x}_j}f)(\mathbf{x}; z) - (d_{\mathbf{x}_j}f)(\mathbf{x}; z')\|_{\mathcal{L}(\mathbb{R}^{3m}; \mathbb{R}^3)} &= \left| \tau(r_0^{-1}(z - \mathbf{x}_j^0)) - \tau(r_0^{-1}(z' - \mathbf{x}_j^0)) \right| \\ &\leq r_0^{-1} C(\tau) |z - z'|. \end{aligned} \quad (3.10)$$

For all $(\mathbf{x}; z) \in \mathbb{R}^{3m} \times \mathbb{R}^3$, the total differential of f w.r.t. z is given by, for $z^\epsilon \in \mathbb{R}^3$,

$$(d_z f)(\mathbf{x}; z) \cdot z^\epsilon = z^\epsilon + r_0^{-1} \sum_{j=1}^m ((d\tau)(r_0^{-1}(z - \mathbf{x}_j^0)) \cdot z^\epsilon)(\mathbf{x}_j - \mathbf{x}_j^0). \quad (3.11)$$

For $\delta > 0$, set $\Omega(\delta) := B(\mathbf{x}_1^0; \delta] \times \cdots \times B(\mathbf{x}_m^0; \delta]$. Recall that the class \mathcal{V} (cf. Definition 3.1) depends on a parameter η_0 (see (3.1)). By (3.11), we can find some $\delta_0 \in]0; r_0/2]$, depending on $C(\tau)$ and r_0 , such that

$$\forall \mathbf{x} \in \Omega(\delta_0), \quad \sup_{z \in \mathbb{R}^3} \|(d_z f)(\mathbf{x}; z) - I_3\|_{\mathcal{L}(\mathbb{R}^3)} \leq \min(\eta_0; 1 - \eta_0) < 1. \quad (3.12)$$

We note that the requirement $\delta_0 \in]0; r_0/2]$ yields $\Omega(\delta_0) \subset \mathcal{U}$, by (3.6) and the triangle inequality. Thanks to (3.12), for each $\mathbf{x} \in \Omega(\delta_0)$, $f(\mathbf{x}; \cdot)$ is a \mathcal{C}^∞ -diffeomorphism of \mathbb{R}^3 . We denote by $f^{(-1)}(\mathbf{x}; \cdot)$ its inverse. Furthermore, for all $\mathbf{x} \in \Omega(\delta_0)$, for all $(z; z') \in (\mathbb{R}^3)^2$,

$$(1 - \eta_0) |z - z'| \leq |f(\mathbf{x}; z) - f(\mathbf{x}; z')| \leq (1 + \eta_0) |z - z'|. \quad (3.13)$$

By a direct computation using (3.9) and (3.11), one can express the derivative $d_z f^{(-1)}$ and $d_{\mathbf{x}} f^{(-1)}$ in terms of $d_z f$ (see Appendix A for details). In particular, one can check that $f \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^3; \mathbb{R}^3)$ and $f^{(-1)} \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^3; \mathbb{R}^3)$.

Consider the map $F : \Omega(\delta_0) \times \mathbb{R}^{3p} \longrightarrow \mathbb{R}^{3p}$, defined by $F(\mathbf{x}; \mathbf{y}) = (f(\mathbf{x}; y_1); \cdots; f(\mathbf{x}; y_p))$. For each $\mathbf{x} \in \Omega(\delta_0)$, $F(\mathbf{x}; \cdot)$ is a \mathcal{C}^∞ -diffeomorphism of \mathbb{R}^{3p} and we denote its inverse

map by $F^{(-1)}(\mathbf{x}; \cdot)$. Explicit expressions for $d_x F$, $d_y F$, $d_x F^{(-1)}$, and $d_y F^{(-1)}$, are provided in Appendix A. In particular, we observe that $F \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3p})$ and $F^{(-1)} \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3p})$.

We introduce the map $U : \Omega(\delta_0) \ni \mathbf{x} \mapsto U_{\mathbf{x}} \in \mathcal{B}_0$ (with $\mathcal{B}_0 = \mathcal{L}(\mathcal{W}_0)$ and $\mathcal{W}_0 = L^2(\mathbb{R}^{3p})$), with values in the set of unitary operators on \mathcal{W}_0 , defined by, for $\theta \in \mathcal{W}_0$,

$$(U_{\mathbf{x}}\theta)(\mathbf{y}) = |\text{Det}(d_y F)(\mathbf{x}; \mathbf{y})|^{1/2} \cdot \theta(F(\mathbf{x}; \mathbf{y})) . \quad (3.14)$$

For $\mathbf{x} \in \Omega(\delta_0)$, the map $U_{\mathbf{x}}$ is the unitary implementation of $F(\mathbf{x}; \cdot)$ on $L^2(\mathbb{R}^{3p})$ and

$$(U_{\mathbf{x}}^{-1}\theta)(\mathbf{y}) = \left| \text{Det}(d_y F^{(-1)})(\mathbf{x}; \mathbf{y}) \right|^{1/2} \cdot \theta(F^{(-1)}(\mathbf{x}; \mathbf{y})) . \quad (3.15)$$

Denoting by A^T the transposed of a linear map A and by $\text{Det} A$ its determinant, we see by a direct computation that, as differential operators on $\Omega(\delta_0) \times \mathbb{R}^{3p}$,

$$U_{\mathbf{x}}^{-1} D_{\mathbf{x}} U_{\mathbf{x}} = D_{\mathbf{x}} + J_1(F) D_{\mathbf{y}} + J_2(F) , \quad (3.16)$$

$$U_{\mathbf{x}} D_{\mathbf{x}} U_{\mathbf{x}}^{-1} = D_{\mathbf{x}} + J_1(F^{(-1)}) D_{\mathbf{y}} + J_2(F^{(-1)}) , \quad (3.17)$$

$$U_{\mathbf{x}}^{-1} D_{\mathbf{y}'} U_{\mathbf{x}} = J_3(F) D_{\mathbf{y}} + J_4(F) , \quad (3.18)$$

$$U_{\mathbf{x}} D_{\mathbf{y}'} U_{\mathbf{x}}^{-1} = J_3(F^{(-1)}) D_{\mathbf{y}} + J_4(F^{(-1)}) , \quad (3.19)$$

where, for $\mathbf{x} \in \Omega(\delta_0)$ and $\mathbf{y} \in \mathbb{R}^{3p}$, with $G = F$ and $G^{(-1)} = F^{(-1)}$ or with $G = F^{(-1)}$ and $G^{(-1)} = F$,

$$J_1(G)(\mathbf{x}; \mathbf{y}) := \left((d_{\mathbf{x}} G)(\mathbf{x}; \mathbf{y}') \right)^T (\mathbf{x}; \mathbf{y}' = G^{(-1)}(\mathbf{x}; \mathbf{y})) ,$$

$$J_2(G)(\mathbf{x}; \mathbf{y}) := -i \left| \text{Det}(d_y G^{(-1)})(\mathbf{x}; \mathbf{y}) \right|^{1/2} D_{\mathbf{x}} \left(\left| \text{Det}(d_{\mathbf{y}'} G)(\mathbf{x}; \mathbf{y}') \right|^{1/2} \right) \Big|_{\mathbf{y}' = G^{(-1)}(\mathbf{x}; \mathbf{y})} ,$$

$$J_3(G)(\mathbf{x}; \mathbf{y}) := \left((d_{\mathbf{y}'} G)(\mathbf{x}; \mathbf{y}') \right)^T (\mathbf{x}; \mathbf{y}' = G^{(-1)}(\mathbf{x}; \mathbf{y})) ,$$

$$J_4(G)(\mathbf{x}; \mathbf{y}) := -i \left| \text{Det}(d_y G^{(-1)})(\mathbf{x}; \mathbf{y}) \right|^{1/2} D_{\mathbf{y}'} \left(\left| \text{Det}(d_{\mathbf{y}'} G)(\mathbf{x}; \mathbf{y}') \right|^{1/2} \right) \Big|_{\mathbf{y}' = G^{(-1)}(\mathbf{x}; \mathbf{y})} .$$

Thanks to the computation written in Appendix A, we note that, on one hand, for $G \in \{F; F^{(-1)}\}$,

$$J_1(G) \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathcal{L}(\mathbb{R}^{3p}; \mathbb{R}^{3m})) , \quad iJ_2(G) \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3m}) , \quad (3.20)$$

$$J_3(G) \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathcal{L}(\mathbb{R}^{3p})) , \quad iJ_4(G) \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3p}) , \quad (3.21)$$

and, on the other hand, if we denote by J the multiplication operator by any component of $J_k(G)$, $k \in \{1; 2; 3; 4\}$ and $G \in \{F; F^{(-1)}\}$,

$$J \in \mathcal{C}^\omega(\Omega(\delta_0); \mathcal{B}_0) . \quad (3.22)$$

Furthermore (cf. Appendix A), there exists $C > 0$ such that, for $G \in \{F; F^{(-1)}\}$, for all $(\mathbf{x}; \mathbf{y}) \in \Omega(\delta_0) \times \mathbb{R}^{3p}$, $J_3(G)(\mathbf{x}; \mathbf{y})$ is invertible and

$$C^{-1} \geq \|J_3(G)(\mathbf{x}; \mathbf{y})\|_{\mathcal{L}(\mathbb{R}^{3p})} + \|(J_3(G)(\mathbf{x}; \mathbf{y}))^{-1}\|_{\mathcal{L}(\mathbb{R}^{3p})} \geq C . \quad (3.23)$$

Now, using (3.18),

$$D_{y'}U_x = U_x(J_3(F)D_y - J_4(F))$$

thus, for all $x \in \Omega(\delta_0)$, we see, by induction and by (3.21), that U_x leaves \mathcal{W}_n invariant, for all $n \in \mathbb{N}$. Similarly, we can show, using (3.19) and (3.21), that, for all $x \in \Omega(\delta_0)$, U_x^{-1} also leaves \mathcal{W}_n invariant, for all $n \in \mathbb{N}$. By the same arguments, we can check, starting from (3.16) and (3.18), that U preserves the space $W^{2,2}(\Omega(\delta_0) \times \mathbb{R}^{3p})$.

3.4 Potential regularization by a twist.

We first consider the action by conjugation of the twist U on the potential part \underline{V} of Hamiltonians \underline{H} in the class $\text{Ham}_2(\underline{\Omega})$.

Proposition 3.3. *Take $x^0 \in \mathcal{U} \subset \mathbb{R}^{3m}$. We choose $r_0 > 0$ and $\delta_0 \in]0; r_0/2]$ such that (3.6) and (3.12) hold true. Recall that the neighbourhood $\Omega(\delta_0) := B(x_1^0; \delta_0] \times \cdots \times B(x_m^0; \delta_0]$ of x^0 is included in \mathcal{U} .*

Then, for any $v \in \mathcal{V}$ and any $\{c; c'\} \in \mathcal{P}$, the map $\Omega(\delta_0) \ni x \mapsto U_x v(X_c - X_{c'}) U_x^{-1} \in \mathcal{B}_1$ is well-defined and real analytic.

Remark 3.4. *We point out that the Coulomb potential $|\cdot|^{-1}$ does belong to \mathcal{V} . Indeed, (3.2), for $v = v_0 = |\cdot|^{-1}$, was proved in [J] and the multiplication by this v_0 belongs to $\mathcal{L}(\mathcal{W}^{1,2}(\mathbb{R}^3); L^2(\mathbb{R}^3))$ by Hardy's inequality (1.2).*

Remark 3.5. *A proof of Proposition 3.3 is essentially available in [Hu, J]. However, the proof in [Hu] is restricted to Coulomb interactions. We provide below a more transparent version of the proof in [J].*

Proof of Proposition 3.3: Let $x \in \Omega(\delta_0)$. Recall that, for $k \in \{0; 1\}$, U_x and U_x^{-1} leave \mathcal{W}_k invariant. Recall that, for all $\{c; c'\} \in \mathcal{P}$, the multiplication by $v(X_c - X_{c'})$ belongs to \mathcal{B}_1 . Therefore the map $w : \Omega(\delta_0) \ni x \mapsto U_x v(X_c - X_{c'}) U_x^{-1} \in \mathcal{B}_1$ is well-defined. We treat all possible cases for $\{c; c'\} \in \mathcal{P}$.

If $\pi_1(c) = \pi_1(c') = n$, then $w : \Omega(\delta_0) \ni x \mapsto v(X_c - X_{c'})$ is a constant.

Assume that $\pi_1(c) = e$ and $\pi_1(c') \in \{e; n\}$ or that $\pi_1(c') = e$ and $\pi_1(c) \in \{e; n\}$. Since $\Omega(\delta_0) \subset \mathcal{U}$, $X_c - X_{c'}$ does not vanish on $\Omega(\delta_0)$. Thus $w : \Omega(\delta_0) \ni x \mapsto v(X_c - X_{c'})$ is real analytic, by composition.

Assume that $\pi_1(c) = e$ and $\pi_1(c') = i$. Let $j = \pi_2(c)$ and $k = \pi_2(c')$. Then, for $x \in \Omega(\delta_0)$, $U_x v(X_c - X_{c'}) U_x^{-1}$ is the multiplication operator on \mathcal{W}_1 by the function

$$y \mapsto v(x_j - f(x; y_k)) = v(f(x; x_j^0) - f(x; y_k)),$$

by (3.8). Let $j' \in \mathcal{I}_e$. Using the fact that $d_x f$ does not depend on x (see (3.9)), we have, for $\alpha \in \mathbb{N}^3$ and $y_k \neq x_j^0$,

$$\begin{aligned} & \partial_{x_{j'}}^\alpha \left(v(f(x; x_j^0) - f(x; y_k)) \right) \\ &= (\partial^\alpha v)(f(x; x_j^0) - f(x; y_k)) \cdot \left(\tau(r_0^{-1}(x_j^0 - x_{j'}^0)) - \tau(r_0^{-1}(y_k - x_{j'}^0)) \right)^{|\alpha|}. \end{aligned} \tag{3.24}$$

Using successively (3.10), (3.13), and (3.2), we obtain

$$\begin{aligned}
& \left| \partial_{x_j}^\alpha \left(v(f(x; x_j^0) - f(x; y_k)) \right) \right| \\
& \leq |(\partial^\alpha v)(f(x; x_j^0) - f(x; y_k))| \cdot \left(r_0^{-1} C(\tau) |x_j^0 - y_k| \right)^{|\alpha|}, \\
& \leq |(\partial^\alpha v)(f(x; x_j^0) - f(x; y_k))| \cdot \left((r_0(1 - \eta_0))^{-1} C(\tau) |f(x; x_j^0) - f(x; y_k)| \right)^{|\alpha|}, \\
& \leq C_1^{1+|\alpha|} \cdot (|\alpha|!) \cdot \left| v_0(f(x; x_j^0) - f(x; y_k)) \right|,
\end{aligned}$$

for some $C_1 > 0$. Now, by the structure of v_0 and by (3.13) and (3.1), we derive that, for some positive constants C_2 and C_3 ,

$$\begin{aligned}
\left| \partial_{x_j}^\alpha \left(v(f(x; x_j^0) - f(x; y_k)) \right) \right| & \leq C_2^{1+|\alpha|} \cdot (|\alpha|!) \cdot \eta(f(x; x_j^0) - f(x; y_k)), \\
& \leq C_3^{1+|\alpha|} \cdot (|\alpha|!) \cdot \eta(|x_j^0 - y_k|). \tag{3.25}
\end{aligned}$$

Now, one can check, by induction on $|\alpha|$ and by the fact that the multiplication by $\eta(|\cdot|)$ belongs to \mathcal{B}_1 , that, on $\Omega(\delta_0)$, the $\partial_{x_j}^\alpha$ partial derivative of w exists and maps each $x \in \Omega(\delta_0)$ to the multiplication operator on \mathcal{W}_1 by the almost everywhere defined function (3.24) of y . Thus, $w \in \mathcal{C}^\infty(\Omega(\delta_0); \mathcal{B}_1)$. Using (3.25) again, we obtain that

$$\sup_{x \in \Omega(\delta_0)} \| (D_{x_j}^\alpha w)(x) \|_{\mathcal{B}_1} \leq C_4^{1+|\alpha|} \cdot (|\alpha|!), \tag{3.26}$$

for some $C_4 > 0$. By (2.2), w is real analytic w.r.t. x_j . Since this holds true for all j' , w is real analytic.

Similarly, we can treat the case $\pi_1(c) = i$ and $\pi_1(c') = e$.

Assume that $\pi_1(c) = n$ and $\pi_1(c') = i$. Let $\ell = \pi_2(c)$ and $k = \pi_2(c')$. For $x \in \Omega(\delta_0)$, $w(x)$ is the multiplication operator in \mathcal{B}_1 by the function

$$y \mapsto v(R_\ell - f(x; y_k)) = v(f(x; R_\ell) - f(x; y_k)), \tag{3.27}$$

by (3.8). We redo the computation (3.24) - (3.25), with x_j^0 replaced by R_ℓ . Furthermore, we also can check that the $\partial_{x_j}^\alpha$ partial derivative of w exists and maps each $x \in \Omega(\delta_0)$ to the multiplication operator on \mathcal{W}_1 by the almost everywhere defined $\partial_{x_j}^\alpha$ partial derivative of (3.27). Using the new estimate (3.25), we derive a bound like (3.26). As above, we conclude that w is real analytic.

Again, we have a similar treatment of the case $\pi_1(c) = i$ and $\pi_1(c') = n$.

We are left with the case $\pi_1(c) = i$ and $\pi_1(c') = i$. We set $k = \pi_2(c)$ and $k' = \pi_2(c')$. By assumption, $k \neq k'$. Using (3.8) again, we see that, for $x \in \Omega(\delta_0)$, $w(x)$ is the multiplication operator in \mathcal{B}_1 by the function

$$y \mapsto v(f(x; y_k) - f(x; y_{k'})). \tag{3.28}$$

Once again, we perform the computation (3.24) - (3.25) and follow the above arguments, using this time that the multiplication by $y \mapsto \eta(|y_k - y_{k'}|)$ belongs to \mathcal{B}_1 . This yields the real analyticity of w . \square

3.5 Twisted regular differential calculus.

Next, we let act the twist U by conjugation on the regular, differential part $\underline{P} \in \text{Diff}_2(\underline{\Omega})$ of Hamiltonians \underline{H} in the class $\text{Ham}_2(\underline{\Omega})$ (cf. Definitions 2.1 and 3.2). It turns out that this action preserves this class $\text{Diff}_2(\underline{\Omega})$.

Proposition 3.6. *Let $\mathbf{x}^0 \in \mathcal{U} \cap \underline{\Omega} \subset \mathbb{R}^{3m}$. Take $r_0 > 0$ and $\delta_0 \in]0; r_0/2]$ such that (3.6) and (3.12) hold true, and such that $\Omega(\delta_0) := B(\mathbf{x}_1^0; \delta_0] \times \cdots \times B(\mathbf{x}_m^0; \delta_0] \subset \underline{\Omega}$. We recall that $\Omega(\delta_0) \subset \mathcal{U}$. Let us define $U^* : \Omega(\delta_0) \ni \mathbf{x} \mapsto U_{\mathbf{x}}^{-1} \in \mathcal{B}_0$.*

Let $\underline{P} \in \text{Diff}_2(\underline{\Omega})$ (see Definition 2.1). Then the operator composition $P = U \underline{P} U^$ defines a differential operator on $\Omega(\delta_0) \times \mathbb{R}^{3p}$ that belongs to $\text{Diff}_2(\Omega(\delta_0))$.*

Remark 3.7. *In [Hu, J], it was proven that the twisted Laplacian $U \Delta U^*$ is elliptic, when $m = 1$. Proposition 3.6 shows that the conjugation by the twist of an elliptic operator is also elliptic. This property is actually used in [KMSW, MS] in a semiclassical framework.*

Remark 3.8. *In the proof of Proposition 3.6 below, we essentially follow [Hu, J]. Observing that U is a Fourier integral operator (cf. [GS, H 4]), one can use composition rules for such operators to derive the relationship between the principal symbols of \underline{P} and P , given in (3.30) below.*

Proof of Proposition 3.6: Take $\underline{P} \in \text{Diff}_2(\underline{\Omega})$, that is of the form (2.4). We first observe that, for $(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p}$ with $|\alpha| + |\beta| \leq 2$, $U c_{\alpha\beta} U^*$ is, as differential operator, the multiplication operator by the function

$$\Omega(\delta_0) \times \mathbb{R}^{3p} \ni (\mathbf{x}; \mathbf{y}) \mapsto c_{\alpha\beta}(\mathbf{x}; F(\mathbf{x}; \mathbf{y})) ,$$

which belongs to $\mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{C})$. Using (3.17) and (3.19) componentwise and taking advantage of (3.20) and (3.21) for $G = F^{(-1)}$, we see that $P = U \underline{P} U^*$ has the form (2.4) on $\Omega(\delta_0) \times \mathbb{R}^{3p}$ with coefficients in $\mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{C})$.

We are left with the proof of the ellipticity of P . To this end, we compute its principal symbol σ_P . Let us denote by “ \cdot ” the scalar product in \mathbb{R}^d , for any $d \in \mathbb{N}^*$. It is well-known that the total symbol of P is the map $S_P : \Omega(\delta_0) \times \mathbb{R}^{3p} \times \mathbb{R}^{3m} \times \mathbb{R}^{3p} \longrightarrow \mathbb{C}$ defined by

$$S_P(\mathbf{x}; \mathbf{y}; \xi; \eta) = e^{-i\mathbf{x} \cdot \xi - i\mathbf{y} \cdot \eta} P(e^{i\mathbf{x} \cdot \xi + i\mathbf{y} \cdot \eta}) . \quad (3.29)$$

Using this, we obtain in Appendix A that, for $(\mathbf{x}; \mathbf{y}; \xi; \eta) \in \Omega(\delta_0) \times \mathbb{R}^{3p} \times \mathbb{R}^{3m} \times \mathbb{R}^{3p}$,

$$\sigma_P(\mathbf{x}; \mathbf{y}; \xi; \eta) = \sigma_{\underline{P}}(\mathbf{x}; F(\mathbf{x}; \mathbf{y}); \xi + J_1(F^{(-1)})(\mathbf{x}; \mathbf{y}) \eta; J_3(F^{(-1)})(\mathbf{x}; \mathbf{y}) \eta) . \quad (3.30)$$

Since \underline{P} is elliptic, there exists $\underline{C} > 0$ such that (2.6) holds true with P replaced by \underline{P} and C replaced by \underline{C} . Take $(\xi; \eta)$ satisfying $|\xi|^2 + |\eta|^2 \geq 1$. Then

$$\underline{C}^{-1} \sigma_P(\mathbf{x}; \mathbf{y}; \xi; \eta) \geq \left| \xi + J_1(F^{(-1)})(\mathbf{x}; \mathbf{y}) \eta \right|^2 + \left| J_3(F^{(-1)})(\mathbf{x}; \mathbf{y}) \eta \right|^2 . \quad (3.31)$$

By (3.20), we can find $M > 0$ such that, for all $(\mathbf{x}; \mathbf{y}) \in \Omega(\delta_0) \times \mathbb{R}^{3p}$,

$$\left\| J_1(F^{(-1)})(\mathbf{x}; \mathbf{y}) \right\|_{\mathcal{L}(\mathbb{R}^{3p}; \mathbb{R}^{3m})} \leq M .$$

Now let $S = \sqrt{1 + 4M^2}$. Consider first the case where $S|\eta| \leq (|\xi|^2 + |\eta|^2)^{1/2}$. We have $2M|\eta| \leq |\xi|$. Thus

$$|\xi + J_1(F^{(-1)})(x; y)\eta|^2 \geq \frac{|\xi|^2}{4}$$

and, using (3.23), we obtain from (3.31) the lower bound

$$\underline{C}^{-1}\sigma_P(x; y; \xi; \eta) \geq \min(1/4; C^{-2})(|\xi|^2 + |\eta|^2).$$

If, now, $S|\eta| \geq (|\xi|^2 + |\eta|^2)^{1/2}$, it follows from (3.23) and (3.31) that

$$\underline{C}^{-1}\sigma_P(x; y; \xi; \eta) \geq C^{-2}S^{-2}(|\xi|^2 + |\eta|^2).$$

This yields (2.6). Thus $P \in \text{Diff}_2(\Omega(\delta_0))$. \square

3.6 Elliptic regularity for twisted Hamiltonians.

Now, we conjugate by the twist U any Hamiltonians \underline{H} of the class $\text{Ham}_2(\underline{\Omega})$ (cf. Definition 3.2) and show that we can apply Theorem 2.2 to the twisted operator.

Theorem 3.9. *Let $\underline{\Omega}$ be an open subset of \mathbb{R}^{3m} and $x^0 \in \mathcal{U} \cap \underline{\Omega}$. Then, one can find a neighbourhood Ω_0 of x^0 and a unitary operators valued map $U : \Omega_0 \ni x \mapsto U_x \in \mathcal{B}_0$ such that the following holds true:*

For any Hamiltonian $\underline{H} \in \text{Ham}_2(\underline{\Omega})$ on $\underline{\Omega} \times \mathbb{R}^{3p} \subset \mathbb{R}^{3(m+p)}$, for any $\Psi \in W^{2,2}(\underline{\Omega} \times \mathbb{R}^{3p})$ such that $\underline{H}\Psi = 0$, the map $U\Psi : \Omega_0 \ni x \mapsto U_x\Psi(x; \cdot) \in \mathcal{W}_2$ is well-defined and real analytic. Furthermore, for any $\alpha \in \mathbb{N}^{3m}$ with $1 \leq |\alpha| \leq 2$, the map $U(D_x^\alpha \Psi) : \Omega_0 \ni x \mapsto U_x(D_x^\alpha \Psi)(x; \cdot) \in \mathcal{W}_{2-|\alpha|}$ is well-defined and real analytic.

Proof: Given $x^0 \in \mathcal{U} \cap \underline{\Omega}$, take $\Omega_0 = \Omega(\delta_0)$ satisfying the assumptions of Proposition 3.6 and consider the twist $U : \Omega_0 \rightarrow \mathcal{B}_0$ defined by (3.14). Recall that $U^* : \Omega_0 \ni x \mapsto U_x^{-1} \in \mathcal{B}_0$. Take $\underline{H} = \underline{P} + \underline{V} \in \text{Ham}_2(\underline{\Omega})$ and $\Psi \in W^{2,2}(\underline{\Omega} \times \mathbb{R}^{3p})$ such that $\underline{H}\Psi = 0$. Applying Proposition 3.3 to each term of \underline{V} and Proposition 3.6 to \underline{P} , we see that $P = U\underline{P}U^*$ and $W = U\underline{V}U^*$ satisfy the assumptions of Theorem 2.2 on Ω_0 , yielding the real analyticity of the map $U\Psi : \Omega_0 \ni x \mapsto U_x\Psi(x; \cdot) \in \mathcal{W}_2$.

By (3.17), we have, for $x \in \Omega_0$, setting $J_1 := J_1(F^{(-1)})$ and $J_2 := J_2(F^{(-1)})$,

$$\begin{aligned} U_x(D_x\Psi)(x; \cdot) &= U_x D_x U_x^{-1} \cdot U_x \Psi(x; \cdot) \\ &= D_x(U_x \Psi(x; \cdot)) + J_1(x; \cdot) \cdot D_y(U_x \Psi(x; \cdot)) + J_2(x; \cdot)(U_x \Psi(x; \cdot)). \end{aligned}$$

By (3.20), this implies that, for $\alpha \in \mathbb{N}^{3m}$ with $|\alpha| = 1$, $U(D_x^\alpha \Psi) : \Omega_0 \ni x \mapsto U_x(D_x^\alpha \Psi)(x; \cdot) \in \mathcal{W}_1$ is well-defined and real analytic. Letting act $U_x D_x U_x^{-1}$ on the above equality and using (3.17), (3.20), and (3.21), we obtain that, for $\alpha \in \mathbb{N}^{3m}$ with $|\alpha| = 2$, $U(D_x^\alpha \Psi) : \Omega_0 \ni x \mapsto U_x(D_x^\alpha \Psi)(x; \cdot) \in \mathcal{W}_0$ is well-defined and real analytic. \square

Remark 3.10. *We observe that the present framework (the one in Theorem 3.9) is closer to the one in [KMSW] than the one in Hunziker's original paper [Hu]. This comes from the fact that, here and in [KMSW], the considered Hamiltonian contains derivatives w.r.t. the "external" variable x , whereas it is not the case in [Hu]. We further point out that a variant of Hunziker's twist is used in a similar way in [MM].*

4 Proof of the main results.

In this section, we provide a proof for our main results, namely Theorem 1.3, Theorem 1.4, and Theorem 1.5. We first prove the result on the current density C .

Proof of Theorem 1.3: Let $x^0 \in \mathbb{R}^3 \setminus \{R_1, \dots, R_L\}$. Notice that, if we set $m = 1$ and $p = N - 1$, then $\mathcal{U} = \mathbb{R}^3 \setminus \{R_1, \dots, R_L\}$, by (3.5). Note that $\underline{H} := H - E \in \text{Ham}(\mathbb{R}^3)$ and $\underline{H}\psi = 0$. Applying Theorem 3.9 to $\underline{\Omega} = \mathbb{R}^{3m} = \mathbb{R}^3$ and $\mathbf{x}^0 = x^0 \in \mathcal{U} \cap \underline{\Omega}$, there exists some neighbourhood Ω_0 of x^0 and a map $U : \Omega_0 \ni x \mapsto U_x \in \mathcal{B}_0$, the values of which are unitary operators on \mathcal{W}_0 , such that, for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq 1$, the map $U(D_x^\alpha \psi) : \Omega_0 \ni x \mapsto U_x(D_x^\alpha \psi)(x; \cdot) \in \mathcal{W}_{2-|\alpha|}$ is well-defined and real analytic. Let $\alpha \in \mathbb{N}^3$ with $|\alpha| = 1$. For $x \in \Omega_0$, we can write, since U_x is unitary,

$$\langle (D_x^\alpha \psi)(x; \cdot), \psi(x; \cdot) \rangle = \langle U_x(D_x^\alpha \psi)(x; \cdot), U_x \psi(x; \cdot) \rangle.$$

As the scalar product of real analytic, \mathcal{W}_0 -valued maps, the map $\Omega_0 \ni x \mapsto \langle (D_x^\alpha \psi), \psi \rangle$ is also real analytic. This gives the desired result by (1.5). \square

Next we come to the proof of Theorem 1.4 on the densities ρ_k . We also comment on the limitation on the analyticity domain in this result (cf. Remark 4.2).

Proof of Theorem 1.4: Take an integer k such that $0 < k < N$. Let $\underline{x}^0 = (x_1^0; \dots; x_k^0) \in \mathcal{U}_k^{(1)}$. Setting $m = k$ and $p = N - k$, we observe that $\mathcal{U}_k^{(1)}$ coincide with the set \mathcal{U} , defined in (3.5). Since $-\Delta = -\Delta_{\underline{x}} - \Delta_y \in \text{Diff}_2(\mathbb{R}^{3m})$ and $V - E$ is a potential of the type \underline{V} in (3.3) on $\mathbb{R}^{3m} \times \mathbb{R}^{3p}$, by Remark 3.4, we may apply Theorem 3.9 to $\underline{H} = H - E$, $\underline{\Omega} = \mathbb{R}^{3m}$, $\Psi = \psi$, and $\mathbf{x}^0 = \underline{x}^0 \in \mathcal{U} \cap \underline{\Omega}$. Thus there exist some neighbourhood Ω_0 of \underline{x}^0 and a map $U : \Omega_0 \rightarrow \mathcal{B}_0$, with values in the set of unitary operators on \mathcal{W}_0 , such that the map $U\psi : \Omega_0 \ni \underline{x} \mapsto U_{\underline{x}}\psi(\underline{x}; \cdot) \in \mathcal{W}_2$ is well-defined and real analytic. For $\underline{x} \in \Omega_0$, the density $\rho_k(\underline{x})$ is, by definition (see (1.3)), the squared \mathcal{W}_0 -norm of $\psi(\underline{x}; \cdot)$, which is also the one of $U_{\underline{x}}\psi(\underline{x}; \cdot)$, since $U_{\underline{x}}$ is unitary on \mathcal{W}_0 . Thus, ρ_k is real analytic on Ω_0 . \square

Remark 4.1. In the $k = 1$ case, the above proof reproduces the one of [J] on the density $\rho = \rho_1$. Thanks to Hunziker's twist, the proof in the general case is essentially identical.

Remark 4.2. Theorem 1.4 ensures the real analyticity of ρ_k on $\mathcal{U}_k^{(1)}$. We actually can give two obstacles to the extension of this property to a larger set by the method used in Section 3.

If we try to apply this method near a point $\underline{x}^0 \in \mathcal{C}_k \cup \mathcal{R}_k$, we have to work near the singularity of a term of the form $|x_j - R_\ell|^{-1}$ or $|x_j - x_{j'}|^{-1}$ with $1 \leq j \neq j' \leq k$ and such a singular term would be unaffected by any twist. Thus, we cannot apply Theorem 2.2 to the twisted equation to get the result of Theorem 3.9.

The second obstacle takes place in the construction of the twist near such a point $\underline{x}^0 \in \mathcal{C}_k \cup \mathcal{R}_k$. To explain this, let us use the notation of Subsection 3.3 and consider a point \mathbf{x}^0 such that, either $\mathbf{x}_j^0 = R_\ell$ or $\mathbf{x}_j^0 = \mathbf{x}_{j'}^0$, for $j \neq j'$. Assume that we have a function f defined on a neighbourhood of such \mathbf{x}^0 times \mathbb{R}^3 such that, for \mathbf{x} close to \mathbf{x}^0 , $f(\mathbf{x}; \cdot)$ is a diffeomorphism on \mathbb{R}^3 that satisfies (3.8). If $\mathbf{x}_j^0 = \mathbf{x}_{j'}^0$, then, for \mathbf{x} close to \mathbf{x}^0 with different

x_j and $x_{j'}$, we would have $x_j = f(x; x_j^0) = f(x; x_{j'}^0) = x_{j'}$, which is a contradiction. If $x_j^0 = R_\ell$, then, for x close to x^0 with $x_j \neq R_\ell$, we would have $x_j = f(x; x_j^0) = f(x; R_\ell) = R_\ell$, which is again a contradiction.

Finally, we prove Theorem 1.5 concerning the density matrices γ_k . Again we provide a comment on the limitation on the analyticity domain in this result (cf. Remark 4.4).

Proof of Theorem 1.5: Let k be an integer between 0 and N . Let $\underline{x}^0 = (x_1^0; \dots; x_k^0)$ and $\underline{x}'^0 = (x_1'^0; \dots; x_k'^0)$ two points in $(\mathbb{R}^3)^k$ such that $(\underline{x}^0; \underline{x}'^0) \in \mathcal{U}_k^{(2)}$. Setting $m = 2k$ and $p = N - k$, we note that $\mathcal{U} = \mathcal{U}_k^{(2)}$. Denoting by y the variable in $\mathbb{R}^{3(N-k)} = \mathbb{R}^{3p}$, we consider the Hamiltonians H_1 and H_2 on $(\mathbb{R}^{3k})^2 \times \mathbb{R}^{3(N-k)} = \mathbb{R}^{3m} \times \mathbb{R}^{3p}$ defined by

$$H_1 = -\Delta_{\underline{x}} - \Delta_{\underline{x}'} - \Delta_y + V(\underline{x}; y) \quad \text{and} \quad H_2 = -\Delta_{\underline{x}} - \Delta_{\underline{x}'} - \Delta_y + V(\underline{x}'; y)$$

and the almost everywhere defined functions $\Psi_1 : \mathbb{R}^{3m} \times \mathbb{R}^{3p} \ni (\underline{x}; \underline{x}'; y) \mapsto \psi(\underline{x}; y)$ and $\Psi_2 : \mathbb{R}^{3m} \times \mathbb{R}^{3p} \ni (\underline{x}; \underline{x}'; y) \mapsto \psi(\underline{x}'; y)$. We note that $H_1 \in \text{Ham}_2(\mathbb{R}^{3m})$ and $H_2 \in \text{Ham}_2(\mathbb{R}^{3m})$. From $H\psi = E\psi$, we derive that $H_1\Psi_1 = E\Psi_1$ and $H_2\Psi_2 = E\Psi_2$, in the distributional sense. Furthermore, for any bounded subset B of \mathbb{R}^{3m} , we observe that

$$\Psi_1 \in W^{2,2}(B \times \mathbb{R}_y^{3p}) \quad \text{and} \quad \Psi_2 \in W^{2,2}(B \times \mathbb{R}_y^{3p}).$$

Let B_0 be a bounded neighbourhood of $(\underline{x}^0; \underline{x}'^0)$.

Now we apply Theorem 3.9 to $\underline{\Omega} = B_0$ and $\underline{x}^0 = (\underline{x}^0; \underline{x}'^0) \in \mathcal{U} \cap \underline{\Omega}$. Therefore there exist a bounded neighbourhood Ω_0 of $(\underline{x}^0; \underline{x}'^0)$ and a map $U : \Omega_0 \rightarrow \mathcal{B}_0$, with values in the set of unitary operators on \mathcal{W}_0 , such that the maps $U\Psi_1 : \Omega_0 \ni (\underline{x}; \underline{x}') \mapsto U_{\underline{x}; \underline{x}'}\Psi_1(\underline{x}; \underline{x}'; \cdot) \in \mathcal{W}_2$ and $U\Psi_2 : \Omega_0 \ni (\underline{x}; \underline{x}') \mapsto U_{\underline{x}; \underline{x}'}\Psi_2(\underline{x}; \underline{x}'; \cdot) \in \mathcal{W}_2$ are well-defined and real analytic. Now, by definition of γ_k (see (1.4)), we have, for $(\underline{x}; \underline{x}') \in \Omega_0$,

$$\gamma_k(\underline{x}; \underline{x}') = \langle \Psi_1(\underline{x}; \underline{x}'; \cdot), \Psi_2(\underline{x}; \underline{x}'; \cdot) \rangle = \langle U_{\underline{x}; \underline{x}'}\Psi_1(\underline{x}; \underline{x}'; \cdot), U_{\underline{x}; \underline{x}'}\Psi_2(\underline{x}; \underline{x}'; \cdot) \rangle,$$

since $U_{\underline{x}; \underline{x}'}$ is unitary. As the scalar product of real analytic, \mathcal{W}_0 -valued maps on Ω_0 , γ_k is also real analytic on Ω_0 . \square

Remark 4.3. We note that our proofs of Theorems 1.3, 1.4, and 1.5, have a common structure. They all use an appropriate twist and Theorem 3.9. Differences between them occur in the used set of variables.

Remark 4.4. According to the first obstacle mentioned in Remark 4.2, the method of Section 3 requires to work on $\mathcal{U}_k^{(1)} \times \mathcal{U}_k^{(1)}$. Now, because of the second obstacle mentioned there, we also need to exclude the set $\mathcal{C}_k^{(2)}$ and, therefore, work on $\mathcal{U}_k^{(2)} = (\mathcal{U}_k^{(1)} \times \mathcal{U}_k^{(1)}) \setminus \mathcal{C}_k^{(2)}$.

5 Extensions.

In Section 3, we worked in a larger framework than the one that would be needed to treat the physical operator H in (1.1). It is thus natural to expect that our main results on H extend to a larger class of Hamiltonians. This is the case, as we shall see in this Section.

We consider again a system of N electrons, interacting to one another and moving under the influence of L fixed nuclei. Let $\mathcal{I} := \{1; \dots; N\}$ be the set of indices for the electronic variables $x \in \mathbb{R}^{3N}$ and let $\mathcal{I}_n := \{1; \dots; L\}$ be the one of the nuclear variables $R_\ell \in \mathbb{R}^3$. It is convenient to introduce the disjoint union of those sets, namely

$$\mathcal{I} \sqcup \mathcal{I}_n := \{(a; j); a \in \{e; n\}, j \in \mathcal{I}_a\}.$$

For $c = (a; j) \in \mathcal{I}_e \sqcup \mathcal{I}_n$, let $\pi_1(c) = a$ and $\pi_2(c) = j$. We define $X_c = x_j$ if $\pi_1(c) = e$ and $X_c = R_j$ if $\pi_1(c) = n$. Let

$$\tilde{\mathcal{P}} := \{\{c; c'\}; (c; c') \in (\mathcal{I}_e \sqcup \mathcal{I}_n)^2 \text{ and } c \neq c'\}$$

be the set of all possible particle pairings. We consider a potential \tilde{V} where

$$\forall x \in \mathbb{R}^{3N}, \quad \tilde{V}(x) = \sum_{\{c; c'\} \in \tilde{\mathcal{P}}} V_{\{c; c'\}}(X_c - X_{c'}), \quad (5.1)$$

with $V_{\{c; c'\}} \in \mathcal{V}$ (cf. Definition 3.1), for all $\{c; c'\} \in \tilde{\mathcal{P}}$.

Now, let $\tilde{P} = \tilde{P}(x; D_x)$ a differential operator on \mathbb{R}^{3N} of the form

$$\tilde{P} = \sum_{\substack{\alpha \in \mathbb{N}^{3N} \\ |\alpha| \leq 2}} \tilde{c}_\alpha(x) D_x^\alpha, \quad (5.2)$$

where the coefficients \tilde{c}_α belong to $\mathcal{C}_b^\infty(\mathbb{R}^{3N}; \mathbb{C})$. Let $\tilde{H} = \tilde{P} + \tilde{V}$ and assume that there exists $\tilde{\psi} \in W^{2,2}(\mathbb{R}^{3N})$ such that $\tilde{H}\tilde{\psi} = 0$. For $0 < k < N$, we still denote by ρ_k , γ_k , and \tilde{C} the objects that are respectively defined by (1.3), (1.4), and (1.5), with ψ replaced by $\tilde{\psi}$.

Theorem 5.1. *Under the above assumptions on \tilde{H} and $\tilde{\psi}$, the results of Theorems 1.3, 1.4, and 1.5, are valid.*

Proof: We can follow the above proofs of Theorems 1.3, 1.4, and 1.5. In the later, we use the elliptic operators

$$H_1 = -\Delta_{\underline{x}'} - \tilde{P}(\underline{x}; y; D_{\underline{x}}; D_y) + \tilde{V}(\underline{x}; y) \quad \text{and} \quad H_2 = -\Delta_{\underline{x}} - \tilde{P}(\underline{x}'; y; D_{\underline{x}'}; D_y) + \tilde{V}(\underline{x}'; y)$$

and the almost everywhere defined functions $\Psi_1 : \mathbb{R}^{3m} \times \mathbb{R}^{3p} \ni (\underline{x}; \underline{x}'; y) \mapsto \tilde{\psi}(\underline{x}; y)$ and $\Psi_2 : \mathbb{R}^{3m} \times \mathbb{R}^{3p} \ni (\underline{x}; \underline{x}'; y) \mapsto \tilde{\psi}(\underline{x}'; y)$. \square

Remark 5.2. *We point out that the present class of operators includes Laplace-Beltrami operators associated to appropriate metrics. One can also replace D_x by $D_x - A(x)$, where A is a real analytic vector potential in $\mathcal{C}_b^\infty(\mathbb{R}^{3N}; \mathbb{R}^3)$, and add an external electric potential V_e in $\mathcal{C}_b^\infty(\mathbb{R}^{3N}; \mathbb{C})$, that is also real analytic.*

Notice that the class \mathcal{V} in Definition 3.1 contains non-radial pair potentials. Among the admissible functions η used to define \mathcal{V} , we have $0 < t \mapsto t^{-1}$ but we can also choose $0 < t \mapsto t^{-1}(\ln(t))^{-\epsilon}$, with $\epsilon > 0$. We observe that this class \mathcal{V} is contained in the class of pair potentials used in [FS].

Appendix.

A Basic computations.

In this section, we provide explicit formulae for several objects related to the twist U in Subsection 3.3 and the computation of the principal symbol of some differential operator, that is needed in Subsection 3.5.

First of all, let us notice that, from the expressions of $d_x f$ and $d_z f$, there exists some τ , r_0 , and x^0 dependent constant M , such that, denoting by I_3 the identity on \mathbb{R}^3 ,

$$\forall x \in \mathbb{R}^{3m}, \forall z \in \mathbb{R}^3, |z| \geq M \implies ((d_x f)(x; z) = 0 \text{ and } (d_z f)(x; z) - I_3 = 0). \quad (\text{A.1})$$

Using (3.12), we can find some $c > 0$ such that

$$\forall (x; z) \in \Omega(\delta_0) \times \mathbb{R}^3, c^{-1} \leq \|((d_z f)(x; z))^{-1}\|_{\mathcal{L}(\mathbb{R}^3)} + \|(d_z f)(x; z)\|_{\mathcal{L}(\mathbb{R}^3)} \leq c. \quad (\text{A.2})$$

For $x \in \Omega(\delta_0)$, $j \in \llbracket 1; m \rrbracket$, and $z \in \mathbb{R}^3$, we have

$$f^{(-1)}(x; f(x; z)) = z.$$

By differentiation of this equality and by using (3.9), we obtain, with equalities in $\mathcal{L}(\mathbb{R}^3)$, in $\mathcal{L}(\mathbb{R}^3)$, and in $\mathcal{L}(\mathbb{R}^{3m}; \mathbb{R}^3)$, respectively,

$$(d_z f^{(-1)})(x; f(x; z)) = ((d_z f)(x; z))^{-1}, \quad (\text{A.3})$$

$$(d_{x_j} f^{(-1)})(x; f(x; z)) = -\tau(r_0^{-1}(z - x_j^0))(d_z f^{(-1)})(x; f(x; z)), \quad (\text{A.4})$$

$$(d_x f^{(-1)})(x; f(x; z)) = -(d_z f^{(-1)})(x; f(x; z)) \circ \sum_{j'=1}^m \tau(r_0^{-1}(z - x_{j'}^0)) dx_{j'}. \quad (\text{A.5})$$

Now, using (A.1), (A.2), (A.3), (A.4), and (A.5), it is straightforward to check that

$$f \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^3; \mathbb{R}^3) \quad \text{and} \quad f^{(-1)} \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^3; \mathbb{R}^3). \quad (\text{A.6})$$

From the very definition of F , we note that, for $x \in \Omega(\delta_0)$, $x^\epsilon \in \mathbb{R}^{3m}$, and $y \in \mathbb{R}^{3p}$,

$$(d_x F)(x; y) \cdot x^\epsilon = ((d_x f)(x; y_1) \cdot x^\epsilon; \dots; (d_x f)(x; y_p) \cdot x^\epsilon) \in \mathbb{R}^{3p}$$

and the k -th component (with $k \in \llbracket 1; p \rrbracket$) in the latter expression is given by

$$(d_x f)(x; y_k) \cdot x^\epsilon = \sum_{j=1}^m \tau(r_0^{-1}(y_k - x_j^0)) x_j^\epsilon \in \mathbb{R}^3,$$

by (3.9). For $x \in \Omega(\delta_0)$, $y \in \mathbb{R}^{3p}$, $z \in \mathbb{R}^3$, and $k \in \llbracket 1; p \rrbracket$, we have

$$(d_{y_k} F)(x; y) \cdot z = (0; \dots; 0; (d_z f)(x; y_k) \cdot z; 0; \dots; 0) \in \mathbb{R}^{3p}$$

(the term containing z being in the k th position) and, for $y^\epsilon \in \mathbb{R}^{3p}$,

$$\begin{aligned} (d_y F)(x; y) \cdot y^\epsilon &= \sum_{k=1}^p (0; \dots; 0; (d_z f)(x; y_k) \cdot y_k^\epsilon; 0; \dots; 0) \in \mathbb{R}^{3p} \\ &= y^\epsilon + r_0^{-1} \sum_{k=1}^p \sum_{j=1}^m \left(0; \dots; 0; ((d\tau)(r_0^{-1}(y_k - x_j^0)) \cdot y_k^\epsilon)(x_j - x_j^0); 0; \dots; 0 \right), \end{aligned}$$

by (3.11), each term containing $d\tau$ being in the k th position. Since we can write, for $x \in \Omega(\delta_0)$ and $y \in \mathbb{R}^{3p}$,

$$F^{(-1)}(x; y) = (f^{(-1)}(x; y_1); \dots; f^{(-1)}(x; y_p)) \in \mathbb{R}^{3p},$$

we have, for $x^\epsilon \in \mathbb{R}^{3m}$,

$$\begin{aligned} (d_x F^{(-1)})(x; F(x; y)) \cdot x^\epsilon \\ = ((d_x f^{(-1)})(x; f(x; y_1)) \cdot x^\epsilon; \dots; (d_x f^{(-1)})(x; f(x; y_p)) \cdot x^\epsilon) \in \mathbb{R}^{3p}, \end{aligned}$$

where $d_x f^{(-1)}$ is given by (A.5). Finally, for $y^\epsilon \in \mathbb{R}^{3p}$, we have

$$\begin{aligned} (d_y F^{(-1)})(x; F(x; y)) \cdot y^\epsilon \\ = \sum_{k=1}^p (0; \dots; 0; (d_z f^{(-1)})(x; f(x; y_k)) \cdot y_k^\epsilon; 0; \dots; 0) \in \mathbb{R}^{3p}, \end{aligned}$$

where $d_z f^{(-1)}$ is explicited in (A.3).

We derive from (A.6) that

$$F \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3p}) \quad \text{and} \quad F^{(-1)} \in \mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R}^{3p}). \quad (\text{A.7})$$

Next, we compute the terms $J_k(G)$, for $1 \leq k \leq 4$ and $G \in \{F; F^{(-1)}\}$, that appear in (3.16), (3.17), (3.18), and (3.19).

Let $G \in \{F; F^{(-1)}\}$. Let us consider the smooth functions $\rho_\pm \in \mathcal{C}^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R})$ defined by

$$\rho_+(x; y) = |\text{Det}(d_y G)(x; y)|^{1/2} \quad \text{and} \quad \rho_-(x; y) = |\text{Det}(d_y G^{(-1)})(x; y)|^{1/2}.$$

From the identity $G^{(-1)}(x; G(x; y)) = (x; y)$, we derive that $\rho_+(x; y)\rho_-(x; G(x; y)) = 1$. Let $\varphi \in \mathcal{C}^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{R})$. We set

$$(U_G \varphi)(x; y) = \rho_+(x; y) \varphi(x; G(x; y)) \quad \text{and} \quad (U_G^{(-1)} \varphi)(x; y) = \rho_-(x; y) \varphi(x; G^{(-1)}(x; y)).$$

Since

$$\begin{aligned} d_x (U_G^{(-1)} \varphi)(x; y) &= \varphi(x; G^{(-1)}(x; y)) d_x \rho_-(x; y) \\ &\quad + \rho_-(x; y) (d_x \varphi)(x; G^{(-1)}(x; y)) \\ &\quad + \rho_-(x; y) (d_y \varphi)(x; G^{(-1)}(x; y)) (d_x G^{(-1)})(x; y), \end{aligned}$$

we have, for $\mathbf{x}^\epsilon \in \mathbb{R}^{3m}$,

$$\begin{aligned} d_{\mathbf{x}}(U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y})\mathbf{x}^\epsilon &= \varphi(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \nabla_{\mathbf{x}}\rho_{-}(\mathbf{x}; \mathbf{y}) \cdot \mathbf{x}^\epsilon \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{x}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \cdot \mathbf{x}^\epsilon \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \cdot (d_{\mathbf{x}}G^{(-1)})(\mathbf{x}; \mathbf{y})\mathbf{x}^\epsilon, \end{aligned}$$

with scalar products in \mathbb{R}^{3m} , \mathbb{R}^{3m} , and \mathbb{R}^{3p} , respectively. This yields

$$\begin{aligned} \nabla_{\mathbf{x}}(U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y}) &= \varphi(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \nabla_{\mathbf{x}}\rho_{-}(\mathbf{x}; \mathbf{y}) \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{x}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (d_{\mathbf{x}}G^{(-1)})^T(\mathbf{x}; \mathbf{y})(\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})). \end{aligned}$$

Thus, using $\rho_{+}(\mathbf{x}; \mathbf{y})\rho_{-}(\mathbf{x}; G(\mathbf{x}; \mathbf{y})) = 1$,

$$\begin{aligned} (U_G \nabla_{\mathbf{x}} U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y}) &= \rho_{+}(\mathbf{x}; \mathbf{y})\varphi(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{x}}\rho_{-})(\mathbf{x}; G(\mathbf{x}; \mathbf{y})) + (\nabla_{\mathbf{x}}\varphi)(\mathbf{x}; \mathbf{y}) \\ &+ (d_{\mathbf{x}}G^{(-1)})^T(\mathbf{x}; G(\mathbf{x}; \mathbf{y}))(\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; \mathbf{y}). \end{aligned}$$

Since

$$\begin{aligned} d_{\mathbf{y}}(U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y}) &= \varphi(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) d_{\mathbf{y}}\rho_{-}(\mathbf{x}; \mathbf{y}) \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (d_{\mathbf{y}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) (d_{\mathbf{y}}G^{(-1)})(\mathbf{x}; \mathbf{y}), \end{aligned}$$

we have, for $\mathbf{y}^\epsilon \in \mathbb{R}^{3p}$,

$$\begin{aligned} d_{\mathbf{y}}(U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y})\mathbf{y}^\epsilon &= \varphi(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \nabla_{\mathbf{y}}\rho_{-}(\mathbf{x}; \mathbf{y}) \cdot \mathbf{y}^\epsilon \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \cdot (d_{\mathbf{y}}G^{(-1)})(\mathbf{x}; \mathbf{y})\mathbf{y}^\epsilon, \end{aligned}$$

with scalar products in \mathbb{R}^{3p} . This yields

$$\begin{aligned} \nabla_{\mathbf{y}}(U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y}) &= \varphi(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})) \nabla_{\mathbf{y}}\rho_{-}(\mathbf{x}; \mathbf{y}) \\ &+ \rho_{-}(\mathbf{x}; \mathbf{y}) (d_{\mathbf{y}}G^{(-1)})^T(\mathbf{x}; \mathbf{y})(\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; G^{(-1)}(\mathbf{x}; \mathbf{y})). \end{aligned}$$

Thus, using $\rho_{+}(\mathbf{x}; \mathbf{y})\rho_{-}(\mathbf{x}; G(\mathbf{x}; \mathbf{y})) = 1$,

$$\begin{aligned} (U_G \nabla_{\mathbf{y}} U_G^{(-1)}\varphi)(\mathbf{x}; \mathbf{y}) &= \rho_{+}(\mathbf{x}; \mathbf{y})\varphi(\mathbf{x}; \mathbf{y}) (\nabla_{\mathbf{y}}\rho_{-})(\mathbf{x}; G(\mathbf{x}; \mathbf{y})) \\ &+ (d_{\mathbf{x}}G^{(-1)})^T(\mathbf{x}; G(\mathbf{x}; \mathbf{y}))(\nabla_{\mathbf{y}}\varphi)(\mathbf{x}; \mathbf{y}). \end{aligned}$$

This proves (3.16), (3.17), (3.18), and (3.19). Now, taking into account (A.7), we obtain (3.20) and (3.21). Using (A.2) and (A.3), we see that (3.23) holds true.

Finally, we compute the principal symbol σ_P appearing in the proof of Proposition 3.6. In the vector space of polynomials in $\mathbb{R}^{3m} \times \mathbb{R}^{3p}$ with $\mathcal{C}_b^\infty(\Omega(\delta_0) \times \mathbb{R}^{3p}; \mathbb{C})$ -coefficients, we denote by \equiv the equality of polynomials modulo polynomials of degree less than 2. Then,

for $(x; y; \xi; \eta) \in \Omega(\delta_0) \times \mathbb{R}^{3p} \times \mathbb{R}^{3m} \times \mathbb{R}^{3p}$, we have, by (3.29), (3.17), and (3.19),

$$\begin{aligned}
& \sigma_P(x; y; \xi; \eta) \\
& \equiv S_P(x; y; \xi; \eta) \\
& \equiv \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| = 2}} c_{\alpha\beta}(x; F(x; y)) \left(\xi + J_1(F^{(-1)})(x; y) \eta + J_2(F^{(-1)})(x; y) \right)^\alpha \\
& \quad \times \left(J_3(F^{(-1)})(x; y) \eta + J_4(F^{(-1)})(x; y) \right)^\beta \\
& \equiv \sum_{\substack{(\alpha; \beta) \in \mathbb{N}^{3m} \times \mathbb{N}^{3p} \\ |\alpha| + |\beta| = 2}} c_{\alpha\beta}(x; F(x; y)) \left(\xi + J_1(F^{(-1)})(x; y) \eta \right)^\alpha \left(J_3(F^{(-1)})(x; y) \eta \right)^\beta
\end{aligned}$$

and the last term is actually $\sigma_P(x; y; \xi; \eta)$, since it is homogeneous of degree 2. Thus

$$\sigma_P(x; y; \xi; \eta) = \sigma_P(x; F(x; y); \xi + J_1(F^{(-1)})(x; y) \eta; J_3(F^{(-1)})(x; y) \eta),$$

yielding (3.30).

B A global pseudo-differential calculus.

In this section, we describe a global pseudo-differential calculus that is suited for Section 2. It is the one developed in the beginning of Chapter 18 in [Hö2].

With the notation of Subsection 2.2, we set $N = m + p$. For $k \in \mathbb{Z}$, the class S^k in [Hö2] (p. 65-75) is the set of smooth, complex valued functions σ on \mathbb{R}^{6N} such that, for all $(\alpha; \beta) \in (\mathbb{N}^{3N})^2$, there exists a constant $C_{\alpha; \beta} > 0$ such that, for all $(x; y; \xi; \eta) \in \mathbb{R}^{6N}$,

$$(1 + |\xi|^2 + |\eta|^2)^{|\beta|/2} |\partial_{x; y}^\alpha \partial_{\xi; \eta}^\beta \sigma(x; y; \xi; \eta)| \leq C_{\alpha; \beta} (1 + |\xi|^2 + |\eta|^2)^{k/2},$$

where $\partial_{x; y}^\alpha$ (resp. $\partial_{\xi; \eta}^\beta$) stands for the α (resp. β) partial derivative w.r.t. $(x; y)$ (resp. $(\xi; \eta)$). Any such symbol σ defines an operator $\sigma(x; y; D_x; D_y)$ from the Schwarz space \mathcal{S} on \mathbb{R}^{3N} into itself in the following linear way: for any $u \in \mathcal{S}$, for any $(x; y) \in \mathbb{R}^{3N}$,

$$(\sigma(x; y; D_x; D_y)u)(x; y) = (2\pi)^{-3N} \int_{\mathbb{R}^{3N}} e^{i(x; y) \cdot (\xi; \eta)} \sigma(x; y; \xi; \eta) \hat{u}(\xi; \eta) d\xi d\eta,$$

where $(x; y) \cdot (\xi; \eta)$ is the usual scalar product in \mathbb{R}^{3N} of the vectors $(x; y)$ and $(\xi; \eta)$ and where \hat{u} denotes the Fourier transform of u . It turns out that, for any $s \in \mathbb{R}$, this operator $\sigma(x; y; D_x; D_y)$ extends to a bounded operator from $W^{s, 2}(\mathbb{R}^{3N})$ to $W^{s-k, 2}(\mathbb{R}^{3N})$. If σ is the constant function equal to one on \mathbb{R}^{3N} then $\sigma \in S^0$ and $\sigma(x; y; D_x; D_y)$ is the identity operator I .

For $(k_1; k_2) \in \mathbb{Z}^2$, for $\sigma_1 \in S^{k_1}$ and $\sigma_2 \in S^{k_2}$, the composition $\sigma_1(x; y; D_x; D_y) \sigma_2(x; y; D_x; D_y)$ is given by $\sigma(x; y; D_x; D_y)$ where $\sigma \in S^{k_1+k_2}$ and $\sigma = \sigma_1 \sigma_2 + r$ with $r \in S^{k_1+k_2-1}$.

Now, we are in a position to construct the parametrix Q needed in Subsection 2.2. We observe that $\widehat{P} = \sigma(x; y; D_x; D_y)$ where its symbol σ belongs to S^2 . Furthermore, one can write $\sigma = \sigma_P + r$ with $\sigma_P \in S^2$ and $r \in S^1$. Let $\tau_P \in \mathcal{C}_c^\infty(\mathbb{R}^{3N})$ such that $\tau(\xi; \eta) = 1$ if $|\xi|^2 + |\eta|^2 \leq 1$. By the ellipticity assumption of \widehat{P} (see (2.6)), the symbol

$$q_1 := (1 - \tau_P) \cdot \sigma_P^{-1} \in S^{-2}$$

and, setting $Q_1 = q_1(x; y; D_x; D_y)$, there exist symbols $r_1, r_2, r_3 \in S^{-1}$ such that

$$\begin{aligned} Q_1 \widehat{P} &= (\sigma q_1)(x; y; D_x; D_y) + r_1(x; y; D_x; D_y) \\ &= (\sigma_P q_1)(x; y; D_x; D_y) + r_2(x; y; D_x; D_y) \\ &= I + r_3(x; y; D_x; D_y). \end{aligned}$$

Let $Q = (I - r_3(x; y; D_x; D_y))Q_1$. Then, $Q = q(x; y; D_x; D_y)$ with $q \in S^{-2}$ and there exists $r_4 \in S^{-2}$ such that $Q \widehat{P} = I - r_4(x; y; D_x; D_y)$. Setting $R := r_4(x; y; D_x; D_y)$, we have $Q \widehat{P} = 1 - R$ and (2.10) holds true.

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